1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Definition: A **linear equation** in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the **coefficients** $a_1, \ldots, a_n$ are constants.

Example: A line of the form $y = mx + b$ is a linear equation. You can rewrite it as $y - mx = b$ to see that it fits the definition above.

Example: The classic equation of a plane in x, y, z-coordinates is

$$ax + by + cz = d,$$

where a, b, c, d are constants. This is an example of a linear equation.

Definition: A **system of linear equations** (also called a **linear system**) is a collection of one or more linear equations involving the same variables.

Example: The following is a linear system in the variables x_1, x_2, x_3:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 3 \\ 7x_2 - 4x_3 &= 10 \\ x_3 &= 1. \end{aligned}$$

To solve this system, we could first note from the third equation that $x_3 = 1$, then substitute this into the second equation to solve for x_2:

$$\begin{aligned} 7x_2 - 4(1) &= 10 \\ \Longrightarrow 7x_2 &= 14 \\ \Longrightarrow x_2 &= 2. \end{aligned}$$

Now we substitute $x_2 = 2$ and $x_3 = 1$ into the first equation to solve for x_1:

$$\begin{aligned} 2x_1 + 3(2) + 1 &= 3 \\ \Longrightarrow 2x_1 &= -4 \\ \Longrightarrow x_1 &= -2. \end{aligned}$$

We conclude that the solution to the system is $(x_1, x_2, x_3) = (-2, 2, 1)$.

Definition: A **solution** of a linear system in the variables $x_1, x_2, \ldots, x_n$ is a list of numbers $s_1, s_2, \ldots, s_n$ which makes all equations of the system true when substituted for the variables $x_1, x_2, \ldots, x_n$, respectively. The set of all possible solutions of a linear system is called its **solution set**. Two linear systems are called **equivalent** if they have the same solution set.

Fact: A system of linear equations has
(i) no solutions, or
(ii) exactly one solution, or
(iii) infinitely many solutions.

Definition: We say a system is **consistent** if it has either one or infinitely many solutions. We say a system is **inconsistent** if it has no solution.

Note: The question of whether or not a system is consistent is an "existence" question: does a solution exist? If a solution exists (i.e. if the system is consistent), we might ask if such a solution is "unique." This corresponds to determining if there is one solution or if there are infinitely many solutions. Questions of existence and uniqueness are fundamental in mathematics.

Definition: We can encode the information of a linear system in a rectangular array called a **matrix**. The **coefficient matrix** of the linear system above is

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 7 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

The **augmented matrix** of the same system is

$$\begin{bmatrix} 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The **size** of a matrix is the number of rows and columns it has, and it is expressed in the form **rows** $\times$ **columns**. For example, the coefficient matrix above is 3×3 and the augmented matrix is 3×4.

Example: We now show the steps to take in solving the same system using the augmented matrix notation. Let r_1 denote the first row, r_2 the second row, and r_3 the third row.

$$\begin{bmatrix} 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{array}{rcl} r_1 & \mapsto & r_1 - r_3 \\ r_2 & \mapsto & r_2 + 4r_3 \end{array} \quad \begin{bmatrix} 2 & 3 & 0 & 2 \\ 0 & 7 & 0 & 14 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$r_2 \mapsto \tfrac{1}{7} r_2 \quad \begin{bmatrix} 2 & 3 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$r_1 \mapsto r_1 - 3r_2 \quad \begin{bmatrix} 2 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$r_1 \mapsto \tfrac{1}{2} r_1 \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

From here, we simply reinterpret the matrix as a linear system to find

$$\begin{aligned} x_1 &= -2 \\ x_2 &= 2 \\ x_3 &= 1, \end{aligned}$$

which we can express more compactly as $(x_1, x_2, x_3) = (-2, 2, 1)$. This is exactly the same solution we found before.

Definition: In the preceding example we performed a sequence of what are called **elementary row operations**. There are three kinds of elementary row operations, which we summarize below.
(i) (Replacement) Replace one row by the sum of itself and a multiple of another row.
(ii) (Interchange) Swap two rows.
(iii) (Scaling) Multiply all entries in a row by a nonzero constant.

