Finite Elements Project

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Abstract

This is the final project of the course **Numerical Methods**, given at Phelma in the Academic Year 2017/2018. In this project we will study a process of material elaboration involving heating by electrode. We 'll study the steady state of this process. The objective is to model the physical phenomena which take place in this process with *finite element method*.

1 Statement of the Problem

The study configuration is considered cylindrical. A scheme of the study geometry is given in Figure 1: The process is constituted by:

- a cylindrical crucible;
- the elaborated material; the geometry of the study domain occupied by the material is cylindrical;
- 2 electrodes.

The electrodes are in graphite. An electrical potential difference is applied between the top electrode and the bottom electrode included in the crucible: ΔU. This electrical potential difference is *continuous*. An electrical current pass through the material placed in the crucible. We suppose that the contact between the electrodes and elaborated material is perfect. **Joule effect heat the material**. The material of the crucible is an insulating material. The crucible is not model and will be replaced by an adapted boundary condition. Electrical problem has to be solved in the electrodes and in the elaborated material. The heat transfer has to be solved in the material only.

2 Equations of the physical phenomenon and Boundary Conditions

The objective of this part is to present the physical equations of the process in the steady state and boundary conditions. In this process two physical phenomena occur:

- electrical phenomenon;
- thermal phenomenon.

2.1 Presentation of the study domain

Describe the study domain. Precise where each phenomenon is solved.

Figure 1: Problem Scheme

The study domain is the one described in Figure 1, right part.

- We will solve the **thermal problem** only in the material only (yellow part), i.e. where $0.1 < r < 0.3$ and $0 < h < 0.3$.
- We will solve the **electrical problem** in the electrodes and in the elaborated material, i.e. for $0 \geq r < 0.3$ and $0 < h < 0.3$.

2.2 Electrical problem

Give the partial differential equation of the electrical problem. Give the boundary conditions of the electrical problem. Give the expression of the current density and of the Joule power density.

The Electrical problem can be modeled employing the fact that $E = -\nabla U$. We also now that the electrical flux is defined through $\vec{J} \cdot \vec{E}$. Using the fact that the divergence of the Electrical flux is 0 we obtain that:

$$\nabla \cdot (-\sigma \nabla U) = 0.$$

We focus now on the boundary conditions: we know that around the crucible there's insulator material, then we'll have that $\frac{\partial U}{\partial x} = 0$ on the boundary, i.e. $\nabla U \cdot \vec{n} = 0$. We can also take into consideration that ΔU is fixed and so we can write the final system as

$$\begin{cases} \nabla \cdot (-\sigma \nabla U) = 0, & 0 < x < 0.1,\ 0.02 < z < 0.42 \\ \nabla U \cdot \vec{n} = 0, & x \in \partial V \\ U = \Delta U, & 0 < x < 0.02,\ z = 0.4 \\ U = 0, & 0 < x < 0.04,\ z = 0. \end{cases}$$

2.3 Thermal problem

- Give the partial differential equation of the thermal problem.
- Give the boundary conditions of the thermal problem.

The Fourier Law tells us that:

$$q = -k \nabla T \tag{1}$$

where

q local heat flux density,

k material's conductivity,

∇T is the temperature gradient.

From the Gauss-Green theorem we know that

$$\iint_S q(x, y, z)\, dS = Q$$

where the first integral is all over the surface defined by $0.1 < r < 0.3$ and $0 < h < 0.3$ and Q is the heat generated by the Joule effect.

By the way, employing the divergence theorem, the left hand side of the equation can be rewritten as

$$\iiint \nabla \cdot q(x, y, z)\, dx\, dy\, dz = \iint_S q(x, y, z)\, dS$$

and then the following identity holds

$$\iiint \nabla \cdot q(x, y, z)\, dx\, dy\, dz = Q. \tag{2}$$

Using Fourier Law (1) and plugging it into the above equation we obtain

$$\nabla \cdot (-k \nabla T) = Q. \tag{3}$$

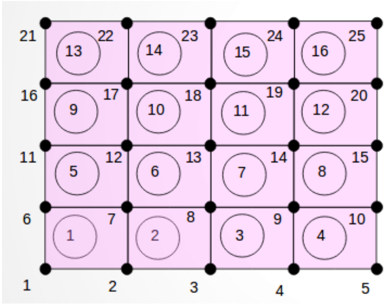

Figure 2: Discretization of the domain.

