Assignment 1 (20%)

You will need very few and simple calculation in order to solve the problems of this assignment.

Let

$$A = \begin{bmatrix} 3 & -1 & 5 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 3 & 5 & -7 & -2 \\ 0 & -5 & 3 & 1 & -3 \\ 0 & 0 & -2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

a) Explain why it would be possible to determine all eigenvalues of A without any calculations and then state what these eigenvalues are.

Since the matrix is a triangular matrix, the eigenvalues are stated on the diagonal, which means they are 0, 1, 3, and 4.

b) State whether or not *A* is invertible and state the reasoning behind your answer.

The determinant is 0, which means that the matrix in not invertible. The determinant can easily be computed as the product of the entries on the diagonal since A is a triangular matrix.

c) Supply a substantiated answer to whether or not A is diagonalizable.

Since A has n distinct eigenvalues, A is diagonalizable.

d) Let B be a $n \times n$ matrix where $\det(B) \neq 0$. What is $rank(B)$ and $dim\ Nul\ B$?

Since the determinant of B is not 0, B is invertible. Since it is invertible it will have pivots in all columns and will only have the trivial solution in its nulbasis. This means *rank B = n* and *dim Nul B = 0*.

e) Supply a substantiated answer to whether $C\bar{x} = \bar{b}$ is consistent for every $\bar{b}$ in $\mathbb{R}^4$.

Since C has a pivot in each row, it will Span $\mathbb{R}^4$ which means it will be consistent for every $\bar{b}$ in $\mathbb{R}^4$

f) Does $C\bar{x} = \bar{0}$ have a non-trivial solution? Explain why/why not.

Since the vector has more columns than rows, we will necessarily have a free variable. Since we have a free variable $C\bar{x} = \bar{0}$ will have a nontrivial solution.

Assignment 2 (10%)

Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } \bar{b} = \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}$$

a) Find the inverse of A using elementary row operations and the identity matrix.

```
>> A=[1 -1 2;-1 2 1;1 -1 1]

A =

     1    -1     2
    -1     2     1
     1    -1     1

>> C=[A eye(3)]

C =

     1    -1     2     1     0     0
    -1     2     1     0     1     0
     1    -1     1     0     0     1

>> rref(C)

ans =

     1     0     0    -3     1     5
     0     1     0    -2     1     3
     0     0     1     1     0    -1

>> Ainv=[ans(:,4) ans(:,5) ans(:,6)]

Ainv =

    -3     1     5
    -2     1     3
     1     0    -1
```

b) Use the inverse of A to solve $A\bar{x} = \bar{b}$

```
>> b=[5 6 2]'
```

```
b =

     5
     6
     2

>> x=Ainv*b

x =

     1
     2
     3
```

Assignment 3 (5%)

Consider the following linear system

$$x_1 + 2x_2 + x_3 = 1$$

$$-2x_1 + x_2 + x_3 = -5$$

$$2x_1 - x_2 - 2x_3 = a$$

For which values of a is the system consistent?

```
>> syms a

>> A=[1 2 1 1;-2 1 1 -5;2 -1 -2 a]


A =

[  1,  2,  1,  1]

[ -2,  1,  1, -5]

[  2, -1, -2,  a]

>> A(3,:)=A(2,:)+A(3,:)

A =

[  1, 2,  1,     1]

[ -2, 1,  1,    -5]

[  0, 0, -1, a - 5]
```

```
>> A(3,:)=-1*A(3,:)

A =

[  1, 2, 1,     1]

[ -2, 1, 1,    -5]

[  0, 0, 1, 5 - a]
```

So the system is consistent for all values of a.

Assignment 4 (25%)

$$A = \begin{bmatrix} a & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$$

a) Express the eigenvalues of A as a function of a

```
>> syms a l

>> A=[a-l -2 0;-2 9-l 0;5 8 3-l]

A =

[ a - l,    -2,     0]

[    -2, 9 - l,     0]

[     5,     8, 3 - l]

>> det(A)

ans =

27*a - 23*l - 12*a*l + a*l^2 + 12*l^2 - l^3 - 12
```

So $-\boldsymbol{\lambda^3 + 12\lambda^2 - 23\lambda + a\lambda^2 - 12a\lambda - 23\lambda - 27a - 12 = 0}$

b) Calculate a so that $\lambda = 5$ is an eigenvalue of A

$$-5^3 + 12 \times 5^2 - 23 \times 5 + a5^2 - 12a5 - 23 \times 5 - 27a - 12 = 0 \Leftrightarrow \boldsymbol{a = 6}$$

c) Using the value of a found in question (b), determine the remaining eigenvalue(s)

```
A =

     6    -2     0
    -2     9     0
     5     8     3

>> eig(A)
```

```
ans =

     3
    10
     5
```

