4 Vector Spaces

4.1 Vector Spaces and Subspaces

We define a vector space below. You will recognize all of these properties as things that hold true in $\mathbb{R}^n$, but it turns out that they hold in a variety of other mathematical objects, as well.

Definition A **vector space** V is a nonempty set of objects, called *vectors*, on which the operations of *addition* and *scalar multiplication* are defined. In addition, the following axioms must hold for all vectors $\bar{u}, \bar{v}, \bar{w}$ and scalars c and d:
(i) The sum $\bar{u} + \bar{v}$ is in V;
(ii) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$;
(iii) $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$;
(iv) There is a **zero** vector $\bar{0} \in V$ such that $\bar{u} + \bar{0} = \bar{u}$;
(v) For each $\bar{u} \in V$, there is a vector $-\bar{u} \in V$ such that $\bar{u} + (-\bar{u}) = \bar{0}$;
(vi) The scalar multiple of $\bar{u}$ by c is in V, i.e $c\bar{u} \in V$;
(vii) $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$;
(viii) $(c + d)\bar{u} = c\bar{u} + d\bar{u}$;
(ix) $c(d\bar{u}) = (cd)\bar{u}$;
(x) $1\bar{u} = \bar{u}$.

Example Each space $\mathbb{R}^n$, where $n \geq 1$, is an example of a vector space.

Example For each $n \geq 0$, let $\mathbb{P}_n$ denote the set of polynomials of degree at most n. So each element $p(x) \in \mathbb{P}_n$ has the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the coefficients a_i and the variable x represent real numbers. The *degree* of p is the highest power of p whose coefficient is nonzero. If p is constant and nonzero, i.e. if $p(x) = a_0 \neq 0$, then p has degree 0. If $p(x) = 0$, then p is the zero polynomial (its degree is undefined). If we have another polynomial $q(x) \in \mathbb{P}_n$, so

$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0,$$

then

$$p(x) + q(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0),$$

which we see is also in $\mathbb{P}_n$. We also have that the scalar multiple $cp(x)$ is given by

$$(cp)(x) = ca_n x^n + ca_{n-1} x^{n-1} + \ldots ca_1 x + ca_0,$$

which we see is in $\mathbb{P}_n$. We have thus verified axioms i, iv, and vi. The rest are left to you.

Example Let V be the set of all real-valued functions (i.e. the outputs are real numbers) defined on the real line $\mathbb{R}$. If $f(x)$ and $g(x)$ are in V, then $(f + g)(x)$ is the function $f(x) + g(x)$, and $(cf)(x)$ is given by $cf(x)$. It is left to you to verify all of the axioms.

Definition A **subspace** of a vector space V is a subset H of V that has three properties:
(i) The zero vector is in H;
(ii) H is closed under vector addition, i.e. if the vectors $\bar{u}$ and $\bar{v}$ are in H, then so is $\bar{u} + \bar{v}$;
(iii) H is closed under scalar multiplication, i.e. for each vector $\bar{u}$ in H and each scalar c, we have that $c\bar{u} \in H$.

Example The set consisting of the zero vector in a vector space V is a subspace of V.

Example Let $\mathbb{P}$ be the set of all polynomials with real coefficients. Then $\mathbb{P}$ is a subspace of the space of all real-valued functions defined on the entire real line $\mathbb{R}$. For each $n \geq 0$, we have also that $\mathbb{P}_n$ is a vector subspace of $\mathbb{P}$.

Example Note well that $\mathbb{R}^2$ is **not** a subspace of $\mathbb{R}^3$. This is because each vector in $\mathbb{R}^2$ has two coordinates, while each vector in $\mathbb{R}^3$ has three coordinates. However, there is a *subspace* of $\mathbb{R}^3$ (in fact, there are many subspaces) that behaves exactly like $\mathbb{R}^2$. Specifically, the set of all vectors of the form $\begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$ is a subspace of $\mathbb{R}^3$ that acts like $\mathbb{R}^2$.

Example Note that a plane in $\mathbb{R}^3$ that does not go through the origin cannot be a subspace.

