Assignment 1 (20%)

You will need very few and simple calculation in order to solve the problems of this assignment.

a) The result of an SVD displays

$$U = \begin{bmatrix} -0.2534 & -0.3430 & 0.9045 \\ -0.4897 & -0.7609 & -0.4257 \\ -0.8343 & 0.5508 & -0.0249 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -0.4132 & 0.4230 & 0.2244 & -0.7746 \\ -0.5756 & -0.2919 & -0.7189 & 0.2582 \\ 0.4397 & 0.8557 & -0.0881 & -0.2582 \\ -0.5519 & -0.0606 & 0.6520 & 0.5164 \end{bmatrix}$$

What is the size of the matrix which has been decomposed?

b) Explain why it would be possible to determine λ_3 of A if λ_1 were 4 and λ_2 were 2 and then state what λ_3 is if this were the case.

$$A = \begin{bmatrix} 8 & 3 & -5 \\ 3 & 3 & -2 \\ -5 & -2 & 1 \end{bmatrix}$$

c) Explain why the following matrix is not invertible

$$\begin{bmatrix} 2 & 4 & 8 & 7 & 1 & 3 \\ 0 & 5 & 3 & 5 & 5 & 6 \\ 0 & 0 & 3 & 3 & 3 & 9 \\ 0 & 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

d) For what value(s) of k are the columns of the following matrix linearly dependent?

$$\begin{bmatrix} 2 & -10 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 2k-6 \end{bmatrix}$$

e) Explain why $Ax = b$ formed from the following matrix in echelon form has a solution for each b

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Assignment 2 (5%)

Determine a basis for the null space of matrix A by setting up the solution in parametric form

$$A = \begin{bmatrix} -1 & 2 & 1 & 4 \\ 1 & 2 & 2 & 6 \end{bmatrix}$$

Assignment 3 (15%)

A 2 x 2 matrix A is given as

$$A = \begin{bmatrix} a & 2 \\ 1 & 2 \end{bmatrix}$$

a) Express the eigenvalues of A as a function of a
b) Calculate a so that $\lambda = 4$ is an eigenvalue of A
c) Using the value of a found in question (b), determine the remaining eigenvalue(s) and the eigenspaces

Assignment 4 (20%)

Let,

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, and\ x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix},$$

a) Show that $\{u_1, u_2, u_3\}$ is a linearly independent set
b) Show that $\{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$
c) Express x as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3\}$.
d) Determine the (shortest) distance between x and the subspace spanned by $\{u_1, u_2, u_3\}$

Assignment 5 (20%)

For a specific sensor, the following corresponding values of time t and output y have been measured

t	y
0	1
1	3
2	7
3	11
4	16
5	24

It is assumed that the system can be approximated by a model of the form $y_1(t) = \beta_0 + \beta_2 t^2$ or of the form $y_2(t) = \delta_0 + \delta_1 t + \delta_2 t^2$

a) Determine the design matrix and observation vector of both models
b) Determine the parameters of both models
c) Supply a substantiated answer to which model is the best fit for the measured data

Assignment 6 (20%)

Let the matrix A be given by

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$$

a) Compute a full singular value decomposition of the matrix A
b) How do the SVDs of A and A^T relate to each other (it is allowed to calculate the SVDs in Matlab in order to answer this question)

Assignment 1 (20%)

You will need very few and simple calculation in order to solve the problems of this assignment.

Let

$$A = \begin{bmatrix} 3 & -1 & 5 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 3 & 5 & -7 & -2 \\ 0 & -5 & 3 & 1 & -3 \\ 0 & 0 & -2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

a) Explain why it would be possible to determine all eigenvalues of A without any calculations and then state what these eigenvalues are.

b) State whether or not *A* is invertible and state the reasoning behind your answer.

c) Supply a substantiated answer to whether or not A is diagonalizable.

d) Let B be a $n \times n$ matrix where $\det(B) \neq 0$. What is $rank(B)$ and $dim\ Nul\ B$?

e) Supply a substantiated answer to whether $C\bar{x} = \bar{b}$ is consistent for every $\bar{b}$ in $\mathbb{R}^4$.

f) Does $C\bar{x} = \bar{0}$ have a non-trivial solution? Explain why/why not.

Assignment 2 (10%)

Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } \bar{b} = \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}$$

a) Find the inverse of A using elementary row operations and the identity matrix.
b) Use the inverse of A to solve $A\bar{x} = \bar{b}$

Assignment 3 (5%)

Consider the following linear system

$$x_1 + 2x_2 + x_3 = 1$$

$$-2x_1 + x_2 + x_3 = -5$$

$$2x_1 - x_2 - 2x_3 = a$$

For which values of a is the system consistent?

Assignment 4 (25%)

$$A = \begin{bmatrix} a & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$$

a) Express the eigenvalues of A as a function of a
b) Calculate a so that $\lambda = 5$ is an eigenvalue of A
c) Using the value of a found in question (b), determine the remaining eigenvalue(s)
d) Determine the eigenspaces for each of the eigenvalues found in (c)
e) Using the value of a found in (b), find the characteristic equation of A^T and compare it to the characteristic equation of A. What can you conclude about the characteristic equations of A and A^T?

