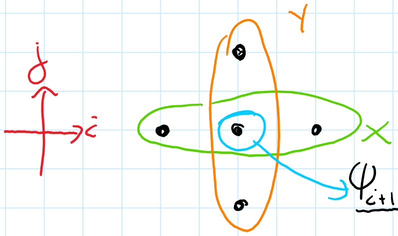
$$\nabla^2 \varphi = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$\varphi$ varies linearly from $0 \to 1$ on sides & is 1 on top 0 on bottom

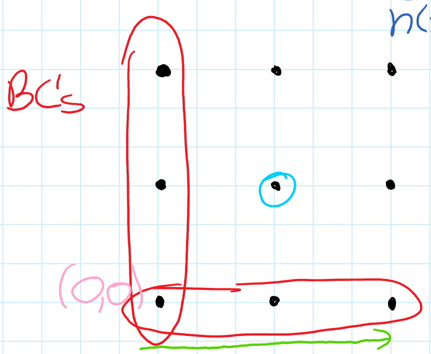



$$\frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{\Delta x^2} + \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{\Delta y^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

this is applied to all interior nodes

exterior/BCs are all Dirichlet - top is 1, bottom is 0, walls vary linearly

$$\varphi = \frac{N \cdot \Delta y}{h(x)} = \frac{y_{pos}}{h(x)}$$




$$\begin{bmatrix} 1 & 0 & \cdots & & & \\ 0 & 1 & 0 & \cdots & & \\ \vdots & & 0 & & & \\ \vdots & & \vdots & & & \\ \end{bmatrix} \begin{bmatrix} \varphi_{0,0} \\ \varphi_{1,0} \\ \vdots \\ \varphi_{i,j-1} \\ \varphi_{i-1,j} \\ \varphi_{i,j} \\ \varphi_{i+1,j} \\ \vdots \\ \varphi_{i,j+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}$$

$-N$ points in index, $+\frac{1}{\Delta y^2}$, ...

$+\frac{1}{\Delta x^2}$, $+\left(\frac{-2}{\Delta x^2} - \frac{2}{\Delta y^2}\right)$, $+\frac{1}{\Delta x^2}$ ...

$+N$ points in index, $+1/\Delta x^2$, ...

$N$ = size of square matrix A, A = N×N size

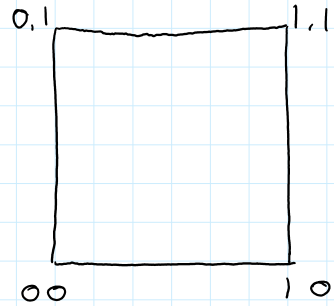
Then when A is assembled, apply BCs to F vector, $\varphi = 1, 0,$ or $\frac{y_{pos}}{h(x)}$

Need $x = f(\xi, \eta)$ & $y = f(\xi, \eta)$

$x = \xi \qquad \Rightarrow \qquad \xi = x$

$y = 0.2 + 0.1\eta(1 - \cos(\pi\xi)) \qquad \eta = \frac{y - 0.2}{0.1(1-\cos(\pi\xi))} = \frac{10y - 2}{1-\cos(\pi\xi)}$

$y = 0.2 + 0.1\eta(1-\cos(\pi\xi))$ $\quad \eta = \dfrac{y-0.2}{0.1(1-\cos(\pi\xi))} = \dfrac{10y-2}{1-\cos(\pi\xi)}$


$x = \xi$ & $\xi = x$

at $\xi = 0$, $x = 0$ & $y = 0.2$ $\quad \eta = 1$?

$\xi = 1$, $x = 1$ & $y = 0.4$ $\quad \eta = 1$?

