Lorentz Transformations: a Group Theory approach

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"The fact that mathematics does such a good job
of describing the Universe is a mystery that we don't understand.
And a debt that we will probably never be able to repay."

William Thomson, 1^{st} Baron Kelvin

Abstract

In this work we will focus on Lorentz transformations, that are coordinate transformations between two coordinate frames that move at constant velocity relative to each other.We recall that the term "Lorentz transformations" only refers to transformations between inertial frames, usually in the context of special relativity. They supersede the *Galilean transformation* of Newtonian mechanics, which assumes an absolute space and time. The Galilean transformation is a good approximation only at relative speeds much smaller than the speed of light. Lorentz transformations have a number of unintuitive features that do not appear in Galilean transformations.

In this work we will derive these transformations focusing on Group Theory properties.

1 Introduction

Traditionally, two postulates are put at the beginning of Special Relativity, from which all other results can be derived:

(i) The Principle of Relativity

(ii) The constancy of the speed of light

From these principles the Lorentz transformation may be derived in numerous ways. This approach does not concentrate on a single Lorentz transformation but works with the totality of all transformations admitted by the principle of relativity. We therefore here set out to derive the Lorentz transformation in a manner that takes into account this central role of principle (i), and take (ii) only to decide between the numbers $-1, 0$, and 1.

2 The Lorentz Transformations

Consider a number of labs in free flight. Such a lab defines an inertial system I. Each (pointlike) event may be recorded by noting its coordinates x, y, z with respect to a rectangular Cartesian coordinate system anchored in I together with the reading t of a clock attached to 1. We shall term this setup an inertial reference frame, and we restrict to positively oriented coordinate axes at the moment. It is useful to consider t, x, y, z as four coordinates $x^i = (x^O, x^l, x^2, x^3) := (t, x, y, z)$. Time thus appears, at first in a purely formal manner, as a fourth ('*zeroth*') coordinate.

Our next task is to find the relation between different inertial frames. If I is *inertial*, then from experience we know that a reference frame I is again inertial if with respect to I it is

1. parallely displaced by $\mathbf{a}$
2. rotated by α
3. moving at constant velocity $\mathbf{v}$
4. time delayed by a^O.

Here α is the rotation vector and a^O is the time lag between the clocks attached to the two systems; parallel displacement and rotation refer to Euclidean Geometry, valid by experience in every inertial system.[1] We are thus looking for the transformation[2]

$$x^{\bar{i}} = f^i(x^k). \tag{1}$$

The infinite number of transformations must be restricted to the requirement that straight world lines with respect to I have to be transformed into straight world lines with respect to $\bar{I}$ by (1). It is reasonable also to require that finite coordinate values are always transformed to finite ones: it is then well-known that transformations with these properties are given by *affine transformations*:

$$x^{\bar{i}} = L^i_k x^k + a^i, \qquad i \in \{0, 1, 2, 3\}. \tag{2}$$

References

[1] Roman U Sexl and Helmuth K Urbantke. *Relativity, groups, particles: special relativity and relativistic symmetry in field and particle physics*. Springer Science & Business Media, 2012.

[1] One does not, however, obtain new inertial systems by considering systems accelerated against I.

[2] Formally, the relation between inertial frames $I, \bar{I}$ is described by specifying, for each event x, the relation between its coordinates x^i with respect to I and its coordinates $x^{\bar{i}}$ with respect to I.