Ex. Partiell integration

$$\int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\, dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4} dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

$$\int \ln x \, dx = \int 1\cdot \ln x\, dx = x\cdot\ln x - \int x\cdot\frac{1}{x}\, dx = x\cdot \ln x - \int 1\, dx = x\cdot\ln x - x + C$$

för att få $2x$ i täljaren

$$\int \arctan x\, dx = \int 1\cdot\arctan x\, dx = x\cdot\arctan x - \int x\cdot\frac{1}{1+x^2}\, dx = x\cdot\arctan x - \frac{1}{2}\int\frac{2x}{1+x^2}dx = x\cdot\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

$$\int \tan x\, dx = \int \sin x\cdot\frac{1}{\cos x}\, dx = -\cos x\cdot\underbrace{\frac{1}{\cos x}}_{\cos x^{-1}} - \int -\cos x\cdot\frac{-1}{\cos^2 x}\cdot -\sin x = -1 + \int\frac{\sin x}{\cos x}\, dx = -1 + \int \tan x\, dx$$ Dåligt

$$\int \tan x\, dx = -\ln|\cos x| \quad \text{ty,} \quad \frac{d}{dx}(-\ln|\cos x|) = \frac{-1}{\cos x}\cdot -\sin x = \frac{\sin x}{\cos x} = \tan x$$

Variabelsubstitution

Formeln för variabelomvandling.

Kedjeregeln ger $\frac{d}{dx}(G(y(x)) = G'(y(x))\cdot y'(x) = g(y)\frac{dy}{dx}$, integrering map x ger: $G(y(x)) = \int g(y)\frac{dy}{dx}\, dx = \int g\, dy$

Ex

$$\int 2x\cdot\sin x^2\, dx = \left[\begin{matrix} t = x^2 & \Rightarrow x = \sqrt{t} \\ \frac{dt}{dx} = 2x & \Rightarrow dt = 2x\,dx\end{matrix}\right] = \int \sin t\, dt = -\cos t + C = -\cos x^2 + C$$

$$\int_0^1 x\sqrt{1-x^2}\, dx = \left[\begin{matrix} 1-x^2 = t, & x=0 \Leftrightarrow t=1, \; x=1 \Leftrightarrow t=0 \\ \frac{dt}{dx} = -2x & \Rightarrow \underbrace{dt = -2x\,dx}_{= -2(x\,dx),\; x\,dx = \frac{dt}{-2}}\end{matrix}\right] = \frac{-1}{2}\int_1^0 \sqrt{t}\, dt = \frac{1}{2}\int_0^1 t^{\frac{1}{2}}\, dt = \frac{1}{2}\left.\frac{t^{\frac{3}{2}}}{3/2}\right|_0^1 = \frac{1}{3}$$

$$\int \sin^2 x\cdot\cos x\, dx = \left[\begin{matrix}\sin x = t \\ \frac{dt}{dx} = \cos x \Rightarrow dt = \cos x\, dx\end{matrix}\right] = \int t^2\, dt = \frac{t^3}{3} + C = \frac{\sin^3 x}{3} + C$$

$$\int \sin^2 x\cdot\cos^3 x\, dx = \int \sin^2 x\cdot\cos^2 x\cdot\cos x\, dx = \int \sin^2 x(1-\sin^2 x)\cos x\, dx = \left[\begin{matrix}\sin x = t \\ \frac{dt}{dx} = \cos x \Rightarrow dt = \cos x\, dx\end{matrix}\right] = \int t^2(1-t^2)\, dt = \int t^2 - t^4\, dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}\, dx = \left[\begin{matrix} x = \sin t, & x = 0 \Leftrightarrow \sin t = 0 \\ \frac{dv}{dt} = \cos t & x = \frac{1}{2} \Leftrightarrow \sin t = \frac{\pi}{6}\end{matrix}\right] = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2 t}\,\frac{dv}{dt}\, dt = \int_0^{\frac{\pi}{6}} |\cos t|\cdot\cos t\, dt = \int_0^{\frac{\pi}{6}} \cos^2 t\, dt = \int_0^{\frac{\pi}{6}} \frac{1+\cos 2t}{2}\, dt = \left.\frac{t}{2} + \frac{\sin 2t}{4}\right|_0^{\frac{\pi}{6}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2}\cdot\frac{1}{4} = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

Tips

$x = a\cdot\sin t$, $\sqrt{a^2 - a^2\sin^2 t} = a\sqrt{1-\sin^2 t}$

$\sqrt{a^2 - x^2}$

$a\sqrt{1-\left(\frac{x}{a}\right)^2}$, $\frac{x}{a} = \sin t$, $\sin^2 t + \cos^2 t = 1$ $\quad$ $\sqrt{a^2+x^2}$, $x = a\tan t$, ...