Definition: We say that two matrices are **row equivalent** if there is a sequence of elementary row operations that transforms one into the other.

Fact: Two linear systems are equivalent (i.e they have the same solution set) if their augmented matrices are row equivalent. In other words, performing elementary row operations on an augmented matrix of a linear system does not change the solution set of the system.

Examples: The following are to be worked on during class (time-permitting):

a) Determine if the following system is consistent:

$$\begin{aligned} x_2 + 4x_3 &= 2 \\ x_1 - 3x_2 + 2x_3 &= 6 \\ x_1 - 2x_2 + 6x_3 &= 9. \end{aligned}$$

b) Give a solution of the following system (if one exists). Is it unique?

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + 6x_2 + 9x_3 &= 12. \end{aligned}$$

c) (Page 9, Practice Problem 4)
For what values of h and k is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k. \end{aligned}$$

1.2 Row Reduction and Echelon Forms

Definition: A matrix is in **echelon form** if it has the following three properties:
(i)All nonzero rows are above all rows of zeros.
(ii) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
(iii) All entries in a column below a leading entry are zeros.

Definition: A matrix is in **reduced echelon form** if, in addition to the three properties mentioned above, the matrix also satisfies:
iv) The leading entry in each nonzero row is 1.
v) Each leading 1 is the only nonzero entry in its column.

Definition: We say that an echelon matrix U is **an echelon form of the matrix** A if U is row equivalent to A. Similarly, we say that a reduced echelon matrix U is **the reduced echelon form of the matrix** A if U is row equivalent to A.

The significance of putting the augmented matrix of a linear system in echelon form is explained by the following theorem.

Theorem: (Existence Theorem)
A linear system is consistent if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix}$$

with b nonzero.

Example: The augmented matrix of the linear system used as the main example in the preceding section,

$$\begin{bmatrix} 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

is in echelon form. Since it has no row of the form mentioned in the theorem, we know immediately that this system is consistent.

Example: Recall that we performed a sequence of row operations on the preceding matrix to get

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

which is in reduced echelon form. This allowed us easily to see the solutions of this system, which is the main advantage of putting the matrix in this form. This brings us to the following important theorem.

Theorem: (Uniqueness of Reduced Echelon Form)
Each matrix is row equivalent to one and only one reduced echelon matrix

Definition: A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

The Row Reduction Algorithm Here we describe an algorithm for turning any matrix into an equivalent (reduced) echelon matrix.

1. Begin with the leftmost nonzero column. This is a pivot column, with the pivot position at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Apply steps 1-3 to the submatrix of all entries below and to the right of the pivot position. Repeat this process until there are no more nonzero rows to modify. (At this point we have reached an echelon form of the matrix.)
5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot using row operations. If a pivot is not 1, make it 1 by a scaling operation. (This step produces the reduced echelon form of the matrix.)

Solutions of Linear Systems

Definition: Let A be the coefficient matrix of a linear system. The pivot columns in the matrix correspond to what we call **basic variables**. The nonpivot columns correspond to what we call **free variables.**

Example: Suppose the augmented matrix of a linear system has been reduced to the following form:

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\blacksquare$ represents any nonzero number, and $*$ represents any number (including 0). The basic variables of this system are x_1, x_2, and x_4. The only free variable is x_3.

Theorem: (Uniqueness Theorem)
If a linear system is consistent, then the solution set contains either
(i) a unique solution, when there are no free variables, or
(ii) infinitely many solutions, when there is at least one free variable.

Examples: The following are to be worked on during class (time-permitting):
a) Suppose the following matrix is the augmented matrix of a linear system in the variables x_1, x_2, and x_3. Row reduce the matrix to <u>echelon form</u> to determine if it is consistent.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

If it is consistent, find the <u>reduced echelon form</u> and write the solution set using free variables as parameters.

b) (Exercise 20 on page 22)
Choose h and k such that the following system has
(i) no solution,
(ii) a unique solution, and
(iii) many solutions.

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ 2x_1 + hx_2 &= k. \end{aligned}$$

1.3 Vector Equations

Definition: A matrix with only one column is called a **column vector** (or **vector**). We say that two vectors are **equal** if and only if their corresponding entries are equal.

