Linear algebra guide

Knowledge questions

The first set of questions are usually non-calculation questions that seek to clarify your knowledge about different topics in linear algebra:

Echelon Form: A matrix is in echelon form if
1) all nonzero rows are above any rows of all zeroes
2) the leading entry of a nonzero row is always to the right of the leading coefficient of the row above it

A pivot position: In a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position. If a matrix has a pivot in every row, it will be consistent.

Free Variables: **Free variables** are indicated by all zero rows – although some of the zero rows might be "hidden". Free variables are also indicated by **non-pivot columns**.

Solution: Sometimes it is asked whether an equation has a solution. This is the same as asking whether the system is **consistent** (has solution) or **inconsistent** (has no solution).

Trivial solution: If a system is consistent, i.e. has a solution, two options occur: The **trivial** or the **nontrivial** solution. The former is the zero vector. $A\boldsymbol{x} = 0$ is certain to have a non-trivial solution if m < n

Unique solution: If a system only has one solution this solution is called a **unique solution**. This would rule out free variables.

Eigenvalue/vector: If you are given a factorization in the form PDP^{-1}, then the diagonal matrix contains the **eigenvalues** and the columns of matrix P are made up of the corresponding **eigenvectors**. Also, in a triangular matrix, the eigenvalues are the entries on the main diagonal.
In a symmetric matrix, the sum of the diagonals is equal to the sum of the eigenvalues.

Invertible matrix: You will most likely be asked to determine whether or not a matrix is invertible. This will be a direct reference to the **Invertible Matrix Theorem** and whether one (and hence all) of these criteria are met (invertible) or not met (not invertible).

Linear Dependence: A set of vectors is said to be **linearly dependent** if one of the vectors in the set can be defined as a **linear combination** of the other vectors. The columns of a matrix are linearly independent if the associated homogeneous equation has only the trivial solution, and vice versa. If a matrix has free variable/non-pivot columns, then the columns are dependent. So the easiest method is to rref and check for free variables. Also, if m < n the columns are not linearly independent = linearly dependent, i.e. if the number of rows is inferior to the number of columns, then the columns are linearly dependent.

Orthogonal basis: If you need to show that a set of vectors form an orthogonal basis, you need to find the inner product (=dot product) of each pair of vectors.

Sum of two vectors: Sometimes you will be asked to express a vector ***v*** as the sum of two vectors. All vectors can be expressed as the sum of two orthogonal vectors. You can find these two vectors by projecting ***v*** on to both of these vectors. See problem below.

How to check if…

If a vector is a solution to Ax = b. Insert in x's place and multiply with A. Then set up in augmented matrix with b and check whether it is consistent

A scalar is an **eigenvalue** of a matrix if $A - \lambda I = 0$ has a nontrivial solution

A vector x is an **eigenvector** of A, if the following holds true Ax = λx, i.e. multiply A and x and see if it is possible to extract a scalar (which will then be an eigenvalue)

How to find

Distance: The shortest distance between a vector and a subspace, is found by projecting the vector on to the subspace, and then finding the distance between the vector and the projection: $\|v - proj(v, W)\|$, where v is the vector and W is the subspace.

Best approximation: You might be given a system that does not have a solution and asked to determine which of two vectors is the best approximation. Here you need to calculate the distances from the vectors formed from Ax_1 to b and Ax_2 to b and see which is smallest, the one produced by x_1 or x_2.

Eigenvalue: To manually find an eigenvalue, you need to solve the equation you get from $\det(A - \lambda I) = 0$. This is called the **characteristic equation**.

Eigenvectors: First you need to find the eigenvalues. Then you subtract the eigenvalues from the diagonal by using $A - \lambda I$. Then you find reduced echelon form of the resulting matrix. This is then placed in parametric form and the resulting vector or vectors will be the eigenvectors. You will get the number of vectors corresponding to the multiplicity of the eigenvalues.

Determinant: Manually, it will be best to use cofactor expansion along the column or row with the most 0 entries. If A is a triangular matrix, the determinant is the product of the entries on the diagonal.

