Ex. prev lecture

Distribution of two balls in three cells.

1. $\{ab|-|-\}$ 4. $\{a|b|-\}$ 7. $\{-|a|b\}$
2. $\{-|ab|-\}$ 5. $\{a|-|b\}$ 8. $\{b|-|a\}$
3. $\{-|-|ab\}$ 6. $\{b|a|-\}$ 9. $\{-|b|a\}$

$[S]/[\Omega]$

$A = \{1,2,3\}$ – one cell is mutually occupied.

B – The first cell is not empty. $\Rightarrow \{1,4,5,6,8\}$

Every cell is occupied. $\Leftrightarrow$ impossible event

C – "a" is in the first cell $\Rightarrow \{1,4,5\}$

Def

1) Events A and B are independent iff (if and only if) $P(A \cap B) = P(A) \cdot P(B)$
2) Let A and B be events such that at least one of $P(A)$ or $P(B)$ is non-zero. A and B are independent iff $P(A|B) = P(A)$ if $P(B) \neq 0$, and $P(B|A) = P(B)$ if $P(A) \neq 0$.

Ex

$P(A) = \frac{3}{9} = \frac{1}{3}$

$P(B) = \frac{5}{9}$

$P(A \cap B) = \frac{1}{9} \neq \frac{1}{3} \cdot \frac{5}{9}$

$P(C) = \frac{3}{9} = \frac{1}{3}$

$P(A \cap C) = \frac{1}{9} = P(A) \cdot P(C)$, A and C are independent

Ex

Given that: $P(1) = \frac{1}{2}$

$P(2) = \ldots = P(9) = \frac{1}{16}$

$P(A) = P(1) + P(2) + P(3) = \frac{1}{2} + \frac{1}{16} + \frac{1}{16} = \frac{5}{8}$

$P(C) = P(1) + P(4) + P(5) =$ — " — $= \frac{5}{8}$

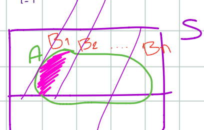
$P(A \cap C) = P(1) = \frac{1}{2} \neq P(A) \cdot P(C)$

Law of total probability

Theorem

Let $B_1, B_2, \ldots, B_n$ be such that $B_1 \cup \ldots \cup B_n = S$ and $B_i \cap B_j \neq \emptyset$, $i \neq j$, with $P(B_i) > 0$ for every i. Then for any event A; $P(A) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)$

Pic



Ex

B_1 – a is in the first cell

B_2 – a — " — second cell

B_3 – a — " — third cell.

$P(B) = P(B|B_1) \cdot P(B_1) + \ldots = 1 \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{5}{9}$

Theorem (Bayes)

Let $B_1, B_2, \ldots, B_n$ be a collection of mutually exclusive events such that $\bigcup_{i=1}^{n} B_i = S$ and $P(B_i) > 0$ for all i.

Let A be an event such that $P(A) > 0$.

$$P(B_j | A) = \frac{P(A|B_j) \cdot P(B_j)}{\sum_{i=1}^{n} P(A|B_i) P(B_i)}$$

Def. Random variables

X is the random variable if it is the function from the sample space S to the real numbers.

Def

The random number is discrete if it takes finite or at most countable number of values. Ie: its values can (in principel) be listed.

Remark I

$\mathbb{N} = \{0, 1, 2, 3 \dots\}$

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2 \dots\}$

$\mathbb{Q} = \{ p/q : p \in \mathbb{N}, q \in \mathbb{Z}, q \neq 0 \}$

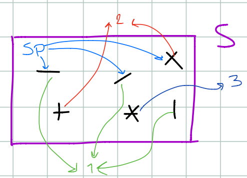
$\mathbb{R} = \mathbb{Q} \cup$ Irrational numbers.

Remark II

"Random function" is a better name for X.

The function is determenistic, but we chose events at random.

Pic



X is the random variable that takes a sample point and returns the number of intersecting lines.

$X(-) = 1$ $\quad X(\times) = 2$

$X(/) = 1$ $\quad X(\ast) = 3$

$X(|) = 1$

$X(+) = 2$

$P(-) = \dots = P(\ast) = \frac{1}{6}$

$\tilde{P}(X(s) = 1) = \tilde{P}(X = 1) = P(-) + P(/) + P(|) = \frac{1}{2}$

argument of "function X"

$P(X = 2) = \frac{1}{3}$

$P(X = 3) = \frac{1}{6}$

Def

Given any event A, define indicator random variable $\mathbb{1}_A$ equal 1 on event A and 0 otherwise.

Remark

If $A_1, A_2, \dots, A_n$ are events then $X = \sum_{i=1}^{n} \mathbb{1}_{A_i}$ is the number of the events that occure.

Ex

$S = \{HH, HT, TH, TT\}$, coin tosses of fair coin

A: One head in two tosses $A = \{HT, TH\}$

$\mathbb{1}_A(HH) = 0$, $\mathbb{1}_A(HT) = 1, \dots$

Ex

B - The first cointoss is head.

$X = \mathbb{1}_A + \mathbb{1}_B \mid X(HH) = 1$

$X(HT) = 2$

$X(TH) = 1$

$X(TT) = 0$

Def: Discrete density

Let X be a discrete r.v. The function f given by $f(x) = P(X = x)$, for $x \in \mathbb{R}$ is called "density function" for X.

Theorem of density function:

f is a discrete density iff:

1) $\forall x: f(x) \geq 0$
2) $f(x) > 0$ only in a countable number of points
3) $\sum_{\text{all } x} f(x) = 1$

Ex

Select one of the integers 1 through 10 at random and define a r.v. X as the number of its divisors. $P(1)=\ldots=P(10)$

Integer:	1	2	3	4	5	6	7	8	9	10
Divisors	1	2	2	3	2	4	2	4	3	4

Number of divisors "x"	1	2	3	4
Value of $f(x)$	0.1	0.4	0.2	0.3

$f(x)=P(X=x)$

$f(0)=0$

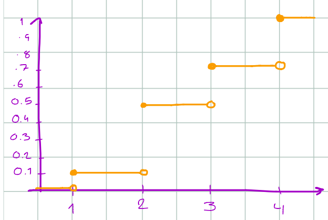
$f(2)=0.1+0.1+0.1+0.1=0.4$

Pic

Def: cumulative distribution

Let X be a discrete r.v. with density f. The cumulative distribution function for X denoted by F_X is defined by $F(x)=P(X\leq x),\ x\in\mathbb{R}$

Ex



$P(X\leq 1.5)=P(X=1)$

$P(X\leq 2)=P(X=1)+P(X=2)$

$$F(x)=\begin{cases}0, & x<1\\ 0.1, & 1\leq x<2\\ 0.5, & 2\leq x<3\\ 0.7, & 3\leq x<4\\ 1, & x\geq 4\end{cases}$$