Example: The following are vectors in $\mathbb{R}^2$ (a.k.a. the plane consisting of ordered pairs of real numbers):

$$\bar{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \qquad \bar{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

They are not equal because their corresponding entries do not match.

Definition: The **sum** of two vectors $\bar{u}$ and $\bar{v}$ in $\mathbb{R}^2$, denoted $\bar{u}+\bar{v}$, is obtained by adding the corresponding entries of $\bar{u}$ and $\bar{v}$. Given a real number c, the **scalar multiple** of $\bar{u}$ by c, denoted $c\bar{u}$, is obtained by multiplying each entry in $\bar{u}$ by c.

Example: If $\bar{u}$ and $\bar{v}$ are as in the preceding example, then

$$\begin{aligned} \bar{u}+\bar{v} &= \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+5 \\ 5+3 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}, \text{ and} \\ 6\bar{u} &= 6\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6\cdot 3 \\ 6\cdot 5 \end{bmatrix} = \begin{bmatrix} 18 \\ 30 \end{bmatrix}. \end{aligned}$$

Note: It is often helpful to identify a vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ with a geometric point (a,b) in the plane in order to get a picture of what we are working with. There are some nice pictures in the textbook showing what this looks like.

Rule: (Parallelogram Rule for Addition)
If $\bar{u}$ and $\bar{v}$ in $\mathbb{R}^2$ are thought of as points in the plane, then $\bar{u}+\bar{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\bar{0}$, $\bar{u}$, and $\bar{v}$.

Note: By $\bar{0}$, we mean the zero vector, or the vector whose entries are all zero. In $\mathbb{R}^2$, we have $\bar{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

These ideas generalize to higher-dimensional spaces. More specifically, we can define $\mathbb{R}^n$ as follows.

Definition: For each positive integer n, we let $\mathbb{R}^n$ denote the collection of *ordered n-tuples* with each entry in $\mathbb{R}$. We often write these elements as $n \times 1$ matrices. We define addition and scalar multiplication of vectors in $\mathbb{R}^n$ in the same way as we do for $\mathbb{R}^2$. That is, we go coordinate-by-coordinate.

Example: If $u_1, u_2, ..., u_n \in \mathbb{R}$, then

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n.$$

Example: If $\bar{u}$ and $\bar{v}$ are in $\mathbb{R}^n$ (with entries denoted $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, respectively), and $c \in \mathbb{R}$, then

$$\begin{aligned} \bar{u}+\bar{v} &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}, \text{ and} \\ c\bar{u} &= c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}. \end{aligned}$$

Properties: (Algebraic Properties of $\mathbb{R}^n$)

For all vectors $\bar{u}$, $\bar{v}$ and $\bar{w}$ in $\mathbb{R}^n$ and all scalars c and d in $\mathbb{R}$, the following hold:
(i) (Commutativity) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$,
(ii) (Associativity) $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$,
(iii) $\bar{u} + \bar{0} = \bar{0} + \bar{u} = \bar{u}$,
(iv) $\bar{u} + (-\bar{u}) = -\bar{u} + \bar{u} = 0$,
(v) $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$,
(vi) $(c + d)\bar{u} = c\bar{u} + d\bar{u}$,
(vii) $c(d\bar{u}) = (cd)\bar{u}$,
(viii) $1\bar{u} = \bar{u}$.

Note: The properties above follow for vectors in $\mathbb{R}^n$ because they hold in each coordinate. For example, to show the commutativity property, we would take two arbitrary vectors

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in $\mathbb{R}^n$, then add them to see

$$\bar{u} + \bar{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \underbrace{\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}}_{\text{by vector addition}} = \underbrace{\begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}}_{\text{by commutativity in } \mathbb{R}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \bar{v} + \bar{u}.$$

You should try to establish some of the other properties on your own by using steps similar to those above. This is an example of the type of proof you may be asked to give on a quiz or exam.

