

Finite Elements Project

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Abstract

This is the final project of the course **Numerical Methods**, given at Phelma in the Academic Year 2017/2018. In this project we will study a process of material elaboration involving heating by electrode. We'll study the steady state of this process. The objective is to model the physical phenomena which take place in this process with *finite element method*.

1 Statement of the Problem

The study configuration is considered cylindrical. A scheme of the study geometry is given in Figure 1: The process is constituted by:

- a cylindrical crucible;
- the elaborated material; the geometry of the study domain occupied by the material is cylindrical;
- 2 electrodes.

The electrodes are in graphite. An electrical potential difference is applied between the top electrode and the bottom electrode included in the crucible: ΔU . This electrical potential difference is *continuous*. An electrical current pass through the material placed in the crucible. We suppose that the contact between the electrodes and elaborated material is perfect. **Joule effect heat the material**. The material of the crucible is an insulating material. The crucible is not model and will be replaced by an adapted boundary condition. Electrical problem has to be solved in the electrodes and in the elaborated material. The heat transfer has to be solved in the material only.

2 Equations of the physical phenomenon and Boundary Conditions

The objective of this part is to present the physical equations of the process in the steady state and boundary conditions. In this process two physical phenomena occur:

- electrical phenomenon;
- thermal phenomenon.

2.1 Presentation of the study domain

Describe the study domain. Precise where each phenomenon is solved.



Figure 1: Problem Scheme

The study domain is the one described in Figure 1, right part.

- We will solve the **thermal problem** only in the material only (yellow part), i.e. where $0.1 < r < 0.3$ and $0 < h < 0.3$.
- We will solve the **electrical problem** in the electrodes and in the elaborated material, i.e. for $0 \geq r < 0.3$ and $0 < h < 0.3$.

2.2 Electrical problem

Give the partial differential equation of the electrical problem. Give the boundary conditions of the electrical problem. Give the expression of the current density and of the Joule power density.

The Electrical problem can be modeled employing the fact that $E = -\nabla U$. We also know that the electrical flux is defined through $\vec{J} \cdot \vec{E}$. Using the fact that the divergence of the Electrical flux is 0 we obtain that:

$$\nabla \cdot (-\sigma \nabla U) = 0.$$

We focus now on the boundary conditions: we know that around the crucible there's insulator material, then we'll have that $\frac{\partial U}{\partial x} = 0$ on the boundary, i.e. $\nabla U \cdot \vec{n} = 0$. We can also take into consideration that ΔU is fixed and so we can write the final system as

$$\begin{cases} \nabla \cdot (-\sigma \nabla U) = 0, & 0 < x < 0.1, 0.02 < z < 0.42 \\ \nabla U \cdot \vec{n} = 0, & x \in \partial V \\ U = \Delta U, & 0 < x < 0.02, z = 0.4 \\ U = 0, & 0 < x < 0.04, z = 0. \end{cases}$$

2.3 Thermal problem

- Give the partial differential equation of the thermal problem.
- Give the boundary conditions of the thermal problem.

The Fourier Law tells us that:

$$\vec{q} = -k \nabla T \quad (1)$$

where

q local heat flux density,

k material's conductivity,

∇T is the temperature gradient.

From the Gauss-Green theorem we know that

$$\iint_S q(x, y, z) dS = Q$$

where the first integral is all over the surface defined by $0.1 < r < 0.3$ and $0 < h < 0.3$ and Q is the heat generated by the Joule effect.

By the way, employing the divergence theorem, the left hand side of the equation can be rewritten as

$$\iiint \nabla \cdot q(x, y, z) dx dy dz = \iint_S q(x, y, z) dS$$

and then the following identity holds

$$\iiint \nabla \cdot q(x, y, z) dx dy dz = Q. \quad (2)$$

Using Fourier Law (1) and plugging it into the above equation we obtain

$$\nabla \cdot (-k \nabla T) = Q. \quad (3)$$

3 Principle of the Modeling

In this project, in a first step, each equation will be developed and test. In a second step, the model with the two coupling equations will be developed and test. For numerical modeling the finite element method is used. In this project, describe the steps of the calculation and the variables used. Take time to define how you will present your numerical results.

4 Numerical Modeling of Heat Transfer Problem with Finite Elements Method

4.1 Study Domain and Mesh

The variable that we are going to use will be the *length and the height of the domain*, as well as the *number of the point* we want in our discretization.

A simple algorithm that takes care of that could be the one in Listing 1, where we have put in input of our function mesh the number of points in which we want to discretize our domain nx, ny .

Listing 1: Mesh

```

1 function [M] = points( lx,ly,nx,ny)
2
3 x=0; %initialize all the variables to zero
4 y=0;
5 i=0;
6 dy=ly/ny;
7 dx=lx/nx;
8 % use the loops for both x and y
9 for y=0:dy:ly
10     for x=0:dx:lx
11         M(i,:)= [x,y]; %save the point
12         i=i+1; %update the counter
13     end
14 end
15 end

```

We stress the fact that in `for` loop we take care also of the fact that if the point considered is a boundary point we have to come back and restart in *another line*.

The algorithm presented in Listing 2 takes care of numbering in a proper way the elements basis and the points we have generated with the alorithm `points`.

Listing 2: Mesh of the Domain

```

1 function [E]=mesh(lx,ly,nx,ny,M)
2
3 for e=1:nx*(ny-1)
4     if M(e,1)~=lx
5         E(e,:)=(e,e+1,e+nx+1,e+nx);
6     end
7 end

```

4.2 Galerkins Formulation of Heat Transfer Equation

We use the canonical change of variable, from cartesian to cylindrical, i.e.,

$$\begin{aligned}
 (x, y, z) &\rightarrow (r, \theta, z) \\
 (x, y, z) &\mapsto (r \cos(\theta), r \sin(\theta), z)
 \end{aligned} \quad (4)$$

The change of variable given by Eq.4 gives as determinant of the Jacobian matrix r , in such a way that the integration must be performed changing $dx dy dz$ into $r dr d\theta dz$.

Moreover we have that

$$T = \sum_{j=1}^N \alpha_j(\xi, \eta, \zeta) \cdot T_j$$

and so the differential becomes

$$\nabla T = \sum_{j=1}^N \nabla \alpha_j(\xi, \eta, \zeta) \cdot T_j$$

where through elementary calculations we can see that

$$\nabla \alpha_j = [JacJ]^{-1} \begin{bmatrix} \frac{\partial \alpha}{\partial \xi} \\ \frac{\partial \alpha}{\partial \eta} \\ \frac{\partial \alpha}{\partial \zeta} \end{bmatrix}$$