Inverse: The easiest way to find the inverse manually is by setting up a matrix $[A \quad I]$ and row reduce until you reach reduced echelon form. The resulting matrix will now give you the inverse: $[I \quad A^{-1}]$. Also see the invertible matrix theorem

Solution: To solve a matrix equation of the form A**x** = **b**, construct the augmented matrix [A b] and then row reduce. Then place in parametric form, stating potential free variables. Not that A**x** = **b** only has a solution if b is a linear combination of the columns in A.

Spaces, basis, dimension and rank

Vector space: Basically just a bunch of vectors

Subspace: Is a smaller space where we are able to add and multiply without leaving the space

Null space: The set of all solutions of the homogenous equation A**x** = **0.** The null space contains x's and will contain the set of all vectors that satisfy the equation. This can also be called the solution set.

Columns space: Is the set of all linear combinations of the columns of A.

Row Space: is the set of all linear combinations of the row vectors of A

Eigenspace: The null space of $A - \lambda I$

Basis: A linearly independent set of vectors that spans a subspace. The basis will contain the minimum amount of vectors needed to fully span the subspace. When writing the solution set in parametric form, the vectors are linearly independent, this providing a basis for the null space. The pivot columns of a matrix form a basis for the column space. A basis for the row space are all the nonzero rows in echelon form.

Dimension: The number of vectors in a basis. dim Nul A = # free variables. Dim Col A = " pivots in A.

Rank: Is the dimension of the column space . The following holds true n = rank A + dim Nul A

Problems

Gram Schmidt
Problem: You have a vector or a set of vectors, but what you need is actually an orthogonal set of vectors.
Solution: Use Gram Schmidt to obtain such a set

Is used especially in problems like SVD and diagonalization.

The process is outlined in section 6.4 and the guide is presented as Theorem 11.

Orthogonal Projections
This is an example of "**Sum of two vectors**" issues mentioned above.

Let

$$u_1 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, u_2 = \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}, u_3 = \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}, u_4 = \begin{bmatrix}-1\\1\\1\\-2\end{bmatrix}, and\ v = \begin{bmatrix}4\\2\\-1\\0\end{bmatrix},$$

a) Show that $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$ using the inner product.

The inner product of all vectors must be equal to 0

$$u_1 \cdot u_2 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix} = 0 \ \wedge u_1 \cdot u_3 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix} = 0 \wedge u_1 \cdot u_4 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\\-2\end{bmatrix} = 0 \wedge$$
$$u_2 \cdot u_3 = \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix} = 0 \wedge u_2 \cdot u_4 = \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\\-2\end{bmatrix} = 0 \wedge u_3 \cdot u_4 = \begin{bmatrix}1\\1\\-2\\-1\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\\-2\end{bmatrix} = 0$$

b) Write *v* as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3, u_4\}$.

$$v = \left(proj(v, u_1) + proj(v, u_2)\right) + \left(proj(v, u_3) + proj(v, u_4)\right)$$

$$= \left(\frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) + \left(\frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right)$$

$$= \left(\frac{7}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{7} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) + \left(\frac{8}{7} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} - \frac{3}{7} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 17/7 \\ 9/7 \\ 12/7 \\ 2/7 \end{bmatrix} + \begin{bmatrix} 11/7 \\ 5/7 \\ -19/7 \\ -2/7 \end{bmatrix}$$

This is an example of “Distance” mentioned above

c) Determine the (shortest) distance between *v* and the subspace spanned by $\{u_1, u_2, u_3\}$
   Let W denote the subspace spanned by $\{u_1, u_2, u_3\}$

$$\|v - proj(v, W)\| = \left\| \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{7}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{7} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{8}{7} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} \right) \right\| = \left\| \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 25/7 \\ 17/7 \\ -4/7 \\ -6/7 \end{bmatrix} \right\| = 1.1339$$

<u>The best fitted line</u>

Problem: You have a set of observations that you would like to express in a (linear) model
Solution: Find the best fitted line

Method:

Step 1:
You have two sets of data: the values of the independent variables and the values of the dependent variables. Set up the independent values in a vector, which you can call y. This is called the observation matrix.

The independent values you set up in a matrix X whose first column are all ones and second column are the independent variables. This is called the design matrix.

You should now have a design matrix X and an observation vector y.

