

Finite Elements Project

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Abstract

This is the final project of the course **Numerical Methods**, given at Phelma in the Academic Year 2017/2018. In this project we will study a process of material elaboration involving heating by electrode. We'll study the steady state of this process. The objective is to model the physical phenomena which take place in this process with *finite element method*.

All the source code, matlab *m files* and figures can be found on the Github repository <https://github.com/ErikPillon/NumericalMethods>

1 Statement of the Problem

The study configuration is considered cylindrical. A scheme of the study geometry is given in Figure 1: The process is constituted by:

- a cylindrical crucible;
- the elaborated material; the geometry of the study domain occupied by the material is cylindrical;
- 2 electrodes.

The electrodes are in graphite. An electrical potential difference is applied between the top electrode and the bottom electrode included in the crucible: ΔU . This electrical potential difference is *continuous*. An electrical current pass through the material placed in the crucible. We suppose that the contact between the electrodes and elaborated material is perfect. **Joule effect heat the material**. The material of the crucible is an insulating material. The crucible is not model and will be replaced by an adapted boundary condition. Electrical problem has to be solved in the electrodes and in the elaborated material. The heat transfer has to be solved in the material only.

2 Equations of the physical phenomenon and Boundary Conditions

The objective of this part is to present the physical equations of the process in the steady state and boundary conditions. In this process two physical phenomena occur:

- electrical phenomenon;
- thermal phenomenon.

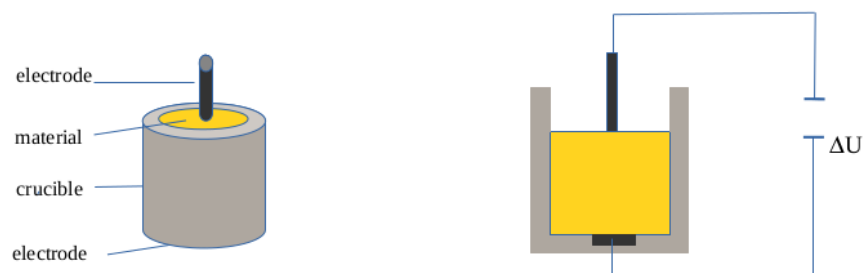


Figure 1: Problem Scheme

2.1 Presentation of the study domain

Describe the study domain. Precise where each phenomenon is solved.

The study domain is the one described in Figure 1, right part.

- We will solve the **thermal problem** only in the material only (yellow part), i.e. where $0.1 < r < 0.3$ and $0 < h < 0.3$.
- We will solve the **electrical problem** in the electrodes and in the elaborated material, i.e. for $0 \geq r < 0.3$ and $0 < h < 0.3$.

2.2 Electrical problem

Give the partial differential equation of the electrical problem. Give the boundary conditions of the electrical problem. Give the expression of the current density and of the Joule power density.

The Electrical problem can be modeled employing the fact that $E = -\nabla U$. We also know that the electrical flux is defined through $\vec{J} \cdot \vec{E}$. Using the fact that the divergence of the Electrical flux is 0 we obtain that:

$$\nabla \cdot (-\sigma \nabla U) = 0.$$

We focus now on the boundary conditions: we know that around the crucible there's insulator material, then we'll have that $\frac{\partial U}{\partial x} = 0$ on the boundary, i.e. $\nabla U \cdot \vec{n} = 0$. We can also take into consideration that ΔU is fixed and so we can write the final system as

$$\begin{cases} \nabla \cdot (-\sigma \nabla U) = 0, & 0 < x < 0.1, 0.02 < z < 0.42 \\ \nabla U \cdot \vec{n} = 0, & x \in \partial V \\ U = \Delta U, & 0 < x < 0.02, z = 0.4 \\ U = 0, & 0 < x < 0.04, z = 0. \end{cases}$$

2.3 Thermal problem

- Give the partial differential equation of the thermal problem.
- Give the boundary conditions of the thermal problem.

A medium through which electric current is conducted may involve the conversion of electrical or chemical energy into heat (or thermal energy), like in our case (i.e., Joule effect). In heat conduction analysis, such conversion processes are characterized as heat (or thermal energy) generation. For example, the temperature of a resistance wire rises rapidly when electric current passes through it as a result of the electrical energy being converted to heat at a rate of $I^2 R$, where I is the current and R is the electrical resistance of the wire. Note that heat generation is a volumetric phenomenon. That is, it occurs throughout the body of a medium. Therefore, the rate of heat generation in a medium is denoted per unit volume. The rate of heat generation in a medium may vary with time as well as position within the medium. When the variation of heat generation with position is known, the total rate of heat generation in a medium of volume V can be determined from

$$Q = \dot{q} = \int_V q_{gen} dV.$$

In our case, where there's no time dependence on the process of heat generation, as in the case of electric resistance heating throughout a homogeneous material, the relation reduces to

$$Q = \int_V q dV.$$

The Fourier Law tells us that:

$$q = -k \nabla T \quad (1)$$

where

q local heat flux density,

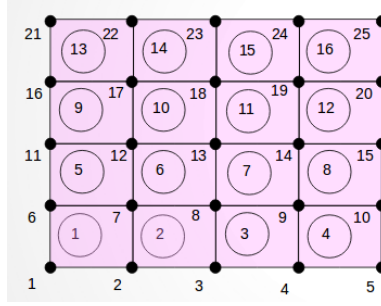


Figure 2: Discretization of the domain.

k material's conductivity,

∇T is the temperature gradient.

