Hur man gör om en n:te ordningens ODE till 1a ordningens. Inför en vektorbehant.

Ex

$x^2y'' - xy' + 2y = \sin x$

$y'' - \frac{1}{x}y' + \frac{2}{x^2}y = \frac{\sin x}{x^2}$

Inför $u = \begin{bmatrix} y \\ y' \end{bmatrix}$

$u' = \begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ \frac{\sin x}{x^2} - \frac{y'}{x} - \frac{2y}{x^2} \end{bmatrix} = \begin{bmatrix} u_2 \\ \frac{\sin x}{x^2} - \frac{u_2}{x} - \frac{2u_1}{x^2} \end{bmatrix} = F(u, x)$

Komplexa tal

är 2-vektorer som utrustats med en produkt: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$. Dessutom finns konjugering av komplexa tal.

Om $z = x + iy$, $\bar{z} = x - iy$

Räkneregler

$\overline{z + w} = \bar{z} + \bar{w}$

$\overline{zw} = \bar{z}\,\bar{w}$

$z\bar{z} = (x+iy)(x-iy) = x^2 + iyx - iyx + (-1)y^2 = x^2 + y^2 = |z|^2$

$z + \bar{z} = x + iy + x - iy = 2x = 2 \cdot \operatorname{Re} z$

$z - \bar{z} = (x+iy) - (x-iy) = x + iy - x + iy = 2i \cdot \operatorname{Im} z$

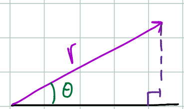
Ex

$\frac{1+3i}{2+3i} = \frac{(2-3i)(1+3i)}{(2-3i)(2+3i)} = \frac{2+6i-3i+9}{2^2+3^2} = \frac{11+3i}{13} = \frac{11}{13} + i\frac{3}{13}$

Skalärprodukt

$\operatorname{Re} z\bar{w} = \operatorname{Re}(x+iy)\overline{(u+iv)} = \operatorname{Re}(x+iy)(u-iv) = \operatorname{Re}(xu + iuy - ixv + yv) = xu + yv$

Polära koordinater



$r = |z|$

$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

$x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Vi har tidigare visat att $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$. Multiplikation $re^{i\theta}Re^{i\varphi} = rRe^{i(\theta+\varphi)}$ $\begin{cases} |zw| = |z||w| \\ \arg(zw) = \arg(z) + \arg(w) \end{cases}$

Ex

$\frac{(1+i\sqrt{3})^{17}}{(1-i\sqrt{3})^{17}}$

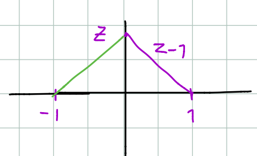
$1 + i\sqrt{3} = \left[\begin{matrix}\text{Räkna ut} \\ \text{längden.}\end{matrix} \ (\sqrt{3})^2 + 1^2 = 4 \Rightarrow |1+\sqrt{3}| = 2\right] = 2\Big(\underset{\substack{\uparrow \\ \cos\theta}}{\tfrac{1}{2}} + \underset{\substack{\uparrow \\ \sin\theta}}{\tfrac{i\sqrt{3}}{2}}\Big) = 2e^{i\frac{\pi}{3}}$

$1 - i\sqrt{3} = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2e^{-i\frac{\pi}{3}}$

$\frac{(1+i\sqrt{3})^{17}}{(1-i\sqrt{3})^{17}} = \frac{2^{17}e^{i\frac{17\pi}{3}}}{2^{17}e^{-i\frac{17\pi}{3}}} = e^{i\frac{34\pi}{3}} = e^{i\frac{4\pi}{3}} = e^{-\frac{2\pi}{3}}$ $\left(2\pi = \frac{6\pi}{3}\right)$

Geometri

Vad är den geometriska betydelsen av $|z-1| = |z+1|$?

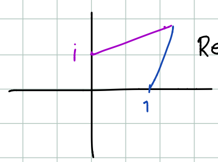


$|z-1| = \text{avst}(z, 1)$

$|z+1| = |z-(-1)| = \text{avst}(z, -1)$

Samma avstånd: $\operatorname{Re} z = 0$

Vad är den geometriska betydelsen av $|z-1| = |z-i|$?



$\text{Re } z = \text{Im } z$

Alt. lösn

$z = x + iy$

$|z-1| = |z-i|$

$|z-1|^2 = |z-i|^2$

$|x+iy-1|^2 = |x+iy-i|^2$

$(x-1)^2 + y^2 = x^2 + (y-1)^2$

$x^2 - 2x + 1 + y^2 = x^2 - 2y + 1 + y^2 \Rightarrow -2x = -2y \Leftrightarrow x = y$

Ex

$|z-1| = 2|z+1|$?

$z = (x + iy)$

$|z-1|^2 = 4|z+1|^2$

$|x+iy-1|^2 = 4|x+iy+1|^2$

$(x-1)^2 + y^2 = 4((x+1)^2 + y^2)$

$x^2 - 2x + 1 + y^2 = 4(x^2 + 2x + 1) + 4y^2$

$3x^2 + 10x + 3 + 3y^2 = 0$

$x^2 + \frac{10}{3}x + 1 + y^2 = 0$

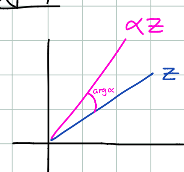
$\underbrace{\left(x + \frac{5}{3}\right)^2}_{x^2 + \frac{10}{3}x + \frac{25}{9}} + y^2 = -1 + \frac{25}{9} = \frac{16}{9} \Rightarrow$ resultatet är en cirkel med centrum i $-\frac{5}{3}$ och radie $\sqrt{\frac{16}{9}} = \frac{4}{3}$.