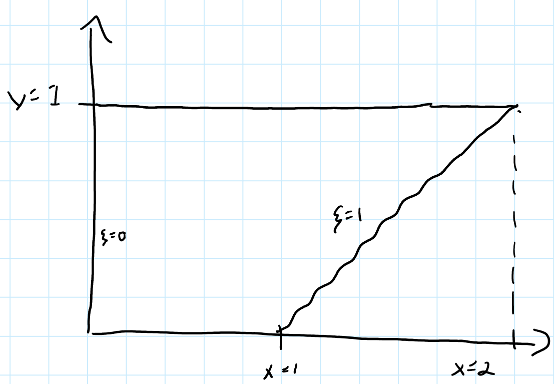
$$y = 0.2\eta\xi + 0.1(1-\cos(\pi\xi))$$

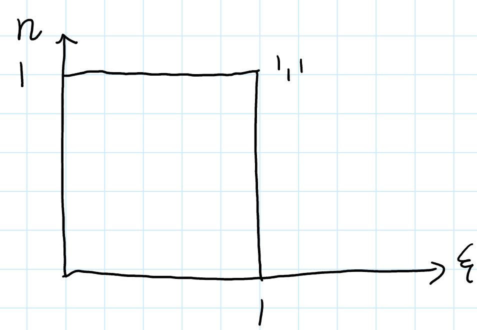
$$y - 0.1(1-\cos(\pi\xi)) = 0.2\eta\xi$$

$$\frac{y-0.1(1-\cos(\pi\xi))}{0.2\xi}$$

$$\frac{5y - 0.5(1-\cos(\pi\xi))}{\xi} = \eta$$

as $\xi \uparrow$




$\eta$ & $y$ match, so $\eta = y$ (vertical)

$\xi = \dfrac{x}{\ldots}$ ← use a normalization factor that moves based on

$\xi = \dfrac{x}{1+y}$ ← use a normalization factor that moves based on height/x-pos

$x = \xi$

$\eta = \dfrac{y}{h(x)} \rightarrow \dfrac{y}{0.2+0.1(1-\cos(\pi\xi))}$

$x = \xi \qquad \eta = \dfrac{y}{0.2+0.1(1-\cos(\pi\xi))}$

$y = (0.2+0.1(1-\cos(\pi\xi))\eta$

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(u_\xi \xi_x + u_\eta \eta_x) \\
&= u_{\xi x}\xi_x + u_\xi \xi_{xx} + u_{\eta x}\eta_x + u_\eta \eta_{xx} \\
&= (\xi_x u_{\xi\xi} + \eta_x u_{\xi\eta})\,\xi_x + u_\xi \xi_{xx} + (\xi_x u_{\eta\xi} + \eta_x u_{\eta\eta})\,\eta_x + u_\eta \eta_{xx} \\
&= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{yy} &= \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial y}(u_\xi \xi_y + u_\eta \eta_y) \\
&= u_{\xi y}\xi_y + u_\xi \xi_{yy} + u_{\eta y}\eta_y + u_\eta \eta_{yy} \\
&= (\xi_y u_{\xi\xi} + \eta_y u_{\xi\eta})\,\xi_y + u_\xi \xi_{yy} + (\xi_y u_{\eta\xi} + \eta_y u_{\eta\eta})\,\eta_y + u_\eta \eta_{yy} \\
&= u_{\xi\xi}\cancel{\xi_y^2} + 2u_{\cancel{\xi}\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \cancel{\xi_{yy}} + u_\eta \cancel{\eta_{yy}}
\end{aligned}$$

$\dfrac{\varphi_{g-1} - 2\varphi_g + \varphi_{g+1}}{\Delta\xi^2}$

$\dfrac{\varphi_{g+1} - \varphi_{g-1}}{2\Delta\xi} \cdot \dfrac{\varphi_{g+c} - \varphi_{g-d}}{2\Delta\eta}$