d) Determine the eigenspaces for each of the eigenvalues found in (c)

```
>> % l = 3
>> rref(A-3*eye(3))

ans =

     1     0     0
     0     1     0
     0     0     0

>> %Eigenspace =
>> [0 0 1]'

ans =

     0
     0
     1

>> % l = 5
>> rref(A-5*eye(3))

ans =

    1.0000         0   -0.2222
         0    1.0000   -0.1111
         0         0         0

>> %Eigenspace =
>> rats(ans)

ans =

       1             0            -2/9
       0             1            -1/9
       0             0              0

>> %Eigenspace =
>> [2/9 1/9 1]'
```

```
ans =

    0.2222
    0.1111
    1.0000

>> rats(ans)

ans =

      2/9
      1/9
       1

>> % l = 10
>> rref(A-10*eye(3))

ans =

    1.0000         0    0.6364
         0    1.0000   -1.2727
         0         0         0

>> rats(ans)

ans =

       1             0            7/11
       0             1          -14/11
       0             0              0

>> %Eigenspace =
>> [-7/11 14/11 1]'

ans =

   -0.6364
    1.2727
    1.0000

>> rats(ans)
```

```
ans =

      -7/11
      14/11
        1
```

e) Using the value of a found in (b), find the characteristic equation of A^T and compare it to the characteristic equation of A. What can you conclude about the characteristic equations of A and A^T?

```
>> At=A'

At =

     6    -2     5
    -2     9     8
     0     0     3

>> det(At-l*eye(3))

ans =

- l^3 + 18*l^2 - 95*l + 150

>> det(A-l*eye(3))

ans =

- l^3 + 18*l^2 - 95*l + 150
```

It turns out that the characteristic equations of A is identical to the characteristic equation of A^T.

Assignment 4 (15%)

Let

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \bar{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad and\ \bar{y} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

And let $H = span\{\bar{u}_1, \bar{u}_2\}$, let $A = [\bar{u}_1 \quad \bar{u}_2]$, and let $H^{\perp}$ denote the subspace that is orthogonal to H

a) Show that $\{u_1, u_2\}$ is an orthogonal basis for H.

```
>> u1=[1 1 0]'

u1 =

     1
     1
     0

>> u2=[1 -1 2]'

u2 =

     1
    -1
     2

>> dot(u1,u2)

ans =

     0
```

It follows that $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ is an orthogonal subspace of H.

b) Find a basis for $H^{\perp}$. Hint: $H = Col\ A \Longrightarrow H^{\perp} = (Col\ A)^{\perp} = Nul\ A^{T}$

```
>> A=[u1 u2]'

 =

     1     1     0

     1    -1     2

>> rref(A)

ans =

     1     0     1

     0     1    -1

>> Basis=[-1 1 1]'
```

Basis =

-1

1

```
1
```

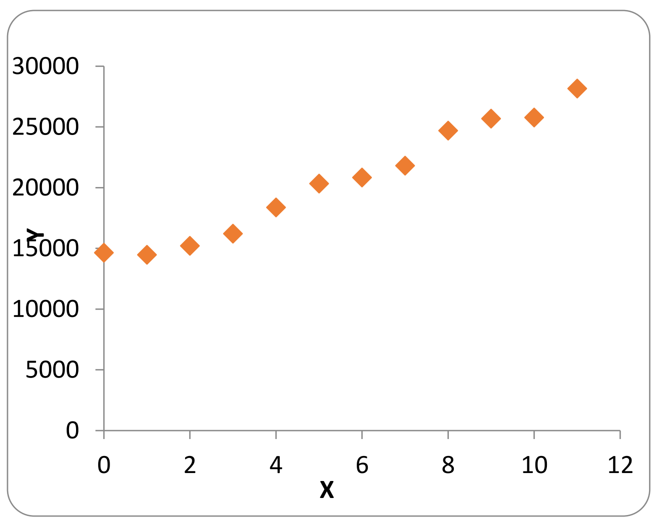
c) Calculate the orthogonal projection of $\bar{y}$ onto H

```
>> y=[1 3 4]'
   y =
         1
         3
         4
   >> yhat = (dot(u1,y)/dot(u1,u1))*u1 + (dot(u2,y)/dot(u2,u2))*u2
   yhat =
         3
         1
         2
```

Assignment 5 (15%)

The number of students enrolled at a university study (5+ years) in the period 2005-2016 is as follows (source: DST.dk):

Year	Index year (t)	Number of students (y)
2005	0	14652
2006	1	14480
2007	2	15216
2008	3	16215
2009	4	18378
2010	5	20347
2011	6	20845
2012	7	21822
2013	8	24707
2014	9	25689
2015	10	25787
2016	11	28168



It is believed that the number of students can be modelled by either a linear function or a quadratic function:

$y_1(t) = \beta_0 + \beta_1 t$ or $y_2(t) = \delta_0 + \delta_1 t + \delta_2 t^2$

a) By creating design matrices, find the parameters of both models and state the regression functions

```
>> t=[0 1 2 3 4 5 6 7 8 9 10 11]';

>> y=[14652 14480 15216 16215 18378 20347 20845 21822 24707 25689
25787 28168]';

>> X1=[ones(12,1) t];

>> X1tX1=X1'*X1

X1tX1 =

    12    66

    66   506

>> X1ty=X1'*y

X1ty =

      246306

     1543203

>> [X1tX1 X1ty]

ans =

          12          66      246306

          66         506     1543203

>> rref(ans)

ans =

   1.0e+04 *

    0.0001         0    1.3275

         0    0.0001    0.1318

>> b1=ans(:,3);
```

