AI 44 & 45.

Visa att $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

$z = x+iy$, $w = u+iv$

$\overline{\left(\frac{z}{w}\right)} = \overline{\left(\frac{x+iy}{u+iv}\right)} = \overline{\frac{(u-iv)(x+iy)}{(u-iv)(u+iv)}} = \frac{\overline{xu+yv+i(uy-xv)}}{u^2+v^2} = \frac{xu+yv-i(uy-xv)}{u^2+v^2}$

$\frac{\bar{z}}{\bar{w}} = \frac{x-iy}{u-iv} = \frac{(x-iy)(u+iv)}{(u-iv)(u+iv)} = \frac{xu+yv+i(xv-yu)}{u^2+v^2}$

57) Låt $x_1, x_2, \ldots x_n$ vara rötterna till $x^n = 1$. Visa att $\sum_{k=1}^{n} x_k = 0$

Bevis

ekv är: $x^n - 1 = 0$

$$x^n - 1 = (x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n) = x^n + \underbrace{x^{n-1}(-x_1-x_2\ldots-x_n)}_{\substack{=0 \\ \text{Jmfr med VL}}} + \text{lägre ordn. termer} \Rightarrow x_1 + x_2 + \ldots x_n = 0$$

Bevis 2

Geometrisk summa.

$$S = 1 + c + c^2 \ldots + c^n$$
$$cS = c + c^2 + c^3 \ldots + c^{n+1}$$
$$S - cS = 1 - c^{n+1}$$
$$(1-c)S = 1 - c^{n+1}$$
$$S = \frac{1-c^{n+1}}{1-c} \qquad c \neq 1$$

Lös ekv $x^n = 1$, $x = re^{i\theta} \Rightarrow x^n = r^n e^{i\theta n} = 1e^{i0}$

$$\begin{cases} r^n = 1 & \Rightarrow r = 1 \\ n\theta = 0 + 2k\pi & \Rightarrow \theta = \frac{2k\pi}{n} \end{cases}$$

$x_k = 1 \cdot e^{\frac{2\pi}{n}(k)i}$ $\quad x_k = \left(e^{\frac{2\pi}{n}i}\right)^k = c^k \Rightarrow \sum_{k=1}^{n} x_k = c + c^2 + c^3 \ldots + c^n = c(1 + c + c^2 + \ldots + c^{n-1}) = c\,\frac{1-c^n}{1-c} = e^{\frac{2\pi}{n}i} \cdot \frac{1 - \overbrace{e^{\frac{2\pi i}{n}n}}^{=1}}{1-c} = 0$

$x^n = 1$

AII 22

Hitta alla komplexa nollställen till $\sin(z) = 0$.

$\sin(z) = \frac{e^{iz} - e^{-iz}}{2} = 0$

$e^{iz} = e^{-iz} = \frac{1}{e^{iz}}$

$e^{2iz} = 1$

$z = x+iy \Rightarrow e^{2iz} = e^{2xi-2y} = e^{2ix} \cdot e^{-2y} = 1 \cdot e^{i0}$

$e^{-2y} = 1$

$2x = 0 + 2k\pi$

$y = 0$

$x = k\pi$

26)

$\cos z = \frac{e^{iz} + e^{-iz}}{2} = [z = x+iy] = \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(x) - i\sin(x))}{2} = \frac{\cos(x)}{2}(e^{-y} + e^{y}) + \frac{i\sin(x)}{2}(e^{-y} - e^{y})$

$\mathrm{Re}(\cos(z)) = \frac{\cos(x)}{2}(e^{-y} + e^{y})$

$\mathrm{Im}(\cos(z)) = \frac{i\sin(x)}{2}(e^{-y} - e^{y})$

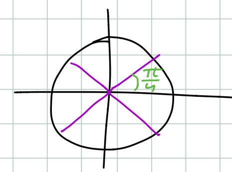
33) $P(z) = z^4 + 1$

Faktorisera $P(z)$ i reella faktorer.

