Repetition

$F(x) = f_0 + f_1 x + f_2 x^2 + \ldots \longleftrightarrow \langle f_0, f_1, f_2, \ldots \rangle$

$F(0) = f_0$

$F'(x) = f_1 + 2 f_2 x + 3 f_3 x^2 + \ldots$

$F'(0) = f_1$

$F''(x) = 2 f_2 + 3 \cdot 2 \cdot f_3 x + \ldots$

$\frac{F''(0)}{2} = f_2$

$\frac{F^{(n)}(0)}{n} = f_n$

Fibonacci

$f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$

$\langle 0, 1, 1, 2, 3, 5, 8, \ldots \rangle$

$F(x) = f_0 + f_1 x + f_2 x^2 + \ldots$

$\langle f_0, f_1, f_2, f_3, \ldots \rangle$

$\langle f_0, f_1, f_0 + f_1, f_1 + f_2, \ldots \rangle$

$$\begin{array}{rcl}
\langle 0, 1, 0, 0, \ldots \rangle & \longleftrightarrow & x \\
\langle 0, f_0, f_1, f_2, \ldots \rangle & \longleftrightarrow & x F(x) \\
+ \ \langle 0, 0, f_0, f_1, f_2, \ldots \rangle & \longleftrightarrow & x \cdot x \cdot F(x) \\
\hline
\underbrace{\langle 0, f_0 + 1, f_0 + f_1, f_1 + f_2, \ldots \rangle}_{\text{Fibonacci} = F(x)} & \longleftrightarrow & x + x \cdot F(x) + x^2 \cdot F(x)
\end{array}$$

$F(x) = x + x F(x) + x^2 F(x)$

$F(x) = \frac{x}{1 - x - x^2}$

Counting Problems

$\binom{n}{k}$ - number of ways we can choose k out of n elements

n elements

Choosing k elements w/o repetition

$\langle \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, \ldots \rangle \longleftrightarrow 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n = (1+x)^n$

a_1 $\quad \langle 1, 1, 0, 0, \ldots \rangle \longleftrightarrow (1 + x)$

$a_1 \ a_2$ $\quad \langle 1, 2, 1, 0, 0, \ldots \rangle \longleftrightarrow 1 + 2x + x^2 = (1+x)(1+x)$

a_2 $\quad \langle 1, 1, 0, 0, \ldots \rangle \longleftrightarrow (1 + x)$

A_1 $\quad F(x)$ - generating function for counting problem

A_2 $\quad G(x) \ldots$

$A_1 \cup A_2$

$C(x) = F(x) G(x) = c_0 + c_1 x + c_2 x^2 + \ldots$

$c_n = f_n g_0 + f_{n-1} g_1 + \ldots + f_0 \cdot g_n$

With repetition

n element set

k elements

a_1 $\quad \langle 1, 1, 1, 1, \ldots \rangle \longleftrightarrow 1 + 1x + 1x^2 + \ldots = \frac{1}{1-x}$

a_2 $\quad \langle 1, 1, 1, \ldots \rangle \longleftrightarrow \frac{1}{1-x}$

$a_1 a_2 \longrightarrow \frac{1}{(1-x)^2}$ $\qquad$ $a_1 \dots a_n \longrightarrow \frac{1}{(1-x)^n} \longleftrightarrow \langle ?, ?, \dots \rangle$

$G(0) = 1$

$G'(x) = ((1-x)^n)' = -n(1-x)^{n-1} \cdot (-1) = n(1-x)^{-(n+1)}$

$G''(x) = n \cdot -(n+1)(1-x)^{-n-2} \cdot (-1) = n(n-1)(1-x)^{-(n+2)}$

$G^{(k)}(x) = n(n+1) \cdot \ldots \cdot (n+k-1)(1-x)^{-(n+k)}$

$\frac{G^{(k)}(0)}{k!} = \frac{n(n+1)\cdot\ldots\cdot(n+k-1)}{k!} = \frac{(n+k-1)\cdot\ldots\cdot(n+1)\cdot n}{k!} \cdot \frac{(n-1)!}{(n-1)!} = \frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k}$

Choosing k from n w/ repetition: $\frac{1}{(1-x)^n} \longleftrightarrow \left\langle \binom{n-1}{0}, \binom{n}{1}, \binom{n+1}{2}, \dots, \binom{n+k-1}{k}, \dots \right\rangle$

Ex

Choosing n fruits from a bag

- The number of apples must be even.
- The numbers of bananas must be a multiple of 5
- There can be atmost 4 oranges
- There can be atmost 1 pear.

