2 Matrix Algebra

2.1 Matrix Operations

Definition If A is an $m \times n$ matrix, then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i,j)**-entry of** A. The **diagonal entries** of A are $a_{11}, a_{22}, a_{33}, \ldots$ and they form the **main diagonal** of A. A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero (note that the diagonal entries can also be zero). Two matrices are **equal** if all of their entries are the same. The **sum** $A+B$ of two $m \times n$ matrices is the matrix whose entries are the sums of corresponding entries in A and B. That is, if $C = A+B$, then $c_{ij} = a_{ij} + b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. If r is a real number, then rA is the **scalar multiple** of A where each entry of A is multiplied by r.

As outlined in the following theorem, many properties we saw for vector addition and scalar multiplication also hold for addition and scalar multiplication of matrices.

Theorem Let A, B and C be matrices of the same size, and let r and s be scalars. Then:
(a) $A+B = B+A$;
(b) $(A+B)+C = A+(B+C)$;
(c) $A+0 = A$ (we think of 0 as the matrix whose entries are all zero);
(d) $r(A+B) = rA+rB$;
(e) $(r+s)A = rA+sA$;
(f) $r(sA) = (rs)A$.

Matrix Multiplication Recall that we saw for an $m \times n$ matrix A that we can define a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\bar{x}) = A\bar{x}$. We define matrix multiplication in such a way that it corresponds to composition of linear maps. That is, if A is an $m \times n$ matrix and B is an $n \times p$ matrix, where A and B correspond to matrix transformations $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^p \to \mathbb{R}^n$, respectively, then we define the product AB in such a way that $T_1 \circ T_2 : \mathbb{R}^p \to \mathbb{R}^m$ is given by

$$T_1 \circ T_2(\bar{x}) = T_1(T_2(\bar{x})) = T_1(B\bar{x}) = AB\bar{x}.$$

Definition If A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\bar{b}_1, \ldots, \bar{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\bar{b}_1, \ldots, A\bar{b}_p$. That is

$$AB = A[\ \bar{b}_1 \quad \bar{b}_2 \quad \ldots \quad \bar{b}_p] = [\ A\bar{b}_1 \quad A\bar{b}_2 \quad \ldots \quad A\bar{b}_p].$$

It is worth saying again, *matrix multiplication corresponds to composition of matrix transformations.*

Example Compute the matrix AB, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}, \quad \text{and } B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}.$$

By the definition of multiplication given above,

$$\begin{aligned} AB &= \begin{bmatrix} A\begin{bmatrix} 2 \\ 3 \end{bmatrix} & A\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \cdot 2 + 2 \cdot 3 \\ 2 \cdot 2 + 3 \cdot 3 \\ 3 \cdot 2 + 4 \cdot 3 \end{bmatrix} & \begin{bmatrix} 1 \cdot (-1) + 2 \cdot 1 \\ 2 \cdot (-1) + 3 \cdot 1 \\ 3 \cdot (-1) + 4 \cdot 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 8 & 1 \\ 13 & 1 \\ 18 & 1 \end{bmatrix}. \end{aligned}$$

Note The first column of AB is $A\bar{b}_1$, which is just shorthand for a linear combination of the columns of A using the weights in $\bar{b}_1$. In general, each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Example If A is 3×5 and B is 5×2, what are the sizes of AB and BA (assuming they are defined)?
- The product AB is defined since the number of columns of A matches the number of rows of B (5). The resulting matrix AB is a 3×2 matrix.
- The product BA is not defined, since the number of columns of B (2) does not match the number of rows of A (3).

Row-Column Rule for Computing AB If the product AB is defined, then the (i, j)-entry of AB is the dot product of the ith row of A with the jth column of B. That is, if A is $m \times n$ and B is $n \times p$, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Example Compute the product AB in the example above using the row-column rule. (We basically just skip the first two steps in the computation above.)

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + 2 \cdot 3 & 1 \cdot (-1) + 2 \cdot 1 \\ 2 \cdot 2 + 3 \cdot 3 & 2 \cdot (-1) + 3 \cdot 1 \\ 3 \cdot 2 + 4 \cdot 3 & 3 \cdot (-1) + 4 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 1 \\ 13 & 1 \\ 18 & 1 \end{bmatrix}. \end{aligned}$$

Now we list a few (probably unsurprising) properties of matrix multiplication.

Theorem Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.
(a) $A(BC) = (AB)C$;
(b) $A(B + C) = AB + AC$;
(c) $(B + C)A = BA + CA$;
(d) $r(AB) = (rA)B = A(rB)$ for any scalar r;
(e) $I_m A = A = AI_n$.

Note It is very important to observe that, in general, the *commutative* law does *not* hold for matrix multiplication. That is, it is often the case that $AB \neq BA$, as in the following example.