Example We can use a collection of vectors in a vector space V to *create* a subspace via their span. For example, let $\bar{v}_1$ and $\bar{v}_2$ be vectors in a vector space V, and let $H = \text{Span}\{\bar{v}_1, \bar{v}_2\}$. We show that H is a subspace of V:

First note that $\bar{0} \in H$, since $\bar{0} = 0\bar{v}_1 + 0\bar{v}_2$. We also have closure under addition, since if we take any two vectors $\bar{u}$ and $\bar{w}$ in H, we can write

$$\bar{u} = s_1\bar{v}_1 + s_2\bar{v}_2 \quad \text{and} \quad \bar{w} = t_1\bar{v}_1 + t_2\bar{v}_2$$

so that

$$\bar{u} + \bar{v} = (s_1 + t_1)\bar{v}_1 + (s_2 + t_2)\bar{v}_2 \in H.$$

Likewise, for any scalar c, we have

$$c\bar{u} = (cs_1)\bar{v}_1 + (cs_2)\bar{v}_2 \in H.$$

In general, we have the following theorem. The proof is analogous to the argument given in the preceding example.

Theorem If $\bar{v}_1, \ldots, \bar{v}_p$ are in a vector space V, then $\text{Span}\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a subspace of V.

Example Let a and b be arbitrary scalars and let H be the set of all vectors of the form $(a + 2b, b, a - b, a)$. We show that H is a subspace of $\mathbb{R}^4$. First we write the vectors in H as column vectors:

$$\begin{bmatrix} a + 2b \\ b \\ a - b \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

So we see that H consists of the collection of all linear combinations of two vectors, hence is a subspace of $\mathbb{R}^4$.

4.2 Null Spaces, Column Spaces, and Linear Transformations

Definition The **null space** of an $m \times n$ matrix A, denoted Nul$A$, is the set of all solutions of the homogeneous equation $A\bar{x} = \bar{0}$. In set notation, we write

$$\text{Nul}A = \{\bar{x} : \bar{x} \text{ is in } \mathbb{R}^n \text{ and } A\bar{x} = \bar{0}\}.$$

Example We let A be the matrix

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 1 & 0 \end{bmatrix}$$

and determine if the vectors

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \bar{v}_2 = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$$

are in Nul$A$. First note that

$$A\bar{v}_1 = \begin{bmatrix} 2+2+0 \\ 4+1+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

so $\bar{v}_1 \notin$ Nul$A$. However,

$$A\bar{v}_2 = \begin{bmatrix} 2-8+6 \\ 4-4+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so that $\bar{v}_2 \in$ Nul$A$.

Theorem The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\bar{x} = \bar{0}$ of m homogeneous equations in n variables is a subspace of $\mathbb{R}^n$.

Proof The proof of this fact requires verifying the three properties of subspaces, namely: (i) $\bar{0} \in$ Nul$A$; (ii) Nul$A$ is closed under addition; (iii) Nul$A$ is closed under scalar multiplication.

For (i), note that $A\bar{0} = \bar{0}$, so $\bar{0} \in$ Nul$A$.

For (ii), if $\bar{u}$ and $\bar{v}$ are in Nul$A$, then $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v} = \bar{0} + \bar{0} = \bar{0}$, so $\bar{u} + \bar{v} \in$ Nul$A$.

For (iii), if c is a scalar and $\bar{u} \in$ Nul$A$, then $A(c\bar{u}) = cA\bar{u} = c\bar{0} = \bar{0}$, so $c\bar{u} \in$ Nul$A$.

Example Let H be the set of vectors $(a, b, c, d) \in \mathbb{R}^4$ that satisfy $a + b = c$ and $2a - b = c + d$. To show that H is a vector subspace of $\mathbb{R}^4$, simply rearrange the equations to get a homogeneous system:

$$\begin{aligned} a + b - c &= 0 \\ 2a - b - c - d &= 0. \end{aligned}$$

It follows by the theorem that H is a vector subspace of $\mathbb{R}^4$.

Note The null space of a matrix A is defined *implicitly*. We know the null space always exists (since it contains at least the zero vector), but we have no idea from the definition what it looks like. To get an *explicit* description of Nul$A$, we typically write the set of vectors $\bar{x}$ satisfying $A\bar{x} = \bar{0}$ in parametric vector form.

Definition The **column space** of an $m \times n$ matrix A, denoted Col$A$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \bar{a}_1 & \dots & \bar{a}_n \end{bmatrix}$, then

$$\text{Col}A = \text{Span}\{\bar{a}_1, \dots, \bar{a}_n\}.$$

Since the span of a set of vectors is a subspace, we have the following theorem.

Theorem The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Another way to think of the column space is as the set of all vectors $\bar{b}$ in $\mathbb{R}^m$ such that $\bar{b} = A\bar{x}$ for some $\bar{x}$ in $\mathbb{R}^n$. In terms of linear transformations, the column space of the matrix A is the range of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\bar{x}) = A\bar{x}$. From this vantage point we see that the column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\bar{x} = \bar{b}$ is consistent for each $\bar{b}$ in $\mathbb{R}^m$. This happens if and only if $T(\bar{x})$ is onto. What is the difference between the null space and column space of a matrix A? We illustrate with an example.

