Assignment 1

The answers to the problems of this assignment do not require any calculations but can be inferred from the information given (provided you have the relevant knowledge!)

a) Represent each linear system of equations in matrix form

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 1 \\ 4x_1 + 7x_2 + x_3 = 3 \\ 7x_1 + 10x_2 - 4x_3 = 4 \end{cases}$$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 4 & 7 & 1 & 3 \\ 7 & 10 & -4 & 4 \end{bmatrix}$$

$$\begin{cases} 3x_1 + 3x_2 + x_3 = -4.5 \\ x_1 + x_2 + x_3 = 0.5 \\ -2x_1 - 2x_2 = 5 \end{cases}$$

$$\begin{bmatrix} 3 & 3 & 1 & -4.5 \\ 1 & 1 & 1 & 0.5 \\ -2 & -2 & 0 & 5 \end{bmatrix}$$

$$\begin{cases} 2x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 + x_3 = 13 \\ 4x_1 - 4x_3 = 4 \end{cases}$$

$$\begin{bmatrix} 0 & 2 & -3 & 1 \\ 3 & 6 & 1 & 13 \\ 4 & 0 & -4 & 5 \end{bmatrix}$$

In b) - e) refer to the following coefficient matrix

$$\begin{bmatrix} 1 & 3 & 6 & -1 & 0 \\ 0 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) Determine which columns have pivot positions
Pivots are found in columns 1, 2, and 4.

c) What is the rank of the matrix?
The rank is the dimensions of the column space which is 3.

d) Are there any free variables, and if so which?
All zero rows indicate free variables, so x_3 and x_5 are free

e) Explain why the matrix is *not* invertible.
The matrix cannot be invertible since it has a column of zeros.

f) Let $det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 5$. Find

$det\left(\begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix}\right) = 5$ since row interchange does not affect the determinant

Assignment 2

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$

a) What value(s) of *k*, if any, will make $AB = BA$?

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix} = \begin{bmatrix} -7 & 18+3k \\ -4 & -9+3k \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 12 \\ -6-k & -9+k \end{bmatrix}$$
$$AB - BA = \begin{bmatrix} -7 & 18+3k \\ -4 & -9+k \end{bmatrix} - \begin{bmatrix} -7 & 12 \\ -6-k & -9+k \end{bmatrix}, so$$
$$\begin{bmatrix} 0 & 6+3k & 0 \\ 2+k & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2+k & 0 & 0 \\ 0 & 2+k & 0 \end{bmatrix} \sim \begin{bmatrix} k & 0 & -2 \\ 0 & k & -2 \end{bmatrix}$$

So, *AB* = *BA* when k = -2.

b) For what value(s) of *k* are the columns of the following matrix linearly dependent?

$$\begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ 0 & -3 & k+3 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & -3 & k+3 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & k-3 \end{bmatrix}$$

Thus, the columns of the matrix are linearly dependent when k = 3, since we then get a free variable.

c) Find the value(s) of *k* for which the matrix $\begin{bmatrix} k^2 & 2k \\ 8 & k \end{bmatrix}$ is singular (i.e. not invertible).

The matrix is singular when det = 0.

$\begin{vmatrix} k^2 & 2k \\ 8 & k \end{vmatrix} = k^3 - 16k = k(k^2 - 16)$ so the matrix is singular when $k = \begin{cases} \pm 4 \\ 0 \end{cases}$

Assignment 3

Find the value(s) of α for which the determinant of the following matrix is zero

$$\begin{bmatrix} a & \sqrt{2} & 0 \\ \sqrt{2} & a & \sqrt{2} \\ 0 & \sqrt{2} & a \end{bmatrix}$$

$$\begin{vmatrix} a & \sqrt{2} & 0 \\ \sqrt{2} & a & \sqrt{2} \\ 0 & \sqrt{2} & a \end{vmatrix} = a\begin{vmatrix} a & \sqrt{2} \\ \sqrt{2} & a \end{vmatrix} - \sqrt{2}\begin{vmatrix} \sqrt{2} & 0 \\ \sqrt{2} & a \end{vmatrix} = a(a^2-2) - 2a = a^3 - 4a = a(a^2-4)$$

So the determinant is zero when $a = \begin{cases} \pm 2 \\ 0 \end{cases}$

Assignment 4

Let

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, and\ v = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix},$$

<u>Solution with error, but finding a new u_3 (i.e. u_3 is not accepted as orthogonal):</u>

a) Show that $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$

You can solve this question by finding the inner product between all four venctors. Here you will see that u_4 is not orthogonal but the others are. It is also possible to check whether $A^TA = I$ with some scale. The resulting matrix will indicate which vector or vectors are not orthogonal. E.g.

```
>> u1 = [1 2 1 1]';u2=[-2 1 -1 1]';u3=[1 1 -2 -1]';u4=[-1 1 1 -1]';y=[4 2 -1 0]';
>> A =[u1 u2 u3 u4];
>> A'*A
ans =
     7     0     0     1
     0     7     0     1
     0     0     7    -1
     1     1    -1     4
```

Here you see that u4 is not orthogonal to the others.

b) Write *v* as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3, u_4\}$.