Step 2:
Find,
$X^T X$ and $X^T y$

Step 3:
Setup the equation $X^T x = X^T y$ which you then setup as an augmented matrix $[X^T x \quad X^T y]$

Step 4:
Row reduce the augmented matrix of Step 3 to reduced form. The value in the first row will give you the intercept and the value in the second row will give you the slope.

Step 5: Setup the parameters that you found in step 4 in an equation

Example:

Measurements of the deflection (mm) of particleboard from stress levels of relative humidity are displayed below.

Stress level (%)	Deflection (mm)
54	16.473
54	18.693
61	14.305
61	15.121
68	13.505
68	11.640
75	11.168
75	12.534
75	11.224

Find the best fitted least-squares line to describe the data above

Step 1:

We find the design matrix: $X = \begin{bmatrix} 1 & 54 \\ 1 & 54 \\ 1 & 61 \\ 1 & 61 \\ 1 & 68 \\ 1 & 68 \\ 1 & 75 \\ 1 & 75 \\ 1 & 75 \end{bmatrix}$ and the observation matrix: $y = \begin{bmatrix} 16.473 \\ 18.693 \\ 14.305 \\ 15.121 \\ 13.505 \\ 11.640 \\ 11.168 \\ 12.534 \\ 11.224 \end{bmatrix}$

Step 2:
$X^T X = \begin{bmatrix} 9 & 591 \\ 591 & 39{,}397 \end{bmatrix}$ and $X^T y = \begin{bmatrix} 124.7 \\ 8{,}023.3 \end{bmatrix}$

Step 3:
$X^T x = X^T y \rightarrow \begin{bmatrix} 9 & 591 & 286 \\ 591 & 39397 & 1{,}202.8 \end{bmatrix}$

Step 4:
$\begin{bmatrix} 9 & 591 & 286 \\ 591 & 39397 & 1{,}202.8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 32.0487 \\ 0 & 1 & -0.2771 \end{bmatrix}$

Step 5:
$y = -0.2771x + 32.0487$

Now sometimes you will be asked to determine the least-squares error of the least squares line found. Here you need to compute all the predicted values of your x-values. Then you determine the difference between all the observed values and the predicted values and finally you find the "length" of the vector of differences

Example

Input	Observed (y)	Predicted ($\hat{y}$)	$y - \hat{y}$
54	16.473	17.0853	-0.6123
54	18.693	17.0853	1.6077
61	14.305	15.1456	-0.8406
61	15.121	15.1456	-0.0246
68	13.505	13.2059	0.2991
68	11.640	13.2059	-1.5659
75	11.168	11.2662	-0.0982
75	12.534	11.2662	1.2678
75	11.224	11.2662	-0.0422

$\underline{\|y - \hat{y}\| = 2.7977}$

NOTE: Often you will be asked to also create a quadratic function. The method is the same as above, except the design matrix will now have three columns where the third column consists of the x-values squared. Also note that the parameter vector now contains three entries.

<u>Diagonalizing Matrices</u>

There are different kinds of factorization. The one mentioned here is from section 5.3. and applies only to square matrices, i.e. n x n matrices. Requirements: A has n linearly independent eigenvectors:

A = PDP^{-1} where D as a diagonal matrix with eigenvalues on diagonal, and the columns of P consist of the eigenvectors.

Here is an example that also includes how to check whether vectors are eigenvenctors.

Example

Let $A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

a) Show that v_1 and v_2 are eigenvectors of A with associated eigenvalues λ_1 and λ_2, respectively

```
A =

     2     0    -2
     1     3     2
     0     0     3

>> v1=[-1 1 0]'

v1 =

    -1
     1
     0

>> v2=[-2 0 1]'

v2 =

    -2
     0
     1

>> A*v1

ans =

    -2
     2
     0
```

```
>> A*v2

ans =

    -6
     0
     3
```

Since A*v1 and A*v2 are merely scales of v1 and v2, 2 and 3, respectively, it follows that v1 and v2 are eigenvectors of A with associated eigenvalues of 2 and 3, respectively.

b) Determine the eigenspaces of λ_1 and λ_2
   Since A is a 3 x 3 matrix, the dimension of the eigenspace is between 1 and 3, both included. That means there either is one more eigenvalue or 2 or 3 has a multiplicity of two.

```
>> nulbasis¹(A-2*eye(3))

ans =

   -1
    1
    0

>> nulbasis(A-3*eye(3))

ans =

    0   -2
    1    0
    0    1
```