$\dfrac{d^2 c}{d\xi\, d\eta}$

$\xi = x$

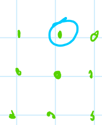
$\xi_x = 1 \qquad \xi_y = 0$

$\xi_{xx} = 0 \qquad \xi_{yy} = 0$

$\eta = \dfrac{y}{0.2+0.1(1-\cos(\pi x))}$

$\eta_y = \dfrac{1}{0.2+0.1(1-\cos(\pi x))}$

$\eta_{yy} = 0$



$\eta = y(0.2+0.1(1-\cos(\pi x)))^{-1}$

$\eta = y(0.2+0.1-0.1\cos(\pi x))^{-1}$

$\eta_x = \dfrac{-0.1\pi y\sin(\pi x)}{(0.2+0.1(1-\cos(\pi x)))^2}$

$$\eta_x = \frac{-0.1\pi y \sin(\pi x)}{(0.2+0.1(1-\cos(\pi x)))^2}$$

$$\eta_{xx} = -\frac{0.1\pi y\left(\pi\cos(\pi x)(0.2+0.1(1-\cos(\pi x))) - 2\pi\sin^2(\pi x)\right)}{(0.2+0.1(1-\cos(\pi x)))^3}$$

$$U_{\xi\xi} = \frac{\psi_L - 2\psi + \psi_R}{\Delta\xi^2}\cdot 1^2 + \overset{1/2}{\cancel{2}}\left(\frac{\psi_{LD}+\psi_{UR}-\psi_{LU}-\psi_{RD}}{\cancel{4}\Delta x\,\Delta y}\right)\cdot 1\cdot \overbrace{\left(\frac{-0.1\pi y\sin(\pi x)}{(0.2+0.1(1-\cos(\pi x)))^2}\right)}^{C_1} +$$

$$\frac{\psi_U - 2\psi + \psi_D}{\Delta\eta^2}\underbrace{\left(\frac{-0.1\pi y\sin(\pi x)}{(0.2+0.1(1-\cos(\pi x)))^2}\right)^2}_{C_2} + 0 + \frac{\psi_U - \psi_D}{2\Delta\eta}\underbrace{\eta_{xx}}_{C_3}$$

$$U_{\eta\eta} = \frac{\psi_U - 2\psi + \psi_D}{\Delta\xi^2}\overbrace{\left(\frac{1}{(0.2+0.1(1-\cos(\pi x)))}\right)^2}^{C_4}$$

$$U_{\xi\xi} + U_{\eta\eta} = 0$$

$$\frac{\psi_L - 2\psi + \psi_R}{\Delta\xi^2} + \frac{\psi_{LD}+\psi_{RU}-\psi_{LU}-\psi_{RD}}{\Delta x\,\Delta y}\,\frac{C_1}{2} + \frac{\psi_U - 2\psi + \psi_D}{\Delta\eta^2}\,C_2 + \frac{\psi_U - \psi_D}{2\Delta\eta}\,C_3 +$$

$$\frac{\psi_U - 2\psi + \psi_D}{\Delta\xi^2}\,C_4$$

| | |
|---|---|
| $\frac{C_1}{2\Delta\xi\Delta\eta}$ | $\psi_{LD}$ |
| $\frac{C_2}{\Delta\eta^2} - \frac{C_3}{2\Delta\eta} + \frac{C_4}{\Delta\xi^2}$ | $\psi_D$ |
| $\frac{-C_1}{2\Delta\xi\Delta\eta}$ | $\psi_{RD}$ |
| $\frac{1}{\Delta\xi^2}$ | $\psi_L$ |
| $\frac{-2}{\Delta\xi^2} - \frac{2}{\Delta\eta^2}C_2 - \frac{2}{\Delta\xi^2}C_4$ | $\psi$ |
| $\frac{1}{\Delta\xi^2}$ | $\psi_R$ |
| $\frac{-C_1}{2\Delta\xi\Delta\eta}$ | $\psi_{LU}$ |
| $C_2 + C_3$ ... $C_4$ | $\psi_U$ |

$\frac{-C_1}{2\Delta\xi\Delta\eta}$ $\psi_{LU}$

$\frac{C_2}{\Delta\eta^2} + \frac{C_3}{2\Delta\eta} + \frac{C_4}{\Delta\xi^2}$ $\psi_U$

$\frac{C_1}{2\Delta\xi\Delta\eta}$ $\psi_{RU}$