From the Gauss-Green theorem we know that

$$\iint_S q(x, y, z) dS = Q$$

where the first integral is all over the surface defined by $0.1 < r < 0.3$ and $0 < h < 0.3$ and Q is the heat generated by the Joule effect.

By the way, employing the divergence theorem, the left hand side of the equation can be rewritten as

$$\iiint \nabla \cdot q(x, y, z) dx dy dz = \iint_S q(x, y, z) dS$$

and then the following identity holds

$$\iiint \nabla \cdot q(x, y, z) dx dy dz = Q. \quad (2)$$

Using Fourier Law (1) and plugging it into the above equation we obtain

$$\nabla \cdot (-k \nabla T) = Q. \quad (3)$$

The final system of equation will be

$$\begin{cases} \nabla \cdot (-k \nabla T) = Q, \\ -k \nabla T \cdot \vec{n} = h(T - T_r) \end{cases} \quad (4)$$

3 Principle of the Modeling

In this project, in a first step, each equation will be developed and test. In a second step, the model with the two coupling equations will be developed and test. For numerical modeling the finite element method is used. In this project, describe the steps of the calculation and the variables used. Take time to define how you will present your numerical results.

4 Numerical Modeling of Heat Transfer Problem with Finite Elements Method

4.1 Study Domain and Mesh

We want to discretize our domain in the same way described in figure 2.

The variable that we are going to use will be the *length and the height of the domain*, as well as the *number of the point* we want in our discretization.

A simple algorithm that takes care of that could be the one in Listing 1, where we have put in input of our function `mesh` the number of points in which we want to discretize our domain `nx, ny`.

Listing 1: Mesh

```

1 function [M,N] = points(lx,ly,nx,ny)
2 % ----- OUTPUT -----
3 % M: matrix of [x,y] components of the point
4 % N: number of points generated
5
6 x=0, y=0; %initialize all the variables to zero
7 i=1; %set counter
8 dy=ly/ny; dx=lx/nx;
9 % use the loops for both x and y
10 for y=0:dy:ly
11     for x=0:dx:lx
12         M(i,:)= [x,y]; %save the point
13         i=i+1; %update the counter
14     end
15 end
16 N = i;
17 end

```

We stress the fact that in `for` loop we take care also of the fact that if the point considered is a boundary point we have to come back and restart in *another line*.

The algorithm presented in Listing 2 takes care of numbering in a proper way the elements basis and the points we have generated with the alorithm `points`.

Listing 2: Build relationship for local point and overall point

```

1 function [E,e] = mesh(lx,ly,nx,ny,M)
2 % ----- INPUT -----
3 % lx, ly : caractéristica lengths of the problem
4 % nx, ny : number of points in each direction
5 % M      : Matrix in which are stored the x and y coordinates of the points
6 % ----- OUTPUT -----
7 % E : Matrix in which are stored all the points of each element
8 % e : number of elements
9
10
11
12 dx = lx/nx; dy = ly/ny;
13 e = 1;
14 x=0; y=0;
15 for iter=1:length(M)
16     if M(iter,1)<lx & M(iter,2)<ly
17         E(e,:) = [iter,iter+1,iter+1+nx,iter+nx];
18         e = e+1;
19     end
20 end
21 % we want to return the number of the elements we've generated
22 e = length(E);

```

In Listing ?? we give an algorithm for the mesh of the boundary elements.

Listing 3: Algorithm for boundary elements

```

1 function [b_Ele,N] = b_elements(lx,ly,nx,ny)
2 % M: matrix of [x,y] components of the point
3 % N: number of points generated
4
5 x=0, y=0; %initialize all the variables to zero
6 i=0; %set counter
7 dy=ly/ny; dx=lx/nx;
8

```

```

9 % bottom points
10 while x < lx
11     i = i+1;
12     b_Ele(i,:) = [i,i+1];
13     x = x+dx;
14     i = i+1;
15 end
16
17 % right-boundary point
18 while y < ly
19     i = i+1;
20     b_Ele(i,:) = [nx*i,(nx+1)*i];
21     y = y+dy;
22 end
23
24 % top points
25 while x > 0
26     i = i+1;
27     b_Ele(i,:) = [i-1,i];
28     x = x-dx;
29 end
30 N = i;

```

4.2 Galerkin's Formulation of Heat Transfer Equation

4.2.1 Projection of the partial differential equation on an element of the basis of the functions α_i

We want to evaluate at this stage the projection of our unknown function T into the element basis β_i , i.e., $\forall i$ we want

$$\iiint_{\Omega} \beta_i \nabla \cdot (-\kappa \nabla T) d\Omega = \iiint_{\Omega} \beta_i Q d\Omega \quad (5)$$

