Taylor-utveckling

Ex

e^{3x} kring $x=1$. (Potenser av $x-1$)

$$e^{3x} = e^{3(x-1)+3} = e^3 \cdot e^{3(x-1)} = e^3 \cdot \left(1 + \frac{3(x-1)^1}{1} + \frac{3(x-1)^2}{2} + \frac{3(x-1)^3}{3!} + \frac{3(x-1)^4}{4!} + \dots\right)$$

Utveckling av $\ln(1+x)$ kring $x=0$.

$$1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x}$$

$$n \to \infty \Rightarrow 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x} \quad \text{om } |x|<1.$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + C \qquad |x|<1$$

$$0 = \ln(1) = 0 + 0 + \dots + C \Rightarrow C = 0$$

Olika former för resttermen

Integralform: $r(x) = (-1^n)\int_a^x \frac{(t-x)^n}{n!} \cdot f^{(n+1)}(t)\,dt$

Lagrange: $r(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \qquad a<\xi<x$

Bevis

En generaliserad medelvärdessats.

Om f och g är kontinuerliga samt att g inte ändrar tecken existerar ett ξ s.a. $\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$

För fallet $g>0$.

Låt $m = \min f(x)$, $M = \max f(x)$. $a \le x \le b$, $m \le f(x) \le M$. $m\,g(x) \le f(x) \le M\,g(x)$

$$\int_a^b m\,g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \int_a^b M\,g(x)\,dx$$

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le M\int_a^b g(x)\,dx$$

$$m \le \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \le M$$

Enligt satsen om mellanliggande värden existerar $\exists \xi \Rightarrow f(\xi) = \frac{\int_a^b fg\,dx}{\int_a^b g\,dx}$

Bevis av Lagrange

$$(-1)^n \int_a^x \frac{(t-x)^n}{n!} f(t)\,dt = f^{(n+1)}(\xi)(-1)^n \int_a^x \frac{(t-1)^n}{n!}\,dt = \left[f^{(n+1)}(\xi)(-1)^n \frac{(t-x)^{n+1}}{(n+1)!}\right]_a^x = f^{(n+1)}(\xi)(-1)^{n+1}\frac{(a-x)^{n+1}}{(n+1)!} = f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!}$$

Def

f är "stort O" av g när $x \to a$. (Detta skrivs: $f(x) = O(g(x))$.) om $\left|\frac{f(x)}{g(x)}\right|$ är begränsad när $x \to a$

Restterm $r(x) = O(x-a)^{n+1}$

$O(x^k)$ betyder termer x^k och högre potenser.

Ex

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{1-(1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}\dots)}{x^2} = \lim_{x\to 0}\frac{1-(1-\frac{x^2}{2}+O(x^4))}{x^2} = \lim_{x\to 0}\frac{1-1+\frac{x^2}{2}+O(x^4)}{x^2} = \lim_{x\to 0}\frac{1}{2}+O(x^2) = \frac{1}{2}$$

Ex

$$\frac{\sin^2 x}{1-\cos x} = \frac{(x-O(x^3))^2}{1-(1-\frac{x^2}{2}+O(x^4))} = \frac{x^2-2x\,O(x^3)+O(x^3)^2}{\frac{x^2}{2}+O(x^4)} = \frac{x^2+O(x^4)+O(x^6)}{\frac{x^2}{2}+O(x^4)} = \frac{x^2+O(x^4)}{\frac{x^2}{2}+O(x^4)} = \frac{x^2(1+O(x^2))}{x^2(\frac{1}{2}+O(x^2))} \to \frac{1}{\frac{1}{2}} = 2$$

Ex

$$\frac{\sin(x)\arctan(x)-x^2}{(\cos(x)-1)^2} = \frac{(x-\frac{x^3}{6}+O(x^5))(x-\frac{x^3}{3}+O(x^5))-x^2}{(1-\frac{x^2}{2}+O(x^4)-1)^2} = \frac{\cancel{x^2}-\frac{x^4}{6}+O(x^5)-\frac{x^4}{3}+\frac{x^6}{18}+O(x^8)+O(x^6)+O(x^8)+O(x^{10})-\cancel{x^2}}{\frac{x^4}{4}+O(x^6)+O(x^8)} = \frac{-\frac{1}{2}x^4+O(x^6)}{\frac{x^4}{4}+O(x^6)} = \frac{x^4(-\frac{1}{2}+O(x^2))}{x^4(\frac{1}{4}+O(x^2))} = \frac{-\frac{1}{2}}{\frac{1}{4}} = -2$$

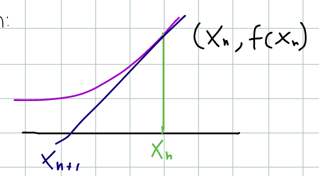
Approximation av e^x

$$r(x) = \frac{e^\xi \cdot x^{n+1}}{(n+1)!} = e^\xi \frac{x\cdot x\cdot x\cdot x\dots}{1\cdot 2\cdot 3\cdot 4\cdots(n-1)\cdot n} \to 0 \quad \forall x.$$

Om $f(x) = P(x) + O(x^{n+1})$, p är ett polynom av grad n så är p Maclaurinpolynomet

Feluppskattning i Newtons metod

Metoden:



Tangentens ekv: $y - f(x) = f'(x_n)(x - x_n)$

$y = 0$

$-f(x_n) = f'(x_n)(x_{n+1} - x_n)$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Låt x^* vara lösningen till $f(x)=0$. Lagrange restterm ger då att:

$0 = f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + f''(\xi)\frac{(x^* - x_n)^2}{2} =$

$-f'(x_n)(x^* - x_{n+1}) = f''(\xi)\frac{(x^* - x_n)^2}{2}$

$-f'(x_n)(x_{n+1} - x_n) + f'(x_n)(x^* - x_n) + f''(\xi)\frac{(x^* - x_n)^2}{2} =$

$f'(x_n)(-x_{n+1} + x_n + x^* - x_n) + f''(\xi)\frac{(x^* - x_n)^2}{2} = 0$

$\iff$

$x^* - x_{n+1} = -\frac{f''(\xi)}{2f'(x_n)}(x^* - x_n)^2$

$|x^* - x_{n+1}| \leqslant \frac{\max|f''|}{2\min|f'|}(x^* - x_n)^2$

En àposterioriuppskattning

Om vi att $f(x_n) \approx f(x^*) = 0$, vad vet vi då om $x_n - x^*$?

$f(x_n) - f(x^*) = f'(\xi)(x_n - x^*)$

$|x_n - x^*| \leqslant \frac{|f(x_n) - f(x^*)|}{\min|f'|}$