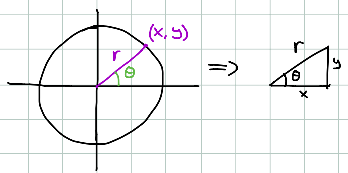
Ex

Längden av en cirkelbåge, cirkeln $x^2+y^2=r^2$

Parameterisercs:



Längden av en båge:

$x' = -r\sin\theta$

$y' = r\cos\theta$

$l = \int_{\theta_1}^{\theta_2} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta$

$= \int_{\theta_1}^{\theta_2} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$

$= \int_{\theta_1}^{\theta_2} \sqrt{r^2}\, d\theta$

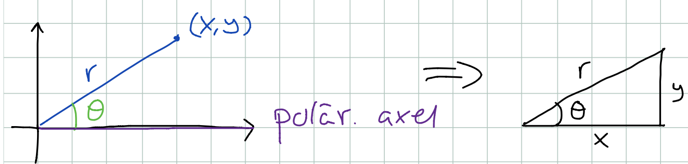
$= r\int_{\theta_1}^{\theta_2} d\theta$

$= r(\theta_2 - \theta_1)$

Hela cirkeln: $0 \leq \theta \leq 2\pi$

$r(2\pi - 0) = 2\pi r$

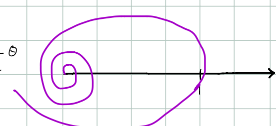
Polära koordinater



$y = r\sin\theta$

$x = r\cos\theta$

Logaritmisk spiral: $r = e^{-\theta}$



Längden av bågen: $0 \leq \theta \leq 2\pi$

$x = r\cos\theta = e^{-\theta}\cdot\cos\theta$

$y = r\sin\theta = e^{-\theta}\cdot\sin\theta$

$x' = -e^{-\theta}\cdot\cos\theta - e^{-\theta}\cdot\sin\theta$

$y' = -e^{-\theta}\cdot\sin\theta + e^{-\theta}\cdot\cos\theta$

$l = \int_0^{2\pi} \sqrt{(-e^{-\theta}\cos\theta - e^{-\theta}\sin\theta)^2 + (-e^{-\theta}\sin\theta + e^{-\theta}\cos\theta)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{e^{-2\theta}\cos^2\theta + \cancel{2e^{-2\theta}\cos\theta\cdot\sin\theta} + e^{-2\theta}\sin^2\theta - \cancel{2e^{-2\theta}\sin\theta\cos\theta} + e^{-2\theta}\cos^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{e^{-2\theta}(\cos^2\theta + \sin^2\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$

$= \int_0^{2\pi} e^{-\theta}\sqrt{2}\, d\theta$

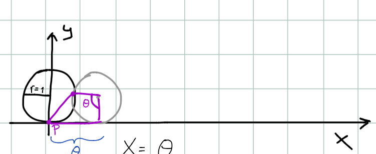
$= -\sqrt{2}\cdot e^{-\theta}\Big|_0^{2\pi}$

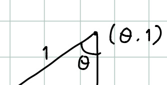
$= \sqrt{2}(-e^{-2\pi} + e^0)$

$= \sqrt{2}\left(1 - \frac{1}{e^{2\pi}}\right)$

Cykloiden

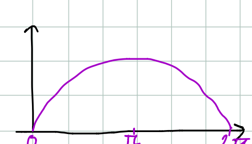
Den kurva en punkt på en cirkel beskriver om cirkeln rullar på en linje.





