Assignment 1

The answers to the problems of this assignment do not require any calculations but can be inferred from the information given (provided you have the relevant knowledge!)

Let $A = \begin{bmatrix} 2 & 3 & 5 & -7 & -2 \\ 4 & -5 & 3 & 1 & -3 \\ 3 & 7 & -2 & 4 & 5 \\ 2 & 2 & -7 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 9 & 5 & 1 \\ 0 & -3 & 1 & 4 \\ 0 & 0 & 0 & -7 \end{bmatrix}$

a) Explain why the columns of A are not linearly independent

The columns of A are not linearly independent because the number of columns is more than the number of entries in each column, i.e. m < n (A has 5 columns and each column has 4 entries (4 rows))

b) Explain why $A\boldsymbol{x} = \mathbf{0}$ has a non-trivial solution

$A\boldsymbol{x} = \mathbf{0}$ has a non-trivial solution because A has more columns than the number of rows

c) Explain why $B\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b}$ in $\mathbb{R}^3$

$B\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b}$ in $\mathbb{R}^3$ because B has a pivot position in every row.

d) Explain whether or not AB^T is well-defined

AB^T is not well-defined since the number of columns in A do not match the number of rows in B^T

Assignment 2

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -4 & -1 \\ -1 & -3 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

a) Find the inverse of A using elementary row operations on the augmented matrix $[A \quad I]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & -4 & -1 & 0 & 1 & 0 \\ -1 & -3 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & -4 & -1 & 0 & 1 & 0 \\ 0 & -2 & 3 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & -3 & 1 & 0 \\ 0 & -2 & 3 & 1 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & -3 & 1 & 0 \\ 0 & 0 & 29/7 & 13/7 & -2/7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & -3 & 1 & 0 \\ 0 & 0 & 1 & 13/29 & -2/29 & 7/29 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & 0 & -35/29 & 21/29 & 28/29 \\ 0 & 0 & 1 & 13/29 & -2/29 & 7/29 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 16/29 & 2/29 & -7/29 \\ 0 & 1 & 0 & 5/29 & -3/29 & -4/29 \\ 0 & 0 & 1 & 13/29 & -2/29 & 7/29 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 11/29 & 5/29 & -3/29 \\ 0 & 1 & 0 & 5/29 & -3/29 & -4/29 \\ 0 & 0 & 1 & 13/29 & -2/29 & 7/29 \end{bmatrix} \text{ so } A^{-1} = \begin{bmatrix} 11/29 & 5/29 & -3/29 \\ 5/29 & -3/29 & -4/29 \\ 13/29 & -2/29 & 7/29 \end{bmatrix}$$

b) Use the inverse of A to solve $Ax = b$

$$x = A^{-1}b = \begin{bmatrix} 11/29 & 5/29 & -3/29 \\ 5/29 & -3/29 & -4/29 \\ 13/29 & -2/29 & 7/29 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 11/29 & 6(\frac{5}{29}) & 4(-\frac{3}{29}) \\ 5/29 & 6(-\frac{3}{29}) & 4(-\frac{4}{29}) \\ 13/29 & 4(-\frac{2}{29}) & 4(\frac{7}{29}) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Assignment 3

Find the value(s) of a for which the determinant of the following matrix is -18 by co-expanding on the second row

$$\begin{bmatrix} 1 & 5 & -a \\ a & -a & a \\ 2 & 13 & -7 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 5 & -a \\ a & -a & a \\ 2 & 13 & -7 \end{vmatrix} = -18 \Leftrightarrow -a\begin{vmatrix} 5 & -a \\ 13 & -7 \end{vmatrix} - a\begin{vmatrix} 1 & -a \\ 2 & -7 \end{vmatrix} - a\begin{vmatrix} 1 & 5 \\ 2 & 13 \end{vmatrix} = -18 \Leftrightarrow$$

$$-a(-35 + 13a) - a(-7 + 2a) - 3a = -18 \Leftrightarrow -35a + 13a^2 + 7a - 2a^2 - 3a + 18 = 0 \Leftrightarrow$$

$$35a - 13a^2 + 7a - 2a^2 - 3a + 18 = 0 \Leftrightarrow -15a^2 + 39a + 18 = 0 \Leftrightarrow a = \begin{cases} 3 \\ -2/5 \end{cases}$$

Assignment 4

Let

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, and\ v = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix},$$

a) Show that $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$ using the inner product.