Now we can identify the element basis β_i with the element basis α_i and so we can write

$$\iiint_{\Omega} \alpha_i \nabla \cdot (-\kappa \nabla T) d\Omega = \iiint_{\Omega} \alpha_i Q d\Omega$$

4.2.2 Give the weak formulation of the Galerkin's method. Introduce the boundary conditions in the formulation.

Using the differential identity

$$\nabla \cdot (-\alpha_i \kappa \nabla T) = -\alpha_i \nabla \cdot (\kappa \nabla T) - \kappa \nabla \alpha_i \nabla T$$

we can plug the above equation into (5) obtaining

$$\iiint_{\Omega} \nabla \cdot (-\alpha_i \kappa \nabla T) d\Omega + \iiint_{\Omega} \kappa \nabla \alpha_i \nabla T d\Omega = \iiint_{\Omega} \alpha_i Q d\Omega$$

By the way, we have, thanks to the divergence theorem, that

$$\iiint_{\Omega} \nabla \cdot (-\alpha_i \kappa \nabla T) d\Omega = - \iint_{\Gamma} \alpha_i \kappa \nabla T \cdot \vec{n} d\Gamma.$$

Then finally we have the **weak formulation of the problem** (5)

$$\iiint_{\Omega} \kappa \nabla \alpha_i \nabla T d\Omega - \iint_{\Gamma} \alpha_i \kappa \nabla T \cdot \vec{n} d\Gamma = \iiint_{\Omega} \alpha_i Q d\Omega.$$

Introducing the boundary conditions, see (4), we have

$$\begin{cases} -\kappa \nabla T \cdot \vec{n} = h(T - T_r), & \text{on the free surface} \\ -\kappa \nabla T \cdot \vec{n} = 0, & \text{on all the non-free surface} \end{cases}$$

and then

$$\forall i \iint_{\Gamma} \alpha_i h(T - T_r) d\Gamma + \iiint_{\Omega} \kappa \nabla \alpha_i \nabla T d\Omega = \iiint_{\Omega} \alpha_i Q d\Omega \quad (6)$$

4.2.3 Precise the expression of the elementary volume, the elementary surface for respectively the volume integral and surface integral.

The idea is to transform an integral all over the volume Ω in many sums all over the small elements e . The elements must be disjoint pairwise and the union of the small elements e must be give the original volume Ω . In symbols we have

$$\iiint_{\Omega} [\dots] d\Omega = \sum_e \iiint_{\omega_e} [\dots] d\Omega_e$$

Similarly we'll have that the integral all over the surface is made up by the sum by the sum of all the small surfaces:

$$\iint_{\Gamma} [\dots] d\Gamma = \sum_f \iint_{\gamma_f} [\dots] d\Gamma_f$$

We use the canonical change of variable, from cartesian to cylindrical, i.e.,

$$\begin{aligned} (x, y, z) &\rightarrow (r, \theta, z) \\ (x, y, z) &\mapsto (r \cos(\theta), r \sin(\theta), z) \end{aligned} \quad (7)$$

The change of variable given by Eq.7 gives as determinant of the Jacobian matrix the element r , in such a way that the integration must be performed changing $dx dy dz$ into $r dr d\theta dz$.

Then we have that

$$2\pi \iint_{\Omega} \kappa \nabla \alpha_i \nabla T r dr dz - 2\pi \int_{\Gamma} \alpha_i \kappa \nabla T \cdot \vec{n} dz = 2\pi \iint_{\Omega} \alpha_i Q r dr dz.$$

In particular we will perform this transformation

$$\iint_e f(x, y) dx dy = \int_{-1}^1 \int_{-1}^1 f(x(u, v), y(u, v)) \det(Jac) du dv$$

where

$$\begin{cases} x(u, v) = \sum_{i=1}^N \alpha_i(u, v) \cdot x_i \\ y(u, v) = \sum_{i=1}^N \alpha_i(u, v) \cdot y_i \end{cases} \quad \text{and} \quad \det(Jac) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^N \frac{\partial \alpha_i(u, v)}{\partial u} \cdot x_i & \sum_{i=1}^N \frac{\partial \alpha_i(u, v)}{\partial u} \cdot y_i \\ \sum_{i=1}^N \frac{\partial \alpha_i(u, v)}{\partial v} \cdot x_i & \sum_{i=1}^N \frac{\partial \alpha_i(u, v)}{\partial v} \cdot y_i \end{vmatrix}$$

and therefore the integral we have to evaluate become a summation with a suitable quadrature formula

$$\iint_e f(x, y) dx dy = \sum_{i \in \{nodes\}} f(x(u_i, y_i), y(u_i, y_i)) \det(J(u_i, y_i)) w_i$$