Definition: Given a set of vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_p \in \mathbb{R}$, the vector $\bar{y}$ given by

$$\bar{y} = c_1\bar{v}_1 + \cdots + c_p\bar{v}_p$$

is called a **linear combination** of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p$ with **weights** $c_1, c_2, \ldots, c_p$.

Example: Let $\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Some linear combinations of $\bar{v}_1$ and $\bar{v}_2$ include

$$\begin{aligned} \bar{0} &= 0\bar{v}_1 + 0\bar{v}_2, \\ \begin{bmatrix} 3 \\ 0 \end{bmatrix} &= \bar{v}_1 + 2\bar{v}_2, \\ \begin{bmatrix} -5 \\ -1 \end{bmatrix} &= -2\bar{v}_1 - 3\bar{v}_2, \text{ and} \\ \begin{bmatrix} 5 \\ 1 \end{bmatrix} &= 2\bar{v}_1 + 3\bar{v}_2. \end{aligned}$$

In class we will see how this can be visualized in the plane.

It is often the case that we wish to know if some vector $\bar{b}$ can be formed as a linear combination of some other set of vectors $\bar{a}_1, \ldots, \bar{a}_n$. The process for figuring this out is given by the following.

Fact: A vector equation

$$x_1\bar{a}_1 + x_2\bar{a}_2 + \cdots + x_n\bar{a}_n = \bar{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_n & \bar{b} \end{bmatrix}.$$

More specifically, if the $\bar{a}_i$'s are in $\mathbb{R}^m$ with

$$\bar{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \bar{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \bar{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}, \quad \text{and } \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

then you would row reduce the matrix

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \uparrow \\ \bar{a}_1 & \bar{a}_2 & & \bar{a}_n & \bar{b} \end{array} \end{array}$$

to determine if there is some set of weights $x_1, \ldots, x_n$ that work.

Example: Let

$$\bar{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \bar{a}_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}, \quad \bar{a}_3 \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \quad \text{and } \bar{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}.$$

We determine if $\bar{b}$ is a linear combination of $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, i.e. if there is some set of weights x_1, x_2, x_3 such that $x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3 = \bar{b}$. By the above, we translate this question to the matrix setting.

$$\begin{array}{ccc} \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} & r_1 \leftrightarrow r_3 & \begin{bmatrix} 1 & -2 & 1 & 7 \\ -1 & 8 & 5 & 3 \\ 2 & 0 & 6 & 10 \end{bmatrix} \\ & \begin{array}{c} r_2 \mapsto r_2 + r_1 \\ r_3 \mapsto r_3 - 2r_1 \end{array} & \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 0 & 4 & 4 & -4 \end{bmatrix} \\ & r_3 \mapsto r_3 - \frac{4}{6}r_2 & \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}. \end{array}$$

Since the symbol $\blacksquare$ denotes something nonzero, we see that there's no solution, i.e. that $\bar{b}$ is not a linear combination of $\bar{a}_1, \bar{a}_2, \bar{a}_3$.

Example: Let

$$\bar{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{a}_2 = \begin{bmatrix} -4 \\ 6 \\ -4 \end{bmatrix}, \quad \bar{a}_3 \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \quad \text{and } \bar{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}.$$

We determine if $\bar{b}$ is a linear combination of $\bar{a}_1, \bar{a}_2, \bar{a}_3$. We reduce the corresponding matrix to echelon form

$$\begin{bmatrix} 1 & -4 & -6 & 11 \\ 0 & 6 & 7 & -5 \\ 1 & -4 & 5 & 9 \end{bmatrix} \quad r_3 \mapsto r_3 - r_1 \quad \begin{bmatrix} 1 & -4 & -6 & 11 \\ 0 & 6 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix},$$

and we see that there is a solution. Now we find what weights x_1, x_2, x_3 work by finding the reduced echelon form:

$$\begin{array}{cc}
r_3 \mapsto \frac{1}{11} r_3 & \left[\begin{array}{cccc} 1 & -4 & -6 & 11 \\ 0 & 6 & 7 & -5 \\ 0 & 0 & 1 & -\frac{2}{11} \end{array}\right] \\
\begin{array}{c} r_2 \mapsto r_2 - 7r_3 \\ r_1 \mapsto r_1 + 6r_3 \end{array} & \left[\begin{array}{cccc} 1 & -4 & 0 & \frac{109}{11} \\ 0 & 6 & 0 & -\frac{41}{11} \\ 0 & 0 & 1 & -\frac{2}{11} \end{array}\right] \\
r_2 \mapsto \frac{1}{6} r_2 & \left[\begin{array}{cccc} 1 & -4 & 0 & \frac{109}{11} \\ 0 & 1 & 0 & -\frac{41}{66} \\ 0 & 0 & 1 & -\frac{2}{11} \end{array}\right] \\
r_1 \mapsto r_1 + 4r_2 & \left[\begin{array}{cccc} 1 & 0 & 0 & \frac{245}{33} \\ 0 & 1 & 0 & -\frac{41}{66} \\ 0 & 0 & 1 & -\frac{2}{11} \end{array}\right],
\end{array}$$

so $(x_1, x_2, x_3) = \left(\frac{245}{33}, -\frac{41}{66}, -\frac{2}{11}\right)$. Since there are no free variables, this is the *unique* solution.

Definition: If $\bar{v}_1, \ldots, \bar{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\bar{v}_1, \ldots, \bar{v}_p$ is denoted by $\text{Span}\{\bar{v}_1, \ldots, \bar{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned by** $\bar{v}_1, \ldots, \bar{v}_p$. In other words, the span of $\bar{v}_1, \ldots, \bar{v}_p$ is all vectors that can be written in the form

$$c_1\bar{v}_1 + \cdots + c_p\bar{v}_p,$$

with $c_1, \ldots, c_p$ scalars.

In class we will see how the span of a set of vectors can be interpreted geometrically.

Examples: The following are to be worked on in class (time-permitting):

(a) Prove property (v) on the list of Algebraic Properties of $\mathbb{R}^n$. That is, show that for two vectors $\bar{u}$ and $\bar{v}$ in $\mathbb{R}^n$ and a scalar $c \in \mathbb{R}$, we have

$$c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}.$$

(b) (Practice Problem 2 on page 31)
For what value(s) of h will $\bar{y}$ be in $\text{Span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ if

$$\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \bar{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.$$

1.4 The Matrix Equation $A\bar{x} = \bar{b}$

Definition: If A is an $m \times n$ matrix with columns $\bar{a}_1, \ldots, \bar{a}_n$, and if $\bar{x} \in \mathbb{R}^n$, the **the product of** A **and** $\bar{x}$, denoted by $A\bar{x}$, is

$$A\bar{x} = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\bar{a}_1 + x_2\bar{a}_2 + \cdots + x_n\bar{a}_n.$$

Example:

$$\begin{aligned} \begin{bmatrix} 4 & 1 & 2 \\ 8 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} &= 2\begin{bmatrix} 4 \\ 8 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 16 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 12 \end{bmatrix} \\ &= \begin{bmatrix} 17 \\ 28 \end{bmatrix}. \end{aligned}$$

Building a bit on the main result from the preceding section, we have the following theorem.

Theorem: If A is an $m \times n$ matrix, with columns $\bar{a}_1, \ldots, \bar{a}_n \in \mathbb{R}^m$ and if $\bar{b} \in \mathbb{R}^m$, the **matrix equation**

$$A\bar{x} = \bar{b}$$

has the same solution set as the vector equation

$$x_1\bar{a}_1 + x_2\bar{a}_2 + \cdots + x_n\bar{a}_n = \bar{b}$$

which, in turn, has the same solution set as the system of linear equations with augmented matrix

$$\begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \ldots & \bar{a}_n & \bar{b} \end{bmatrix}.$$

Fact: The equation $A\bar{x} = \bar{b}$ has a solution if and only if $\bar{b}$ is a linear combination of the columns of A.