   As we can see, 3 has multiplicity 2 and yields two eigenvector. Thus, an eigenspace for λ_1 = 2 is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$ and an eigenspace for 3 is $\left\{\begin{matrix}0 & -2\\1 & 0\\0 & 1\end{matrix}\right\}$

c) Orthogonally diagonalize A where $A = PDP^{-1}$ and the columns of P are normalized
   We will need to find the inverse of P whose columns are the eigenvectors of A.

```
P =

    -1      0     -2
     1      1      0
     0      0      1
```

[1] The nulbasis function is an add in to Matlab. If you do not have it, you need to rref A-λI and find the vectors in parametric form

```
>> [P eye(3)]

ans =

    -1     0    -2     1     0     0
     1     1     0     0     1     0
     0     0     1     0     0     1

>> rref(ans)

ans =

     1     0     0    -1     0    -2
     0     1     0     1     1     2
     0     0     1     0     0     1
```

So the diagonalization is

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Diagonalisation of symmetric matrices</u>

An important characteristic of symmetric matrices is that any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Method:

Step 1: Find the eigenvalues of A

Step 2: Find the eigenvectors corresponding to the eigenvalues

Step 3: Make sure that the eigenvectors form an orthogonal basis. If not, use Gram Schmidt to obtain orthogonal vectors

Step 4: Normalize all eigenvectors

Step 5: Set up P, D and P^T from the above

Example

Diagonalize the following matrix, if possible

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Step 1: Find eigenvalues

```
>> A = [0 1 1;1 0 1;1 1 0]

A =

     0     1     1
     1     0     1
     1     1     0

>> eig(A)

ans =

   -1.0000
   -1.0000
    2.0000
```

We see that one of the eigenvalues have multiplicity of 2, which means the two resulting eigenvectors will not be orthogonal

Step 2: Find the eigenvectors

```
>> A+1*eye(3)

ans =

     1     1     1
     1     1     1
     1     1     1

>> rref(ans)

ans =

     1     1     1
     0     0     0
     0     0     0
```

So we get $\bar{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

```
>> A-2*eye(3)

ans =

    -2     1     1
     1    -2     1
     1     1    -2

>> rref(ans)

ans =

     1     0    -1
     0     1    -1
     0     0     0
```

So we get $\bar{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Step 3: Make sure all vectors are orthogonal
As noted before **v**$_1$ and **v**$_2$ are not orthogonal, although both are orthogonal to v3. We therefore need to find a vector orthogonal to either **v**$_1$ or **v**$_2$:

$\bar{u}_1 = \bar{v}_1$
$\bar{u}_2 = \bar{v}_2 - proj_{\bar{u}_1}\bar{v}_2$

```
>> u2=v2-proj(v2,u1)

u2 =

   -0.5000
   -0.5000
    1.0000
```

Step 4: Normalise the vectors

$$u_1 = \begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{bmatrix},\quad u_2 = \begin{bmatrix}-1/\sqrt{6}\\ -1/\sqrt{6}\\ 2/\sqrt{6}\end{bmatrix},\quad u_3 = \begin{bmatrix}1/\sqrt{3}\\ 1/\sqrt{3}\\ 1/\sqrt{3}\end{bmatrix}$$

Step 5: Setup

$$A = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

Singular Value Decomposition

Problem: We know how to factor symmetric matrices using the process outlined previously but we need a method to factorise *any* m x n matrix.

Solution: SVD enables us to decompose a matrix into three matrices U, S, and V^T

Theory and method:

$A_{m\,x\,n} = U_{mm} \cdot S_{mn} \cdot V_{nn}^T$

- The vector V consist of orthonormal eigenvectors of A^TA
- The vector U consists of orthonormal eigenvectors of AA^T
- The entries in the diagonal matrix S, **called singular values**, are the square roots of the eigenvalues, and are placed in decreasing order of magnitude. The corresponding vectors in U and V are placed in corresponding columns.