4.2.4 Give the expression of the integrals on the reference element

The expression for each element of the basis is then

$$\sum_j \iiint_e \nabla \alpha_i \nabla \alpha_j \xi d\xi d\eta.$$

Let's notice that we have

$$T = \sum_{j=1}^N \alpha_j(\xi, \eta, \zeta) \cdot T_j$$

and so the differential becomes

$$\nabla T = \sum_{j=1}^N \nabla \alpha_j(\xi, \eta, \zeta) \cdot T_j$$

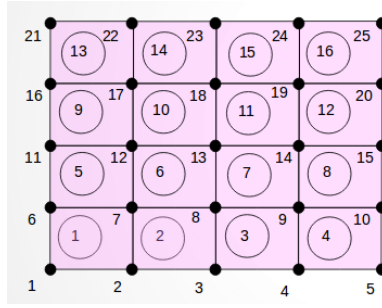


Figure 3: The boundary elements will have 3 boundaries or 2, depending on which one we are considering.

where through elementary calculations we can see that

$$\nabla \alpha_j = [Jac J]^{-1} \begin{bmatrix} \frac{\partial \alpha}{\partial \xi} \\ \frac{\partial \alpha}{\partial \eta} \\ \frac{\partial \alpha}{\partial \zeta} \end{bmatrix}.$$

Then the final expression will be

$$\sum_e \int_{\omega_e} \kappa \nabla \alpha_i \cdot \left(\sum_{j=1}^N \nabla \alpha_j T_j \right) \rho(\xi, \eta) d\xi d\eta + \sum_f \int_{\Gamma} \alpha_i h \left(\sum_{j=1}^N \alpha_j T_j \right) \rho(\xi, \eta) d\xi d\eta = \sum_f \int_{\Gamma} \alpha_i h T_r \rho(\xi, \eta) d\eta + \sum_e \int_{\omega_e} \alpha_i Q \rho(\xi, \eta) d\xi d\eta$$

4.2.5 Detail the expression of the elementary matrix on an element e . Precise the size of each elementary matrix (sub matrix). Precise the expression of each integral and the principle of calculation. For each integral, precise the nature of the element e

The elementary matrix of an element e is given by the inner product $\nabla \alpha_i \nabla \alpha_j$. In particular we'll have a matrix A_{ij}^e in which we will have in position (i, j) the element $\nabla \alpha_i \nabla \alpha_j$. The matrix will be evaluated through a 2-dimensional quadrature formula; in 1-dimension quadrature formula allow us to pass from a continuous integral to a discrete sum of values. First of all we have to rescale the integral extrema to 1 and -1,

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) dx.$$

Then the discretization will become

$$\frac{b-a}{2} \sum_{i=1}^n w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right),$$

where suitable values x_i and weights w_i will be taken.

The matrix A_{ij}^e is a 4×4 matrix, for each inner element. For the outer element it will be a 2×2 matrix.

For the elements 22, 23, 24, 25 of Figure 3 we have also to develop the integral on the boundary element f

$$\sum_f \int_{\Gamma} \alpha_i h \left(\sum_{j=1}^N \alpha_j T_j \right) \rho(\xi, \eta) d\xi d\eta$$

The resulting local matrix will be a 2x2 matrix, given by the product $\alpha_i \cdot \alpha_j$.

4.3 Algorithm of the Finite Element Formulation of Heat Transfer Equation

The algorithm goes as follows:

- Discretize the domain and build the mesh.
- For each element e of the domain:
 - Construct the Jacobian matrix Jac with the functions α_i
 - Evaluate the Jacobian $\det(Jac)$
 - Evaluate all the 16 terms $\nabla \alpha_i \nabla \alpha_j$ with the help of the Gauss quadrature formula.
 - Update the global matrix A with the function Update

- For each element f of the upper boundary:
 - Construct the Jacobian Jac (that is already a scalar value) with the functions α_1 and α_2
 - Evaluate all the 4 terms $\alpha_i \alpha_j$ with the help of the 1D Gauss quadrature formula.
 - Update the global matrix B with the function `Update` modified for the boundary elements
- Solve the linear problem $(A_{ij} + B_{ij})T_i = Q_i$

In the program we make use of the following functions:

- `jacobian(M, Elem, e)` where M is the coordinates matrix and $Elem$ is the mesh matrix, while e is the global element taken into consideration.
- `alpha(xi, eta)` where xi, eta are the local coordinates.
- `update` for updating the global matrix with the 16 values obtained for each element e .
- The functions `points` and `mesh` as already stated in Section 4.1
- The function `Update` for putting the values of the local matrix of each element e (resp. f) in the global matrix A (resp. B).