Assignment 4 (15%)

Let

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \bar{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad and\ \bar{y} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

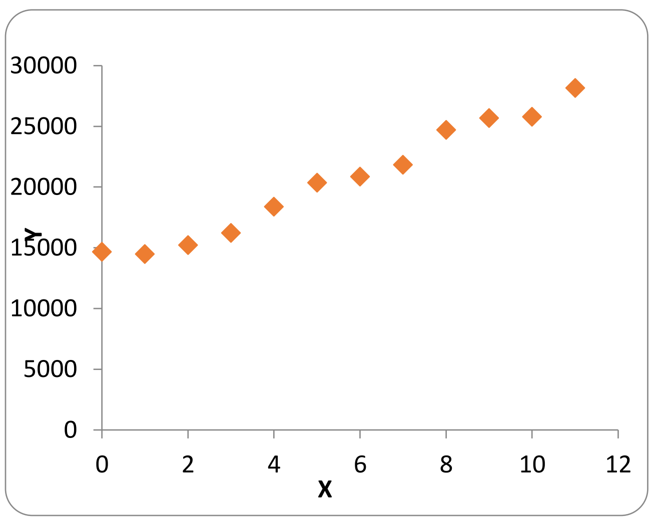
And let $H = span\{\bar{u}_1, \bar{u}_2\}$, let $A = [\bar{u}_1 \quad \bar{u}_2]$, and let $H^\perp$ denote the subspace that is orthogonal to H

a) Show that $\{u_1, u_2\}$ is an orthogonal basis for H.
b) Find a basis for $H^\perp$. Hint: $H = Col\ A \Longrightarrow H^\perp = (Col\ A)^\perp = Nul\ A^T$
c) Calculate the orthogonal projection of $\bar{y}$ onto H

Assignment 5 (15%)

The number of students enrolled at a university study (5+ years) in the period 2005-2016 is as follows (source: DST.dk):

Year	Index year (t)	Number of students (y)
2005	0	14652
2006	1	14480
2007	2	15216
2008	3	16215
2009	4	18378
2010	5	20347
2011	6	20845
2012	7	21822
2013	8	24707
2014	9	25689
2015	10	25787
2016	11	28168



It is believed that the number of students can be modelled by either a linear function or a quadratic function:

$y_1(t) = \beta_0 + \beta_1 t$ or $y_2(t) = \delta_0 + \delta_1 t + \delta_2 t^2$

a) By creating design matrices, find the parameters of both models and state the regression functions
b) Supply a substantiated answer to which model is the best fit for the measured data
c) Use the best fitted model to predict how many students will be enrolled at a university study in 2020.

Assignment 6 (10%)

Compute a full singular value decomposition of the following matrix

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

Assignment 1 (15%)

You will need very few and simple calculation in order to solve the problems of this assignment.

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & -5 & 7 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

a) State which of the statements below are true and which are false (notice: every incorrect answer cancels a correct one). No answer will count as an incorrect answer.

   a. A is not invertible
   b. A is in echelon form
   c. $Nullity\ A = 1$
   d. $Rank\ A = 3$
   e. $Nullity\ A + rank\ A = 6$
   f. The number 0 is an eigenvalue of A
   g. A is in reduced echelon form
   h. There exists a vector $\bar{x} \in \mathbb{R}^4$ such that $A\bar{x} = \bar{b}$ is not consistent
   i. $\det A = 0$

b) Let W be a subspace of $\mathbb{R}^6$ having dimension 4. What is $\dim(W^{\perp})$?

c) Explain why a 3×3 matrix with eigenvalues 1, 2, and -3 is both invertible and diagonalizable.

Assignment 2 (10%)

Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) State the eigenvalues of A and determine a basis for each of the corresponding eigenspaces.
b) Specify whether A is diagonalisable. If so, find matrices P and D such that D is diagonal, P is invertible, and $A = PDP^{-1}$.

Assignment 3 (20%)

Consider the following system

$$3x - 4z = 7 - 5y$$

$$1 + 4z = 2y + 3x$$

$$y = -4 - 6x + 8z$$

a) Write the system in the matrix form $A\bar{x} = \bar{b}$ for $\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
b) Write out the augmented matrix for this system and state its *echelon* form
c) Write out the complete set of solutions (if they exist) in parametric vector form.
d) Calculate the inverse of the coefficient matrix A you found in part (a) using the identity matrix, if it exists, or show that A^{-1} doesn't exist.

Assignment 4 (10%)

Show that for arbitrary real numbers a, b, c and d, the determinant of the following matrix is always zero.

$$\begin{bmatrix} a & 0 & d & c \\ b & 0 & -c & d \\ 0 & c & -b & a \\ 0 & d & a & b \end{bmatrix}$$

Assignment 5 (15%)

Let

$$A = \begin{bmatrix} a & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{bmatrix}$$

a) Find a such that $\det(A) = 24$.
b) Assuming, that you found $a = 1$ in (a), show that the columns of A form a basis for $\mathbb{R}^5$.
c) Show that the columns of A do not form an orthogonal basis, and use the Gram Schmidt process to obtain such an orthogonal basis.

