3 Determinants

3.1 Introduction to Determinants

It was remarked in a previous section that if the determinant $ad - bc$ of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonzero, then A is invertible (and vice-versa). To see why this is true, we assume $a \neq 0$ so that A has a pivot in the first row, then perform some row operations on A:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{c} r_2 \mapsto ar_2 \end{array} \quad \begin{bmatrix} a & b \\ ac & ad \end{bmatrix}$$
$$r_2 \mapsto r_2 - cr_1 \quad \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}.$$

In order for this matrix to be invertible, we need a pivot in the second row, which happens if and only if $ad - bc \neq 0$. Notice that this quantity $ad - bc$ is a number that is formed using every entry of the matrix, and it tells us very useful information about the matrix. This same idea generalizes to larger matrices; i.e. associated to every $n \times n$ matrix A, there is a determinant $\det A$ (sometimes denoted Δ) involving every entry of A that tells us whether this matrix is invertible.

Notation For any square $n \times n$ matrix A, we let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the jth column.

Example Let A be the 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Then

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad \text{and} \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}.$$

Definition For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with the plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}.$$

In summation notation, we write

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}.$$

Remark This definition is *recursive*. For example, computing the determinant of a 3×3 matrix is dependent on knowing how to compute the determinant of a 2×2 matrix. More generally, computing the determinant of an $n \times n$ matrix is dependent on knowing how to compute an $(n-1) \times (n-1)$ determinant, which is dependent on knowing how to compute an $(n-2) \times (n-2)$ determinant, etc. In practice, we usually let a computer handle such a computation for large matrices.

Example We compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Using the definition, we have

$$\begin{aligned}
\det A &= 1 \cdot \det A_{11} - 2 \cdot \det A_{12} + 3 \cdot \det A_{13} \\
&= \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \\
&= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) \\
&= (45 - 48) - 2(36 - 42) + 3(32 - 35) \\
&= -3 - 2(-6) + 3(-3) \\
&= -3 + 12 - 9 \\
&= 0.
\end{aligned}$$

This tells us that A is *not* invertible (though we won't provide an explanation until the next section).

In the definition of the determinant given above, it may seem as though there is something special about expanding along the first row. This turns out to be false. That is, we can compute the determinant by a process called **cofactor expansion** across *any* row or *any* column. The power of this is that it allows for fast computation of the determinant when a particular row or column has lots of zeros.

Notation We often use the notation $|\cdot|$ instead of det to indicate that a determinant is being computed. For example

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}.$$

Definition Given an $n \times n$ matrix $A = [a_{ij}]$, the (i, j)-**cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

With this definition, we see that

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

This is called the **cofactor expansion along the first row** of A.

Theorem The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

Similarly, the expansion down the jth column using cofactors is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Note The plus or minus sign in the (i, j)-cofactor depends on the position of a_{ij} in the matrix. The factor $(-1)^{i+j}$ determines a "checkerboard" pattern as follows:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Example We use cofactor expansion across the third row of the following matrix to simplify the computation of the determinant:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 5 & 3 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

We see that

$$\begin{aligned}
\det A &= 0 \cdot \det A_{31} - 2 \cdot \det A_{32} + 0 \cdot \det A_{33} \\
&= -2 \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} \\
&= -2(2 \cdot 1 - 0 \cdot 5) \\
&= -4.
\end{aligned}$$

We conclude that this matrix is invertible. Likewise, we could have expanded along the third column, as follows:

$$\begin{aligned}
\det A &= 0 \cdot \det A_{13} - 1 \cdot \det A_{23} + 0 \cdot \det A_{33} \\
&= -1 \cdot \det A_{23} \\
&= - \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \\
&= -(2 \cdot 2 - 1 \cdot 0) \\
&= -4.
\end{aligned}$$

Notice that we get the same answer.

