Def

A r.v. X that takes values in $\{0,1,2 \dots\}$ is said to be a Poisson r.v with parameter $\lambda > 0$, if $P(X=i) = \frac{e^{-\lambda} \cdot \lambda^i}{i!}$

Remark

$\sum_{i=0}^{\infty} P(X=i) = 1$

$\sum_{i=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \left\{ \begin{matrix} \text{Taylor} \\ \text{series} \end{matrix} \right\} = e^{-\lambda} \cdot e^{\lambda} = 1$

$X \sim Bin(n,p)$, $n \to \infty$, p is small $\Rightarrow$ $\lambda = np$ - moderate $\Rightarrow$ $p = \frac{\lambda}{n}$

Then $P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i} = \frac{n(n-1)\cdot \dots \cdot (n-i+1)}{n^i} \cdot \frac{\lambda^i}{i!} \cdot \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$

For n large

$$\left.\begin{aligned} &\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda} \\ &\frac{n(n-1)\cdot \dots \cdot (n-i+1)}{n^i} \approx 1 \\ &\left(1-\frac{\lambda}{n}\right)^i \approx 1 \end{aligned}\right\} \approx \frac{e^{-\lambda} \cdot \lambda^i}{i!}$$

Expectation & Variance

Intuiton: $X \sim Bin(n,p)$, $E[X] = np$, $Var[X] = np(1-p)$

$Y \sim P_o(\lambda)$, $E[Y] \text{"="} E[X] = np = \lambda$

$Var[Y] \text{"="} Var[X] = np(1-p) \approx np = \lambda$

Proof

$E[Y] = \sum_{i=0}^{\infty} \frac{i e^{-\lambda} \cdot \lambda^i}{i!} = \lambda \sum_{i=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{i-1}}{(i-1)!} = \{j = i-1\} = \lambda \underbrace{\sum_{j=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^j}{j!}}_{=1} = \lambda$

$E[Y^2] = \sum_{i=0}^{\infty} \frac{i^2 e^{-\lambda} \cdot \lambda^i}{i!} = \lambda \sum_{i=0}^{\infty} \frac{i \cdot e^{-\lambda} \cdot \lambda^{i-1}}{(i-1)!} = \{j = i-1\} = \lambda \sum_{j=0}^{\infty} \frac{(j+1) e^{-\lambda} \cdot \lambda^j}{j!} = \lambda \sum_{j=0}^{\infty} \frac{j \cdot e^{-\lambda} \cdot \lambda^j}{j!} + \lambda \sum_{j=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^j}{j!} = \lambda^2 + \lambda$

$Var[Y] = E[Y^2] - E^2[Y] = \lambda^2 + \lambda - \lambda^2 = \lambda$

Stochastic Process

Def

A stochastic process X_t is a collection of r.v.'s. That is for each t in index T, X_t is a random variable

Note

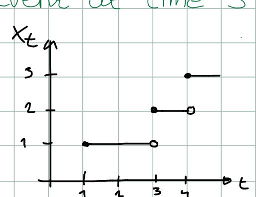
Markov Chain, $T = \{0,1,2 \dots\}$

Pair of r.v's, $T = \{1,2\}$

Poisson process, $T = \mathbb{R}$

Def

Any realization of X_t is called sample path. For instance, if events are occurring randomly in time and X_t represents the number of events that occure in $[0,t]$, then the sample path of X_t, which corresponds to the initial event occurring at t_1 the next event at time 3 and the third at time 4, is given below.



Def

A stochastic process $N_t, t \geq 0$ is said to be a counting process if N_t represents the total number of "events" that have occured up to time t.

1) $N_t \geq 0$ for every t
2) N_t is integered valued.
3) $N_t - N_s$, $t > s$, represents the number of events that have happened in the time interval $[s, t]$.

Def

A counting process is said to posses independent increments if the number of events that occur in disjoint time intervals are independent.

Def

A counting process is said to posses stationary increments if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval.

Def

The counting process N_t is said to be a poisson process having rate $\lambda > 0$, if:

1) $N_0 = 0$
2) The process has independent increments.
3) The number of events in any interval of length t [time] is given by $P(N_{t+s} - N_s = i) = \frac{e^{-\lambda t} \cdot (\lambda t)^i}{i!}$

Ex

Suppose that earthquakes occur in Japan in accordance with the assumptions above, with $\lambda = 2$ and with one week as the unit of time.
Find the probability that atleast three earthquakes occur during the next two weeks.

$$P(N_2 \geq 3) = 1 - P(N_2 = 0) - P(N_2 = 1) - P(N_2 = 2) = 1 - e^{-4} - \frac{e^{-4} \cdot 4^1}{1!} - \frac{e^{-4} \cdot 4^2}{2!} \approx 0.763$$

Def

A continuous r.v. with density $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$, $F(x) = 1 - e^{-\lambda x}$, $E[X] = \frac{1}{\lambda}$, $Var[X] = \frac{1}{\lambda^2}$ is called exponential

Ex

Suppose that earthquakes occur in Japan in accordance with the assumptions above, with $\lambda = 2$ and with one week as the unit of time.
Find the CDF of the time, starting from now, until the next earthquake.
Let X denote the amount of time, in weeks, 'til the next earthquake.

$P(X > t) = P(N_t = 0) = e^{-\lambda t}$

$F(x) = P(X \leq t) = 1 - P(X > t) = 1 - e^{-\lambda t}$

The waiting time between poisson events is exponential in accordance to λ.