Assignment 6 (15%)

As part of their final project, two ICT students are working on a data warehouse support system. The major workload is the warehouse orders. Thus, the key business metric is identified as number of order lines. The students want to find a method to predict CPU utilization based on the number of order lines entered into the system and have collected 31 samples of CPU utilization and number of order line entries (the data is also uploaded as ‘ALI2018_6.mat’):

Sample #	CPU Utilisation	Order lines per day
1	27.01	16483
2	32.43	13142
3	21.74	12015
4	20.56	11986
5	2.85	1119
6	1.41	0
7	1.45	0
8	46.38	12259
9	21.95	6531
10	29.55	14086
11	30.04	12797
12	28.08	13141
13	3.26	454
14	1.62	1
15	29.41	5971
16	40.02	10901
17	29.86	14271
18	28.34	13728
19	34.82	12938
20	3.22	1158
21	1.43	0
22	34.22	11450
23	23.58	5311
24	33.66	17073
25	23.36	11336
26	26.76	7340
27	4.31	11330
28	2.62	0
29	33.44	10679
30	29.19	12803
31	28.11	12827

The students believe that the relationship can be modelled by either a linear function or a quadratic function:

$y_1(x) = \beta_0 + \beta_1 x$ or $y_2(x) = \delta_0 + \delta_1 x + \delta_2 x^2$

a) By creating design matrices, find the parameters of both models and state the regression functions
b) Supply a substantiated answer to which model is the best fit for the measured data
c) Use the best fitted model to predict the CPU Utilisation at 20000 order lines per day.

Assignment 7 (15%)

Consider the following matrix

$$A = \begin{bmatrix} -18 & 13 & -4 \\ 2 & 19 & -4 \\ -14 & 11 & -12 \\ -2 & 21 & 4 \end{bmatrix}$$

a) Compute a full singular value decomposition of A.
b) Use this decomposition of A, with no calculations, to write a basis for $Col\ A$, and use the values of the singular values to determine the rank of A.

Applied Linear Algebra Final Exam

R. Brooks

VIA University College

June 14, 2019

Please state all answers in the 'ALI-exam.ipynb'. If you have handwritten answer which is scanned, please state 'In paper' or similar so the examiner knows where to find the answer. Also, please include all ipynb files when you hand in.

Assignment 1 (12 %)

Determine whether statements a-d are true or false. State the reasons for your answers. **The points for this problem are given entirely for your reasons.**

a) Two matrices are row equivalent if they have the same number of rows.

b) If a system of linear equations has no free variables, then it has a unique solution.

c) There exists a 5×6 matrix A and a 5×1 vector $\vec{b}$ for which $\mathrm{A}\vec{\mathrm{x}} = \vec{b}$ has only one solution.

d) A is diagonalizable where A is a 9×9 matrix with three distinct eigenvalues of which two of the eigenvalues have multiplicity 3; and one eigenvalue has multiplicity 2.

Assignment 2 (8 %)

Let A be the matrix $A = \begin{bmatrix} 4 & 8 & -2 \\ -6 & 2 & 10 \\ -2 & 6 & 6 \end{bmatrix}$, and let $\vec{b}$ be the vector $\vec{b} = \begin{bmatrix} 2 \\ 18 \\ 15 \end{bmatrix}$

a) Determine whether $\vec{b}$ is in the span of the columns of A.

b) Let $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ denote the columns of the matrix A. Is the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent or linearly dependent? If it is linearly dependent, find a linear dependency relation.

Assignment 3 (25 %)

a) Evaluate the following determinant and write your answer as a polynomial in x.

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix}$$

b) Find all values of λ for which the following determinant will equal 0.

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix}$$

c) Evaluate the following determinant by using cofactor expansion.

$$\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$$

d) Let A $= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ b \end{bmatrix}$. Find the values of a and b for which the linear system whose augmented matrix is of the form $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ has one solution, infinitely many solutions, and no solutions, and then find the value of a for which the determinant of A is 1.

Assignment 4 (10%)

Let $W \subseteq \mathbb{R}^4$ be the subspace of vectors (x_1, x_2, x_3, x_4) satisfying

$$2x_1 - x_3 + x_4 = 0$$

Find an orthonormal basis for W.

Assignment 5 (15 %)

Let A be the matrix A $= \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$

a) Show that A is diagonalizable.

b) Find an invertible matrix X and a diagonal matrix D such that $X^{-1}AX = D$.

c) Next, find an expression for A^k, where k is an arbitrary positive integer.

Assignment 6 (20 %)

For this assignment, you will need to load the file "Smoking_and_Cancer.xlsx". The data was collected in 1960 from the National Cancer Institute and provides death rates (per 100.000) for lung cancer as a function of the per capita numbers of cigarettes sold. The National Cancer Institute believe that the relationship can be modelled by either a linear function or a quadratic function:

$$f_1(x) = \beta_0 + \beta_1 x \text{ or } f_2(x) = \delta_1 x + \delta_2 x^2$$

a) Fit the two proposed models for lung cancer.
b) Determine which of the two models provides the best fit for the measured data.
c) Use the best fitted model to predict death rate of that specific cancer when the number of cigarettes sold per capita reaches 50.