Example Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \qquad \text{and } B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 5 & 11 \end{bmatrix}, \text{ and} \\ BA &= \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 4 & 1 \end{bmatrix}. \end{aligned}$$

If A is a square $n \times n$ matrix, then it makes sense to multiply A with itself. In general,

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A}_{k \text{ times}}.$$

Definition If A is an $m \times n$ matrix, then the **transpose** of A, denoted A^T, is the $n \times m$ matrix whose columns are formed from the corresponding rows of A.

Example If

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \\ 7 & 8 \end{bmatrix},$$

then

$$A^T = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 8 \end{bmatrix}.$$

Theorem Let A and B denote matrices whose sizes are appropriate for the following sums and products.
(a) $(A^T)^T = A$;
(b) $(A + B)^T = A^T + B^T$;
(c) $(rA)^T = rA^T$ for scalars r;
(d) $(AB)^T = B^T A^T$.

Example The following is to be worked on in class (time-permitting):

(Practice Problem 1 on Page 100)
Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \bar{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Compute $(A\bar{x})^T$, $\bar{x}^T A^T$, $\bar{x}\bar{x}^T$, and $\bar{x}^T\bar{x}$. Is $A^T\bar{x}^T$ defined?

2.2 The Inverse of a Matrix

Definition An $n \times n$ matrix A is said to be **invertible** (or **nonsingular**) if there is an $n \times n$ matrix C such that

$$AC = CA = I_n.$$

Computing matrix inverses is of great interest in linear algebra. In the case of 2×2 matrices, computing the inverse is quite straightforward.

Theorem Let A be the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then the inverse of A, denoted A^{-1}, is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Note The quantity $ad - bc$ is called the **determinant** of A, and we write

$$\det A = ad - bc.$$

So a 2×2 matrix is invertible if and only if its determinant is nonzero. This idea generalizes to larger matrices, as we will see later in the course.

Example We let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and compute A^{-1}. By the theorem, we have

$$\begin{aligned} A^{-1} &= \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ &= -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}. \end{aligned}$$

Now we confirm that $AA^{-1} = I_2$:

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1(-2) + 2\left(\frac{3}{2}\right) & 1(1) + 2\left(-\frac{1}{2}\right) \\ 3(-2) + 4\left(\frac{3}{2}\right) & 3(1) + 4\left(-\frac{1}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} -2 + 3 & 1 - 1 \\ -6 + 6 & 3 - 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

You should check on your own that $A^{-1}A = I_2$ as well.

Theorem If A is an invertible $n \times n$ matrix, then for each $\bar{b} \in \mathbb{R}^n$, the matrix $A\bar{x} = \bar{b}$ has a *unique* solution given by $\bar{x} = A^{-1}\bar{b}$.

Proof The solution exists, since if we substitute $A^{-1}\bar{b}$ for $\bar{x}$, we get

$$A(A^{-1}\bar{b}) = AA^{-1}\bar{b} = I_n\bar{b} = \bar{b}.$$

It is unique, since if $\bar{u}$ is any solution, then

$$A\bar{u} = \bar{b} \quad \Longrightarrow \quad A^{-1}A\bar{u} = A^{-1}\bar{b} \quad \Longrightarrow \quad I_n\bar{u} = A^{-1}\bar{b} \quad \Longrightarrow \quad \bar{u} = A^{-1}\bar{b}.$$

Example We use the inverse in the example above to solve the linear system

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6. \end{aligned}$$

We think of this in matrix terms as $A\bar{x} = \bar{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \bar{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

The theorem tells us that $\bar{x} = A^{-1}\bar{b}$ is a solution, i.e. that

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} -2(5) + 1(6) \\ \frac{3}{2}(5) - \frac{1}{2}(6) \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ \frac{9}{2} \end{bmatrix}. \end{aligned}$$

Theorem Let A and B be invertible $n \times n$ matrices. Then:
(a) A^{-1} is invertible, with

$$(A^{-1})^{-1} = A;$$

(b) The product AB is invertible, with

$$(AB)^{-1} = B^{-1}A^{-1};$$

(c) The transpose of A is also invertible, i.e. A^T is invertible, with

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof The proof of part (a) is immediate. For part (b), note that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n, \text{ and}$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n.$$

For (c), we use the general fact from section 2.1 that $(AB)^T = B^TA^T$. Now observe that

$$A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n, \text{ and}$$

$$(A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n.$$

This completes the proof.

Elementary Matrices An **elementary matrix** is a matrix that is obtained by performing a single elementary row operation (replacement, swap, or scaling) on an identity matrix. For example, the matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

is an elementary matrix since it is obtained from I_3 via the single elementary row operation $r_3 \mapsto r_3 - 2r_1$.

