Def

Let $X_1, X_2, \ldots$ be a sequence of r.v's with c.d.f's $F_1, F_2, \ldots$ Let X be a r.v. with cdf F. We say that X_n converges in distribution to X if $\lim_{n\to\infty} F_n(x) = F(x)$, for every x. (Every x at which F is continuous.)

Theorem - Continuity

Let F_n be a sequence of c.d.f's with the corresponding mgf's M_n. Let F be a cdf with mgf M. If $M_n(t) \to M(t)$ when $n \to \infty$ for all t, then $F_n(x) \to F(x)$ when $n \to \infty$ for every x.

Theorem - CLT

Let $X_1, X_2, \ldots$ be a sequence of i.i.d r.v's having mean 0, variance σ^2, c.d.f F and mgf M. Let $S_n = \sum_{i=1}^{n} X_i$, then $\lim_{n\to\infty} \underbrace{P\left(\frac{S_n}{\sigma\sqrt{n}} \leq x\right)}_{F_{\frac{S_n}{\sigma\sqrt{n}}}(x)} = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$, c.d.f of $N(0,1)$

Proof

Let $Z = \frac{S_n}{\sigma\sqrt{n}}$. We will show that $M_{Z_n}(t) \xrightarrow[n\to\infty]{} e^{\frac{t^2}{2}}$

$M_{S_n}(t) = M_{X_1 + \ldots X_n}(t) = (M(t))^n$

$M_{Z_n}(t) = \{E[e^{tx}]\} = E[e^{tZ_n}] = E[e^{t \cdot \frac{S_n}{\sigma\sqrt{n}}}] = M_{S_n}\left(\frac{t}{\sigma\sqrt{n}}\right) = \left(M\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$

Remark - Tayler exp.

$f(s) = f(0) + f'(0)s + \frac{1}{2}f''(0)s^2 + \varepsilon$, $\frac{\varepsilon}{s^2} \to 0$ when $s \to 0$

$e^x \approx 1 + x + \frac{x^2}{2}$, when x is small.

$\sin(x) \approx x$ — '' —

$\cos(x) \approx 1 - \frac{x^2}{2}$, — '' —

$\ln(1-x) \approx -x - \frac{x^2}{2}$ — '' —

Use Taylor expantion fer mgf M.

$M(s) = M(0) + M'(0)s + \frac{1}{2}M''(0)s^2 + \varepsilon$

$M'(0) = E[X_i] = 0$

$M''(0) = E[X_i^2]$

$Var[X_i] = E[X_i^2] - E^2[X_i] \Rightarrow$

$M(t) = E[e^{tX_i}]$

$M''(0) = Var[X_i] = \sigma^2$

$M_{Z_n}(t) = \left(M\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n \approx \left(1 + \frac{1}{2}\sigma^2 \cdot \left(\frac{t}{\sigma\sqrt{n}}\right)^2\right)^n = \left(1 + \frac{t^2}{2n}\right)^n$

assuming ε small

Remark

$\lim_{n\to\infty}\left(1 + \frac{a_n}{n}\right)^n = e^a$, where $a_n \to a$, $n \to \infty$

$M_{Z_n}(t) \approx \left(1 + \frac{t^2}{n}\right)^n \to e^{\frac{t^2}{2}}$

Def - Method of moments (MM)

1. Calculate low-order moments $E[X], E[X^2], E[X^3]$. Finding expressions for the moments in terms of the parameters
2. Invert the expressions from step one to find new expressions for the parameters in terms of the moments.
3. Insert the sample moments into the expressions from step 2.