Assignment 7 (10 %)

Do a full singular value decomposition of matrix A below. This includes the following steps:

1) Finding the eigenvalues of either $A^T A$ or AA^T
2) Determining the corresponding eigenvectors
3) Finding the columns of U or V depending on which method you used in step 1, where U or V are matrices made up of the eigenvectors of AA^T and $A^T A$, respectively
4) Setting up $A = U\Sigma V^T$
5) Testing that $A = U\Sigma V^T$

$$\mathrm{A} = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix}$$

Applied Linear Algebra

Richard Brooks

June, 2020

Please state all answers in the "ALI_exam.ipynb"-file. If you have handwritten answers (that are scanned), please state "In paper" or similar so the examiner knows where to find the answer. Also, please include all ipynb files when you hand in.

Assignment 1 (15%)

The questions in this assignment require little or no calculations. **The points for this assignment are given entirely for your reasons**.

a. The following matrix has three eigenvectors

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

It is know that $\lambda_1 = -1$ and that it has multiplicity 2. Explain how this information enables you to easily find the last eigenvalue and state its value.

b. Identify the matrices that are in reduced echelon form, are in echelon form, and are not in any echelon form and state your reasons for identifying them as such.

$$A = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} -8 & -4 & -8 & -9 & -8 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} C = \begin{bmatrix} 1 & 1 & -4 \\ 1 & 0 & -10 \end{bmatrix} D = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

c. If A is *any* m by n matrix with $m > n$, explain why AA^T is necessarily singular. *Hint*: In your answer, consider the rank of AA^T.

d. The determinant of A is 0. Explain how you can deduce this simply by observing the matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

e. Below you see a matrix A and its echelon form. Explain how you from this information can deduce that dim Col $A = 2$ and dim Nul $A = 3$.

$$A = \begin{bmatrix} 2 & 4 & -5 & 2 & -3 \\ 3 & 6 & -8 & 3 & -5 \\ 0 & 0 & 9 & 0 & 9 \\ -3 & -6 & -7 & -3 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 & 1 & -4 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

f. Give an example of each of the following or explain why no such example can exist:

- An inconsistent linear system in three variables, with a coefficient matrix of rank two.
- A consistent linear system with three equations and two unknowns, with a coefficient matrix of rank one.
- A consistent linear system with three equations and two unknowns, with a coefficient matrix of rank larger than one
- A linear system of two equations in three unknowns, with an invertible coefficient matrix.
- A linear system in three variables, whose geometrical interpretation is three planes intersecting in a line.

Assignment 2 (15%)

a. Let the matrix A be given by

$$A = \begin{bmatrix} 3-2q & 1 \\ 4 & 3+2q \end{bmatrix},$$

where q is a scalar. Calculate q so that

$$A^2 = \begin{bmatrix} 29 & 6 \\ 24 & 125 \end{bmatrix}$$

b. Let B be an invertible $n \times n$ matrix. Reduce the expression $B^2 B^T B B^{-1} \left(B^{-1}\right)^T B \left(B^{-1}\right)^2$ as much as possible and account for the rules used in each step of the reduction.

c. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$. What value(s) og k, if any, will make $AB = BA$?

Assignment 3 (10%)

a. Do the three lines $x_1 + 2x_2 = 5$, $3x_1 - 2x_2 = 1$ and $2x_1 + 4x_2 = 10$ have a common point of intersection? If yes, find the point; if no, explain the reason.

b. Let the *augmented* matrix of a system of linear equations be

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & \alpha & 2\alpha \\ 1 & \alpha & 2 & -2 \end{bmatrix}$$

Find the value(s) of α for which the system of linear equations has (i) three basic variables; (ii) two basic variables and one free variable.

Assignment 4 (10%)

a. Calculate the following determinant (using suitable properties of the determinant to simplify the calculation):

$$\begin{vmatrix} x & y & z & 1 \\ 1 & -2 & 3 & 1 \\ 2 & -3 & 1 & 1 \\ 4 & -6 & 3 & 1 \end{vmatrix}$$

b. Let A and B be 4×4 square matrices such that $\det(A) = 3$ and $\det(B) = -2$. Compute $\det(2A)$, $\det\left(A^3\right)$, $\det\left(A^{-1}\right)$, $\det\left(A^2B^3\right)$ and $\det\left(A^3B^{-2}\right)$

Assignment 5 (10%)

Diagonalize the following matrices, if possible. If it is not possible, supply a substantiated explanation of why this is the case.

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Assignment 6 (25%)

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$. The positions (in metres) were: 0, 26.4, 89.7, 186.0, 314.1, 477.3, 666.0, 883.5, 1.141.2, 1.413.3, 1.715.1, 1.715.1, and 2.427.6.

It is assumed that the system can be approximated by a model of the form $y_1(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ or of the form $y_2(t) = \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3$ or of the form $y_3(t) = \delta_0 + \delta_1 t + \delta_2 t^2$

a. Determine the design matrix and observation matrix of all three models.

b. Determine the parameters of all three models

c. State, with substantiation, which of the three models is the best fit for the measured data

d. Use the best fitted model to estimate the velocity of the plane when $t = 4.5$ seconds.

Assignment 7 (15%)

Consider the following matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

a. Do a full singular value decomposition of matrix A. This includes the following steps:

   1) Finding the eigenvalues of either $A^T A$ or AA^T
   2) Determining the corresponding eigenvectors
   3) Finding the columns of V and from this deriving the columns of U or vice versa depending on which method you used in (1).
   4) Setting up $A = U\Sigma V^T$
   5) Testing that $A = U\Sigma V^T$

b. Find orthonormal bases for the four fundamental subspaces: Col A; Nul A^T; Col A^T; Nul A associated with A (It is actually possible to derive these from the SVD).

