Moment generating functions

$$M_X(t) = E[e^{tX}] = \begin{cases} \sum e^{tx} \cdot p(x), & p\text{ - discrete density} \\ \int_{-\infty}^{\infty} e^{tx} p(x)\, dx, & p\text{ - continuous density} \end{cases}$$

$Z \sim N(0,1)$

$$M_Z(t) = E[e^{Zt}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2-2tx}{2}\right)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-t)^2}{2} + \frac{t^2}{2}} dx = \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-t)^2}{2}} dx =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-t)^2}{2}} d(x-t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-t)^2}{2}} dx$$

$X \sim N(\mu, \sigma^2)$

$\frac{X-\mu}{\sigma} = Z \sim N(0,1)$

$X = \sigma Z + \mu \sim N(\mu, \sigma^2)$

$$M_X(t) = E[e^{tX}] = E[e^{t(\sigma Z + \mu)}] = E[e^{(t\sigma Z)} e^{t\mu}] = e^{t\mu} \overbrace{E[e^{t\sigma Z}]}^{M_Z(t\sigma)} = e^{t\mu} M_Z(t\sigma) = e^{t\mu} e^{\frac{(t\sigma)^2}{2}} = e^{\left(\frac{\sigma^2 t^2}{2} + \mu t\right)}$$

$$\boxed{M^{(n)}(0) = E[X^n]}$$

Proposition

X, Y are independant r.v's with mgf.s $M_X(t)$ and $M_Y(t)$ respectively. Then $M_{X+Y}(t) = M_X(t) M_Y(t)$

Proof

$$M_{X+Y}(t) = E[e^{(X+Y)t}] = E[e^{Xt} \cdot e^{Yt}] = E[e^{Xt}] \cdot E[e^{Yt}] = M_X(t) \cdot M_Y(t)$$

Corollary

X, Y are independent, $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right)} \cdot e^{\left(\frac{\sigma_2^2 \cdot t^2}{2} + \mu_2 t\right)} = e^{\frac{(\sigma_1^2 + \sigma_2^2) t}{2} + (\mu_1 + \mu_2) t}$$

$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Proposition

Mgf determines the distribution uniquely.

H/W: Check that $Y_1 + Y_2 \sim Po(\lambda_1 + \lambda_2)$ if $Y_1 \sim Po(\lambda_1)$ and $Y_2(\lambda_2)$

Misc

$Y \sim Po(\lambda)$, $Y \overset{\text{in distribution}}{=} Y_1 + Y_2$, $Y_1, Y_2 \sim Po\left(\frac{\lambda}{2}\right)$

Theorem - Markov's Inequality

If X is non-negative r.v., then for any $\varepsilon > 0$, $P(X > \varepsilon) \leq \frac{E[X]}{\varepsilon}$

Proof

O for $\varepsilon > 0$, let: $I = \begin{cases} 1 & , X \geq \varepsilon \\ 0 & \text{otherwise} \end{cases}$

Since $X \geq 0$ $I \leq \frac{X}{\varepsilon}$

Taking expectation $E[I] \leq E\left[\frac{X}{\varepsilon}\right]$, $E[I] = P(X \geq \varepsilon) \leq \frac{E[X]}{\varepsilon}$

Theorem - Chebyshev's inequality

If X is a r.v. with finite $E[X]$ and $Var[X]$, then for any $k>0$: $P(|X-\mu|\geq k) \leq \frac{Var[X]}{k^2}$

Proof

Since $(X-\mu)^2$ is a non-negative r.v. we can apply the markov inequality

$P((X-\mu)^2 \geq k^2) = \frac{E[(X-\mu)^2]}{k^2}$, $k>0$ — $Var[X]$

But since $(X-\mu)^2 \geq k^2$ if and only if $|X-\mu| \geq k$

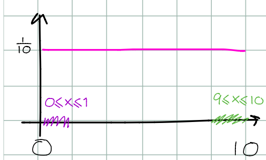
$P(|X-\mu|\geq k) \leq \frac{Var[X]}{k^2}$

$X \sim Uni(0,10)$

$E[X] = 5$, $Var[X] = \frac{25}{3}$

$P(|X-5|\geq 4) \leq \frac{\frac{25}{3}}{16} \approx 0.52$ (with Chubby)

$P(|X-5|\geq 4) = \frac{1}{10} + \frac{1}{10} = 0.2$ (w/o chubby)



density of $X \sim Uni(0,10)$

$|X-5| \geq 4 \sim \begin{cases} 10 \geq x \geq 9 \\ \text{or} \\ 1 \geq x \geq 0 \end{cases}$

Theorem - Weak law of large numbers

Let $X_1, X_2, \dots$ be a sequence of i.i.d r.v., each having: $E[X_i] = \mu$. Then for any $\varepsilon > 0$:

$P(|\underbrace{\frac{X_1+\dots+X_n}{n}}_{\bar{X}_n} - \mu| \geq \varepsilon) \to 0$ as $n \to \infty$

Proof

Additional assumption: $Var[X_i] = \sigma^2$

$E[\frac{X_1+\dots+X_n}{n}] = \mu$, $Var[\frac{X_1+\dots+X_n}{n}] = \frac{\sigma^2}{n}$

$P(|\frac{X_1+\dots+X_n}{n} - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \longrightarrow 0$ when $n \to \infty$

constants

Theorem

Let $X_1, X_2, \dots$ be a sequence of i.i.d r.v's, each having $E[X_i] = \mu$, $Var[X] = \sigma^2$. Then

$P(\frac{X_1+\dots X_n - n\mu}{\sigma\sqrt{n}} \leq a) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}} dx = \Phi(a)$

Remark

You can also say that: n

Proof

Proposition

$Y_1, Y_2, \dots$ - i.i.d rv having CDF $F_{Y_i}(x)$ and mgf $M_{Y_i}(t)$ and let Y be a rv with CDF $F_Y(t)$ and mgf $M_Y(t)$.

If $M_{Y_1+\dots+Y_n}(t) \xrightarrow[n\to\infty]{} M_Y(t)$ then $F_{Y_1+\dots+Y_n}(x) \to F_Y(x)$ for every "suitable" x.

To be continued....