4.4 Programming and calculation

4.4.1 Construct the program: main program and functions

Listing 4: main program

```

1  clc; clear all; close all;
2  % this is the main file for the thermal part
3
4  %% discretization of the domain and building of the mesh for the thermal part
5  % initialization of the variables lx, ly, nx, ny
6  % THESE VALUES MUST BE CHANGED!
7  lx = 0.3; ly = 0.5; nx = 10; ny = 10;
8  dx = lx/nx; dy = ly/ny;
9
10 % creation of the points and of the mesh
11 [M, N_points] = points(lx, ly, nx, ny);
12 [Elem, N_elements] = mesh(lx, ly, nx, ny, M);
13 [b_Elem, N_b_elements] = b_elements_thermal(lx, ly, nx, ny);
14
15 %% initialization of the points for the gauss quadrature formula points
16 w=[0.347855 0.652145 0.652145 0.347855];
17 u=[-0.861136 -0.339981 0.339981 0.861136];
18
19 %% initialization of the variables
20 k = 1;
21 h = 1;
22
23 %% creation of the big matrices
24 % this matrix has to be filled with all the terms of interactions between
25 % the internal elements
26 Interaction = zeros(N_points);
27
28 %% evaluation of the integral on the variuos elements of the basis
29 for e = 1:N_elements
30     % creation of the Jacobian matrix
31     [Jac] = jacobian(M, Elem, e);
32     J = det(Jac); %jacobian
33
34     %% evaluation of the integral
35     % evaluation of the 4x4 matrix gai_gaj

```



```

36 % here we will need the gauss points
37 gai_gaj = double_integral(M, Elem, e, dx, u, w);
38 Contribution = k*gai_gaj*J;
39 % update the global matrix values; it will recollect the global number
40 % from the local number and the element e
41 Interaction = Update(Interaction, Elem, e, Contribution);
42 end
43
44 % this matrix must be filled with the terms given by the boundary effects
45 Bound = zeros(N_points);
46 for f = 1:N_b_elements
47     [Jac] = jacobian(M, Elem, e);
48     J = det(Jac);
49     %% evaluation of the 1 dimensional integral
50     % evaluation of the 2x2 matrix ai_aj
51     % here we will need the gauss points
52     ai_aj = integral_1D(M, b_Elem, f, dx, u, w);
53     Contribution_boundary = k*ai_aj*J;
54     % update the global matrix values; it will recollect the global number
55     % from the local number and the element e
56     Bound = Update_boundary(Bound, b_Elem, f, Contribution_boundary);
57 end
58 % for testing we use rhs random
59 rhs = 1000*rand(N_points,1);
60 T = (Interaction+Bound)\rhs;

```

Where we used the function jacobian and alpha listed below.

Listing 5: function for the creation of the jacobian matrix for each element

```

1 function [Jac] = jacobian(M, E, e)
2 % ----- INPUT -----
3 % M: matrix of the points with x and y coordinates
4 % E: matrix of the mesh
5 % e: global number of the element we're considering
6 %
7 % ----- OUTPUT -----
8 % Jac: 2x2 matrix with the values of the jacobian matrix
9
10 % we obtain the x and y coordinates of the points involved
11 x_i = M(E(e,:),1);
12 y_i = M(E(e,:),2);
13 % xi and eta points in which evaluate the functions
14 xi = [-1, 1, 1, -1];
15 eta = [-1, -1, 1, 1];
16
17 Jac = zeros(2);
18 for i = 1:4
19     [alfa, dalfa_xi, dalfa_eta] = alpha(xi(i),eta(i));
20     Jac(1,1) = Jac(1,1) +dalfa_xi(i)*x_i(i);
21     Jac(1,2) = Jac(1,1) +dalfa_xi(i)*y_i(i);
22     Jac(2,1) = Jac(1,1) +dalfa_eta(i)*x_i(i);
23     Jac(2,2) = Jac(1,1) +dalfa_eta(i)*y_i(i);
24 end

```

Listing 6: Function for the creation of the alpha functions and the derivatives of the latter

```

1 function [alfa, dalfa_xi, dalfa_eta] = alpha(xi,eta)
2 % ----- INPUT -----
3 % xi, eta: nodes in which the function must be evaluated
4 %
5 %
6 %
7 %
8 %
9 %

```

```

6 % ----- OUTPUT -----
7 % alfa:      4x1 vector with the values of the four alfas evaluated at (xi,eta)
8 % dalfa_xi: 4x1 vector of the partial derivatives of alfas wrt xi evaluated
9 %           at (xi,eta)
10 % dalfa_eta: 4x1 vector of the partial derivatives of alfas wrt eta evaluated
11 %           at (xi,eta)
12
13 % alfa functions
14 alfa(1,:) = 1/4*(1-xi).*(1-eta);
15 alfa(2,:) = 1/4*(1+xi).*(1-eta);
16 alfa(3,:) = 1/4*(1+xi).*(1+eta);
17 alfa(4,:) = 1/4*(1-xi).*(1+eta);
18
19 % evaluation of the derivative respect to u
20 dalfa_xi(1,:) = -1/4.*(1-eta);
21 dalfa_xi(2,:) = 1/4.*(1-eta);
22 dalfa_xi(3,:) = 1/4.*(1+eta);
23 dalfa_xi(4,:) = -1/4.*(1+eta);
24
25 % evaluation of the derivative respect to v
26 dalfa_eta(1,:) = -1/4.*(1-xi);
27 dalfa_eta(2,:) = -1/4.*(1+xi);
28 dalfa_eta(3,:) = 1/4.*(1+xi);
29 dalfa_eta(4,:) = 1/4.*(1-xi);
30
31 end