$z^4 = -1$ saknar reella losningar. Om vi däremot läter $z = re^{i\theta} \Rightarrow z^4 = r^4 e^{i4\theta} = -1 = 1 \cdot e^{i\pi}$

$r^4 = 1 \quad \Rightarrow \quad r = 1$

$4\theta = \pi + 2k\pi \quad \Rightarrow \quad \theta = \frac{\pi}{4} + \frac{k\pi}{2}$



$z = \pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}}$

$$P(z) = \left(z - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(z - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= (z - z_1)(z - \bar{z}_1)(z - z_2)(z - \bar{z}_2)$$

$$= \left(z^2 + z\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)\left(z^2 + z\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)$$

$$= \left(z^2 - \frac{2}{\sqrt{2}} z + 1\right)\left(z^2 + \frac{2}{\sqrt{2}} + 1\right)$$

$$= \left(z^2 - \sqrt{2} z + 1\right)\left(z^2 + \sqrt{2} + 1\right)$$

34) $P(z) = z^4 - 4z^3 + 12z^2 - 16z + 16$, $z_1 = 1 - \sqrt{3}i$

Påstående: $P(z_1) = 0$

Om $P(z_1) = 0 \Leftrightarrow P(\bar{z}_1) = 0$ och detta går att dela med $(z - z_1)(z - \bar{z}_1)$ — $z_1 \bar{z}_1 = |z_1|^2$

$$(z - z_1)(z - \bar{z}_1) = (z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i) = z^2 + z(-1 + \sqrt{3}i - 1 - \sqrt{3}i) + (1^2 + (\sqrt{3})^2) = z^2 - 2z + 4$$

$$\begin{array}{l|l} \multicolumn{1}{l}{z^2 - 2z + 4} & \\ \hline z^4 - 4z^3 + 12z^2 - 16z + 16 & \underline{z^2 - 2z + 4} \\ \hline & 0 \end{array}$$

Divisionen gick jämt ut $\Rightarrow$ z_1 & $\bar{z}_1$ är nollställen. Återstående nollställen ges av att lösa $z^2 - 2z + 4 = 0$ men vi vet redan att det ger $z = 1 \pm \sqrt{3}i$

$$P(z) = (z - 1 - \sqrt{3}i)^2 (z - 1 + \sqrt{3}i)^2 = (z^2 - 2z + 4)^2$$

Tenta 2012-08-28

UPP 7.

$x \in \mathbb{R}$, $\frac{x}{x-i}$ blir en cirkel med $c = \frac{1}{2}$, $r = \frac{1}{2}$

Bevis

$$\left|\frac{x}{x-i} - \frac{1}{2}\right|^2 = \left|\frac{2x - (x-i)}{2(x-i)}\right|^2 = \frac{1}{4}\left|\frac{x+i}{x-i}\right|^2 = \frac{1}{4} \frac{x^2+1}{x^2+1} = \frac{1}{4}$$

2011-03-17

Uppg 6.

Finn real och imaginärdel av $\frac{2 - e^{it}}{2 + e^{it}}$.

$$\frac{(2 - e^{it})(2 + e^{-it})}{(2 + e^{it})(2 + e^{-it})} = \frac{4 - 2e^{it} + 2e^{-it} - 1}{4 + 2e^{it} + 2e^{-it} + 1} = \left[\frac{e^{it} + e^{-it}}{2} = \cos t , \quad \frac{e^{it} - e^{-it}}{2i} = \sin t\right] = \left[-2e^{it} + 2e^{-it} = \frac{-4e^{it} + 4e^{-it}}{2} = -4 \frac{e^{it} - e^{-it}}{2} = 4i \frac{e^{it} - e^{-it}}{2i}\right] = \frac{3 - 4i\sin t}{5 + 4\cos t} =$$

$$\frac{3}{5 + 4\cos t} + i\left(\frac{-4\sin t}{5 + 4\cos t}\right)$$