Example Let A be a general 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

and we let E_1, E_2, and E_3 be the elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice, then, that E_1 corresponds to $r_3 \mapsto r_3 - 2r_1$, E_2 corresponds to $r_2 \longleftrightarrow r_3$, and E_3 corresponds to $r_2 \mapsto 3r_2$. We also have the following products:

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-2a & h-2b & i-2c \end{bmatrix}, \quad E_2A = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}, \quad \text{and} \quad E_3A = \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}.$$

Notice that E_1A is the matrix obtained by performing the row operation $r_3 \mapsto r_3 - 2r_1$ on A. In general, multiplying A on the left by an elementary matrix is the same as performing the corresponding row operation on A. We can also represent a sequence of row operations by multiplication of several elementary matrices. For example,

$$E_2E_1A = \begin{bmatrix} a & b & c \\ g-2a & h-2b & i-2c \\ d & e & f \end{bmatrix}$$

corresponds to performing $r_3 \mapsto r_3 - 2r_1$ followed by $r_2 \longleftrightarrow r_3$ on the matrix A.

As we have observed before, row operations are reversible. It follows that elementary matrices are also invertible. This leads to the following theorem.

Theorem An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. In this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

This theorem leads to a nice algorithm for finding the inverse of an $n \times n$ matrix, assuming such an inverse exists.

Algorithm for Finding A^{-1} Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

Example We find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix},$$

if it exists. We row reduce $\begin{bmatrix} A & I \end{bmatrix}$ as follows:

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & 5 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -4 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & -3 & 0 & -4 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} & \frac{2}{3} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}.
\end{aligned}$$

Hence

$$A^{-1} = \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} & -\frac{1}{6} \\ \frac{4}{3} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Example The following examples are to be worked on in class (time-permitting):

(a) Use the determinant to figure out which of the following are invertible:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

(b) Compute the inverse of

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix},$$

if it exists.

2.3 Characterizations of Invertible Matrices

The following theorem summarizes many things we have already seen. We will be adding things to the list as the course progresses.

Theorem (The Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. Then the following statements are equivalent.
(a) A is invertible.
(b) A is row equivalent to I_n.
(c) A has n pivot positions (i.e. one for each row and column).
(d) The equation $A\bar{x} = \bar{0}$ has *only* the trivial solution.
(e) The columns of A are linearly independent.
(f) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\bar{x}) = A\bar{x}$ is one-to-one.
(g) The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in $\mathbb{R}^n$.
(h) The columns of A span $\mathbb{R}^n$.
(i) The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\bar{x}) = A\bar{x}$ is onto.
(j) There is an $n \times n$ matrix C such that $CA = I$.
(k) There is an $n \times n$ matrix D such that $AD = I$.
(l) A^T is invertible.

Example We decide if the following matrix is invertible:

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ 4 & 4 & 6 \end{bmatrix}.$$

Performing row operations, we see

$$\begin{aligned} A &\sim \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 4 & 4 & 6 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & 0 \\ 0 & -8 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \end{aligned}$$

which has 3 pivots, so by (c) we have that A is invertible.

Definition A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} S(T(\bar{x})) &= \bar{x} \quad \text{for all } \bar{x} \in \mathbb{R}^n, \text{ and} \\ T(S(\bar{x})) &= \bar{x} \quad \text{for all } \bar{x} \in \mathbb{R}^n. \end{aligned}$$

We call S the **inverse** of T, and we write it as T^{-1}.

Theorem Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\bar{x}) = A^{-1}\bar{x}$ is the unique function satisfying the two equations in the above definition.

Proof (We don't prove uniqueness.) Suppose T is invertible. Then T is onto $\mathbb{R}^n$, since for all $\bar{b} \in \mathbb{R}^n$, we have $T(S(\bar{b})) = \bar{b}$. Hence A is invertible by part (i) of the theorem above. For the reverse, suppose that A is invertible and set $S(\bar{x}) = A^{-1}\bar{x}$. Then S is a linear transformation and satisfies the equations in the definition above, since

$$\begin{aligned} S(T(\bar{x})) &= S(A\bar{x}) = A^{-1}A\bar{x} = I_n\bar{x} = \bar{x}, \text{ and} \\ T(S(\bar{x})) &= T(A^{-1}\bar{x}) = AA^{-1}\bar{x} = I_n\bar{x} = \bar{x}. \end{aligned}$$

Examples The following are to be worked on in class (time-permitting):

(a) Determine which of the following matrices are invertible:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 2 \\ 3 & 2 & 3 \end{bmatrix}.$$

(b) (Exercise 13 on Page 115)

An $m \times n$ **upper triangular matrix** is one whose entries *below* the main diagonal are 0's. When is a square upper triangular matrix invertible? Justify your answer.