```

Listing 7: Function for update the global matrix with the values obtained for the local element e

```

1 function [M] = Update(M, Elements_matrix, e, A)
2 % ----- INPUT -----
3 % Interaction      : matrix to be modified
4 % Elements_matrix: matrix of the mesh
5 % e                : number of the global element
6 % A                : matrix of the local contributions
7 %
8 % ----- OUTPUT -----
9 % M: Matrix modified
10
11 if size(A) ~= [4,4]
12     error('Local Matrix dimension are not 4x4')
13     M = zeros(4);
14     return
15 end
16
17
18
19 for i=1:4
20     for j=1:4
21         M(Elements_matrix(e,i),Elements_matrix(e,j)) = M(Elements_matrix(e,i),
22             Elements_matrix(e,j))+A(i,j);
23     end
24 end

```

Listing 8: Function for evaluating the double integral $\iint \nabla \alpha_i \nabla \alpha_j$

```

1 function [A] = double_integral(M, Elem, e, dx, u, w)
2
3 A = zeros(4);
4 r = M(Elem(e,1),1);
5

```

```

6  for i = 1:4
7      for j = 1:4
8          sum = 0;
9          for xi_coord = 1:4
10             for eta_coord = 1:4
11                 [Alpha, dalpha_xi, dalpha_eta] = alpha(u(xi_coord),u(eta_coord));
12                 % the following is the summation of the inner product
13                 % grad(alpha_i).grad(alpha_j)
14                 w1 = w(xi_coord);
15                 w2 = w(eta_coord);
16                 sum=sum+(dalpha_xi(i)*dalpha_xi(j)+dalpha_eta(i)*dalpha_eta(j))*(r+dx/2+u
                     (xi_coord)*dx)*w1*w2;
17             end
18         end
19         A(i,j) = sum;
20     end
21 end
22
23 end

```

5 Model the Electrical Problem with the Finite Elements Method

5.1 Study Domain and Mesh

Since the domain of interest of the electric problem is larger than the thermal problem, we have to slightly modify our algorithm. We then have the algorithm in Listing 9, where we have put in input of our function `mesh_electric` the number of points in which we want to discretize our domain.

Listing 9: Points

```

1  function [M,N] = points_electric()
2  % ----- OUTPUT -----
3  % M: matrix of [x,y] components of the point
4  % N: number of points generated
5
6  % discretization for the electric part
7  dx=0.01; dy=0.01;
8  N = 1; % counter
9
10 % bottom electrode
11 y = 0;
12 while y <= 0.02
13     x = 0;
14     while x <= 0.04
15         M(N,:) = [x,y];
16         x = x+dx;
17         N = N+1;
18     end
19     y = y+dy;
20 end
21
22 % material
23 while y <= 0.4
24     x = 0;
25     while x <= 0.1
26         M(N,:) = [x,y];
27         x = x+dx;
28         N = N+1;
29     end
30     y = y+dy;
31 end

```

```

32
33 % top electrode
34 while y <= 0.1
35     x = 0;
36     while x <= 0.02
37         M(N,:) = [x,y];
38         x = x+dx;
39         N = N+1;
40     end
41     y = y+dy;
42 end

```

Listing 10: Build relationship for local point and overall point

```

1 function [E,e] = mesh_electric(M)
2 % ----- INPUT -----
3 % M : Matrix in which are stored the x and y coordinates of the points
4 % ----- OUTPUT -----
5 % E : Matrix in which are stored all the points of each element
6 % e : number of elements
7
8 dx = 0.01; dy = 0.01;
9 e = 1;
10 y=0;
11 while y <= 0.02
12     x = 0;
13     while x <= 0.04
14         nx = 0.04/dx;
15         E(e,:) = [e,e+1,e+1+nx,e+nx];
16         x = x+dx;
17         e = e+1;
18     end
19     y = y+dy;
20 end
21
22 % material
23 while y <= 0.4
24     x = 0;
25     while x <= 0.1
26         nx = 0.1/dx;
27         E(e,:) = [e,e+1,e+1+nx,e+nx];
28         x = x+dx;
29         e = e+1;
30     end
31     y = y+dy;
32 end
33
34 % top electrode
35 while y <= 0.1
36     x = 0;
37     while x <= 0.02
38         nx = 0.02/dx;
39         E(e,:) = [e,e+1,e+1+nx,e+nx];
40         x = x+dx;
41         e = e+1;
42     end
43     y = y+dy;
44 end
45
46 end

```

5.2 Galerkin's Formulation of the Electrical Equation

The projection of the partial differential equation on a basis element is

$$\iiint_{\omega} \alpha_i \nabla \cdot (-\sigma \nabla U) d\Omega = 0, \quad \forall i$$

and then the **weak formulation for the electric part** becomes

$$\iiint_{\omega} \sigma \alpha_i \cdot \nabla U d\Omega = 0, \quad \forall i. \quad (8)$$

