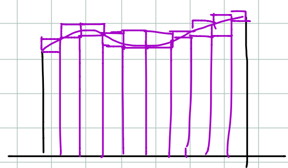
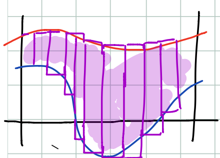
Hur man beräknar.

Numeriska metoder



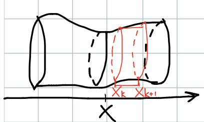
1. Rektangelregeln (mittpunktsregeln) $\int_a^b f(x)\,dx = \sum f\left(\frac{x_k+x_{k-1}}{2}\right)(x_k - x_{k-1})$
2. Trapetsregeln $\int_a^b \frac{f(x_k)+f(x_{k-1})}{2}(x_k - x_{k-1})$
3. Simpsons formel (Den MATLAB kör)

Area



Arean $\approx \sum (f(x_k)-g(x_k))(x_k - x_{k-1}) \approx \int_a^b f(x)-g(x)\,dx$

Volym (Skivformeln)

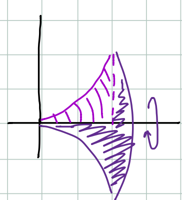


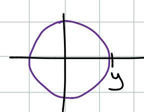
Tvärsnittsarea $A(x)$.

Volymen $\approx \sum A(x_k)(x_{k+1} - x_k)$, detta är Riemannsumman till $\int_a^b A(x)\,dx$

Ex

Ett område i planet begränsas av $y=x^2$, $y=0$ och $x=1$. Detta roteras kring x-axeln. Bestäm volymen.



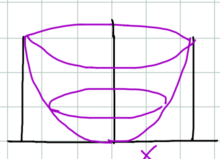


$A(x) = \pi y^2 = \pi (x^2)^2 = \pi x^4$

Volymen blir: $\int_0^1 \pi x^4\,dx = \pi \frac{x^5}{5}\Big|_0^1 = \frac{\pi}{5}$

Ex

Samma område kring y-axeln.

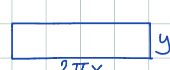


$A(y) = \pi 1^2 - \pi x^2 = \pi - \pi y$

Volymen $= \int_0^1 A(y)\,dy = \int_0^1 \pi - \pi y\,dy = \pi y - \frac{\pi y^2}{2}\Big|_0^1 = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

Rörformeln

Rotationskroppen tänks uppbyggd av tunna cylindriska skal.



Volymen $\approx \sum 2\pi x_k y_k (x_{k+1} - x_k)$, Riemannsumma till: $\int 2\pi x y\,dx$

Rotation kring Y-axeln

$$\int_0^1 2\pi x y\,dx = \int_0^1 2\pi x x^2\,dx = \int_0^1 2\pi x^3\,dx = \frac{2\pi x^4}{4}\Big|_0^1 = \frac{\pi}{2}$$

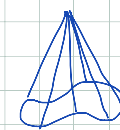
x-axeln

$$\int_0^1 2\pi y(1-x)\,dy = \int_0^1 2\pi y(1-y^{\frac{1}{2}})\,dy = \int_0^1 2\pi y - 2\pi y^{\frac{3}{2}}\,dy = 2\pi\left(\frac{y^2}{2} - \frac{2y^{\frac{5}{2}}}{5}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2} - \frac{2}{5}\right) = \frac{\pi}{5}$$

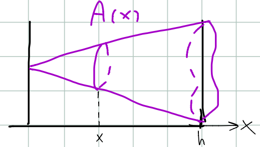
Volymen av en kon

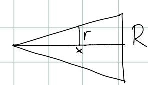
Def

En kon har en spets och en generatris, och består av alla linjer genom spetsen och en punkt i generatrisen.