Theorem If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Example We compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

By the theorem,

$$\det A = 1 \cdot 6 \cdot 1 \cdot 4 \cdot 6 = 144.$$

Note In applications, matrices of size, say, 25×25 are relatively small. Yet computing the determinant of such a matrix using the method outlined in this section would take approximately $25! = 25 \cdot 24 \cdot 23 \cdot \cdots \cdot 2 \cdot 1$ multiplications, which is roughly 1.5×10^{25}. This would take a computer many years to perform (the textbook gives an estimate of over $500,000$ years, assuming the computer performs one trillion multiplications per second). There are other ways to significantly shorten the computation of a determinant.

Example The following example is to be worked on in class (time-permitting):
Compute

$$\begin{vmatrix} 1 & -4 & 2 & -1 \\ 0 & 4 & 0 & 2 \\ 3 & -2 & 0 & 5 \\ 0 & 1 & 0 & 7 \end{vmatrix}.$$

3.2 Properties of Determinants

In the last section, we saw that the determinant of a triangular matrix is very easy to compute (just multiply the entries along the diagonal). As we will see shortly, the determinant of a matrix changes predictably under row operations. Combining these facts, we will find that an efficient way to compute a determinant is to row reduce to echelon form while keeping track of what the row operations do to the determinant.

Theorem (Row Operations)
Let A be a square matrix.
(a) If B is obtained from A via a row replacement, then $\det B = \det A$.
(b) If B is obtained from A via a row interchange, then $\det B = -\det A$.
(c) If B is obtained from A via a row scaling by k, then $\det B = k \cdot \det A$.

Example We compute $\det A$, where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 2 \\ -3 & -5 & 2 \end{bmatrix}.$$

The computation proceeds as follows:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 2 \\ -1 & -2 & 2 \\ -3 & -5 & 2 \end{vmatrix} &= \begin{vmatrix} 1 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 8 \end{vmatrix} \quad \text{(row replacement)} \\ &= -\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 8 \\ 0 & 0 & 4 \end{vmatrix} \quad \text{(row interchange).} \\ &= -1 \cdot 1 \cdot 4 \\ &= -4. \end{aligned}$$

Example We compute $\det A$, where

$$A = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 0 & 3 & 1 & 7 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

We proceed as follows:

$$\begin{aligned} \begin{vmatrix} 3 & 6 & 9 & 12 \\ 0 & 3 & 1 & 7 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} &= 3\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 7 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} \quad \text{(factor 3 out of row 1)} \\ &= 3\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 2 & 1 \end{vmatrix} \quad \text{(row replacement)} \\ &= 3\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -5 \end{vmatrix} \quad \text{(row replacement)} \\ &= 3(1 \cdot 3 \cdot (-1) \cdot (-5)) \\ &= 45. \end{aligned}$$

In general, it is possible to reduce a square matrix A to an echelon form U using only replacements and interchanges (i.e. without any scaling). Applying the the theorem above, we see that

$$\det A = (-1)^r \det U,$$

where r is the number of row interchanges performed (replacements have no effect on the determinant, while interchanges change the sign). This leads to the following formula:

$$\det A = \begin{cases} (-1)^r \text{ (product of pivots of } U) & \text{if } A \text{is invertible} \\ 0 & \text{if } A \text{is not invertible.} \end{cases}$$

We have observed before that echelon forms of a matrix are not unique (although *reduced* echelon forms are). However, the formula above tells us that, up to a plus/minus sign, the *product* of the pivots of the different echelon forms *are* unique. We also get the following theorem.

Theorem A square matrix A is invertible if and only if $\det A \neq 0$.

This theorem is quite useful. It basically tells us that if *any* of the statements in the Invertible Matrix Theorem fail to hold for a square matrix A, then we must have $\det A = 0$.

Example Let A be the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}.$$

After performing the operations $r_2 \mapsto r_2 - 2r_1$ and $r_3 \mapsto r_3 - 3r_1$, we find

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 4 & 5 & 6 & 7 \end{bmatrix},$$

at which point we see that $r_3 = 2r_2$ so that this matrix is *not* invertible (we will be short a pivot once we zero out one of these rows). It follows that $\det A = 0$.

