Lorentz Transformations: a Group Theory approach

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"The fact that mathematics does such a good job of describing the Universe is a mystery that we don't understand. And a debt that we will probably never be able to repay."

William Thomson, 1^{st} Baron Kelvin

Abstract

In this work we will focus on Lorentz transformations, that are coordinate transformations between two coordinate frames that move at constant velocity relative to each other.We recall that the term "Lorentz transformations" only refers to transformations between inertial frames, usually in the context of special relativity. They supersede the *Galilean transformation* of Newtonian mechanics, which assumes an absolute space and time. The Galilean transformation is a good approximation only at relative speeds much smaller than the speed of light. Lorentz transformations have a number of unintuitive features that do not appear in Galilean transformations.

In this work we will derive these transformations focusing on Group Theory properties.

1 Introduction

Traditionally, two postulates are put at the beginning of Special Relativity, from which all other results can be derived:

(i) The Principle of Relativity

(ii) The constancy of the speed of light

From these principles the Lorentz transformation may be derived in numerous ways. This approach does not concentrate on a single Lorentz transformation but works with the totality of all transformations admitted by the principle of relativity. We therefore here set out to derive the Lorentz transformation in a manner that takes into account this central role of principle (i), and take (ii) only to decide between the numbers $-1, 0$, and 1.

2 The Lorentz Transformations

Consider a number of labs in free flight. Such a lab defines an inertial system I. Each (pointlike) event may be recorded by noting its coordinates x, y, z with respect to a rectangular Cartesian coordinate system anchored in I together with the reading t of a clock attached to 1. We shall term this setup an inertial reference frame, and we restrict to positively oriented coordinate axes at the moment. It is useful to consider t, x, y, z as four coordinates $x^i = (x^O, x^l, x^2, x^3) := (t, x, y, z)$. Time thus appears, at first in a purely formal manner, as a fourth ($'zeroth'$) coordinate.

Our next task is to find the relation between different inertial frames. If I is *inertial*, then from experience we know that a reference frame I is again inertial if with respect to I it is

1. parallely displaced by $\mathbf{a}$
2. rotated by α
3. moving at constant velocity $\mathbf{v}$
4. time delayed by a^O.

Here α is the rotation vector and a^O is the time lag between the clocks attached to the two systems; parallel displacement and rotation refer to Euclidean Geometry, valid by experience in every inertial system.[1] We are thus looking for the transformation[2]

$$x^{\bar{i}} = f^i(x^k). \tag{1}$$

The infinite number of transformations must be restricted to the requirement that straight world lines with respect to I have to be transformed into straight world lines with respect to $\bar{I}$ by (1). It is reasonable also to require that finite coordinate values are always transformed to finite ones: it is then well-known that transformations with these properties are given by *affine transformations*:

$$x^{\bar{i}} = L^i_k x^k + a^i, \qquad i \in \{0, 1, 2, 3\}. \tag{2}$$

2.1 Consequences of relativity Principle

Since there are no restrictions on the space-time translations a^i, we will consider here only the homogeneous transformations, eq. (1) with a $i = 0$, and take up translations only later. As we have stated, there are no absolute directions and velocities. As a consequence, the relation between I and $\bar{I}$, and thus the matrix L^i_k must be expressible by the *axial vector* α describing the relative angular orientation between their spatial axes, together with the *polar vector* $\mathbf{v}$ of relative velocity.

If there is only a relative rotation between the systems, L^i_k has to be formed from the rotation vector α alone. In this case, the coefficient L^i_k in eq. (2) has the form

$$L^i_k(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & R^\mu_\nu(\alpha) \end{bmatrix}$$

where $R^\mu_\nu(\alpha)$ is a rotation matrix, which rotates any vector in one sense (active transformation), or equivalently the coordinate frame in the opposite sense (passive transformation).

However, if the systems differ only by uniform rectilinear relative motion, then only v is at our disposal for constructing L^i_k, and the transformation must look like

$$\begin{cases} x^{\bar{0}} = a(v)x^0 + b(v)\mathbf{v}\mathbf{x} \\ \bar{\mathbf{x}} = c(v)\mathbf{x} + \frac{d(v)}{v^2}\mathbf{v}(\mathbf{v}x) + e(v)\mathbf{v}x^0. \end{cases}$$

A first restriction for the unknown functions $a(v), b(v), c(v), d(v)$ and $e(v)$ comes from the condition that the origin of I be moving with velocity $\mathbf{v}$ relative to $\bar{I}$, which means that $\mathbf{x} = \mathbf{v}x^0$ must imply $\bar{x} = 0$, and this is the case if

$$c(v) + d(v) + e(v) = O. \tag{3}$$

[1] One does not, however, obtain new inertial systems by considering systems accelerated against I.

[2] Formally, the relation between inertial frames $I, \bar{I}$ is described by specifying, for each event x, the relation between its coordinates x^i with respect to I and its coordinates $x^{\bar{i}}$ with respect to I.