Ex

$X \sim Bin(n,p)$ Find the method of moments (MM) estimator of p

1. $E[X]=np$
2. $p=\frac{E[X]}{n}$
3. $\hat{p}=\frac{\overline{X}}{n}$

$X \sim Po(\lambda)$

1. $E[X]=\lambda$
2. $\lambda=E[X]$
3. $\lambda=\overline{X}$

$X \in \{0,1,2\}$

$P(X=0)=p$, $P(X=1)=P(X=2)=\frac{1-p}{2}$

1. $E[X]=0\cdot p+1\cdot\frac{1-p}{2}+2\cdot\frac{1-p}{2}=3\left(\frac{1-p}{2}\right)=\frac{3-3p}{2}$
2. $1-p=\frac{2}{3}E[X]=$

   $p=1-\frac{2}{3}E[X]$
3. $\hat{p}=1-\frac{2}{3}\overline{X}$

Def - Likelihood Function

Let $X_1,\dots,X_n$ be i.i.d r.v's with density f. Then $Lik(\theta)=\prod_{i=1}^{n} f_\theta(X_i)$ - likelihood function

$L(\theta)=\log(Lik(\theta))=\sum_{i=1}^{n}\log(f_\theta(x_i))$ - log likelihood function

MLE estimator is the that maximizes likelihood function

Ex

$X \in \{0,1,2\}$

$P(X=0)=p$, $P(X=1)=P(X=2)=\frac{1-p}{2}$ $\Rightarrow$ $0\leq p\leq 1$

$x_1=0,\ x_2=0,\ x_3=1$

$lik(p)=p\cdot p\cdot\frac{1-p}{2}$, $L(p)=\log(lik(p))=2\log(p)+\log\left(\frac{1-p}{2}\right)$, $L'(p)=2\cdot\frac{1}{p}+\frac{2}{1-p}\cdot\frac{1}{2}=\frac{2}{p}-\frac{1}{1-p}=\frac{2-3p}{p(1-p)}$

"$f_\theta(x_1)$"

$2-3p=0 \Rightarrow \hat{p}=\frac{2}{3}$, mle estimate

Ex

$X_1,\dots,X_n$ are coming from an exponential distribution: $Exp(\theta)$. Density of $Exp(\theta)$ is given by: $f_\theta(x)=\theta e^{-\theta x}$

$lik(\theta)=\theta e^{-\theta x_1}\cdot\ldots\cdot\theta e^{-\theta x_n}$

$L(\theta)=\log(\theta e^{-\theta x_1}+\ldots+\theta e^{-\theta x_n})=\{\log\theta+\log e^{-\theta x_1}=\log\theta-\log\theta x_1\}=n\log\theta-\theta\sum_{i=1}^{n}x_i$

$L'(\theta)=\frac{n}{\theta}-\sum_{i=1}^{n}x_i=\frac{n-\theta\sum_{i=1}^{n}x_i}{\theta}$

$L'(\hat{\theta})=0 \iff \hat{\theta}=\frac{n}{\sum_{i=1}^{n}x_i}=\frac{1}{\overline{x}}$

mle estimate

Remark

If $X\sim Exp(\theta)$ then $E[X]=\frac{1}{\theta}$ or $\theta=\frac{1}{E[X]}$

Regarding the exam

1. Generating functions
   How to obtain GF for a given sequence.
   How to obtain a sequence given a GF.
   Counting Problems (like the example with the bag of fruits)

2. Moment Generating Functions
   Find a mgf for a given distribution. Both with given values and generally.
   Find some moments of a given distribution.

3. Limit Theorems
   Proof's of: Markov's, Chebyshev's inequalities, law of large numbers, CLT.

4. Confidence Intervals
   CI for the μ, σ^2 known
   CI for the μ, σ^2 unknown (small sample) $\leftarrow$ t-distribution
   CI for σ^2
   CI for P - population proportion

Ex: How to construct an interval.

$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ approx $N(0,1)$ because of CLT.

$P(L_1 \leqslant \mu \leqslant L_2) = (1-\alpha)$ $\quad [L_1, L_2]$ - $100(1-\alpha)\%$ CI for μ.

$$(1-\alpha) = P\left(L'_1 \leqslant \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leqslant L'_2\right) = P\left(\frac{L'_1\sigma}{\sqrt{n}} \leqslant \overline{X}-\mu \leqslant \frac{L'_2\sigma}{\sqrt{n}}\right) = P\left(\frac{L'_1\sigma}{\sqrt{n}} - \overline{X} \leqslant -\mu \leqslant \frac{L'_2\sigma}{\sqrt{n}} - \overline{X}\right) = P\left(\overline{X} - \frac{L'_2\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{X} - \frac{L'_1\sigma}{\sqrt{n}}\right)$$