Assignment 1
The answers to the problems of this assignment do not require any calculations but can be inferred from the information given (provided you have the relevant knowledge!)

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 5 & -7 & -2 \\ 4 & -5 & 3 & 1 & -3 \\ 3 & 7 & -2 & 4 & 5 \\ 2 & 2 & -7 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 9 & 5 & 1 \\ 0 & -3 & 1 & 4 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

a) Explain why the columns of *A* are not linearly independent
b) Explain why $A\boldsymbol{x} = \mathbf{0}$ has a non-trivial solution
c) Explain why $B\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b}$ in $\mathbb{R}^3$
d) Explain whether or not AB^T is well-defined

Assignment 2

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -4 & -1 \\ -1 & -3 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

a) Find the inverse of *A* using elementary row operations on the augmented matrix $[A \quad I]$
b) Use the inverse of A to solve $A\boldsymbol{x} = \boldsymbol{b}$

Assignment 3
Find the value(s) of α for which the determinant of the following matrix is -18 by co-expanding on the second row

$$\begin{bmatrix} 1 & 5 & -a \\ a & -a & a \\ 2 & 13 & -7 \end{bmatrix}$$

Assignment 4
Let

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, and\ v = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix},$$

a) Show that $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$ using the inner product.
b) Write *v* as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3, u_4\}$.

Assignment 5
Measurements of the deflection (mm) of particleboard from stress levels of relative humidity are displayed below.

Stress level (%)	Deflection (mm)
54	16.473
54	18.693
61	14.305
61	15.121
68	13.505
68	11.640
75	11.168
75	12.534
75	11.224

a) Find the best fitted least-squares line to describe the data above
b) Determine the least-squares error of the least squares line found in (a).

Assignment 6

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -4 & -1 \\ -1 & -3 & 2 \end{bmatrix}$, $v_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

a) Show that v_1 and v_2 are eigenvectors of A with associated eigenvalues λ_1 and λ_2, respectively
b) Determine the eigenspaces of λ_1 and λ_2
c) Orthogonally diagonalize A where $A = PDP^{-1}$ and the columns of P are normalized

Assignment 1 (10%)
You will need very few and simple calculation in order to solve the problems of this assignment.
The following questions refer to the matrices

$$A = \begin{bmatrix} 4 & 9 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 7 \\ 1 & 7 & 7 \end{bmatrix}$$

and the vector

$$\boldsymbol{b} = \begin{bmatrix} -1 \\ 2 \\ 8 \end{bmatrix}$$

a) Which of the matrices are in echelon form?
b) Write the system of linear equations that correspond to the matrix equation $Cx = b$.
c) Is the system of linear equations that correspond to the matric equation $Cx = b$ consistent?
d) Explain which of the following five expressions that makes sense.

$$AB, \quad CA, \quad B^2$$
$$\det(B), \quad \det(C)$$

Assignment 2 (20%)
Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

a) Solve the matrix equation $X + A = 2(X - B)$
b) Determine the rank and the nullity of A.
c) Determine the rank and the nullity of B and find a basis for $Col\ B$ and a basis for $Null\ B$.
d) Find the characteristic polynomial of B and use it to find the eigenvalues of B.

Assignment 3 (10%)

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -4 & -1 \\ -1 & -3 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

a) Find the inverse of *A* using elementary row operations on the augmented matrix $[A \quad I]$
b) Use the inverse of A to solve $Ax = b$

Assignment 4 (5%)
Find the value(s) of α for which the determinant of the following matrix is -18 by co-expanding on the second row

$$\begin{bmatrix} 1 & 5 & -a \\ a & -a & a \\ 2 & 13 & -7 \end{bmatrix}$$

Assignment 5 (20%)

Let

$$\boldsymbol{u_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{u_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{u_3} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \; and \; \boldsymbol{u_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

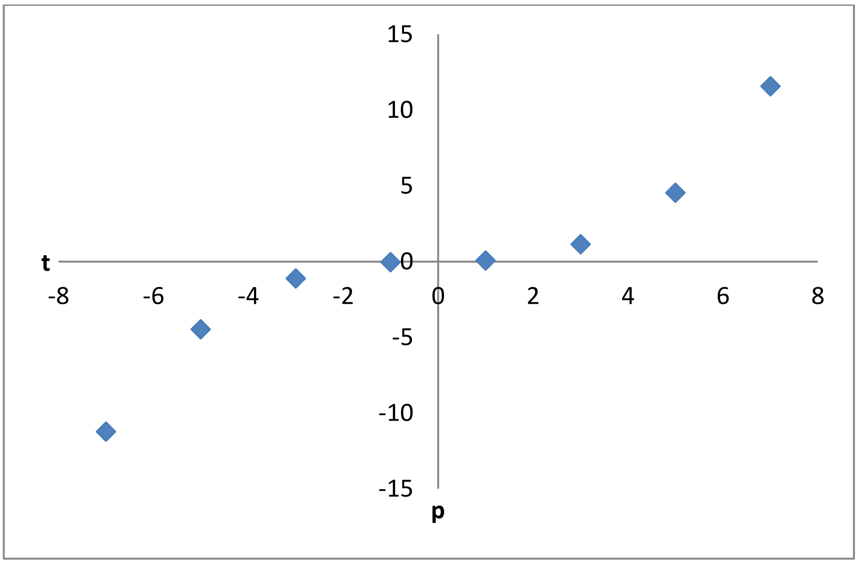
a) Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ is a basis for $\mathbb{R}^4$.
b) Use the Gram-Schmidt process on $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ to obtain an orthogonal basis for $\mathbb{R}^4$.