The final system of equation will be

$$\begin{cases} \nabla \cdot (-k\nabla T) = Q, \\ -\kappa\nabla T \cdot \vec{n} = h(T - T_r) + \varepsilon_{SB}(T^4 - T_r^4) \end{cases} \tag{4}$$

3 Principle of the Modeling

In this project, in a first step, each equation will be developed and test. In a second step, the model with the two coupling equations will be developed and test. For numerical modeling the finite element method is used. In this project, describe the steps of the calculation and the variables used. Take time to define how you will present your numerical results.

4 Numerical Modeling of Heat Transfer Problem with Finite Elements Method

4.1 Study Domain and Mesh

We want to discretize our domain in the same way described in figure 2.

The variable that we are going to use will be the *length and the height of the domain*, as well as the *number of the point* we want in our discretization.

A simple algorithm that takes care of that could be the one in Listing 1, where we have put in input of our function `mesh` the number of points in which we want to discretize our domain `nx,ny`.

Listing 1: Mesh

```
function [M] = points( lx,ly,nx,ny)

x=0; %initialize all the variables to zero
y=0;
i=0;
dy=ly/ny;
dx=lx/nx;
% use the loops for both x and y
for y=0:dy:ly
    for x=0;dx;lx
        M(i,:)= [x,y]; %save the point
        i=i+1; %update the counter
    end
end
end
```

We stress the fact that in `for` loop we take care also of the fact that if the point considered is a boundary point we have to come back and restart in *another line*.

The algorithm presented in Listing 2 takes care of numbering in a proper way the elements basis and the points we have generated with the alorithm `points`.

Listing 2: Mesh of the Domain

```
function [E]=mesh(lx,ly,nx,ny,M)

for e=1:nx*(ny-1)
    if M(e,1)!=lx
       E(e,:)=(e,e+1,e+nx+1,e+nx);
end
end
```

4.2 Galerkins Formulation of Heat Transfer Equation

4.2.1 Projection of the partial differential equation on an element of the basis of the functions α_i

We want to evaluate at this stage the projection of our unknown function T into the element basis β_i, i.e., $\forall i$ we want

$$\iiint_\Omega \beta_i \nabla \cdot (-\kappa \nabla T)\, d\Omega = \iiint_\Omega \beta_i Q\, d\Omega \tag{5}$$

Now we can identify the element basis β_i with the element basis α_i and so we can write

$$\iiint_\Omega \alpha_i \nabla \cdot (-\kappa \nabla T)\, d\Omega = \iiint_\Omega \alpha_i Q\, d\Omega$$

4.2.2 Give the weak formulation of the Galerkins method. Introduce the boundary conditions in the formulation.

Using the differential identity

$$\nabla \cdot (-\alpha_i)\kappa \nabla T) = -\alpha_i \nabla \cdot (\kappa \nabla T) - \kappa \nabla \alpha_i \nabla T$$

we can plug the above equation into (5) obtaining

$$\iiint_\Omega \nabla \cdot (-\alpha_i)\kappa \nabla T\, d\Omega + \iiint_\Omega \kappa \nabla \alpha_i \nabla T\, d\Omega = \iiint_\Omega \alpha_i Q\, d\Omega$$

By the way, we have, thanks to the divergnece theorem, that

$$\iiint_\Omega \nabla \cdot (-\alpha_i)\kappa \nabla T\, d\Omega = -\iint_\Gamma \alpha_i \kappa \nabla T \cdot \vec{n}\, d\Gamma.$$

Then finally we have the **weak formulation of the problem** (5)

$$\iiint_\Omega \kappa \nabla \alpha_i \nabla T\, d\Omega - \iint_\Gamma \alpha_i \kappa \nabla T \cdot \vec{n}\, d\Gamma = \iiint_\Omega \alpha_i Q\, d\Omega.$$

Introducing the boundary conditions, see (4), we have

$$-\kappa \nabla T \cdot \vec{n} = h(T - T_r) + \varepsilon_{SB}(T^4 - T_r^4)$$

and then

$$\forall i \iint_\Gamma \alpha_i h(T - T_r)\, d\Gamma + \iint_\Gamma \alpha_i \varepsilon_{SB}(T^4 - T_r^4)\, d\Gamma + \iiint_\Omega \kappa \nabla \alpha_i \nabla T\, d\Omega = \iiint_\Omega \alpha_i Q\, d\Omega \tag{6}$$

4.2.3 Precise the expression of the elementary volume, the elementary surface for respectively the volume integral and surface integral.

