What's needed for doing *good* Theoretical Physics

Erik Pillon

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Abstract

This work had been inspired by the work https://www.staff.science.uu.nl/ gadda001/goodtheorist/index.html

Introduction

This is a fairly detailed collection of all the subjects, concepts, theorems and tools I'we studied during my educational trip. There is no request of completeness as well as this "few" concepts must be intended as a leading guide for the student that is approaching mathematics and/or physics from scratch.

1 Elementary Math

- Sum and Difference.
- Multiplication and Division.
- Fraction.
- Powers.
- Logarithm and exponential.

2 Logic

- Unary operators: *not*: $\neg$; tautology: $\top$
- Binary operators: and, or, nor, nand, ...
- Equivalence Class
- Axiom of Choice

3 Linear Algebra

- CayleyâĂŞHamilton theorem

4 Single-valued Real Analysis

- Topology, open and closed sets.

5 Mechanics

6 Thermodynamics

7 Algorithms and Computations

- Computational cost.
- List, Stack, Arrays, Queries.
- Bubble sort, Merge Sort, Quick Sort.

8 Multi-valued Analysis

- Partial derivatives, derivatives vector ∇.
- Curl and Divergence.

9 Numerical Analysis

10 Algebra

Even if this (sometimes *very*) abstract subject is considered almost selfcontained and useful only for very narrow fields purposes, I wanted to be very precises in the subsections in order to underline the extreme importance that Algebra and (Lie) Groups in general has assumed in the last few decades. The ones listed belows are nowaday unavoidable topics for future theoreticians willing to completely understand all the Symmetry Properties in Nature, from Special and General Relativity to Classical and Quantum Mechanics as well as Particle Physics.

10.1 Group Theory

- Cayley Diagram
- Generators
- Klein Group
- Cyclic Group
- Abelian Group
- Dehydral Group
- Coset
- Normal Subgroup
- Quotient Group
- Semidirect Product
- Group Representation
- Irreducible Group Class

11 Geometry

It is not uncommon today for a physicist's mathematical education to ignore all but the simplest geometrical ideas, despite the fact that young physicists are encouraged to develop mental 'pictures' and 'intuition' appropriate to physical phenomena. This curious neglect of 'pictures' of one's mathematical tools may be seen as the outcome of a gradual evolution over many centuries. Geometry was certainly extremely important to ancient and medieval natural philosophers; it was in geometrical terms that Ptolemy, Copernicus, Kepler, and Galileo all expressed their thinking. But when Descartes introduced coordinates into Euclidean geometry, he showed that the study of geometry could be regarded as an application of algrebra. Since then, the importance of the study of geometry in the education of scientists has steadily declined, so that at present a university undergraduate physicist or applied mathematician is not likely to encounter much geometry at all.

One reason for this suggests itself immediately: the relatively simple geometry of the three-dimensional Euclidean world that the nineteenth-century physicist believed he lived in can be mastered quickly, while learning the great diversity of analytic techniques that must be used to solve the differential equations of physics makes very heavy demands on the student's time. Another reason must surely be that these analytic techniques were developed at least partly in response to the profound realization by physicists that the laws of nature could be expressed as differential equations, and this led most mathematical physicists genuinely to neglect geometry until relatively recently.

However, two developments in this century have markedly altered the balance between geometry and analysis in the twentieth-century physicist's outloook. The first is the development of the theory of relativity, according to which the Euclidean three-space of the nineteenth-century physicist is only an approxi- mation to the correct description of the physical world. The second development, which is only beginning to have an impact, is the realization by twentieth-century mathematicians, led by Cartan, that the relation between geometry and analysis is a two-way street: on the one hand analysis may be the foundation of the study of geometry, but on the other hand the study of geometry leads naturally to the development of certain analytic tools (such as the Lie derivative and the exterior calculus) and certain concepts (such as the manifold, the fiber bundle, and the identification of vectors with derivatives) that have great power in applications of analysis. In the modern view, geometry remains subsidiary to analysis. For example, the basic concept of differential geometry, the differentiable manifold, is defined in terms of real numbers and differentiable functions. But this is no disadvantage: it means that concepts from analysis can be expressed geometri- cally, and this has considerable heuristic power.

Because it has developed this intimate connection between geometrical and analytic ideas, modern differential geometry has become more and more important in theoretical physics, where it has led to a greater simplicity in the mathematics and a more fundamental understanding of the physics. This revolution has affected not only special and general relativity, the two theories whose content is most obviously geometrical, but other fields where the geometry involved is not always that of physical space but rather of a more abstract space of variables: electromagnetism, thermodynamics, Hamiltonian theory, fluid dynamics, and elementary particle physics.