$\begin{cases} x = \theta - \sin\theta \\ y = 1 - \cos\theta \end{cases}$

Längden av bågen $0 \leq \theta \leq 2\pi$:



$x' = 1 - \cos\theta$

$y' = \sin\theta$

$l = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{4\,\frac{1-\cos\theta}{2}}\, d\theta$

$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$

$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$

$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta$

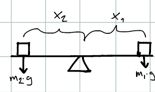
$= 2 - \cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\Big|_0^{2\pi}$

$= 4(-\cos\pi - (-\cos(0)))$

$= 4(1+1) = 8$

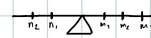
Tyngdpunkten av en rak stång

1. Gungbräda



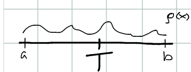
jämvikt: $x_1 \cdot m_1 = x_2 \cdot m_2$

Fler tyngder:



jämvikt: $\sum x_k m_k = \sum y_k n_k$

Kontinuerlig fördelning:



$\rho(x)$ en massfördelning (kg/m)

massan nära x påverkar vridmomentet med $(x-T)\rho(x)\,dx$

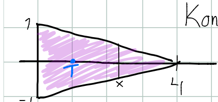
Villkoret för jämvikt: Lika moment vänster som höger kring tyngdpunkten T.

$$\underbrace{\int_a^T (T-x)\rho(x)\,dx}_{\text{Till V.}} = \underbrace{\int_T^b (x-T)\rho(x)\,dx}_{\text{Till H.}}$$

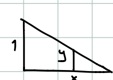
$$0 = -\int_a^T (T-x)\rho(x)\,dx + \int_T^b (x-T)\rho(x)\,dx$$
$$= \int_a^T (x-T)\rho(x)\,dx + \int_T^b (x-T)\rho(x)\,dx$$
$$= \int_a^b (x-T)\rho(x)\,dx$$
$$= \int_a^b x\rho(x) - T\rho(x)\,dx$$
$$= \int_a^b x\cdot\rho(x)\,dx - \int_a^b T\rho(x)\,dx$$
$$= \int_a^b x\cdot\rho(x)\,dx - T\int_a^b \rho(x)\,dx \Rightarrow T = \frac{\int_a^b x\,\rho(x)\,dx}{\underbrace{\int_a^b \rho(x)\,dx}_{\text{Totala massan}}}$$

Ex

Tyngdpunkten för en triangel.



Konstant densitet $\rho \frac{kg}{m^2}$

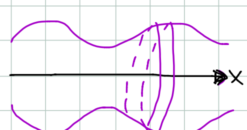


$\frac{y}{1} = \frac{4-x}{4} = 1 - \frac{x}{4}$ $\qquad \rho(x) = 2(1-\frac{x}{4})\rho$

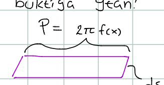
Tyngdpunkten är $(T, 0)$ där $T = \dfrac{\int_0^4 x\cdot 2(1-\frac{x}{4})\rho\,dx}{\rho \cdot \text{Arean}} = \dfrac{1}{2\cdot\frac{4}{2}}\int_0^4 2x - \frac{x^2}{2}\,dx = \frac{1}{4}\left(x^2 - \frac{x^3}{6}\right)\Big|_0^4 = \frac{1}{4}\left(16 - \frac{64}{6}\right) = \frac{16}{4\cdot 3} = \frac{4}{3}$

Rotationsyta

$y = f(x)$ roteras kring x-axeln, vilken area har den buktiga ytan?



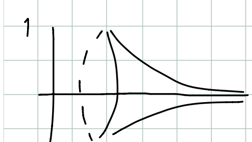
Tänks bestå av band:



$$\text{Arean} = \int_a^b 2\pi f(x)\,ds = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Ex

Kurvan $y = \frac{1}{x}$, $x \geq 1$ roteras kring x-axeln



Volymen (Skivformeln) $= \int_1^\infty \pi \frac{1}{x^2}\,dx = -\pi\frac{1}{x}\Big|_1^\infty = 0 + \frac{\pi}{1} = \pi$

Arean: $\int_1^\infty 2\pi\frac{1}{x}\sqrt{1+(-\frac{1}{x^2})^2}\,dx = \int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > \int_1^\infty \frac{2\pi}{x}\,dx = \infty$