Multiplikation med α, $|\alpha| = 1$

$|\alpha z| = |\alpha||z| = |z|$

$\arg(\alpha z) = \arg\alpha + \arg z$

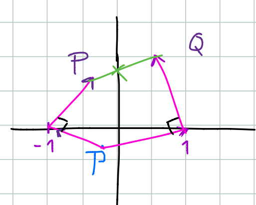
Vridning $\arg(\alpha)$ runt O.



Sjörövarskatten

Inför ett komplext talplan: Palmer i ± 1.

Galgen T



$P = -1 + (-i)(-1 - T)$

$Q = 1 + i(1 - T)$

Skatten $= \frac{P+Q}{2} = \frac{1}{2}(-1 + i + iT + 1 + i - iT) = \frac{2i}{2} = i$

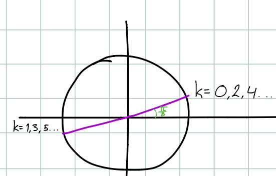
Kvadratrötter

Lös $z^2 = 1 + i$

$z = re^{i\theta}$

$z^2 = r^2 e^{2i\theta} = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$$\begin{cases} r^2 = \sqrt{2} \\ 2\theta = \frac{\pi}{4} + 2k\pi \end{cases} \Rightarrow \begin{matrix} r = \sqrt[4]{2} \\ \theta = \frac{\pi}{8} + k\pi \end{matrix}$$



$z = \pm\sqrt[4]{2}\, e^{i\frac{\pi}{8}}$

Lösn. 2

$z = x + iy$

$z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = 1 + i$

$$\left.\begin{matrix} x - y^2 = 1 \\ 2xy = 1 \end{matrix}\right\} \quad y = \frac{1}{2x} \Rightarrow x^2 - \frac{1}{4x^2} = 1 \Rightarrow 4x^4 - 4x^2 - \frac{1}{4} = 0$$

$[x^2 = u]$

$u^2 - u - \frac{1}{4} = 0$

$u = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2} (\pm) \sqrt{\frac{1}{2}} =$ $(u = x^2 > 0)$

$x = \pm \sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}}$

$y = \pm \frac{1}{2\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}}}$

Ex

$z^2 = 2+3i$

$z = re^{i\theta}$

$z^2 = r^2 e^{2i\theta} = 2+3i = \sqrt{13}\left(\frac{2}{\sqrt{13}} + i\frac{3}{\sqrt{13}}\right) = \sqrt{13}\, e^{i \arctan\frac{3}{2}}$

$$\left.\begin{array}{l} r^2 = \sqrt{13} \quad \Rightarrow \quad r = \sqrt[4]{13} \\ 2\theta = \arctan\frac{3}{2} + 2k\pi \Rightarrow \theta = \frac{1}{2}\arctan\frac{3}{2} + k\pi \end{array}\right\} z = \pm\sqrt[4]{13}\, e^{i\frac{1}{2}\arctan\frac{3}{2}}$$

Högre rötter

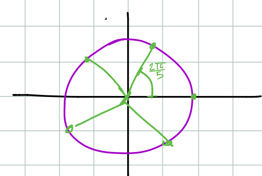
Lös $z^5 = 1$

$z = re^{i\theta}$

$z^5 = r^5 e^{5i\theta} = 1 = 1 \cdot e^{i \cdot 0}$

$r^5 = 1 \quad \Rightarrow \quad r = 1$

$5\theta = 0 + 2k\pi \Rightarrow \theta = \frac{2}{5}k\pi$



Ex

$z^3 = 8i$

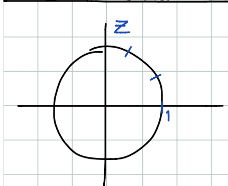
$z = re^{i\theta}$

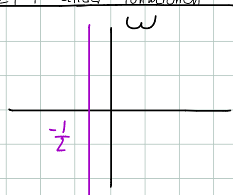
$z^3 = r^3 e^{3i\theta} = 8i = 8e^{i\frac{\pi}{2}}$

$r^3 = 8 \qquad r = 2$

$3\theta = \frac{\pi}{2} + 2k\pi \qquad \theta = \frac{\pi}{6} + \frac{2k\pi}{3}$

Vad är bilden av $|z| = 1$ under funktionen $w = \frac{1}{z-1}$?





Punkter med $|1| = e^{i\theta} = z$

$w = \frac{1}{e^{i\theta} - 1} = \frac{1}{\cos\theta + i\sin\theta - 1} = \frac{\cos\theta - 1 - i\sin\theta}{(\cos\theta - 1)^2 + \sin^2\theta} = \frac{\cos\theta - 1 - i\sin\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta} = \frac{\cos\theta - 1 - \sin\theta}{2 - 2\cos\theta} =$

$\frac{\cos\theta - 1}{2(1-\cos\theta)} - i\frac{\sin\theta}{2(1-\cos\theta)} = -\frac{1}{2} - i\frac{\sin\theta}{2(1-\cos\theta)}$

$\frac{\sin\theta}{2(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2 \cdot 2\cos^2\frac{\theta}{2}} = \frac{1}{2}\tan\frac{\theta}{2} \qquad -\infty < \tan\frac{\theta}{2} < \infty$