- Topological space
- trivial topology
- discrete topology
- cofinite topology
- Neighbourhoods
- Closure
- Continuous applications
- Homeomorphisms
- Limit points and isolated points
- Dense set
- Topological subspace
- induced topology
- Product spaces
- Separation axioms

- Hausdorff spaces
- Normal spaces
- Regular spaces
- Countability axioms
- Quotient space
- Open and closed applications
- Relevant examples: sphere, projective space, Moebius strip
- Compactness
- Heine-Borel Theorem
- Tychonoff Theorem
- Bolzano-Weierstrass Theorem
- Connectivity, local connectivity
- Path connectivity
- Simply connected
- Homotopy and fundamental group
- Jordan curve Theorem
- Embedding and immersion.
- Vector fields along a curve
- Tangent vector and line
- Length of an arc
- Parametrization by arc-length
- Inflection points
- Curvature and radius of curvature
- Center of curvature
- Frenet-Serret formula
- Tangent line
- Normal plane.
- Inflection points.
- Osculator plane.
- Curvatures.
- Principal frame.
- Frenet-Serret formula.
- Torsion.
- Fundamental Theorem.
- Differentiable atlas.

- Oriented atlas
- Tangent plane
- Normal versor.
- First fundamental quadratic form: metric and area.
- Tangential curvature and normal curvature of a curve on a surface.
- Curvatures
- normal sections
- Meusnier Theorem.
- Principal curvatures
- Gaussian curvature and mean curvature: Theorem Egregium.
- Geodetics.

References

[1] Manfredo P Do Carmo. "Differential geometry of surfaces". In: *Differential forms and applications*. Springer, 1994, pp. 77–98.

[2] Manfredo P Do Carmo et al. "Differential Geometry". In: *Mathematical Models*. Springer, 2017, pp. 155–180.

[3] Bernard F. Schutz. *Geometrical methods of mathematical physics*.

12 Probability

13 Dynamical Systems

14 Electromagnetism

15 Fluid Dynamics

16 Wave Mechanics

- Wave Equation $\partial_{tt} u - c^2 \Delta u = 0$
- Planar wave
- Poynting Vector

17 Complex Analysis

18 Numerical Analysis for (Partial) Differential Equations

19 Stochastic Processes

- Conditional expectations and conditional laws
- Filtered probability space, filtrations
- Adapted stochastic process (wrt a given filtration)
- Martingale (Markov chains)

- Kolmogorov characterization theorem
- Stopping times
- Definition of martingale process, resp. super, resp. lower, martingale
- Stopping times for martingale processes
- Convergence theorems for martingales
- Markov chains (MC)
- Transition matrix for a MC
- Construction and existence for MC
- Omogeneous MC (with respect to time and space)
- Canonical MC
- Classification of states for a given MC (and associated classes)
- Chapman-Kolmogorov equation
- Recurrent, resp. transient, states (classification criteria)
- Irriducible and recurrent chains
- Invariant (stationary) measures, ergodic measures, limit measures (Ergodic theorem)
- Birth and death processes (discrete time)
- Continuous time MC
- Absolute and stationary distributions
- Probability and rates of transition
- Kolmogorov differential equations
- Stationary laws
- Birth and death processes (first steps in continuous time)
- Queuing theory (first steps in continuous time)
- Point, Counting and Poisson Processes
- Stochastic point processes (SPP) and Stochastic Counting Processes (SCP)
- Stationarity, intensity and composition for SPP and SCP
- Homogeneous Poisson Processes (HPP)
- Non Homogeneous Poisson Processes (nHPP)
- Mixed Poisson Processes (MPP)
- Birth and Death processes (B&D)
- Time-dependent state probabilities
- Stationary state probabilities
- Inhomogeneous B&D processes

References

[1] Paolo Baldi. *Calcolo delle probabilita*. McGraw-Hill, 2007.

[2] Frank Beichelt. *Stochastic processes in science, engineering and finance*. CRC Press, 2006.

20 Differential Geometry

- Differentiable Atlas
- Orientable Atlas
- Tangent plane
- Normal versor
- First Fundamental Form: lengths and area
- Geodesic curvature and normal curvature
- Normal sections and Meusnier Theorem
- Principal Curvatures, Gaussian curvature, Mean curvature: minimal surfaces
- Theorema Egregium
- Geodetics
- Free vector space
- Tensor product of two vector spaces
- Tensor product of n vector spaces
- Tensor Algebra
- Transformation of the componenents of a tensoriale
- Mixed tensors
- Symmetric tensors
- Antysimmetric (alternating) tensors
- Exterior Algebra
- Determinant
- Area and Volume
- Definition and examples
- Classification of 1-manifolds
- Classification of simply-connected 2-manifolds
- Product and quotient spaces
- Differentiable maps
- Tangent space and tangent bundle
- Vector field on a manifold
- Tensor field
- Exterior Algebra on manifolds
- Riemannian Manifolds

- Metric Tensor
- Orientations
- Volume
- Exterior derivative
- De Rham Cohomology
- Homotopy
- Affine connection
- Parallel transport
- Levi-Civita connection
- Geodetics
- Riemann curvature tensor
- Bianchi identities

References

[1] Manfredo P Do Carmo et al. "Differential Geometry". In: *Mathematical Models*. Springer, 2017, pp. 155–180.