The boundary conditions given can be modeled in the following fashion

$$\begin{cases} U = 0, & \text{on the top of the crucible,} \\ U = U_0, & \text{on the bottom of the crucible,} \\ J \cdot n = 0, & \text{on all the other surfaces.} \end{cases}$$

As before, the elementary element of surface and volume are given by the canonical change of variables, from cartesian to cylindrical, i.e.,

$$\begin{aligned} (x, y, z) &\rightarrow (r, \theta, z) \\ (x, y, z) &\mapsto (r \cos(\theta), r \sin(\theta), z) \end{aligned} \quad (9)$$

The change of variable given by Eq.9 gives as determinant of the Jacobian matrix the element r , in such a way that the integration must be performed changing $dx dy dz$ into $r dr d\theta dz$.

The second change of variables will be

$$\begin{aligned} (r, \theta, z) &\rightarrow (\xi, \eta) \\ (r, \theta, z) &\mapsto (\xi(r, z), \eta(r, z)) \end{aligned} \quad (10)$$

and we will have $dx dy dz = \det(Jac) d\xi d\eta$.

5.3 Algorithm of the Finite Element Formulation of the Electrical Equation

The algorithm goes as follows:

- Discretize the domain and build the mesh.
- For each element e of the domain:
 - Construct the Jacobian matrix Jac with the functions α_i
 - Evaluate the Jacobian $\det(Jac)$
 - Evaluate all the 16 terms $\nabla \alpha_i \nabla \alpha_j$ with the help of the Gauss quadrature formula.
 - Update the global matrix A with the function `Update`
- For each element f of the upper boundary:
 - Construct the Jacobian Jac (that is already a scalar value) with the functions α_1 and α_2
 - Evaluate all the 4 terms $\alpha_i \alpha_j$ with the help of the 1D Gauss quadrature formula.
 - Update the global matrix B with the function `Update` modified for the boundary elements
- Solve the linear problem $(A_{ij} + B_{ij})T_i = Q_i$

In the program we make use of the following functions:

- `jacobian(M, Elem, e)` where M is the coordinates matrix and $Elem$ is the mesh matrix, while e is the global element taken into consideration.
- `alpha(xi, eta)` where xi, eta are the local coordinates.
- `update` for updating the global matrix with the 16 values obtained for each element e .
- The functions `points_electric` and `mesh` as already stated in Section 4.1
- The function `Update_electric` for putting the values of the local matrix of each element e (resp. f) in the global matrix A (resp. B).

5.4 Programming and calculation

5.4.1 Construct the program: main program and functions

Listing 11: main program

```
1 clc; clear all; close all;
2 % this is the main file for the electric part
3
4 %% discretization of the domain and building of the mesh for the thermal part
5 % initialization of the variables lx, ly, nx, ny
6 dx = 0.01; dy = 0.01;
7
8 % creation of the points and of the mesh
9 [M, N_points] = points_electric();
10 [Elem, N_elements] = mesh_electric(M);
11 [b_Elem, N_b_elements] = b_elements_electric(lx,ly,nx,ny);
12
13 %% initialization of the points for the gauss quadrature formula
14 w=[0.347855 0.652145 0.652145 0.347855];
15 u=[-0.861136 -0.339981 0.339981 0.861136];
16
17 %% initialization of the variables
18 k = 1;
19 sigma = 1;
20
21 %% creation of the big matrices
22 % this matrix has to be filled with all the terms of interactions between
23 % the internal elements
24 Interaction = zeros(N_points);
25
26 %% evaluation of the integral on the variuos elements of the basis
27 for e = 1:N_elements
28     % creation of the Jacobian matrix
29     [Jac] = jacobian(M, Elem, e);
30     J = det(Jac); %jacobian
31
32     %% evaluation of the integral
33     % evaluation of the 4x4 matrix gai_gaj
34     % here we will need the gauss points
35     gai_gaj = double_integral(M, Elem, e, dx, u, w);
36     Contribution = sigma*gai_gaj*J;
37     % update the global matrix values; it will recollect the global number
38     % from the local number and the element e
39     Interaction = Update(Interaction, Elem, e, Contribution);
40 end
41
42 % this matrix must bel filled with the terms given by the boundary effects
43 Bound = zeros(N_points);
44 for f = 1:N_b_elements
45     [Jac] = jacobian(M, Elem, e);
46     J = det(Jac);
47
48     %% evaluation of the 1 dimensional integral
49     % evaluation of the 2x2 matrix ai_aj
50     % here we will need the gauss points
51     ai_aj = integral_1D(M, b_Elem, f, dx, u, w);
52     Contribution_boundary = k*ai_aj*J;
53     % update the global matrix values; it will recollect the global number
54     % from the local number and the element e
55     Bound = Update_boundary(Bound, b_Elem, f, Contribution_boundary);
56 end
57 % for testing we use rhs random
```

```

57 rhs = 1000*rand(N_points,1);
58 T = (Interaction+Bound)\rhs;

```

Where we used the function jacobian and alpha listed below.