We use the canonical change of variable, from cartesian to cylindrical, i.e.,

$$\begin{aligned}(x, y, z) &\longrightarrow (r, \theta, z)\\ (x, y, z) &\mapsto (r cos(\theta), r sin(\theta), z)\end{aligned} \tag{7}$$

The change of variable given by Eq.7 gives as determinant of the Jacobian matrix the element r, in such a way that the integration must be performed changing $dx\, dy\, dz$ into $r\, dr\, d\theta\, dz$.

Then we have that

$$2\pi \iint_\Omega \kappa \nabla \alpha_i \nabla T\, r\, dr\, dz - 2\pi \int_\Gamma \alpha_i \kappa \nabla T \cdot \vec{n}\, dz = 2\pi \iint_\Omega \alpha_i Q\, r\, dr\, dz.$$

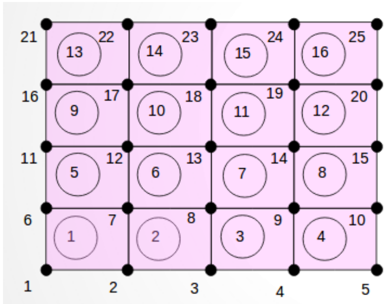

Figure 3: The boundary elements will have 3 boundaries or 2, depending on which one we are considering.

4.2.4 Give the expression of the integrals on the reference element

The expression for each element of the basis is then

$$\sum_j \iiint_e \nabla\alpha_i \nabla\alpha_j \xi \, d\xi \, d\eta.$$

Let's notice that we have

$$T = \sum_{j=1}^{N} \alpha_j(\xi, \eta, \zeta) \cdot T_j$$

and so the differential becomes

$$\nabla T = \sum_{j=1}^{N} \nabla\alpha_j(\xi, \eta, \zeta) \cdot T_j$$

where through elementary calculations we can see that

$$\nabla\alpha_j = \left[JacJ\right]^{-1} \begin{bmatrix} \frac{\partial\alpha}{\partial\xi} \\ \frac{\partial\alpha}{\partial\eta} \\ \frac{\partial\alpha}{\partial\zeta} \end{bmatrix}.$$

Then the final expression will be

$$\sum_e \iint_{\omega_e} \kappa\nabla\alpha_i \cdot \left(\sum_{j=1}^{N} \nabla\alpha_j T_j\right) \xi \, d\xi \, d\eta + \sum_f \int_\Gamma \alpha_i h \left(\sum_{j=1}^{N} \alpha_j T_j\right) d\eta + \sum_f \int \alpha_i \varepsilon \sigma_{SB} T^3 \left(\sum_{j=1}^{N} \alpha_j T_j\right) d\eta = \sum_f \int_\Gamma \alpha_i h T_r \, d\eta + \sum_f \int_\gamma \alpha_i \varepsilon \sigma_{SB} T_r^4 \, d\eta + \sum_e \iiint_{\omega_e} \alpha_i Q \, \xi \, d\xi \, d\eta.$$

4.2.5 Detail the expression of the elementary matrix on an element e. Precise the size of each elementary matrix (sub matrix). Precise the expression of each integral and the principle of calculation. For each integral, precise the nature of the element e

The elementary matrix of an element e is given by the inner product $\nabla\alpha_i \nabla\alpha_j$. In particular we'll have a matrix A^e_{ij} in which we will have in position (i, j) the element $\nabla\alpha_i \nabla\alpha_j$.

The matrix A^e_{ij} is a 4×4 matrix, for each inner element. For the outer element it will be a 3×3 or 2×2 depending on the position.

4.2.6 Detail the elementary second term on an element e. Precise the size of each elementary second vector. Precise the expression of each integral and the principle of calculation. Precise the principle of the construction of the matrix system. For each integral, precise the nature of the element e

4.2.7 Precise the principle of the construction of the matrix and the second term vector of the system