In class, we found the vectors of V and U using the same method. You must either find V and then derive U or find U and then derive V. It is not possible to find U and V independent of each other since this may give you a slight error. Here I show by finding U from V. This is the reasoning: Assume the singular values and V^T have been found (step 1 and 2 below). Now, remember that $V^T = V^{-1}$ (because A^TA is symmetric) We get:

$$A = U \cdot S \cdot V^T = U \cdot S \cdot V^{-1} \Leftrightarrow A \cdot V = U \cdot S$$

If we take one column at a time we get:

$A\bar{v}_1 = \sigma_1 u_1$

$A\bar{v}_2 = \sigma_2 u_2$

$A\bar{v}_3 = \sigma_3 u_3$

.

.

$A\bar{v}_n = \sigma_n u_n$

This means we can compute each column of U from the corresponding columns of V and S:

$\frac{1}{\sigma_1} A\bar{v}_1 = u_1$ and so on

Depending on the size of A, V might contain fewer columns than U. In order to find the rest of the columns, simply use Gram Schmidt. If there are more rows than columns, this will be the case (so if m > n, you will need Gram Schmidt). The steps are:

Step1: Find the eigenvalues of A^TA using either Matlab's eig function or using the characteristic equation

Step 2: Find the orthonormalised eigenvectors of A^TA. These will make up the columns of V

Step 3: Find the columns of U from $\frac{1}{\sigma_i} A\bar{v}_i = u_i$. If m > n, use Gram Schmidt to find the remaining vectors

Step 4: Find V^T and S (by squaring the eigenvalues) and setup $A = U\ S\ V^T$

Step 5: Enjoy a job well-done!

Example

Compute a full singular value decomposition of the following matrix A:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Step 1:

```
A =

     0     1
     1     0
     1     1

>> AtA = A'*A

AtA =

     2     1
     1     2

>> eig(AtA)

ans =

     1
     3
```

Step 2:

```
>> AtA-3*eye(2)

ans =

    -1     1
     1    -1

>> rref(ans)

ans =

     1    -1
     0     0
```

This means that x_2 is free, and since x_1 is 1, we get $\begin{bmatrix}1\\1\end{bmatrix}$. In order to normalize this vector, we divide the vector by the length of the vector. This is where using Matlab's norm function may return awful approximation in terms of rational numbers. I recommend finding the norm by hand. In this case, it is quite easy:

$\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\| = \sqrt{1^2+1^2} = \sqrt{2}$, thus we get:

$$v_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

```
>> AtA-1*eye(2)

ans =

     1     1
     1     1

>> rref(ans)

ans =

     1     1
     0     0
```

This means that x_2 is free, and since x_1 is -1, we get $\begin{bmatrix}-1\\1\end{bmatrix}$. In order to normalize this vector, we divide the vector by the length of the vector. This is where using Matlab's norm function may return awful approximation in terms of rational numbers. I recommend finding the norm by hand. In this case, it is quite easy:

$\left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, thus we get:

$$v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Step 3:

So now we need to find U. From the dimension of A we can see that U will be a 3x3 matrix. This means we can obtain u_1 and u_2 from v_1 and v_2, but we will need to use Mr. Gram Schmidt to obtain u_3.

$$u_1 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{1}} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

In order to find u_3, we need a vector that satisfies the following system:

$$u_1 \cdot u_3 = 0$$
$$u_2 \cdot u_3 = 0$$

$$\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{1}{\sqrt{6}} x_1 + \frac{1}{\sqrt{6}} x_2 + \frac{2}{\sqrt{6}} x_3 = 0$$

$$\frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_2 = 0$$

We will multiply by the denominator to make it easier to work with:

$x_1 + x_2 + 2x_3 = 0$
$x_1 - x_2 = 0$

```
X =

     1     1     2
     1    -1     0

>> rref(X)

ans =

    1.0000         0    1
         0    1.0000    1
```

So we get

$$x_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

We normalize this

$$u_3 = \frac{x_3}{\|x_3\|} = \begin{bmatrix} -1\backslash\sqrt{3} \\ -1\backslash\sqrt{3} \\ 1\backslash\sqrt{3} \end{bmatrix}$$

We normalize to obtain u_3

$$u_3 = \|w_3\| = \begin{matrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{matrix}$$

Step 4:

$$A = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -1/\sqrt{3} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -1/\sqrt{3} \\ \frac{2}{\sqrt{6}} & 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$