12.5-6

Utifrån tangentplan har vi fått s.k. linjära approximationer.

$f(x,y): \mathbb{R}^2 \to \mathbb{R}$

Tangentplanet till funktionsytan $z = f(x,y)$ i $(a,b,f(a,b))$ ges av $f_1(a,b)(x-a) + f_2(a,b)(y-b) - (z - f(a,b)) = 0$

Linjäriseringen ges av: $Lf(a+b, h+k) = f(a,b) + f_1(a,b)h + f_2(a,b)k$

$Lf(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$

Deriverbarhet $\Leftrightarrow$ Linj appx är "bra" (små feltermer)

Sats

f deriverbar om f_1 & f_2 kontinuerliga

Kedjeregel

$f: \mathbb{R} \to \mathbb{R}, \quad \lambda: \mathbb{R} \to \mathbb{R}$

$f(\lambda(t))' = f'(\lambda(t)) \cdot \lambda'(t) = \frac{\partial f}{\partial \lambda} \cdot \frac{d\lambda}{dt}$

Vi vill få bra formler för $\frac{\partial}{\partial s} f(u(s,t), v(s,t))$ & $\frac{\partial}{\partial t}(u(s,t), v(s,t))$, $u(s,t): \mathbb{R}^2 \to \mathbb{R}$ $\quad v(s,t): \mathbb{R}^2 \to \mathbb{R}$.

Typiskt om man vill göra variabelbyte.

Hypotes: $a = b = f(a,b) = 0$ $\qquad$ Vi vill visa att $L(f(u,v)) = Lf \cdot (Lu, Lv)$

$u(0,0) = v(0,0) = 0$

$u(s,t) \approx Lu = \frac{\partial u}{\partial s} s + \frac{\partial u}{\partial t} t$

$v(s,t) \approx Lv = \frac{\partial v}{\partial s} s + \frac{\partial v}{\partial t} t$ $\qquad$ (Betraktar jag linjeriseringar map (u,v) & (s,t).

$f \approx Lf = \frac{\partial f}{\partial s} s + \frac{\partial f}{\partial t} t = \frac{\partial f}{\partial u} u + \frac{\partial f}{\partial v} v \approx \frac{\partial f}{\partial u} Lu + \frac{\partial f}{\partial v} Lv = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial s} s + \frac{\partial u}{\partial t} t\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial s} s + \frac{\partial v}{\partial t} t\right)$

Jämför två uttryck för Lf. ① $Lf = \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s}\right)s + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}\right)t$

② $Lf = \frac{\partial f}{\partial s} s + \frac{\partial f}{\partial t} t$

$f \neq Lf$, men om f diff $\Rightarrow$ $f = Lf +$ liten felterm

$u = Lu +$ — " —

$v = Lv +$ — " —

Feltermerna kommer inte spela någon roll för sammansättningen $\Rightarrow$ s-linjära termerna & t-linjära termerna motsvarar varandra.

Formler

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s}$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}$

Ex

Triggettan mha kedjeregeln.

$f(u,v) = u^2 + v^2$, $\quad u = \cos t$, $v = \sin t$ (ingen s-variabel)

$f(\cos t, \sin t) = 1$

Vill visa att deriv i t är $0 \Rightarrow f(\cos t, \sin t) = $ konst. $f = 1$ $\;$ m test g

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}$

$\left.\begin{array}{l} \frac{\partial f}{\partial u} = 2u, \quad \frac{\partial f}{\partial v} = 2v \\ \frac{\partial u}{\partial t} = -\sin t, \quad \frac{\partial v}{\partial t} = \cos t \end{array}\right\} \frac{\partial f}{\partial t} = 2\cos t(-\sin t) + 2\sin t \cos t = 0$ $\qquad$ $\cos 0 = 1, \quad \sin 0 = 0$

Ytterligare härledning av formeln i linjära fallet: $f(u,v) = Au + Bv$, $\frac{\partial f}{\partial u} = A$, $\frac{\partial f}{\partial v} = B$
$u(s,t) = a \cdot s + bt$, $\frac{\partial u}{\partial s} = a$, $\frac{\partial u}{\partial t} = b$
$v(s,t) = cs + dt$, $\frac{\partial v}{\partial s} = c$, $\frac{\partial v}{\partial t} = d$

Vill ha deriv. av sammansättningen: $f(u(s,t), v(s,t)) = A(as+bt) + B(cs+dt) = (Aa + Bc)s + (Ab + Bd)t$

$\Rightarrow \frac{\partial f}{\partial s} = Aa + Bc$
$\frac{\partial f}{\partial t} = Ab + Bd$ } Insättning av ◯ bekräftar kedjeregeln.

Säg att $f: \mathbb{R}^2 \to \mathbb{R}$
$\lambda: \mathbb{R}^2 \to \mathbb{R}^2$, $\lambda(s,t) = (u(s,t), v(s,t))$

$df = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$ vektor $d\lambda = \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix}$ Kedjeregeln: $d(f \circ \lambda) = df \circ d\lambda$ ← vanliga kedjeregeln

Ex

Beräkna $\frac{\partial f}{\partial x}(x^2+y^2, x+y)$ & $\frac{\partial f}{\partial y}(x^2+y^2, x+y)$ i termer av f_1 och f_2.

