Räkneregler

1. Derivering är en linjär operation. Dvs: om a, b konst $\Rightarrow$ $(af+bg)' = af' + bg'$
2. Produktregeln: $(fg)' = f'g + fg'$
3. Kedjeregeln: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Ex

$\frac{d}{dx}(x^2 + x\sin x) = 2x + 1\sin x + x\underset{(\sin x)'}{\cos x}$

$\frac{d}{dx} xe^{\sin x^2} = 1 \cdot e^{\sin x^2} + xe^{\sin x^2} \cdot (\cos x^2) \cdot 2x = e^{\sin x^2} \cdot (1 + 2x^2 \cos x^2)$

Partiella derivator

$\frac{\partial f}{\partial x}$ = derivera map x men behåll övriga variabler konstanta.

Ex

$\frac{d}{dx} xe^{x^2y} = e^{x^2y} + xe^{x^2y}(2xy)$

$\frac{d}{dy} xe^{x^2y} = xe^{x^2y}(x^2)$.

Bevis

1. Lätt
2. $\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} = g(x+h)\underbrace{\frac{f(x+h)-f(x)}{h}}_{f'(x)} + f(x)\underbrace{\frac{g(x+h)-g(x)}{h}}_{g'(x)} = g(x)f'(x) + f(x)g'(x)$
3. $\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} = f'(g(x)) \cdot g'(x)$

Grundläggande derivator

$\frac{d}{dx}\sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos h + \cos x \sin h - \sin x}{h} = \sin(x) \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} = \sin(x) \cdot \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} + \cos(x) \cdot \frac{\sin h}{h} = \sin x \cdot \frac{\cos^2 h - 1}{h(\cos h + 1)} + \cos(x) \cdot \frac{\sin h}{h} =$

$-\sin x \cdot \frac{\sin^2 h}{h(\cos^2 h + 1)} + \cos x \cdot \frac{\sin h}{h} \to \cos(x)$

$\frac{\sin^2 h}{h} \to 0$, $(\cos^2 h + 1) \to 2$, $\frac{\sin h}{h} \to 1$

$\frac{d}{dx}\cos(x) = \frac{d}{dx}(\sin\frac{\pi}{2} - x) = \cos(\frac{\pi}{2} - x)(-1) = -\sin(x)$

$\frac{d}{dx}\tan(x) = \overset{\text{Kvotregeln}}{\left(\frac{f}{g}\right)' = (f(g^{-1}))' = f'g^{-1} + f(-1)g^{-2}g' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}} = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2(x)} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

Kedjeregeln ger för inversa funktionerna:

$f^{-1}(f(x)) = x$

$f^{-1\prime}(f(x))f'(x) = 1 \Rightarrow (f^{-1})' = \frac{1}{f'}$

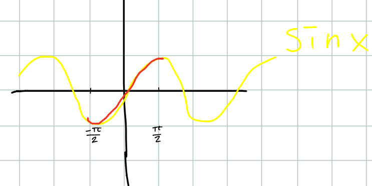
$\frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$ om vi väljer $a = e$ blir $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$ (bevisas ej)

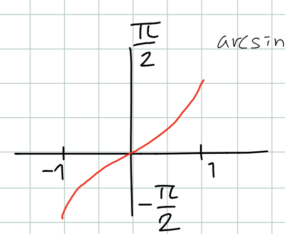
$\frac{d}{dx} e^x = e^x$

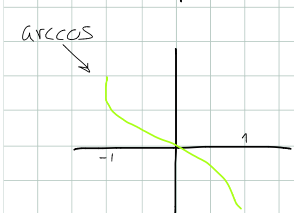
Inversen $\ln y$, $y = e^x$

$\ln'(y) = \ln'(e^x) = \frac{1}{e^x} = \frac{1}{y}$ $\quad \frac{d}{dx}\ln(x) = \frac{1}{x}$

"Inversa" trig. funktioner (Cyklometriska funktioner)







Derivator

$y = \arctan(\tan y) \quad (-\frac{\pi}{2} < y < \frac{\pi}{2})$

$\frac{d}{dx}$: $1 = \arctan'(\tan y)\tan' y = \arctan'(\tan y) \cdot \frac{1}{\cos^2 y}$

$\cos^2 y = \arctan'(\tan y)$

$\sin^2 y + \cos^2 y = 1$

$\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$

$\tan^2 y + 1 = \frac{1}{\cos^2 y}$

$\arctan'(\tan y) = \cos^2 y = \frac{1}{\tan^2 y + 1} \qquad \tan y = x \Rightarrow \arctan'(x) = \frac{1}{1+x^2}$

$y = \arcsin'(\sin y)$

$\frac{d}{dx}$: $1 = \arcsin'(\sin y)\cos y$

$\arcsin'(\sin y) = \frac{1}{\cos y} = \frac{1}{(\pm)\sqrt{1-\sin^2 y}} \Rightarrow \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

Implicit derivering

Vad är tangenten till cirkeln $x^2 + y^2 = 4$ i punkten $(\sqrt{3}, 1)$?

Tänk att $y = y(x)$. Derivera map x.

$2x + 2yy' = 0$

$y' = -\frac{2x}{2y} = \frac{-x}{y} = \sqrt{3} \Rightarrow$ Tangentens ekvation: $\sqrt{3}(x+\sqrt{3}) = (y-1)$

Explicit

$y^2 = 4 - x^2$

$y = \sqrt{4-x^2} = (4-x^2)^{\frac{1}{2}}$

$y' = \frac{1}{2} \cdot (4-x^2)^{-\frac{1}{2}}(-2x) = \frac{1}{2} \cdot \frac{1}{\sqrt{1}}(-2)(-\sqrt{3}) = \sqrt{3}$

Linjär approximation

$f(x+h) - f(x) \approx f'(x)h$

$f(x+h) \approx f(x) + f'(x)h$

Ex $\sin(47°)$

$= \sin(\frac{\pi}{4} + \frac{2\pi}{180}) \approx \sin\frac{\pi}{4} + \cos\frac{\pi}{4} \cdot \frac{2\pi}{180} = \frac{1}{\sqrt{2}}(1 + \frac{\pi}{90})$

Ex: Imp der

Bestäm $\sin y$ där y är lösning till ekvationen $e^y = x + y$, $x = 2 \pm 0.001$

$\sin y = \sin(y(x))$

$\frac{d}{dx}\sin(y(x)) = \cos(y(x))\,y'(x)$

derivera $e^y = x+y$ mapx: $e^y y' = 1 + y' \Leftrightarrow e^y y' - y' = 1 \Leftrightarrow y' = \frac{1}{e^y - 1}$

För $x = 2$ ger matlab: $y = 1{,}1462$

$y' = 0{,}4659$

$\cos y = 0{,}4120$

$\sin y = 0{,}9112$

För $x = 2 \pm 0{,}01$ får vi: $\sin y = 0{,}9112 \pm 0{,}01 \cdot 0{,}4569 \cdot 0{,}01 = 0{,}9112 \pm 0{,}001919$