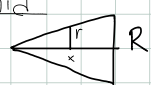
Ex



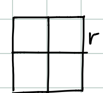


$\frac{r}{x} = \frac{R}{h} \Rightarrow r = \frac{Rx}{h} \Rightarrow A(x) = \pi \frac{R^2x^2}{h^2}$

Pyramid

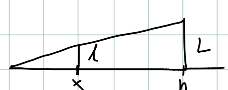


$r = \frac{Rx}{h}$



$A(x) = 2r^2 = 4r^2 = \frac{4R^2x^2}{h^2}$

Allmän kon



$\frac{l}{x} = \frac{L}{h} \Rightarrow l = \frac{Lx}{h}$

Karakteriska längder

KL står i förhållande $\frac{l}{L} = \frac{x}{h}$ areorna i $\left(\frac{l}{L}\right)^2 = \frac{x^2}{h^2}$

Om vi kallar bottenareorna $A(h)$ så har vi $A(x) = \frac{x^2}{h^2}A(h)$

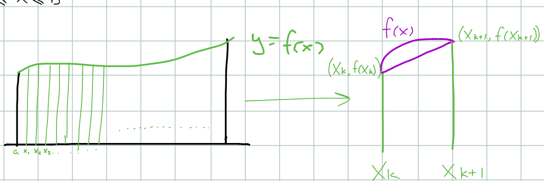
$$\int_0^h A(x)\,dx = \int_0^h \frac{x^2}{h^2}A(h)\,dx = \frac{x^3}{3h^2}A(h)\Big|_0^h = \frac{h^3}{3h^2}A(h) = \frac{A(h)h}{3}$$

Volymen av ett klot

Ett klot kan fås genom att rotera $y = \sqrt{R^2-x^2}$ genom x-axeln. Skivformeln ger: $V = \int_{-R}^{R} \pi y^2\,dx = \pi\int_{-R}^{R}(R^2-x^2)\,dx = \pi\left(R^2x - \frac{x^3}{3}\right)\Big|_{-R}^{R} =$
$\pi\left(R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3}\right) = \frac{4\pi}{3}R^3$

Längden av en kurva

$y = f(x)$, $a \leq x \leq b$



Längden av kurvan kan approximeras med polygon.

$$\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+(f(x_{k+1})-f(x_k))^2} = \begin{bmatrix}\text{medelv.}\\ \text{satsen}\end{bmatrix} = \sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+(f'(\xi_k))^2(x_{k+1}-x_k)^2} = \sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2(1+f'(\xi_k)^2)} =$$

$$\sum_{k=0}^{n-1}(x_{k+1}-x_k)\sqrt{1+f'(\xi_k)^2}$$, Riemannsumma till $\int_a^b \sqrt{1+f'(x)^2}\,dx$

Ex

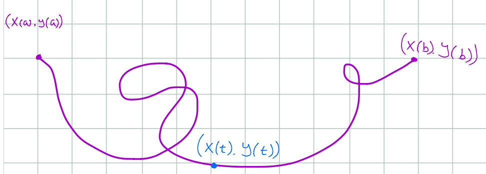
$y = \ln x - \frac{x^2}{8} \qquad 4 \leq x \leq 8$

$\frac{dy}{dx} = \frac{1}{x} - \frac{2x}{8} = \frac{1}{x} - \frac{x}{4}$

$\int_4^8 \sqrt{1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2}\, dx = \int_4^8 \sqrt{1 + \frac{1}{x^2} - \frac{2}{4} + \frac{x^2}{16}}\, dx = \int_4^8 \sqrt{\frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16}}\, dx = \int_4^8 \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\, dx = \int_4^8 \frac{1}{x} + \frac{x}{4}\, dx = \ln x + \frac{x^2}{8} \Big|_4^8 = \ln 8 + 8 - \ln 4 - 4 = \ln 2 + 6$

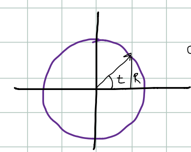
Kurvor, allmänt

Parametriserade kurvor.



Ex

Cirkeln



$\begin{cases} x = R \cdot \cos(t) \\ y = R \cdot \sin(t) \end{cases}, \quad 0 \leq t \leq 2\pi$

Gör en indelning av kurvan: $a = t_0 < t_1 < t_2 \ldots < t_n < b$

Kurvans längd $\approx \sum_{k=0}^{n-1} \sqrt{(x(t_{k+1}) - x(t_k))^2 + (y(t_{k+1}) - y(t_k))^2} = \ldots\ldots \approx$

$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$