Tillståndsmodeller

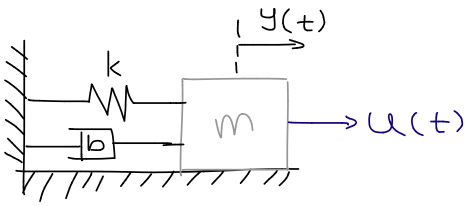
$\dot{x}(t) = Ax(t) + Bu(t)$, $x(t)$: tillståndsvektor $n \times 1$

$y(t) = Cx(t) + Du(t)$ $\quad u(t)$: insignalsvektor $m \times 1$

$y(t)$: utsignalsvektor $p \times 1$

Dimensioner: $A: n \times n$, $B: n \times m$

$C: p \times n$ $\quad D: p \times m$



$\rightarrow$: $m\ddot{y} = u(t) - ky(t) - b\dot{y}(t)$

$(ms^2 + bs + k)\,Y(s) = U(s)$

$G(s) = \frac{1}{ms^2+bs+k}$

Tillståndsekvation

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases} \qquad \begin{aligned} \dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = \frac{u}{m} - \frac{k}{m}y(t) - \frac{b}{m}\dot{y}(t) = \frac{u}{m} - \frac{k}{m}x_1 - \frac{b}{m}x_2 \end{aligned}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \;, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + [0]\,u$$

Notera: Tillståndsvariablerna väljs som $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$, ... upp till grad $n-1$.

2.1

a)

Givet

$\ddot{y}(t) + \dot{y}(t) = u(t)$

u: insignal

y: utsignal

Sökt

Systemet på tillståndsform

Lösning

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = -\dot{y} + u = -x_2 + u \end{aligned}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

c)

Givet

$$\begin{cases} 4\dot{v}(t) + 5v(t) = 2u(t) \\ \dot{y}(t) + 2y(t) = 5v(t) \end{cases}$$

Sökt

Tillståndsmodell

$$\begin{aligned} x_1 &= y \\ x_2 &= v \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= \dot{y} = -2y + 5v = -2x_1 + 5x_2 \\ \dot{x}_2 &= \dot{v} = -\tfrac{5}{4}v + \tfrac{1}{2}u = -\tfrac{5}{4}x_2 + \tfrac{1}{2}u \end{aligned} \quad \Rightarrow \quad \dot{x} = \begin{bmatrix} -2 & 5 \\ 0 & -1.25 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u \;, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

2.4 |

Givet

$x(t)$

$$\dot{x}(t)=\begin{bmatrix}2 & 0\\ -3 & 4\end{bmatrix}x(t)+\begin{bmatrix}1\\2\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}2 & -1\end{bmatrix}x(t)+\begin{bmatrix}1\end{bmatrix}u(t)$$

Sökt

$$G(s)=\frac{Y(s)}{U(s)}$$

$$G(s)=C(sI-A)^{-1}B+D=\begin{bmatrix}2 & -1\end{bmatrix}\left(\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}2 & 0\\ -3 & 4\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}1\end{bmatrix}=$$

$$\begin{bmatrix}2 & -1\end{bmatrix}\left(\begin{bmatrix}s-2 & 0\\ 3 & s-4\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\2\end{bmatrix}+1=\begin{bmatrix}2 & -1\end{bmatrix}\frac{1}{\det(sI-A)}\begin{bmatrix}s-4 & 0\\ 3 & 4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}+1=$$

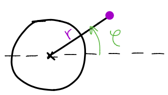
$$\begin{bmatrix}2 & -1\end{bmatrix}\frac{1}{(s-2)(s-4)}\begin{bmatrix}s-4\\ -3+2(s-2)\end{bmatrix}+1=\frac{1}{(s-2)(s-4)}\Big(2(s-4)-1(-3+2(s-2))\Big)+1=$$

$$\frac{1}{(s-2)(s-4)}(-8+3+4)+1=\frac{-1+(s-2)(s-4)}{(s-2)(s-4)}=\frac{s^2-6s+7}{(s-2)(s-4)}$$