Theorem If A is a square matrix, then $\det A^T = \det A$.

Why should we expect this theorem to be true? Recall that a determinant can be computed by cofactor expansion across *any* row or down *any* column. It is for this reason that, if we turn rows into columns (which is what happens when taking the transpose), the determinant does not change.

Theorem If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Warning It is *not* in general true that $\det(A + B) = \det A + \det B$.

Example Let $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. We compute separately $\det AB$ and $(\det A)(\det B)$. First notice that

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix},$$

so that

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 45.$$

Also notice that

$$\det A = 6 \cdot 2 - 1 \cdot 3 = 9$$

and

$$\det B = 4 \cdot 2 - 3 \cdot 1 = 5,$$

so that

$$(\det A)(\det B) = 9 \cdot 5 = 45.$$

We have seen how we can use an $m \times n$ matrix A to define a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ via $T(\bar{x}) = A\bar{x}$. In a similar manner, we can use the determinant to define a linear map on an $n \times n$ matrix A. Fix all except the jth column of $A = [\ \bar{a}_1 \quad \bar{a}_2 \quad \ldots \quad \bar{a}_n\ ]$, and replace the jth column with a variable $\bar{x}$. So we now have the following:

$$\left[\ \bar{a}_1 \quad \ldots \quad \bar{a}_{j-1} \quad \bar{x} \quad \bar{a}_{j+1} \quad \ldots \quad \bar{a}_n\ \right].$$

Now define

$$T : \mathbb{R}^n \to \mathbb{R}$$

by

$$T(\bar{x}) = \det \left[\ \bar{a}_1 \quad \ldots \quad \bar{a}_{j-1} \quad \bar{x} \quad \bar{a}_{j+1} \quad \ldots \quad \bar{a}_n\ \right].$$

It turns out that this map is linear; i.e. it satisfies
(i) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v}$ in $\mathbb{R}^n$, and
(ii) $T(c\bar{x}) = cT(\bar{x})$ for all $c \in \mathbb{R}$ and all $\bar{x} \in \mathbb{R}^n$.

Example The following example is to be worked on in class (time-permitting):

> Page 175 Practice Problem #2
> Use a determinant to decide if $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are linearly independent if
>
> $$\bar{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \qquad \bar{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \qquad \bar{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}.$$

3.3 Cramer's Rule, Volume, and Linear Transformations

Note Cramer's Rule is not particularly efficient for hand calculation of solutions of linear systems, but it is nevertheless a useful theoretical tool.

Notation For an $n \times n$ matrix A and any $\bar{b} \in \mathbb{R}^n$, we let $A_i(\bar{b})$ denote the matrix obtained by replacing the ith column of A with the vector $\bar{b}$.

Theorem Let A be an invertible $n \times n$ matrix. For any $\bar{b}$ in $\mathbb{R}^n$, the unique solution of $\bar{x}$ of $A\bar{x} = \bar{b}$ has entries given by

$$x_i = \frac{\det A_i(\bar{b})}{\det A}, \qquad i = 1, 2, \ldots, n.$$

Example Use Cramer's rule to solve the following linear system:

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ -4x_1 + 2x_2 &= 4. \end{aligned}$$

The coefficient matrix A is given by

$$A = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix},$$

and we have

$$A_1\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, \qquad A_2\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ -4 & 4 \end{bmatrix}.$$

Hence

$$x_1 = \frac{\det \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}}{\det \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}} = \frac{6-4}{4+4} = \frac{1}{4},$$

and

$$x_2 = \frac{\det \begin{bmatrix} 2 & 3 \\ -4 & 4 \end{bmatrix}}{\det \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}} = \frac{8+12}{4+4} = \frac{5}{2}.$$