$\frac{\sin^2 t}{\cos^2 t} + 1 = \frac{1}{\cos^2 t}$

$\tan t + 1 = \frac{1}{\cos^2 t}$

Ex

$$\int \frac{e^x + 2}{e^x - e^{-x}}\, dx = \left[\begin{matrix} e^x = t \\ \frac{dt}{dx} = e^x \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t}\end{matrix}\right] = \int \frac{t+2}{t - \frac{1}{t}}\,\frac{dt}{t} = \int\frac{t+2}{t^2-1}\, dt = \int\frac{t+2}{(t-1)(t+1)}\, dt = \int \frac{A}{t+1} + \frac{B}{t-1}\, dt = \int\frac{A(t-1)+B(t+1)}{(t-1)(t+1)}\, dt = \int\frac{At - A + Bt + B}{(t-1)(t+1)}\, dt$$

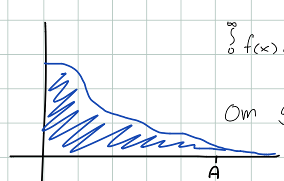
$$\left.\begin{matrix} t^1: A + B = 1 \\ t^0: -A + B = 2\end{matrix}\right\} \quad 2B = 3, \quad \begin{matrix} A = \frac{3}{2} \\ B = -\frac{1}{2}\end{matrix}$$

$$\int -\frac{1}{2}\cdot\frac{1}{t+1} + \frac{3}{2}\cdot\frac{1}{t-1}\, dt = -\frac{1}{2}\ln|t+1| + \frac{3}{2}\ln|t-1| + C = -\frac{1}{2}\ln|e^x+1| + \frac{3}{2}\ln|e^x - 1| + C$$

Arean av begränsade områden

Generaliserade (improper) integraler

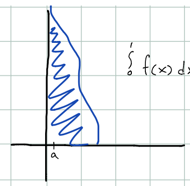
Typ 1:



$$\int_0^\infty f(x)\, dx = \lim_{A\to\infty}\int_0^A f(x)\, dx$$

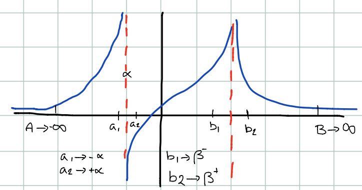
Om gränsvärdet existerar kallas intergralen **konvergent**, om inte **divergent**.

Typ 2:



$$\int_0^1 f(x)\,dx = \lim_{a\to 0^+} \int_a^1 f(x)\,dx$$

Blandad:



Ex

$\int_1^\infty \frac{1}{x^P}dx$ är konvergent $\Leftrightarrow P>1$

$\int_0^1 \frac{1}{x^P}dx$ är konvergent $\Leftrightarrow P<1$

Bevis

$$\int_1^A \frac{1}{x^P}dx = \int_1^A x^{-P}dx = \frac{x^{1-P}}{1-P}\Big|_1^A = \frac{1}{(1-P)A^{P-1}} - \frac{1}{1-P} \to \infty \text{ om } P<1, \to \frac{1}{P-1} \text{ om } P>1$$

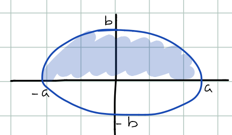
om inte $P=1$: $\int_1^A \frac{1}{x}dx = \ln|x|\Big|_1^A = \ln A \to \infty$ om $A \to \infty$

$$\int_a^1 \frac{1}{x^P}dx = \frac{x^{1-P}}{1-P}\Big|_a^1 = \frac{1}{1-P} - \frac{a^{1-P}}{1-P} \to \begin{cases} \frac{1}{1-P}, & P<1 \\ \infty, & P>1 \end{cases}$$

$P=1$: $\int_a^1 \ln x\,dx = -\ln a \to \infty,\ a\to 0$

Arean av en ellips

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$$

$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\text{Arean} = 2\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx = 2b\int_{-a}^{a}\frac{1}{a}\sqrt{a^2-x^2}\,dx = \frac{2b}{a}\underbrace{\int_{-a}^{a}\sqrt{a^2-x^2}\,dx}_{\text{arean av en halvcirkel}} = \frac{2b}{a}\cdot\frac{1}{2}\cdot\pi a^2 = \pi ab$$