Assignment 6 (15%)

The data

t	p
-7,0	-11,23
-5,0	-4,47
-3,0	-1,12
-1,0	-0,05
1,0	0,06
3,0	1,14
5,0	4,53
7,0	11,56

show some symmetry as can be seen from the following diagram, where the data-points are placed rather symmetrically around origo in the coordinate system.



This suggests that the data can be modelled by an equation of the form

$$p = \beta_0 t + \beta_1 \cdot t^3$$

where β_0 and β_1 are constants.

a) Find the model of this type that produces the least-squares fit of the data.

Assignment 7 (20%)

Given a matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 2 \end{bmatrix}$$

a) Compute a singular value decomposition of A in the form

$$A = USV^T$$

b) Show that the columns of U are eigenvectors of AA^T and determine the corresponding eigenvalues.

Assignment 1 (15%)

You will need very few and simple calculation in order to solve the problems of this assignment.

Let

$$A = \begin{bmatrix} -1 & 0 & 2 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 9 & 0 & 2 \\ 4 & -5 & 0 & 1 \\ -7 & 2 & 0 & 3 \\ 3 & 1 & 0 & -4 \end{bmatrix}$$

a) Explain why it is very easy to calculate the determinant of matrix A and state what this determinant is.

b) State whether or not A^5 is invertible and state the reasoning behind your answer.

c) Let B and C be two $n \times n$ matrices. Suppose CBC is invertible. Explain why it follows that $\det(B) \neq 0$ and $\det(C) \neq 0$.

d) Explain whether or not matrix D is invertible.

e) Supply a substantiated answer to whether or not the columns of D span $\mathbb{R}^4$.

Assignment 2 (10%)

Let

$$A = \begin{bmatrix} 4 & 0 & 4 \\ -9 & -2 & -8 \\ 8 & 4 & 6 \end{bmatrix} \text{ and } \bar{b} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

a) Determine whether $\bar{b}$ is in $Nul\ A$.
b) Determine whether $\bar{b}$ is in $Col\ A$.

Assignment 3 (20%)

Consider the following system

$$x - y + 3z = 1$$

$$y = -2x + 5$$

$$9z - x - 5y + 7 = 0$$

a) Write the system in the matrix form $A\bar{x} = \bar{b}$ for $\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

b) Write out the augmented matrix for this system and state its echelon form

c) Write out the complete set of solutions (if they exist) in parametric vector form.
d) Calculate the inverse of the coefficient matrix A you found in part (a), if it exists, or show that A^{-1} doesn't exist.

Assignment 4 (15%)

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

a) Using co-factor expansion, show that $\det(A) = 0$
b) Find the characteristic polynomial, eigenvalues and eigenvectors of A

Assignment 5 (15%)

Let

$$\bar{x}_1 = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \bar{x}_2 = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \quad and\ \bar{y} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}.$$

And let $H = span\{\bar{x}_1, \bar{x}_2\}$, and let $H^{\perp}$ denote the subspace that is orthogonal to H

a) Use the Gram-Schmidt process to find an orthogonal basis for H.
b) Decompose $\bar{y}$ as $\bar{y} = \hat{y} + \bar{z}$, where $\hat{y}$ is in H and z is in $H^{\perp}$.

Assignment 6 (15%)

Concerns about global warming have led to studies of the relationship between global temperature and the concentration of carbon dioxide (CO_2). Listed below are the concentrations (in parts per million) of CO_2 and temperatures (in ºC) for different years.

CO_2 (x)	Temperature (y)
314	13,9
317	14
320	13,9
326	14,1
331	14
339	14,3
346	14,1
354	14,5
361	14,5
369	14,4

It is believed that the global temperature can be modelled by either a linear function or a quadratic function:

$y_1(x) = \beta_0 + \beta_1 x$ or $y_2(x) = \delta_0 + \delta_1 x + \delta_2 x^2$

a) By creating design matrices, find the parameters of both models and state the regression functions
b) Supply a substantiated answer to which model is the best fit for the measured data
c) Use the best fitted model to predict the global temperature once the CO_2 level reaches 400.

Assignment 7 (10%)

Compute a full singular value decomposition of the following matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Assignment 1 (15%)

You will need very few and simple calculation in order to solve the problems of this assignment.

Consider the matrix

$$A = \begin{bmatrix} 3 & 3 & 1 & 4 & 1 \\ 0 & -15 & 7 & -20 & -5 \\ 0 & 0 & -34 & 50 & -25 \\ 0 & 0 & 0 & -60 & -140 \\ 0 & 0 & 0 & 0 & 910 \end{bmatrix}$$

a) State which of the statements below are true and which are false (notice: every incorrect answer cancels a correct one). Non-answers will count as an incorrect answer.