$v = z_1 + z_2 = \big(proj(v, u_1) + proj(v, u_2)\big) + \big(proj(v, u_3) + proj(v, u_4)\big)$

Technically we would need to do the above computation but it is possible to simply solve for z_2 instead:

$$z_2 = v - z_1 = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \left(proj(v, u_1) + proj(v, u_2)\right)$$

$$= \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{7}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{7} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 17/7 \\ 9/7 \\ 12/7 \\ 2/7 \end{bmatrix} = \begin{bmatrix} 11/7 \\ 5/7 \\ -19/7 \\ -2/7 \end{bmatrix}, so$$

$$y = \begin{bmatrix} 17/7 \\ 9/7 \\ 12/7 \\ 2/7 \end{bmatrix} + \begin{bmatrix} 11/7 \\ 5/7 \\ -19/7 \\ -2/7 \end{bmatrix}$$

c) Determine the (shortest) distance between *v* and the subspace spanned by $\{u_1, u_2, u_3\}$

Let W denote the subspace spanned by $\{u_1, u_2, u_3\}$

$$\|v - proj(v, W)\| = \left\| \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{7}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{7} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{8}{7} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} \right) \right\| = \left\| \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 25/7 \\ 17/7 \\ -4/7 \\ -6/7 \end{bmatrix} \right\| = 1.1339$$

Assignment 5

The following data were measured in an experiment that was aimed at studying the effects of reducing current draw in a magnetic core by electronic means. The researchers measured the current in a magnetic winding with and without the electronics in a paired experiment and the data for the case without electronics is provided in the following table

Supply Voltage	Current Without Electronics (mA)
0.66	7.32
1.32	12.22
1.98	16.34
2.64	23.66
3.3	28.06
3.96	33.39
4.62	34.12
3.28	39.21
5.94	44.21
6.6	47.48

a) Find the best fitted least-squares line to describe the data above

We find the design matrix: $X = \begin{bmatrix} 1 & 0.66 \\ 1 & 1.32 \\ 1 & 1.98 \\ 1 & 2.64 \\ 1 & 3.3 \\ 1 & 3.96 \\ 1 & 4.62 \\ 1 & 3.28 \\ 1 & 5.94 \\ 1 & 6.6 \end{bmatrix}$ and the observation matrix: $y = \begin{bmatrix} 7.32 \\ 12.22 \\ 16.34 \\ 23.66 \\ 28.06 \\ 33.39 \\ 34.12 \\ 39.21 \\ 44.21 \\ 47.48 \end{bmatrix}$

$X^T X = \begin{bmatrix} 10 & 34.300 \\ 34.300 & 150.586 \end{bmatrix}$ and $X^T y = \begin{bmatrix} 286 \\ 1{,}202.8 \end{bmatrix}$

$X^T x = X^T y \rightarrow \begin{bmatrix} 10 & 34.300 & 286 \\ 34.300 & 150.586 & 1{,}202.8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5.5027 \\ 0 & 1 & 6.7342 \end{bmatrix}$

So $\underline{y = 5.5027 + 6.7342x}$

b) Determine the least-squares error of the least squares line found in (a).

Input	Observed (y)	Predicted ($\hat{y}$)	$y - \hat{y}$
0.66	7.32	9.95	-2.63
1.32	12.22	14.39	-2.17
1.98	16.34	18.84	-2.50
2.64	23.66	23.28	0.37

3.3	28.06	27.73	0.33
3.96	33.39	32.17	1.22
4.62	34.12	36.61	-2.49
3.28	39.21	27.59	11.62
5.94	44.21	45.50	-1.29
6.6	47.48	49.95	-2.47

$\|y - \hat{y}\| = 12.98$

Assignment 6

Compute a full singular value decomposition of the following matrix A (also see the SVD file I uploaded):

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\det(A^T A - \lambda) = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda = \begin{cases} 1 \\ 3 \end{cases}$

$\underline{\lambda = 3:}$

$A^T A - 3 * I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$\underline{\lambda = 1:}$

$A^T A - 1 * I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so $w_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$ and $u_2 = \frac{1}{\sqrt{\sigma_2}} A v_2 = \frac{1}{\sqrt{1}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

We need one more vector that is orthogonal to u1 and u2

$x_1 + x_2 + 2x_3 = 0$

$x_1 - x_2 = 0$

We get $u_3 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$