Consider $n=6$

Apples:	6	4	4	2	2	0	0
Bananas:	0	0	0	0	0	5	5
Oranges	0	2	1	4	3	1	0
Pears	0	0	1	0	1	0	1

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Apples

$\langle 1,0,1,0,1,0,\dots \rangle \longleftrightarrow 1 + x^2 + x^4 + \ldots = \frac{1}{1-x^2} = A(x)$

Bananas

$\langle 1,0,0,0,0,\underset{\text{5 banas}}{1},0,\dots \rangle \longleftrightarrow 1 + x^5 + x^{10} + \ldots = \frac{1}{1-x^5} = B(x)$

Oranges

$\langle 1,1,1,1,1,0,0,\dots \rangle \longleftrightarrow 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x} = O(x)$

Reminder

$1 + x + x^2 + \ldots = \frac{1}{1-x}$

$1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$

Pears

$\langle 1,1,0,0,\dots \rangle \longleftrightarrow 1 + x = P(x)$

Combined

$A(x) \cdot B(x) \cdot O(x) \cdot P(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-\cancel{x^5}} \cdot \frac{1-\cancel{x^5}}{1-x} \cdot (1+x) = \frac{\cancel{1+x}}{(1-x)\cancel{(1+x)}(1-x)} = \frac{1}{(1-x)^2} \longleftrightarrow \langle 1,2,3,4,5,\dots \rangle$

Answer: The number of ways to form a bag of n fruits is $n+1$

Moment generating function

Remark

$E[X^k]$: k:th rows moment
First moment : $E[X]$
Second moment: $E[X^2]$
$E([X-E[X]])^k$: k:th central moment
Second central moment: $Var[X]$

Def

The moment generating function $M(t)$ of the r.v. X is defined for all real values of t by:

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum e^{tx} \cdot p(x) & \text{if } X \text{ is discret with density } p \\ \int_{-\infty}^{\infty} e^{tx} p(x)\,dx & \text{if } X \text{ is continuous with density } p \end{cases}$$

$M'(t) = \frac{d}{dt} E[e^{tX}] = E[\frac{d}{dt} e^{tX}] = E[X \cdot e^{tX}]$
$M'(0) = E[X]$

$M''(t) = (M'(t))' = \frac{d}{dt} E[X e^{tX}] = E[\frac{d}{dt} X \cdot e^{tX}] = E[X^2 e^{tX}]$
$M''(0) = E[X^2]$

$M^{(n)}(0) = E[X^n]$

Ex

$X \sim Bin(n, p)$
$M(t) = E[e^{tX}] = \sum_{i=0}^{n} e^{ti} \binom{n}{i} p^i (1-p)^{(n-i)} = \sum_{i=0}^{n} \binom{n}{i} (e^t p)^i (1-p)^{(n-i)} = (p \cdot e^t + 1-p)^n$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$M'(t) = n(p \cdot e^t + 1-p)^{n-1} \cdot p e^t$
$M'(0) = n(p+1-p)^{n-1} \cdot p = np = E[X]$

Ex

$X \sim Po(\lambda)$
$M(t) = E[e^{tX}] = \sum_{k=0}^{\infty} \frac{e^{tk} \cdot e^{-\lambda} \cdot \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t \cdot \lambda)^k}{k!} = \left\{ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \right\} = e^{-\lambda} \cdot e^{e^t \lambda} = e^{\lambda(e^t - 1)}$

$M'(t) = e^{\lambda(e^t-1)} \lambda e^t$
$M'(0) = \lambda = E[X]$

Ex

$X \sim Exp(\lambda)$, $p(x) = \lambda \cdot e^{-\lambda x}$ - density
$M(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx = \frac{\lambda}{\lambda - t}$, $t < \lambda$

$M'(t) = \frac{\lambda}{(\lambda - t)^2}$
$M'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} = E[X]$