Further conditions for the unknown functions now follow from the principle of relativity. Let us exchange the roles of I and $\bar{I}$: then $\bar{I}$ is moving against the latter with velocity $\mathbf{v} = -\mathbf{v}$.

Plugging the equations obtained by putting $\mathbf{v} = -\mathbf{v}$ into eq. (2), we will obtain an identity only if

$$C^2 = 1, \quad a^2 - ebv^2 = 1, \quad e^2 - ebv^2 = 1, \quad e(a+e) = 0, \quad b(a+e) = 0, \tag{4}$$

as is best checked by specializing $=(v, 0, 0)^T$.

The value $c = -1$ would correspond to a 180- rotation contained in (4) and has to be excluded here. From the third equality of eqs.(4) we have $e \neq 0$, hence $a + e = 0$ from the fourth. This satisfies the fifth also, and the second and third become equivalent. Thus we have

$$b = \frac{1-a^2}{av^2}, \qquad c = 1, \qquad d = a - 1, \qquad e = -a. \tag{5}$$

The only yet unknown function $a(v)$ will finally result from the application of the principle of relativity to three inertial frames $I, \bar{I}, \bar{\bar{I}}$, where $\bar{I}$ is moving with $\mathbf{v}$ against I and $\bar{\bar{I}}$ moving with speed $\bar{\mathbf{w}}$ against $\bar{I}$. If here $\mathbf{v}$ and $\bar{\mathbf{w}}$ are proportional, the relation between $\bar{\bar{I}}$ and I has again to be a pure *'boost'* in the same direction.

Putting v$\mathbf{v}$ and $\bar{\mathbf{w}}$ into the 1-directions, the product of the transformations must take the form of

$$\begin{cases} x^{\bar{0}} = a(u)x^0 + \frac{1-a^2(u)}{ua(u)}x^1, \\ x^{\bar{1}} = a(u)x^1 - ua(u)x^0, \\ x^{\bar{2}} = x^2, \qquad x^{\bar{3}} = x^3. \end{cases} \tag{6}$$

for some u. Comparing coefficients, we obtain two expressions for $a(u)$; equating them gives

$$\frac{1-a^2(v)}{v^2a^2(v)} = \frac{1-a^2(\bar{w})}{\bar{w}^2a^2(\bar{w})} = K, \tag{7}$$

Here K is a constant which is the same for each pair of inertial systems-hence it is universal. Solving eq. (7) for $a^2(v)$ we obtain finally the relation between I and $\bar{I}$

$$\begin{cases} x^{\bar{0}} = a(v)(x^0 + K\mathbf{v}\mathbf{x}) \\ \bar{\mathbf{x}} = \mathbf{x} + \frac{a(v)-1}{v^2}\mathbf{v}(\mathbf{v}\mathbf{x}) - a(v)\mathbf{v}x^0 \end{cases} \tag{8}$$

where $a^2(v) := (1 + Kv^2)^{-1}$ We see that the principle of relativity almost completely fixes the transformation, only a universal constant K (and the sign of $a(v)$) remaining undetermined.

3 Invariance of the Speed of Light.

The yet undetermined constant K has the physical dimension of reciprocal velocity squared. To interpret it we remark that the foundamental relation holds

$$(dx^0)^2 + K(d\mathbf{x})^2 = (dx^{\bar{0}})^2 + K(d\bar{\mathbf{x}})^2$$

As a consequence, for any motion $\mathbf{x} = \mathbf{x}(x^0)$ satisfying $(d\mathbf{x}/dx^O)^2 = -1/K$ in one inertial system the analogous relation is true in any other inertial system. Therefore, $c := 1/\sqrt{-K}$ plays the role of a uniquely determined *invariant speed*.[3]

In what follows, we shall most of the time assume performed the rescaling indicated above, and use units where $c = 1$-i.e., speeds are expressed as multiples of c.

[3]It is an experimental question whether such exists in nature, and if so, what is its value.

Then we have

$$K = -1, \qquad a(v) = \frac{1}{\sqrt{1-v^2}} =: \gamma, \qquad \frac{\gamma - 1}{v^2} \equiv \frac{\gamma^2}{\gamma + 1}, \tag{9}$$

and eq. (7) becomes the (special) Lorentz transformation (*'Lorentz boost'*)

$$\begin{cases} x^{\bar{0}} = \gamma(x^0 - \mathbf{v}\mathbf{x}) \\ \bar{\mathbf{x}} = \mathbf{x} + \frac{\gamma^2}{\gamma+1}\mathbf{v}(\mathbf{v}\mathbf{x}) - \gamma\mathbf{v}x^0. \end{cases} \tag{10}$$

By composing space-time translations, space rotations and Lorentz boosts in various ways we get more complicated transformations. Homogeneous ones will be called *(general) Lorentz transformations*, inhomogeneous ones will be called *Poincaré transformations*.

References

[1] Roman U Sexl and Helmuth K Urbantke. *Relativity, groups, particles: special relativity and relativistic symmetry in field and particle physics*. Springer Science & Business Media, 2012.