Tidsdiskreta LTI-system

$y[n] = (h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

- Stabilt om $\sum_{k=-\infty}^{\infty} |h[k]| < +\infty$
- Kausalt om $h[k] = 0 \quad k<0$

Z-transformen

$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}, \; z \neq 0, \; z \in \mathbb{C}$

jmfr med Laplace; $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad s \in \mathbb{C}$

Byt $t \to k$

$0 \neq e^{s} = z$

Egenskaper

Viktigt att ROC överlappar för H och X.

1) $y[n] = (h*x)[n] \leftrightarrow Y(z) = H(z)X(z)$
2) $w[n] = x[-n] \leftrightarrow W(z) = X(z^{-1})$
3) $y[n] = x[n-n_0] \leftrightarrow Y(z) = z^{-n_0} \cdot X(z)$
4) $y[n] = \alpha^n u[n] \leftrightarrow Y(z) = \frac{1}{1-\alpha z^{-1}}, \; |z| > \alpha$
5) $y[n] = a^n x[n] \leftrightarrow Y(z) = X\left(\frac{z}{a}\right)$

<u>Ex</u>: $x[n] = u[n] \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} u[n]z^{-n} = \sum_{n=0}^{\infty} \underbrace{z^{-n}}_{\left(\frac{1}{z}\right)^n} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}, \; \underbrace{|z|>1}_{\text{ROC}}$

Bevis: $W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{n} = \sum_{n=-\infty}^{\infty} x[n](z^{-1})^{n} = X(z^{-1})$

<u>Ex</u> Tenta 2015-30-03, uppg 2

<u>Givet</u>

LTI-system $y[n] = (h*x)[n]$

$h[n] = 2^n u[n]$

<u>Sökt</u>

a) Stabilt?
   Kausalt?

b) Bestäm $H(z)$:s ROC.

c) Om $x[n] = u[n]$, vad blir $y[n]$

<u>Lösning</u>

a) Stabilt $\Leftrightarrow \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{0} 2^k = \sum_{k=0}^{\infty} 2^{-k} = \left\{\begin{matrix}\text{geometrisk}\\ \text{summa}\end{matrix}\right\} = \frac{1}{1-\frac{1}{2}} = 2 < +\infty$

Kausalt $\Leftrightarrow h[k] = 0$ för $k<0$, ei uppfyllt ty $h[-1] = 2^{-1} \Rightarrow$ Ej kausalt.

b) $h[n] = w[-n]$ där $w[n] = 2^{-n}u[n]$.

Egenskap 2 $\Rightarrow H(z) = W(z^{-1}) \overset{4}{\Rightarrow} W(z) = \frac{1}{1-\frac{1}{2z}}, \; |z| > \overset{=\alpha}{\frac{1}{2}} \Rightarrow H(z) = \frac{1}{1-\frac{z}{2}}, \; |z| < 2 \left(= \frac{1}{\alpha}\right)$

c) Egenskap 1) $\Rightarrow Y(z) = H(z)U(z)$

$Z\{u[n]\} = U(z) = \frac{z}{z-1} \quad |z|>1$

$Y(z) = \frac{1}{1-\frac{z}{2}} \cdot \frac{z}{z-1}, \; 1 < |z| < 2 \qquad \left[\frac{z}{z-1} = \frac{1}{1-\frac{1}{z}}\right]$

Hur hittar vi $y[n]$? <u>PBU!</u>

$\frac{1}{(1-\frac{z}{2})(1-\frac{1}{z})} = \frac{A}{(1-\frac{z}{2})} + \frac{B}{1-\frac{1}{z}} \Rightarrow y[n] = A \cdot 2^{-n}u[-n] + Bu[n]$

<u>Ex</u> $y[n] = 2^n u[5-n]$, Vad är $Y(z)$?

$y[n] = 2^n u[5-n] = 2^n u[-(n-5)] = 2^n \cdot w[n-5] = \underbrace{\{w[n] = u[-n]\}}_{\text{egenskap 2}} = 2^5 \cdot 2^{n-5} \cdot w[n-5] = 2^5 \cdot q[n-5] = \underbrace{\{q[n] = 2^n w[n]\}}_{\text{egenskap 4}}$

Egenskap 3 $\Rightarrow Y(z) = 2^5 \cdot Q(z) \cdot z^{-5}$

Egenskap 5 $\Rightarrow Q(z) = W\left(\frac{z}{2}\right)$

Egenskap 2 $\Rightarrow W(z) = U(z^{-1})$

Egenskap 4 $\Rightarrow U(z^{-1}) = \frac{1}{1-z^{-1}}, \; |z|>1$

Baklänges $\Rightarrow$

$W(z) = \frac{1}{1-z}, \; |z|<1$

$Q(z) = \frac{1}{1-\frac{z}{2}}, \; \left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$

$Y(z) = 2^5 \cdot z^{-5} \cdot \frac{1}{1-\frac{z}{2}}, \; |z|<2$

Ex 11.16

Givet

Kausalt LTI-system.

$y[n] = x[n] + y[n-1]$

Sökt

a) Finn $h[n]$

b) $x[n] = \left(\frac{1}{2}\right)^n u[n] \Rightarrow y[n]$?

c) Stabilt?

Lösning

a) LTI $\Rightarrow y[n] = (h * x)[n]$

$Y(z) = H(z)X(z)$

Z-transform av den givna signalen: $Y(z) = X(z) + z^{-1} \cdot Y(z)$

$Y(z)(1 - z^{-1}) = X(z)$

$H(z) = \frac{1}{1 - z^{-1}}$ ROC?

B: $\begin{cases} h[n] = u[n] \\ h[n] = u[-n-1] \end{cases}$ eftersom vi inte vet ROC

Eftersom kausalt $\Rightarrow u[n]$ eftersom $u[-n-1] \neq 0$ för $n < 0$

$n = -2 \Rightarrow u[-(-2)-1] = u[1]$

b) $Y(z) = H(z)X(z)$

Eg 4 $\Rightarrow X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$ $\Rightarrow Y(z) = \frac{1}{1 - \frac{1}{z}} \cdot \frac{1}{1 - \frac{1}{2}z^{-1}}$ ROC $|z| > 1$ & $|z| > \frac{1}{2}$

PBU: $Y(z) = \frac{A}{1 - \frac{1}{z}} + \frac{B}{1 - \frac{1}{2}z^{-1}} \Rightarrow y[n] = A \cdot u[n] + B \cdot \left(\frac{1}{2}\right)^n u[n]$

c) Stabilt $\Leftrightarrow \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} 1 = +\infty \Rightarrow$ Ei stabilt

Note!

Stabila system har ROC $\supset \{z = 1\}$. I vårt fall $\text{ROC}(H) = \{|z| > 1\} \not\supset \{|z| = 1\}$