$\boldsymbol{y_1(t) = 13275 + 1318t}$

```
>> X2=[ones(12,1) t t.^2];

>> X2tX2=X2'*X2
```

```
X2tX2 =

          12          66          506

          66         506         4356

         506        4356        39974

>> X2ty=X2'*y

X2ty =

      246306

     1543203

    12492785

>> [X2tX2 X2ty]

ans =

          12          66          506      246306

          66         506         4356     1543203

         506        4356        39974    12492785

>> rref(ans)

ans =

   1.0e+04 *


    0.0001         0         0    1.3730

         0    0.0001         0    0.1045

         0         0    0.0001    0.0025

>> b2=ans(:,4);
```

$$\boldsymbol{y_2(t) = 13730 + 1045t + 25t^2}$$

b) Supply a substantiated answer to which model is the best fit for the measured data

```
>> y1=X1*b1

y1 =

   1.0e+04 *

    1.3275
```

```
    1.4593
    1.5911
    1.7230
    1.8548
    1.9866
    2.1185
    2.2503
    2.3821
    2.5140
    2.6458
    2.7776

>> norm(y-y1)

ans =

   2.4396e+03

>> y2=X2*b2

y2 =

   1.0e+04 *

    1.3730
    1.4800
    1.5920
    1.7089
    1.8308
    1.9577
    2.0895
    2.2263
    2.3681
    2.5149
    2.6666
    2.8233

>> norm(y-y2)

ans =

   2.2645e+03
```

Since y_2 yields a smaller error, the best model of the two is y_2

c) Use the best fitted model to predict how many students will be enrolled at a university study in 2020.

$$y_2(15) = 13730 + 1045 \times 15 + 25 \times 15^2 = \mathbf{34998} \cong \mathbf{35000}$$

Assignment 6 (10%)

Compute a full singular value decomposition of the following matrix

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

```
>> A=[-1 2 0;2 0 -2;0 -2 1]

A =

    -1     2     0

     2     0    -2

     0    -2     1

>> AtA=A'*A

AtA =

     5    -2    -4

    -2     8    -2

    -4    -2     5

>> eig(AtA)

ans =

     0

     9

     9

>> AtA-9*eye(3)

ans =

    -4    -2    -4

    -2    -1    -2

    -4    -2    -4

>> rref(ans)

ans =
```

```
    1.0000    0.5000    1.0000
         0         0         0
         0         0         0

>> v1=[-1 2 0]'

v1 =

    -1
     2
     0

>> AtA-0*eye(3)

ans =

     5    -2    -4
    -2     8    -2
    -4    -2     5

>> rref(ans)

ans =

    1.0000         0   -1.0000
         0    1.0000   -0.5000
         0         0         0

>> v3=[2 1 2]’

v3 =

     2
     1
     2

>> [v1';v3']

ans =

    -1     2     0
     2     1     2

>> rref(ans)
```

```
ans =

    1.0000         0    0.8000
         0    1.0000    0.4000
>> v2=[-4/5 -2/5 1]'
v2 =
   -0.8000
   -0.4000
    1.0000
>> dot(v1,v2)
ans =
     0
>> dot(v3,v2)
ans =
     0
>> dot(v1,v2)
ans =
     0
>> v1=v1/norm(v1)
v1 =
   -0.4472
    0.8944
         0
>> v2=v2/norm(v2)
v2 =
   -0.5963
   -0.2981
    0.7454
```

```
>> v3=v3/norm(v3)

v3 =

    0.6667

    0.3333

    0.6667

>> V=[v1 v2 v3]

V =

   -0.4472   -0.5963    0.6667

    0.8944   -0.2981    0.3333

         0    0.7454    0.6667

>> S=[sqrt(9) 0 0;0 sqrt(9) 0;0 0 0]

S =

     3     0     0

     0     3     0

     0     0     0

>> u1=(1/3)*A*v1

u1 =

    0.7454

   -0.2981

   -0.5963

>> u2=(1/3)*A*v2

u2 =

    0.0000

   -0.8944

    0.4472

>> dot(u1,u2)

ans =


     0
```

```
>> [u1';u2']

ans =

    0.7454   -0.2981   -0.5963

    0.0000   -0.8944    0.4472

>> rref(ans)

ans =

    1.0000         0   -1.0000

         0    1.0000   -0.5000

>> u3=[1 1/2 1]'

u3 =

    1.0000

    0.5000

    1.0000

>> u3=u3/norm(u3)

u3 =

    0.6667

    0.3333

    0.6667

>> U=[u1 u2 u3]

U =

    0.7454    0.0000    0.6667

   -0.2981   -0.8944    0.3333

   -0.5963    0.4472    0.6667

>> U*S*V'

ans =

   -1.0000    2.0000    0.0000

    2.0000    0.0000   -2.0000

   -0.0000   -2.0000    1.0000
```