Example: We let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ 1 & 12 & 19 \end{bmatrix}, \text{ and } \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

and determine if $A\bar{x} = \bar{b}$ is consistent for all possible $\bar{b}$. We begin by row reducing the augmented matrix:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ -2 & 1 & 2 & b_2 \\ 1 & 12 & 19 & b_3 \end{bmatrix} &\sim \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 5 & 8 & b_2 + 2b_1 \\ 0 & 10 & 16 & b_3 - b_1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 5 & 8 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 - 2(b_2 + 2b_1) \end{bmatrix}, \end{aligned}$$

which has a solution only if the bottom right entry is zero, i.e. only if

$$-5b_1 - 2b_2 + b_3 = 0.$$

This is the equation of a plane through the origin in $\mathbb{R}^3$. This plane is spanned by the column vectors of A.

The following theorem tells us when the column vectors of an $m \times n$ matrix (the columns are vectors in $\mathbb{R}^m$, since there are m rows) can be used to generate all of $\mathbb{R}^m$. This basically summarizes several things we have already seen in different contexts.

Theorem: Let A be an $m \times n$ matrix. Then the following statements are equivalent.
(a) For each $\bar{b} \in \mathbb{R}^m$, the equation $A\bar{x} = \bar{b}$ has a solution.
(b) Each $\bar{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.
(c) The columns of A span $\mathbb{R}^m$.
(d) The matrix A has a pivot position in every row.

Rule: (Row-Vector Rule for Computing $A\bar{x}$)
Assuming the product $A\bar{x}$ is defined, then the ith entry in $A\bar{x}$ is the sum of the product of corresponding entries from row i of A and from the vector $\bar{x}$. In other words, the ith entry in $A\bar{x}$ is the dot product of the vector forming the ith row of A and the vector $\bar{x}$.

Example: Let

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 5 & 0 & 2 \end{bmatrix}, \text{ and } \bar{x} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}.$$

Then

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 4 \\ 5 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} &= \begin{bmatrix} 1 \cdot 2 + (-1) \cdot 1 + 4 \cdot 5 \\ 5 \cdot 2 + 0 \cdot 1 + 2 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 21 \\ 20 \end{bmatrix}. \end{aligned}$$

Notice that the number of columns in A must match the number of rows in $\bar{x}$, for otherwise the dot product would not make sense!

Theorem: If A is an $m \times n$ matrix, $\bar{u}$ and $\bar{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:
(a) $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v}$,
(b) $A(c\bar{u}) = c(A\bar{u})$.

Proof: We prove (a). The proof of (b) is similar and is left to you. Let

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \ \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \text{ and } A = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_n \end{bmatrix}.$$

Then

$$\begin{aligned} A(\bar{u} + \bar{v}) &= \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_n \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \\ &= (u_1 + v_1)\bar{a}_1 + (u_2 + v_2)\bar{a}_2 + \cdots + (u_n + v_n)\bar{a}_n. \end{aligned}$$

The second equality follows straight from the definition of the product of a matrix A with a vector $\bar{x}$ that was given earlier in this section. Expanding and then collecting the u terms and v terms separately, we find that

$$(u_1\bar{a}_1 + u_2\bar{a}_2 + \cdots + u_n\bar{a}_n) + (v_1\bar{a}_1 + v_2\bar{a}_2 + \cdots + v_n\bar{a}_n) \quad = \quad A\bar{u} + A\bar{v},$$

which completes the proof.

Examples: The following are to be worked on in class (time-permitting):

(a) (Practice Problem #1 on page 40)

Let

$$A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}, \quad \bar{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}.$$

It can be shown that $\bar{p}$ is a solution of $A\bar{x} = \bar{b}$. Use this fact to write $\bar{b}$ as a linear combination of the columns of A.

(b) Prove part (b) of the preceding theorem.

1.5 Solution Sets of Linear Systems

Definition: A linear system is said to be **homogeneous** if it can be written in the form

$$A\bar{x} = \bar{0},$$

where A is an $m \times n$ matrix and $\bar{0}$ is the zero vector in $\mathbb{R}^m$. Such a system always has the **trivial solution**, namely $\bar{x} = \bar{0}$ (the zero vector of $\mathbb{R}^n$, not $\mathbb{R}^m$).

Since homogeneous systems always have the trivial solution, the interesting question is whether or not they have nontrivial solutions.