$F(x,y) = f(\underbrace{x^2+y^2}_{u}, \underbrace{x+y}_{v})$

$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_1 \cdot 2x + f_2 \cdot 1$ $\qquad$ $\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = f_1 \cdot 2y + f_2 \cdot 1$

Tredje gången.....

Kedjeregeln enl. boken i en variabel

$f: \mathbb{R}^2 \to \mathbb{R}$, $u = u(t)$, $v = v(t)$, finns inget s.

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}$

"Bevis"

$g(t) = f(u(t), v(t))$

$\frac{\partial f}{\partial t} = \frac{dg}{dt} = \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0} \frac{f(u(t+h), v(t+h)) - f(u(t), v(t))}{h} = \left[\begin{matrix}\text{lägg till, dra bort} \\ f(u(t), v(t+h)\end{matrix}\right] =$

ingen h-beroende i första variabeln

$\lim_{h\to 0} \frac{f(u(t+h), v(t+h) - f(u(t), v(t+h))}{h} + \lim_{h\to 0} \frac{f(u(t), v(t+h) - f(u(t), v(t))}{h}$ $\Rightarrow$ Andra termen är en vanlig sammansätt av 1-var-funktionen $\Rightarrow$ Vanlig kedjeregel

$\Rightarrow$ Andra termen $= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} - \frac{\partial f}{\partial v}\frac{\partial v}{\partial t}$

$\sim$ Med lite våld, ja första termen $\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}$

Ex

$f(u,v) = u^2 v$, $u = \cos(xy)$, $v = e^x$ Vad är $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 2uv(-y(\sin(xy)) + u^2 \cdot e^x = 2\cos(xy)e^x(-y\sin(xy) + \cos^2(xy)e^x$

$\frac{\partial f}{\partial y} =$ P.SS.

Idén är att linjäriseringarna ger goda approximationer som vi fick från geometri.
Vad betyder de?

$\lambda(s,t) = (u(s,t), v(s,t)) : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad d\lambda = \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix}$ ← Matrisen är en linjär avbildningen mellan tangentplanet som def av olika ytor.

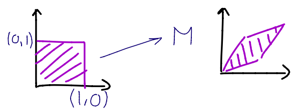
Detta kallas Jacobimatrisen av transformationen $(u(s,t), v(s,t)) : \mathbb{R}^2 \to \mathbb{R}^2$

Def

det|M| = Jacobiderminanten, den mäter hur mycket volymen/arean ändras under avbildningen

Ex

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

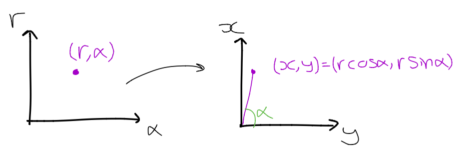


← |det M| = volym/area-ändring dvs arean av parallellogram

Ex

$x = r\cos\alpha$, $y = r\sin\alpha$ } Polära koordinater

Jacobi matrisen M och det M?



$$M = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -r\sin\alpha \\ \sin\alpha & r\cos\alpha \end{pmatrix}$$

$$\det M = (\cos\alpha \cdot r\cos\alpha - (-r\sin\alpha \cdot \sin\alpha)) = r(\cos^2\alpha + \sin^2\alpha) = r$$

12.7 | Gradienter och Riktningsvektorer

Def

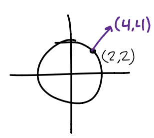
$\nabla f = (f_1, f_2)$ = gradienten till funktionen $f: \mathbb{R}^2 \to \mathbb{R}$

Ex

Jacobimatrisen M för $(u(s,t), v(s,t))$ $\quad M = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}$ $\quad \nabla \equiv \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right)$
$\nabla f = \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j\right)$

Ex

$f(x,y) = x^2 + y^2 \Rightarrow \nabla f = (2x, 2y)$
i punkten $x = y = 2$ kommer $\nabla f(2,2) = (4,4)$



Notera att vektorn $\nabla f(2,2)$ är ortogonal till tangentlinjen till nivåkurvan.
$(2\sqrt{2})^2 = x^2 + y^2$

Ex

$f(x,y) = 5x + 3y - 7$, gradienten i $(1,1)$

$\nabla f = (5,3) \Rightarrow z(1,1) = 5 + 3 - 7 = 1$

Kurvan $5x + 3y = 8$ är en nivåkurva till $z = 1$. Vektorn $(5,3)$ är exakt normalen till kurvan.

Sats

$\nabla f(a,b)$ är ortogonal till tangenten till nivåkurvan $z = f(x,y)$ för fixt x, y. $f(a,b) = f(x,y)$
Om $\nabla f(a,b) \neq 0$

Motexempel

$z = x^3 - y^2$, $(0,0)$ $\Rightarrow \nabla f(3x^2, -2y)$, $z = 0 = x^3 - y^2$ $\Rightarrow x^3 = y^2$

relevant nivåkurva

Blir konstig när man riter....

Bevis

Betrakta $z = c = f(x,y)$. Antag att vi har en parametrisering. $C = f(x(t), y(t)) = $ konst.

$\frac{dC}{dt} = 0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right) \cdot (x', y') = 0$

dot product

tangentvektor/
hastigheten