1. A is not invertible
2. A is in echelon form
3. $Nullity\ A = 1$
4. $Rank\ A = 5$
5. $Nullity\ A + rank\ A = 6$
6. The number -15 is an eigenvalue of A
7. A is in reduced echelon form
8. There exists a vector $\bar{b} \in \mathbb{R}^5$ such that $A\bar{x} = \bar{b}$ is not consistent
9. A is diagonalizable.

b) Three vectors $\bar{u}, \bar{v}, \bar{z} \in \mathbb{R}^7$ are linearly independent. Let $H = span\{\bar{u}, \bar{v}, \bar{z}\}$. Identify the true statement below.

1. $Dim\ H = 7$
2. $Dim\ H = 3$
3. H can be described as a line in $\mathbb{R}^7$

c) Explain why a 4×4 matrix with eigenvalues 1, 2, 3 and -3 is both invertible and diagonalizable.

d) Let V be the subspace of $\mathbb{R}^4$ given by all solutions to the equation $2x_1 - x_2 + 3x_3 = 0$. What is the dimension of V?

Assignment 2 (20%)

Let

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & 1 \\ 3 & 3 & a \end{bmatrix}$$

and consider the following matrix equation

$$A\bar{x} = \begin{bmatrix} -8 \\ 2 \\ b \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

where $a, b \in \mathbb{R}$.

a) Determine the determinant of A using cofactor expansion on the second row.
b) For which values of a, b does the matrix equation have exactly one solution?
c) For which values of a, b does the matrix equation have no solution?
d) For which values of a, b does the matrix equation have an infinite number of solutions?

Assignment 3 (20%)

Consider the following system

$$x - y = 1 - 3z$$

$$y = -2x + 5$$

$$9z - x - 5y + 7 = 0$$

a) Write the system in the matrix form $A\bar{x} = \bar{b}$ for $\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
b) Write out the augmented matrix for this system and state its *echelon* form
c) Write out the complete set of solutions (if they exist) in parametric vector form.
d) Calculate the inverse of the coefficient matrix A you found in part (a) using the identity matrix, if it exists, or show that A^{-1} doesn't exist.

Assignment 4 (15%)

Let A be given by

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

a) Test A for diagonalisability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$
b) Next, find an expression for A^n, where n is an arbitrary positive integer.

Assignment 5 (10%)

Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be a basis for the subspace $W \subseteq \mathbb{R}^4$

a) By using the Gram Schmidt process, find an orthogonal basis for W.
b) Now determine an orthonormal basis for W.

Assignment 6 (10%)

The file 'TV_Viewing.mat' provides sample data on the number of hours of TV viewing per week for different adults. The first column displays the age of the viewer and the second column displays the hours spent viewing TV per week. The TV executives would like to build a model for estimating TV viewing time as a function of age. They believe that the relationship can be modelled by either a linear function or a quadratic function:

$f_1(x) = \beta_0 + \beta_1 x$ or $f_2(x) = \delta_0 + \delta_1 x + \delta_2 x^2$

a) By creating design matrices, find the parameters of both models and state the regression functions
b) Supply a substantiated answer to which model is the best fit for the measured data

Assignment 7 (10%)

Compute a full singular value decomposition of A:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Applied Linear Algebra Re-Exam

Richard Brooks

August 29, 2019

Please state all answers in the "ALI-Reexam.ipynb". If you have handwritten answers (that are scanned), please state In paper or similar so the examiner knows where to find the answer. Also, please include all ipynb files when you hand in.

Assignment 1 (10%)

The questions in this assignment require little or no calculations. **The points for this assignment are given entirely for your reasons**.

a. Identify the matrices that are not in echelon form and explain why they are not in echelon form.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -4 & 1 \\ 2 & 0 & 0 \\ 1 & -3 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -7 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

b. Determine the eigenvalues and their multiplicity of A below and explain why it is possible to answer this question without any further calculations.

$$A = \begin{bmatrix} -1 & 0 & 2 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

c. Explain why the following matrix contains sufficient information to determine whether it is diagonalizable.

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 1 & -3 & 3 \\ 0 & 2 & 1 & -2 & 3 & 0 & 2 \\ 0 & 0 & 3 & -1 & 9 & 11 & 2 \\ 0 & 0 & 0 & 4 & 7 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

d. The following matrix has three eigenvectors

$$\begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

It is know that $\lambda_1 = 8$ and $\lambda_2 = 3$. Explain how this information enables you to easily find λ_3, and then find λ_3.

Assignment 2 (20%)

Let three matrices be given by:

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -2 \\ 1 & 0 \end{bmatrix}$$

a. Show that A is invertible by using the determinant, show that B is invertible by finding its inverse using the identity matrix and row reduction, and show that C is invertible by finding the null space of C.

b. Find solutions to all of the following matrix equations, if they exist

$$AX = B, \quad A^2X + B = 0, \quad AXB = C, \quad AX + BX = C, \quad ACX = 0$$

Assignment 3 (15%)

A matrix A is given by

$$A = \begin{bmatrix} 1 & -1 & 3 & 5 \\ -1 & -3 & 1 & -1 \\ 2 & 6 & -2 & 2 \end{bmatrix}$$

a. Determine bases for the null space, column space and row space of A.

b. Find the number of solutions to the homogenous equation $A\mathbf{x} = \mathbf{0}$.

c. Find a vector $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ can be solved.