Cramer's rule also provides another way to compute inverses of $n \times n$ matrices. If A is an $n \times n$ invertible matrix with inverse $A^{-1} = [\ \bar{x}_1 \quad \bar{x}_2 \quad \ldots \quad \bar{x}_n\ ]$, then

$$AA^{-1} = \begin{bmatrix} A\bar{x}_1 & A\bar{x}_2 & \ldots & A\bar{x}_n \end{bmatrix} = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \ldots & \bar{e}_n \end{bmatrix},$$

so that the jth column of A^{-1} satisfies

$$A\bar{x}_j = \bar{e}_j.$$

Applying Cramer's rule, we find that the (i, j)-entry of A^{-1} is

$$x_{ij} = \frac{\det A_i(\bar{e}_j)}{\det A}.$$

Now recall that we use A_{ji} to denote the submatrix of A obtained by deleting the jth row and the ith column. Since

$$A_i(\bar{e}_j) = \begin{bmatrix} \bar{a}_1 & \ldots & \bar{a}_{i-1} & \bar{e}_j & \bar{a}_{i+1} & \ldots & \bar{a}_n \end{bmatrix},$$

we see that it will be particularly advantageous to compute $\det A_i(\bar{e}_j)$ using cofactor expansion down the jth column (it is all zeros except for a 1 in the jth entry). The matrix left over after deleting the jth row and ith column will be exactly A_{ji}. Hence

$$\det A_i(\bar{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji},$$

where C_{ji} is a cofactor of A. It follows then that

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}.$$

Warning The (i,j)-entry of A^{-1} has the (j,i) cofactor C_{ji} in it. Note well that the order of the indices switches!

Definition We call the matrix of cofactors

$$\begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

the **adjugate** (or **classical adjoint**) of A, and denote this by $\mathrm{adj}A$.

The discussion above boils down to the following theorem.

Theorem Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}A.$$

Example We compute the inverse of $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ using the preceding theorem. We first compute the cofactors:

$$C_{11} = +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0 \qquad C_{31} = +\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1 \qquad C_{22} = +\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2 \qquad C_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0$$

$$C_{13} = +\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 \qquad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1 \qquad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

Hence

$$\mathrm{adj}A = \begin{bmatrix} 3 & 0 & -3 \\ -1 & 2 & 0 \\ -1 & -1 & 3 \end{bmatrix}.$$

We could compute $\det A$ directly to finish the computation, but multiplying $\mathrm{adj}A$ with A is another way about it that provides a check on our work:

$$\begin{aligned} (\mathrm{adj}A) \cdot A &= \begin{bmatrix} 3 & 0 & -3 \\ -1 & 2 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\ &= 3I_3. \end{aligned}$$

Multiplying both sides on the right by A^{-1} and dividing by 3, we get

$$\frac{1}{3}\mathrm{adj}A = A^{-1},$$

from which it follows that $\det A = 3$. We also get that

$$A^{-1} = \frac{1}{3}\mathrm{adj}A = \begin{bmatrix} 1 & 0 & -1 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}.$$

Determinants as Area or Volume A very nice geometric interpretation of the determinant is as an area or volume. Specifically, we can use the determinant of a 2×2 (or of a 3×3) matrix A to find the area of the parallelogram (or volume of the parallelepiped) given by the vectors forming the columns of A.

Theorem If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

We will see the following example (from page 181 of your book) worked out in detail during class.

Example Calculate the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1)$, and $(6,4)$.

Linear Transformations We can use determinants to describe exactly how the size of a region changes under a linear transformation.

Theorem Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If R is a region in $\mathbb{R}^2$ with finite area (a parallelogram, for example), then

$$\{\text{area of } T(R)\} = |\det A| \cdot \{\text{area of } R\}.$$

Likewise, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is determined by a 3×3 matrix A, and if R is a region in $\mathbb{R}^3$ with finite volume, then

$$\{\text{volume of } T(R)\} = |\det A| \cdot \{\text{volume of } R\}.$$

Example The following example will be worked on in class:
Let a and b be positive. We find the area of the region R bounded inside the ellipse whose equation is given by

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$