Listing 12: Function for the creation of the jacobian matrix for each element

```

1 function [Jac] = jacobian(M, E, e)
2 % ----- INPUT -----
3 % M: matrix of the points with x and y coordinates
4 % E: matrix of the mesh
5 % e: global number of the element we're considering
6 %
7 % ----- OUTPUT -----
8 % Jac: 2x2 matrix with the values of the jacobian matrix
9
10 % we obtain the x and y coordinates of the points involved
11 x_i = M(E(e,:),1);
12 y_i = M(E(e,:),2);
13 % xi and eta points in which evaluate the functions
14 xi = [-1, 1, 1, -1];
15 eta = [-1, -1, 1, 1];
16
17 Jac = zeros(2);
18 for i = 1:4
19     [alfa, dalfa_xi, dalfa_eta] = alpha(xi(i),eta(i));
20     Jac(1,1) = Jac(1,1) +dalfa_xi(i)*x_i(i);
21     Jac(1,2) = Jac(1,1) +dalfa_xi(i)*y_i(i);
22     Jac(2,1) = Jac(1,1) +dalfa_eta(i)*x_i(i);
23     Jac(2,2) = Jac(1,1) +dalfa_eta(i)*y_i(i);
24 end

```

Listing 13: Function for the creation of the alpha functions and the derivatives of the latter

```

1 function [alfa, dalfa_xi, dalfa_eta] = alpha(xi,eta)
2 % ----- INPUT -----
3 % xi, eta: nodes in which the function must be evaluated
4 %         - they can be also vector values
5 %
6 % ----- OUTPUT -----
7 % alfa:      4x1 vector with the values of the four alfas evaluated at (xi,eta)
8 % dalfa_xi:  4x1 vector of the partial derivatives of alfas wrt xi evaluated
9 %            at (xi,eta)
10 % dalfa_eta: 4x1 vector of the partial derivatives of alfas wrt eta evaluated
11 %            at (xi,eta)
12
13 % alfa functions
14 alfa(1,:) = 1/4*(1-xi).*(1-eta);
15 alfa(2,:) = 1/4*(1+xi).*(1-eta);
16 alfa(3,:) = 1/4*(1+xi).*(1+eta);
17 alfa(4,:) = 1/4*(1-xi).*(1+eta);
18
19 % evaluation of the derivative respect to u
20 dalfa_xi(1,:) = -1/4.*(1-eta);
21 dalfa_xi(2,:) = 1/4.*(1-eta);
22 dalfa_xi(3,:) = 1/4.*(1+eta);
23 dalfa_xi(4,:) = -1/4.*(1+eta);
24
25 % evaluation of the derivative respect to v
26 dalfa_eta(1,:) = -1/4.*(1-xi);
27 dalfa_eta(2,:) = -1/4.*(1+xi);
28 dalfa_eta(3,:) = 1/4.*(1+xi);

```

```

29 dalfa_eta(4,:) = 1/4.*(1-xi);
30
31 end

```

Listing 14: Function for update the global matrix with the values obtained for the local element e

```

1 function [M] = Update(M, Elements_matrix, e, A)
2 % ----- INPUT -----
3 % Interaction      : matrix to be modified
4 % Elements_matrix: matrix of the mesh
5 % e                : number of the global element
6 % A                : matrix of the local contributions
7 %
8 % ----- OUTPUT -----
9 % M: Matrix modified
10
11 if size(A) ~= [4,4]
12     error('Local Matrix dimension are not 4x4')
13     M = zeros(4);
14     return
15 end
16
17
18
19 for i=1:4
20     for j=1:4
21         M(Elements_matrix(e,i),Elements_matrix(e,j)) = M(Elements_matrix(e,i),
22             Elements_matrix(e,j))+A(i,j);
23     end
24 end

```

Listing 15: Function for evaluating the double integral $\iint \nabla \alpha_i \nabla \alpha_j$

```

1 function [A] = double_integral(M, Elem, e, dx, u, w)
2
3 A = zeros(4);
4 r = M(Elem(e,1),1);
5
6 for i = 1:4
7     for j = 1:4
8         sum = 0;
9         for xi_coord = 1:4
10             for eta_coord = 1:4
11                 [Alpha, dalpha_xi, dalpha_eta] = alpha(u(xi_coord),u(eta_coord));
12                 % the following is the summation of the inner product
13                 % grad(alpha_i).grad(alpha_j)
14                 w1 = w(xi_coord);
15                 w2 = w(eta_coord);
16                 sum=sum+(dalpha_xi(i)*dalpha_xi(j)+dalpha_eta(i)*dalpha_eta(j))*(r+dx/2+u
17                     (xi_coord)*dx)*w1*w2;
18             end
19         end
20         A(i,j) = sum;
21     end
22 end
23 end

```