Fact: The homogeneous equation $A\bar{x} = \bar{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example: Determine if the following homogeneous system has a nontrivial solution. If it does, describe the solution set using the free variable(s).

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ -2x_1 + 5x_2 - 7x_3 &= 0. \end{aligned}$$

We let A be the coefficient matrix of the system, and reduce the augmented matrix $\begin{bmatrix} A & \bar{0} \end{bmatrix}$ to echelon form (notice how we put the second equation first in order to simplify the computation a bit):

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & -3 & 0 \\ -2 & 5 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so we see that x_3 is a free variable, and as a result we have nontrivial solutions. Now we get the reduced echelon form:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives us the equations

$$\begin{aligned} x_1 - \frac{2}{3}x_3 &= 0 \quad \Longrightarrow x_1 = \frac{2}{3}x_3 \\ x_2 - \frac{5}{3}x_3 &= 0 \quad \Longrightarrow x_2 = \frac{5}{3}x_3. \end{aligned}$$

In vector form, the solution $\bar{x}$ of the equation $A\bar{x} = 0$ is written

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_3 \\ \frac{5}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}.$$

Since x_3 can be anything, in geometric terms this solution set describes the line in $\mathbb{R}^3$ extending from the origin through the point $\left(\frac{2}{3}, \frac{5}{3}, 1\right)$.

Definition: Whenever a solution set is described explicitly with vectors (as in the preceding example), we say that the solution is in **parametric vector form.**

Example: Describe all solutions of the homogeneous equation

$$x_1 - 2x_2 - 5x_3 = 0.$$

Writing this system in a matrix, we would see that x_2 and x_3 are free variables, and x_1 is a basic variable with $x_1 = 2x_2 + 5x_3$. Hence the general solution is

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 + 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix},$$

where the right-most part of this equation is the parametric vector form of the solution set. In this case, the entire solution set is a plane in $\mathbb{R}^3$ that contains the two lines extending from $(0,0,0)$ to $(2,1,0)$, and from $(0,0,0)$ to $(5,0,1)$.

Solutions of Nonhomogeneous Systems When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus arbitrary linear combinations of vectors that satisfy the corresponding homogeneous system.

Example: We reconsider the homogeneous system at the beginning of this section, except this time it will be nonhomogeneous (i.e. the right side will not be all zeroes):

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 2 \\ x_1 - x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 - 7x_3 &= -2. \end{aligned}$$

As before, we perform row operations on the augmented matrix $[\ A \quad \bar{b}\ ]$ where A is the same coefficient matrix and

$$\bar{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix},$$

and we find that

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -3 & 2 \\ -2 & 5 & -7 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 1 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$\begin{aligned} x_1 &= 1 + \frac{2}{3}x_3, \text{ and} \\ x_2 &= \frac{5}{3}x_3. \end{aligned}$$

This can be written

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{2}{3}x_3 \\ \frac{5}{3}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix},$$

which has the form we claimed. This is the equation of the line through the point $(1,0,0)$ that is parallel to the line extending from $(0,0,0)$ through $\left(\frac{2}{3}, \frac{5}{3}, 1\right)$.

We summarize the general situation with the following theorem. It might be best to think of the conclusion as shifting all homogeneous solutions by the vector $\bar{p}$.

Theorem: Suppose $A\bar{x} = \bar{b}$ is consistent for some $\bar{b}$, and let $\bar{p}$ be a solution. Then the solution set of $A\bar{x} = \bar{b}$ is the set of all vectors of the form

$$\bar{w} = \bar{p} + \bar{v}_h,$$

where $\bar{v}_h$ is any solution of the homogeneous equation $A\bar{x} = \bar{0}$.

Examples: The following are to be worked on in class (time-permitting):

(a) Write the general solution of

$$x_1 - 2x_2 - 5x_3 = 3$$

in parametric vector form. In geometric terms, what does this solution set look like in comparison to the solution set of the equation $x_1 - 2x_2 - 5x_3 = 0$ that we saw earlier?

(b) (Exercise #20 on page 47) Find the parametric equation of the line through $\bar{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ that is also parallel to $\bar{b} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$.