[2] Bo-Yu Hou and Bo-Yuan Hou. *Differential geometry for physicists*. Vol. 6. World Scientific Publishing Co Inc, 1997.

21 Functional Analysis

- L^p spaces
- Riesz Lemma
- Fredholm Alternative

References

[1] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.

22 Mathematical tools for Physics

- Eigenfunctions for the cube and for the cylinder
- Bessel Equation
- Bessel Functions of the first and second kind: $J_\alpha(x)$ and $Y_n(x)$
- Fourier-Bessel Series
- Eigenfunction for the sphere
- Laplacian in Spherical Coordinates
- Legendre Equation
- Legendre Polynomials
- Rodriguez Formula

- Fourier-Legendre Series
- Recurrence Relations
- Associated Legendre Functions
- Spherical harmonics $Y_{lm}(\theta, \varphi)$
- Perturbation Theory
- Fourier Transform in $\mathbb{R}^n$
- Green Function(s)
- Spectral Representation of Green (homogeneous) Functions
- 1^{st} and 2^{nd} Green Formulas
- Laguerre Polynomials

23 Statistical Mechanics

- Fundamental Assumption of Equilibrium Statistical Mechanics
- Accessible Macrostate
- Liouville Equation $\frac{\partial \rho}{\partial t} = \{H, \rho\}$
- Microcanonical Ensemble
- Stirling's Formula
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Recommended Books

[1] Charles Kittel and Herbert Kroemer. *Thermal physics*. 1998.

24 Solid State Physics

- Boltzman Model
- Einstein Model
- Bebye Model
- Drude Theory

Recommended Books

[1] Charles Kittel. *Introduction to solid state physics*. Wiley, 2005.

25 Nuclear Physics

26 Partial Differential Equations

- Characteristics Method

References

[1] Sandro Salsa. *Equazioni a derivate parziali: Metodi, modelli e applicazioni*. Vol. 98. Springer, 2016.

[2] Sandro Salsa. *Partial differential equations in action: from modelling to theory*. Vol. 99. Springer, 2016.

27 Stochastic Differential Equations

- Itô Integral

28 Advanced Numerical Analysis

29 Analytical Mechanics

30 Quantum Mechanics

- Schrödinger Equation
- Probability Density $\partial_t \psi^* \psi$
- Probability Current Density $\vec{\nabla} \cdot \vec{S}$
- Infinity conditions for the wave function
- Stationary States for a quantum mechanical system
- Klein-Gordon Equation
- Schrödinger Solution as a Markov process
- Simple Harmonic Oscillator
- Ladder Operators a and $a^\dagger$
- Hermite Differential Equation
- 1D Square Well Potential
- Forbidden Regions
- Square Potential Barriers
- Tunneling Effect
- Particle in the box
- Concept of classical limit $\hbar \to 0$
- Gauge Transformations and Landau Gauge
- Landau Levels
- Spherical Harmonics
- Pseudo-vectors/Axial Vectors
- Spin-Orbit Coupling
- Shell Model of the Nucleus
- Loosely bound states
- Isopartner fermions
- Generalized Pauli Principle

- Interacting Harmonic Oscillators
- Schrödinger Equation on a circle
- Dinamics of a particle in a box
- Schrödinger Picture
- Heisenberg Picture

31 Nonequilibrium Statistical Mechanics

Nonequilibrium statistical mechanics deals with the issue of microscopically modelling the speed of irreversible processes that are driven by imbalances. Examples of such processes include chemical reactions or flows of particles and heat. Unlike with equilibrium, there is no exact formalism that applies to non-equilibrium statistical mechanics in general, and so this branch of statistical mechanics remains an active area of theoretical research.

- Einstein-Smoluchowski relation $D = \mu k_B T$
- Stokes-Einstein equation $D = \frac{k_\mathrm{B} T}{6\pi\,\eta\,r}$
- Ornstein-Uhlenbeck process
- GreenâĂŞKubo relations for transport coefficients γ
- Diffusion Tensor D_{ij}

32 Advanced Quantum Theory

- Coupling Basis
- Clebsch-Gordan Coefficients
- Isospin
- Coherent State
- Displacement Operator
- Squeezing Operator
- Cross section amplitude coefficient σ

33 Quantum Field Theory

34 Advanced Quantum Field Theory