2.11 |

Givet

$$\omega_0=\frac{2\pi}{24\cdot 3600}\ \frac{rad}{s}$$



Sökt

a) Tillståndsmodell

b) Linjärisera modellen, hitta eigenvalues

① $\ddot{r}(t)+\frac{k}{r^2(t)}-r(t)\omega^2(t)=u_1(t)$

② $\frac{1}{r(t)}\frac{d}{dt}\left(r^2(t)\omega(t)\right)=u_2(t)$

k: gravitationskonstant

u_1, u_2: radiella resp tangentiella styrkrafter

a) $x_1=r$, $x_2=\dot{r}$, $x_3=\omega$

$$\begin{cases}\dot{x}_1=\dot{r}=x_2\\ \dot{x}_2=\ddot{r}=-\frac{k}{r^2}+r\cdot\omega^2+u_1=\frac{-k}{x_1^2}+x_1x_3^2+u_1\\ \dot{x}_3=\dot{\omega}=\{\text{derivera uttryck } 2\}=\frac{-2\dot{r}\omega+u_2}{r}=-\frac{2x_2x_3}{x_1}+\frac{u_2}{x_1}\end{cases}$$

b) Tillståndsderivator är 0 i arbpkt.

$u_{10}=u_{20}=0$

Linjärisering: $\Delta\dot{x}=A\Delta x+B\Delta u$

$$A=\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \vdots & & \vdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{bmatrix},\quad B=\begin{bmatrix}\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n}\\ \vdots & & \vdots\\ \frac{\partial f_n}{\partial u_n} & \cdots & \frac{\partial f_n}{\partial u_n}\end{bmatrix}$$

(x_0, u_0): arbpkt

$$0=x_{20}$$

$$0=-\frac{k}{x_{10}^2}+x_{10}x_{30}^2+u_{10}^{=0}\Rightarrow x_{10}=\sqrt[3]{\frac{k}{x_{30}^2}}\Rightarrow x_0=\begin{bmatrix}\sqrt[3]{\frac{k}{x_{30}^2}}\\ 0\\ \omega_0\end{bmatrix}$$

$$0=-\frac{2x_{20}x_{30}}{x_{10}}+\frac{u_{20}}{x_{10}}$$

$$A=\begin{bmatrix}0 & 1 & 0\\ \frac{2k}{x_{10}^3}+x_{30}^2 & 0 & 2x_{10}x_{30}\\ \frac{2x_{20}x_{30}}{x_{10}^2}-\frac{u_{20}}{x_{10}^2} & -\frac{2x_{30}}{x_{10}} & -\frac{2x_{20}}{x_{10}}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 3\omega_0^2 & 0 & 2\sqrt[3]{k\omega_0}\\ 0 & -2k^{-1/3}\cdot\omega_0^{5/3} & 0\end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{x_{10}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \left(\frac{k}{\omega_0^2}\right)^{-1/3} \end{bmatrix}$$

$$\det(sI - A) = \det(\lambda I - A) = 0$$

$$\omega_0^{6/3} = \omega_0^2$$

$$\det\begin{pmatrix} \lambda & -1 & 0 \\ -3\omega_0^2 & \lambda & -2(k\omega_0)^{1/3} \\ 0 & 2k^{-1/3}\omega_0^{5/3} & \lambda \end{pmatrix} = 0 \iff \lambda^3 - 3\omega_0^2\lambda + \lambda 4 \cancel{k^{-1/3}} \omega_0^{5/3} (\cancel{k}\omega_0)^{1/3} = 0$$

$$\lambda^3 - 3\omega_0^2\lambda + 4\omega_0^2\lambda = 0$$

$$\lambda^3 + \omega_0^2\lambda = 0$$

$$\lambda(\lambda^2 + \omega_0^2) = 0 \implies \lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{-\omega_0^2} = \pm j\omega_0$$