Assignment 4 (10%)

a. By co-factor expanding on the second row, show that the determinant of the following matrix is 10

$$A = \begin{bmatrix} -6 & a & -1 & 3 \\ 2 & 0 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 8 & b & 1 & -4 \end{bmatrix}$$

b. Explain how you can use $\det A = 10$ to determine the determinant of B, and then find $\det B$.

$$B = \begin{bmatrix} -6 & a & -1 & 3 \\ 4 & 5 & 6 & 0 \\ 4 & 0 & 6 & 0 \\ 12 & b & 7 & -4 \end{bmatrix}$$

Assignment 5 (10%)

Consider the matrix $A\mathbf{x} = \mathbf{b}$ where A and $\mathbf{b}$ are given by

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 4 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a. The equation $A\mathbf{x} = \mathbf{b}$ can not be solved. Show why.

Two guesses of approximate solutions to $A\mathbf{x} = \mathbf{b}$ are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

b. Determine which of the two proposed solutions is the best one in the least squares sense.

Assignment 6 (20%)

Assume that the following corresponding values of time t and output y for a system has been measured

t	y
0	1
2	6
5	17
6	19

It is assumed that the system can be approximated by a model of the form $y_1(t) = \beta_0 + \beta_2 t^2$ or of the form $y_2(t) = \gamma_1 t + \gamma_2 t^2$ or of the form $y_3(t) = \delta_0 + \delta_1 t + \delta_2 t^2$

a. Determine the design matrix and observation matrix of all three models.

b. Determine the parameters of all three models

c. State, with substantiation, which of the three models is the best fit for the measured data

Assignment 7 (15%)

Consider the following matrix:

$$\begin{bmatrix} 9 & 3 & 3 \\ -3 & 9 & 3 \end{bmatrix}$$

a. Do a full singular value decomposition of matrix A. This includes the following steps:

   1) Finding the eigenvalues of either $A^T A$
   2) Determining the corresponding eigenvectors
   3) Finding the columns of V and from this deriving the columns of U
   4) Setting up $A = U\Sigma V^T$
   5) Testing that $A = U\Sigma V^T$

b. Show that the columns of U are eigenvectors of AA^T and determine the corresponding eigenvalues.

Assignment 1
The answers to the problems of this assignment do not require any calculations but can be inferred from the information given (provided you have the relevant knowledge!)

a) Represent each linear system of equations in matrix form

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 1 \\ 4x_1 + 7x_2 + x_3 = 3 \\ 7x_1 + 10x_2 - 4x_3 = 4 \end{cases}$$

$$\begin{cases} 3x_1 + 3x_2 + x_3 = -4.5 \\ x_1 + x_2 + x_3 = 0.5 \\ -2x_1 - 2x_2 = 5 \end{cases}$$

$$\begin{cases} 2x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 + x_3 = 13 \\ 4x_1 - 4x_3 = 4 \end{cases}$$

In b) - e) refer to the following coefficient matrix

$$\begin{bmatrix} 1 & 3 & 6 & -1 & 0 \\ 0 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) Determine which columns have pivot positions
c) What is the rank of the matrix?
d) Are there any free variables, and if so which?
e) Explain why the matrix is *not* invertible.

f) Let $det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 5$. Find

$$det\left(\begin{bmatrix} a & b & c \\ d + 2a & e + 2b & f + 2c \\ g & h & i \end{bmatrix}\right)$$

Assignment 2

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$

a) What value(s) of *k*, if any, will make $AB = BA$?

b) For what value(s) of *k* are the columns of the following matrix linearly dependent?

$$\begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & k \end{bmatrix}$$

c) Find the value(s) of *k* for which the matrix $\begin{bmatrix} k^2 & 2k \\ 8 & k \end{bmatrix}$ is singular (i.e. not invertible).

Assignment 3

Find the value(s) of α for which the determinant of the following matrix is zero

$$\begin{bmatrix} a & \sqrt{2} & 0 \\ \sqrt{2} & a & \sqrt{2} \\ 0 & \sqrt{2} & a \end{bmatrix}$$

Assignment 4

Let

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, and\ v = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix},$$

a) Show that $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$

b) Write *v* as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3, u_4\}$.

c) Determine the (shortest) distance between *v* and the subspace spanned by $\{u_1, u_2, u_3\}$

Assignment 5

The following data were measured in an experiment that was aimed at studying the effects of reducing current draw in a magnetic core by electronic means. The researchers measured the current in a magnetic winding with and without the electronics in a paired experiment and the data for the case without electronics is provided in the following table

Supply Voltage	Current Without Electronics (mA)
0.66	7.32
1.32	12.22
1.98	16.34
2.64	23.66
3.3	28.06
3.96	33.39
4.62	34.12
3.28	39.21
5.94	44.21
6.6	47.48

a) Find the best fitted least-squares line to describe the data above
b) Determine the least-squares error of the least squares line found in (a).

Assignment 6

Compute a full singular value decomposition of the following matrix A:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$