The inner product of all vectors must be equal to 0

$$u_1 \cdot u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 0 \wedge u_1 \cdot u_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} = 0 \wedge u_1 \cdot u_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} = 0 \wedge$$

$$u_2 \cdot u_3 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} = 0 \wedge u_2 \cdot u_4 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} = 0 \wedge u_3 \cdot u_4 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} = 0$$

b) Write *v* as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3, u_4\}$.

$$v = \left(proj(v, u_1) + proj(v, u_2)\right) + \left(proj(v, u_3) + proj(v, u_4)\right)$$

$$= \left(\frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) + \left(\frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right)$$

$$= \left(\frac{7}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{7} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) + \left(\frac{8}{7} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} - \frac{3}{7} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 17/7 \\ 9/7 \\ 12/7 \\ 2/7 \end{bmatrix} + \begin{bmatrix} 11/7 \\ 5/7 \\ -19/7 \\ -2/7 \end{bmatrix}$$

c) Determine the (shortest) distance between *v* and the subspace spanned by $\{u_1, u_2, u_3\}$

Let W denote the subspace spanned by $\{u_1, u_2, u_3\}$

$$\|v - proj(v, W)\| = \left\| \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{7}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{7} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{8}{7} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} \right) \right\| = \left\| \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 25/7 \\ 17/7 \\ -4/7 \\ -6/7 \end{bmatrix} \right\| = 1.1339$$

Assignment 5

Measurements of the deflection (mm) of particleboard from stress levels of relative humidity are displayed below.

Stress level (%)	Deflection (mm)
54	16.473
54	18.693
61	14.305
61	15.121
68	13.505

68	11.640
75	11.168
75	12.534
75	11.224

a) Find the best fitted least-squares line to describe the data above

We find the design matrix: $X = \begin{bmatrix} 1 & 54 \\ 1 & 54 \\ 1 & 61 \\ 1 & 61 \\ 1 & 68 \\ 1 & 68 \\ 1 & 75 \\ 1 & 75 \\ 1 & 75 \end{bmatrix}$ and the observation matrix: $y = \begin{bmatrix} 16.473 \\ 18.693 \\ 14.305 \\ 15.121 \\ 13.505 \\ 11.640 \\ 11.168 \\ 12.534 \\ 11.224 \end{bmatrix}$

$X^T X = \begin{bmatrix} 9 & 591 \\ 591 & 39{,}397 \end{bmatrix}$ and $X^T y = \begin{bmatrix} 124.7 \\ 8{,}023.3 \end{bmatrix}$

$X^T x = X^T y \rightarrow \begin{bmatrix} 9 & 591 & 286 \\ 591 & 39397 & 1{,}202.8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 32.0487 \\ 0 & 1 & -0.2771 \end{bmatrix}$

$y = -0.2771x + 32.0487$

b) Determine the least-squares error of the least squares line found in (a).

Input	Observed (y)	Predicted ($\hat{y}$)	$y - \hat{y}$
54	16.473	17.0853	-0.6123
54	18.693	17.0853	1.6077
61	14.305	15.1456	-0.8406
61	15.121	15.1456	-0.0246
68	13.505	13.2059	0.2991
68	11.640	13.2059	-1.5659
75	11.168	11.2662	-0.0982
75	12.534	11.2662	1.2678
75	11.224	11.2662	-0.0422

$\|y - \hat{y}\| = 2.7977$

Assignment 6

There seems to be an error in this exercise. I don't know what went wrong. But it is not possible to do this assignment as it is. Just skip this assignment. Here I have written how you should do it methodologically.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -4 & -1 \\ -1 & -3 & 2 \end{bmatrix}$, $v_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

a) Show that v_1 and v_2 are eigenvectors of A with associated eigenvalues λ_1 and λ_2, respectively

   To answer this, you need to compute Av_1 and see whether you can obtain $\lambda_1 v_1$ and then Av_2, and see whether you can obtain $\lambda_2 v_2$.

b) Determine the eigenspaces of λ_1 and λ_2
   This would be calculated as

$$Basis = nulbasis(A - \lambda_1 I_3)$$
$$Basis = nulbasis(A - \lambda_2 I_3)$$

c) Orthogonally diagonalize A where $A = PDP^{-1}$ and the columns of P are normalized
   This is impossible with these numbers.