4.5 The Dimension of a Vector Space

Theorem If a vector space V has a basis $\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

This tells us that each linearly independent set in V has *no more than* n vectors. We can then show the following.

Theorem If a vector space V has a basis of n vectors, then *every* basis of V must consist of exactly n vectors.

Definition If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, denoted $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\bar{0}\}$ is *defined* to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Example Let $H = \text{span}\{\bar{v}_1, \bar{v}_2\}$, where

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\bar{v}_1$ and $\bar{v}_2$ are not multiples of one another, the set $\{\bar{v}_1, \bar{v}_2\}$ is linearly independent, hence is a basis for H. It follows that $\dim H = 2$.

Example The standard basis for $\mathbb{R}^n$ is $\mathcal{E} = \{\bar{e}_1, \ldots, \bar{e}_n\}$, which contains n vectors. Hence $\dim \mathbb{R}^n = n$. Likewise, since the standard basis of $\mathbb{P}_n$ is $\{1, x, x^2, \ldots, x^n\}$, we have $\dim \mathbb{P}_n = n + 1$.

Example We find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \subseteq \mathbb{R}^4.$$

Each element of H can be written in the form

$$a \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix},$$

so if we let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \bar{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \text{ and } \qquad \bar{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix},$$

it is clear that $H = \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$. Notice, however, that $\bar{v}_3 = -2\bar{v}_2$. Hence we can remove $\bar{v}_3$ from the set and still have a set that spans. That is

$$H = \text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_4\}.$$

Is this new spanning set linearly independent? We see easily that $\bar{v}_1$ is not a multiple of $\bar{v}_2$, and that $\bar{v}_4$ cannot be a linear combination of $\bar{v}_1$ and $\bar{v}_2$, since the last coordinate of $\bar{v}_4$ is nonzero, while those of $\bar{v}_1$ and $\bar{v}_2$ are zero. It follows that $\{\bar{v}_1, \bar{v}_2, \bar{v}_4\}$ is a basis for H, and that $\dim H = 3$.

Example We can characterize the subspaces of $\mathbb{R}^3$ by dimension:
- 0-dimensional: The only 0-dimensional subspace is the zero subspace.
- 1-dimensional: Subspaces spanned by a single nonzero vector; i.e. they are lines through the origin.
- 2-dimensional: Subspaces spanned by two nonzero vectors that are not multiples of each other; i.e. they are planes through the origin.
- 3-dimensional: Only $\mathbb{R}^3$.

Theorem Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. We also have that H is finite-dimensional, and

$$\dim H \leq \dim V.$$

Theorem (The Basis Theorem) Let V be a p-dimensional vector space with $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Likewise, any set of exactly p elements that spans V is automatically a basis for V.

If we are dealing with a matrix A, we have the following:

The dimension of Nul$A$ is the number of free variables in the equation $A\bar{x} = \bar{0}$, and the dimension of Col$A$ is the number of pivot columns in A.

Example Let A be the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix},$$

which we know from an example in section 4.3 is row equivalent to

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see easily that there are three pivot columns, so $\dim \mathrm{Col}A = 3$. Likewise, since there are two free variables in the equation $A\bar{x} = \bar{0}$, we see that $\dim \mathrm{Nul}A = 2$.

4.6 Rank

Definition Let A be an $m \times n$ matrix. The set of all linear combinations of the *row* vectors of A is called the **row space of** A, which we denote by Row$A$. Since each row has n entries, we see that Row$A$ is a subspace of $\mathbb{R}^n$. Since the columns of A^T are equal to the rows of A, notice also that $\text{Row}A = \text{Col}A^T$.

Typically, to compute a basis for Col$A$, you would row reduce A in order to identify its pivots, and then the pivot columns of A are the basis. This procedure does not hold up for computing the basis of Row$A$, since row operations change row dependence relations (while they do *not* change column dependence relations). Luckily, we have the following theorem.

Theorem If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A, as well as that of B.

The main result of this section is a theorem that relates Row$A$, Col$A$, and Nul$A$. Before getting to it, we work the following example.

Example We find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}.$$

We begin by reducing A to an echelon form

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the preceding theorem, the first three rows of B form a basis of Row$A$ (since they are nonzero). Thus, a basis for Row$A$ is

$$\{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}.$$

For the column space, notice that the pivots of B are in columns 1, 2, and 4. Hence a basis for Col$A$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

To find a basis for Nul$A$, we need the solution of $A\bar{x} = \bar{0}$ in parametric vector form. Since $A \sim B$, this is the same as the solution set of $B\bar{x} = \bar{0}$. Hence we reduce B to *reduced echelon from* (not just echelon form), as follows:

$$\begin{aligned} \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} &\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 3 & -5 & 0 & 10 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= 2x_3 - 3x_5 \\ x_3 &= \text{free} \\ x_4 &= 5x_5 \\ x_5 &= \text{free.} \end{aligned}$$

Hence the solutions of $B\bar{x} = \bar{0}$ have the form

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix},$$

so a basis for Nul$A$ is given by

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

Definition The **rank** of A is the dimension of the column space of A. That is, $\text{rank}A = \dim \text{Col}A$.

Since $\text{Row}A = \text{Col}A^T$, we see that $\dim \text{Row}A = \dim \text{Col}A^T = \text{rank}A^T$.

Theorem (The Rank Theorem) The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, is equal to the number of pivot positions in A and satisfies

$$\text{rank}A + \dim \text{Nul}A = n.$$

Example Suppose you have found two linearly independent solutions to a homogeneous system of 8 equations in 10 variables, and that all solutions are linear combinations of the two solutions. Can you conclude that each associated nonhomogeneous system (with the same coefficients) will have a solution?

Yes! We can represent the homogeneous system by the equation $A\bar{x} = \bar{0}$, where A is the 8×10 matrix of coefficients. The fact that two linearly independent solutions span the solution set tells us that $\dim \text{Nul}A = 2$. It follows that

$$\text{rank}A + 2 = 10,$$

so that

$$\text{rank}A = 8.$$

This means that the column space is 8-dimensional. Since there are exactly 8 rows, this means the column space spans $\mathbb{R}^8$. In other words, for each $\bar{b} \in \mathbb{R}^8$, there is a solution to the matrix equation

$$A\bar{x} = \bar{b}.$$

Example Could a 6×9 matrix A have a two-dimensional null space?

No! If $\dim \text{Nul}A = 2$, then $\text{rank}A = 9 - 2 = 7$. That is, the column space is 7-dimensional. But the vectors in the column space have only 6 entries (since there are 6 rows in A), so this is nonsense.

In light of our new results on dimension and rank, we can add a few more conditions to the Invertible Matrix Theorem.

Theorem (The Invertible Matrix Theorem, continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

(m) The columns of A form a basis of $\mathbb{R}^n$.

(n) $\mathrm{Col}A = \mathbb{R}^n$.

(o) $\dim \mathrm{Col}A = n$.

(p) $\mathrm{rank}A = n$.

(q) $\mathrm{Nul}A = \{\bar{0}\}$.

(r) $\mathrm{dimNul}A = 0$.