Example Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$

We answer the following questions:
(a) For what value k is Col$A$ a subspace of $\mathbb{R}^k$?
(b) For what value k is Nul$A$ a subspace of $\mathbb{R}^k$?
(c) Find a nonzero vector in Col$A$. Is this always possible?
(d) Find a nonzero vector in Nul$A$. Is this always possible?
(e) Let

$$\bar{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

(i) Determine if $\bar{u}$ is in Nul$A$. Is it also in Col$A$?
(ii) Determine if $\bar{v}$ is in Col$A$. Is it also in Nul$A$?

For (a) and (b), we consider that A is a 3×4 matrix. Hence Col$A$ is a subspace of $\mathbb{R}^3$, and Nul$A$ is a subspace of $\mathbb{R}^4$.

For (c), we can simply take any nonzero column of A, say the first column

$$\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}.$$

This is *not* possible in the case that A is the zero matrix.

For (d), first note that there must be a nonzero vector in Nul$A$, since there are more columns than rows (i.e. the number of column vectors is greater than the number of entries in each vector, so there must be some linear dependence). In order to actually find a nonzero vector, we row reduce A to

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Considering this as a homogeneous system ($A\bar{x} = \bar{0}$) in the variables x_1, x_2, x_3, x_4, we see that x_3 must be a free variable. Hence

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}.$$

Picking $x_3 = 1$ (or any nonzero number) gives us a nonzero vector in Nul$A$.

For (e)(i), we simply compute $A\bar{u}$ and check if it is zero:

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \bar{0}.$$

Hence $\bar{u}$ is *not* in Nul$A$. It also could not be in Col$A$, since $\bar{u}$ contains 4 entries, while those in Col$A$ contain 3.

For (e)(ii), we want to see if there's an $\bar{x}$ such that $A\bar{x} = \bar{v}$. From above, we see that A has a pivot in each row, so there must be a solution to this matrix equation. It follows that $\bar{v} \in$ Col$A$. However, since $\bar{v}$ has 3 entries whereas vectors in Nul$A$ have 4, we see that $\bar{v}$ is not in Nul$A$.

Note The table on page 204 of your textbook gives a nice list of contrasting properties of Nul$A$ and Col$A$.

Definition A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\bar{x}$ in V a unique vector $T(\bar{x})$ in W such that
(i) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v} \in V$, and
(ii) $T(c\bar{u}) = cT(\bar{u})$ for all $\bar{u} \in V$.

Notice that this definition is exactly the same as what we saw before, except that we replace $\mathbb{R}^n \to \mathbb{R}^m$ with $V \to W$.

Definition The **kernel** of T is the set of all $\bar{u} \in V$ such that $T(\bar{u}) = 0$. The **range** of T is the set of vectors $\bar{y} \in W$ such that $T(\bar{x}) = \bar{y}$ for some $\bar{x} \in V$. Notice that if T can be written as a matrix transformation $T(\bar{x}) = A\bar{x}$ (which is always the case if $T : \mathbb{R}^n \to \mathbb{R}^m$, but not necessarily otherwise), then

$$\begin{aligned} \text{kernel of } T &= \text{null space of } A, \text{ and} \\ \text{range of } T &= \text{column space of } A. \end{aligned}$$

Note The kernel of T is a subspace of V, whereas the range of T is a subspace of W.

Example Let V be the vector space of real-valued functions f defined on an interval $[a, b]$ with derivatives f' that are also continuous on $[a, b]$. Let W be the vector space of all real-valued continuous functions on $[a, b]$, and let $D : V \to W$ be the transformation that takes a function f to its derivative f', i.e. $D(f) = f'$. By properties of derivatives, we have that

$$\begin{aligned} D(f+g) &= f' + g' = D(f) + D(g), \text{ and} \\ D(cf) &= cf' = cD(f), \end{aligned}$$

so D is a linear map of vector spaces. The kernel of D consists of the constant functions (functions whose derivative is zero), and the range of D is all of W. To see why the range of D is all of W, take an arbitrary function $h \in W$. Then h is a continuous function on $[a, b]$, so we can define $H(x) = \int_a^x h(t)dt$, which is real-valued and defined on $[a, b]$. By the Fundamental Theorem of Calculus, $D(H) = H'(x) = h(x)$, so D is onto.