Symmetric:

$A = A^T$: $\begin{bmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{bmatrix}$

Diagonalize sym. matrix:

1) Find all λ
2) Find eig. vect

$\Big\}$ use eigenvects

3) Make sure you have orthogonal basis
   ↳ if not, use Gram schmidt.
4) Normalize all vectors
5) Setup P, D and P^{-1}.

Singular Value Decomposition

$$A_{m \times n} = U_{m \times m} \, S_{m \times n} \, V^T_{n \times n}$$

V: orthonormal eigenvectors of $A^T A$

U: orthonormal eigenvectors of $A A^T$

S: Has singular values on its diagonal

$\sigma_i = \sqrt{\lambda_i}$ and are placed in decreasing magnitude

$A = U S V^{-1}$

$\therefore$

$AV = USV^{-1} \cdot V$

$\therefore$

$AV = US$

$A\underline{V} = \underline{U}\,\underline{S}$

$A\bar{v}_1 = \sigma_1 \cdot \bar{u}_1$

$A\bar{v}_2 = \sigma_2 \cdot \bar{u}_2$

$\vdots$

$A\bar{v}_n = \sigma_n \cdot \bar{u}_n$

Columns of U are obtained from

$$\boxed{\frac{1}{\sigma_i} A\bar{v}_i = \bar{u}_i}$$

Method:

1) Find λ of $A^T A$
2) Find orthonormalised eigenvectors of $A^T A$
   ↳ these make up columns of V
3) Find columns of U by

$$\bar{u}_i = \frac{1}{\sigma_i} \cdot A \cdot \bar{v}_i$$

If $m > n$, you need to find more vectors
↳ see example.
(also use Gram Schmidt)

4) Find V^T and S and setup $A = USV^T$

Ali 2014.6

$m > n$ (more rows than columns)

$u_1 \cdot u_3 = 0$

$u_2 \cdot u_3 = 0$, $u_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} \frac{\sqrt{6}}{6} x_1 + \frac{\sqrt{6}}{6} x_2 + \frac{\sqrt{6}}{6} x_3 &= 0 \\ \sqrt{2}/2\, x_1 - \frac{\sqrt{2}}{2} x_2 + 0 \cdot x_3 &= 0 \end{aligned}\right\}$$

what is this?

$$\begin{bmatrix} \sqrt{6}/6 & \sqrt{6}/6 & \sqrt{6}/6 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \end{bmatrix} = \begin{bmatrix} \bar{u}_1^T \\ \bar{u}_2^T \end{bmatrix}$$

2016.1:

a) A is a 3×4 matrix.

b) Since A is symmetric, sum of diagonal is same as sum of eigenvalues. So $\lambda_3 = 12 - 4 - 2 = \underline{\underline{6}}$

c) not invertible $\Rightarrow \det(A) = 0 \Rightarrow$ since we have 0 on diagonal $\Rightarrow \det =$ product of diagonal when A is triangular $\Rightarrow$ A is triangular.

d) If $2k - 6 \neq 0$ columns are independent so if $2k - 6 = 0$ columns are dependent:

$$2k - 6 = 0$$

$$2k = 6 \Rightarrow \underline{\underline{k = 3}}$$

e) It does Not have a solution for each $\vec{b}$ since we dont have pivots in all rows of A.

2016.2

Method: we rref/null space to find vectors and setup in p.v. form:

$$\bar{x} = x_3\begin{bmatrix} -1/2 \\ -3/4 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ -5/2 \\ 0 \\ 1 \end{bmatrix}$$

2016.3

a)

$$\det(A - \lambda \cdot I) = 0$$

$$\Updownarrow$$

$$(2-\lambda)\cdot(a-\lambda) - 2 = 0$$

$$\Updownarrow$$

$$\underline{\underline{2a - 2\lambda - a\cdot\lambda + \lambda^2 - 2 = f(a)}}$$

b)

$$2\cdot a - 2\cdot 4 - a\cdot 4 + 4^2 - 2 = 0$$

$$\Updownarrow$$

$$-2a + 6 = 0$$

$$\Updownarrow$$

$$\underline{\underline{a = 3}}$$

$\underline{\lambda = 4:}$

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$\underline{\lambda = 1:}$

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

2016.4

a)

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & -2 \end{bmatrix} \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since we get pivots in all columns, they are lin. independent.

b) Since $U^T U = \begin{bmatrix} v_1 & & 0 \\ & v_2 & \\ 0 & & v_3 \end{bmatrix}$, the vectors form an orthogonal basis

c) $$X = \left(\text{proj}_x v_1 + \text{proj}_x v_2 \right) + \text{proj}_x v_3$$

$$= \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$$

d) Trick question. Since $\bar{v}_1$, $\bar{v}_2$ an $\bar{v}_3$ form a basis for $\mathbb{R}^3$, all vectors will lie in this subspace. So $\underline{\underline{\text{dist} = 0}}$

2016.5

a)

$$X_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 4 \\ 1 & 9 \\ 1 & 16 \\ 1 & 25 \end{bmatrix}, \quad y_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \\ 11 \\ 16 \\ 24 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 1 \\ 3 \\ 7 \\ 11 \\ 16 \\ 24 \end{bmatrix}$$

b)

$$y_1 = 2.27 + 0.88t^2$$

$$y_2 = 1.07 + 1.48t + 0.61t^2$$

c) $\left.\begin{aligned} e_1 &= 1.99 \\ e_2 &= 0.97 \end{aligned}\right\}$ since $e_1 > e_2$, we conclude that $y_2(t)$ is the best model.

2016.6:

a) See output in Notebook

b) U and V are swithed:

$$A^T = V^t S U$$