Assignment 1 (20%)

You will need very few and simple calculation in order to solve the problems of this assignment.

a) The result of an SVD displays

$$U = \begin{bmatrix} -0.2534 & -0.3430 & 0.9045 \\ -0.4897 & -0.7609 & -0.4257 \\ -0.8343 & 0.5508 & -0.0249 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -0.4132 & 0.4230 & 0.2244 & -0.7746 \\ -0.5756 & -0.2919 & -0.7189 & 0.2582 \\ 0.4397 & 0.8557 & -0.0881 & -0.2582 \\ -0.5519 & -0.0606 & 0.6520 & 0.5164 \end{bmatrix}$$

What is the size of the matrix which has been decomposed?

A is a 3 x 4 matrix

b) Explain why it would be possible to determine λ_3 of A if λ_1 were 4 and λ_2 were 2 and then state what λ_3 is if this were the case.

$$A = \begin{bmatrix} 8 & 3 & -5 \\ 3 & 3 & -2 \\ -5 & -2 & 1 \end{bmatrix}$$

Since A is symmetric, the sum of the diagonals equal the sum of the eigenvalues. The sum of the diagonal is 12. This means that the last eigenvalue would be 6.

c) Explain why the following matrix is not invertible

$$\begin{bmatrix} 2 & 4 & 8 & 7 & 1 & 3 \\ 0 & 5 & 3 & 5 & 5 & 6 \\ 0 & 0 & 3 & 3 & 3 & 9 \\ 0 & 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

Since it is a triangular matrix, the determinant can be calculated as the product of the diagonals. Since one of the values of the diagonal is 0, the determinant is zero. It follows from the invertible matrix theorem that the matrix cannot be invertible since its determinant is 0

d) For what value(s) of k are the columns of the following matrix linearly dependent?

$$\begin{bmatrix} 2 & -10 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 2k-6 \end{bmatrix}$$

The columns of a matrix will be linearly independent if it has pivots in all columns. This is true for the above matrix for k != 3.

e) Explain why $Ax = b$ formed from the following matrix in echelon form has a solution for each b

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix does NOT have a solution for each b, since it does not have pivots in all columns.

Assignment 2 (5%)

Determine a basis for the null space of matrix A by setting up the solution in parametric form

$$A = \begin{bmatrix} -1 & 2 & 1 & 4 \\ 1 & 2 & 2 & 6 \end{bmatrix}$$

```
A =

    -1     2     1     4
     1     2     2     6

>> rref(A)

ans =

    1.0000         0    0.5000    1.0000
         0    1.0000    0.7500    2.5000
```

So, $\bar{x} = x_3 \begin{bmatrix} 2 \\ 1 \\ -4 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 5 \\ 0 \\ -2 \end{bmatrix}$

Thus, a nulbasis is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ -2 \end{bmatrix} \right\}$

Assignment 3 (15%)

A 2 x 2 matrix A is given as

$$A = \begin{bmatrix} a & 2 \\ 1 & 2 \end{bmatrix}$$

a) Express the eigenvalues of A as a function of a

```
>> A =

[ a - 1,      2]
```

```
[      1, 2 - l]

det(A)

ans =

2*a - 2*l - a*l + l^2 - 2
```

So we get $2a - 2\lambda - a\lambda + \lambda^2 - 2 = 0$

b) Calculate a so that $\lambda = 4$ is an eigenvalue of A

$2a - 2 \cdot 4 - a \cdot 4 + 4^2 - 2 = 0 \Leftrightarrow 2a - 8 - 4a + 16 - 2 = 0 \Leftrightarrow -2a + 6 = 0 \Leftrightarrow a = 3$

c) Using the value of a found in question (b), determine the remaining eigenvalue(s) and the eigenspaces

```
A =

     3     2
     1     2

>> eig(A)

ans =

     4
     1
```

$\underline{\lambda = 4}$

```
>> A-4*eye(2)

ans =

    -1     2
     1    -2

>> rref(ans)

ans =

     1    -2
     0     0
```

The eigenspace is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$\underline{\lambda = 1}$

```
>> A-1*eye(2)

ans =

     2     2
     1     1

>> rref(ans)

ans =

     1     1
     0     0
```

The eigenspace is $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

Assignment 4 (20%)

Let,

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, and\ x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix},$$

a) Show that $\{u_1, u_2, u_3\}$ is a linearly independent set

```
>> [1 0 1;-1 4 1;2 1 -2]'

ans =

     1    -1     2
     0     4     1
     1     1    -2

>> rref(ans)

ans =

     1     0     0
     0     1     0
     0     0     1
```

```
Since there is a pivot in all columns, the set is linearly
independent.
```

b) Show that $\{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$

```
A =


     1    -1     2

     0     4     1

     1     1    -2

>> dot(A(:,1),A(:,2))

ans =

     0

>> dot(A(:,1),A(:,3))

ans =

     0

>> dot(A(:,2),A(:,3))

ans =

     0
```

Since all the inner products are 0, which display orthogonality, and since the set is linearly independent, it follows that we have an orthogonal set that will span $\mathbb{R}^3$

c) Express x as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3\}$.

$$x = \left(proj(x, u_1) + proj(x, u_2)\right) + \left(proj(x, u_3)\right)$$

```
>> v1=proj(x,u1)

v1 =

    2.5000
         0
    2.5000

>> v2=proj(x,u2)

v2 =
```

```
    1.5000
   -6.0000
   -1.5000

>> z1=v1+v2

z1 =

    4.0000
   -6.0000
    1.0000

>> z2=proj(x,u3)

z2 =

     4
     2
    -4

>> x=z1+z2

x =

    8.0000
   -4.0000
   -3.0000
```

$x = z_1 + z_2$

d) Determine the (shortest) distance between x and the subspace spanned by $\{u_1, u_2, u_3\}$
This distance would be 0, since the subspace is an orthogonal basis for $\mathbb{R}^3$. This also becomes evident when looking at assignment c), where x is expressed in terms of the subspace.

Assignment 5 (20%)

For a specific sensor, the following corresponding values of time t and output y have been measured

t	y
0	1
1	3
2	7

3	11
4	16
5	24

It is assumed that the system can be approximated by a model of the form $y_1(t) = \beta_0 + \beta_2 t^2$ or of the form $y_2(t) = \delta_0 + \delta_1 t + \delta_2 t^2$

a) Determine the design matrix and observation vector of both models

```
>> t=[0 1 2 3 4 5]'

t =

     0
     1
     2
     3
     4
     5

>> y=[1 3 7 11 16 24]'

y =

     1
     3
     7
    11
    16
    24

>> X1=[ones(6,1) t.^2]

X1 =

     1     0
     1     1
     1     4
     1     9
     1    16
     1    25

>> X2=[ones(6,1) t t.^2]

X2 =
```

```
     1     0     0
     1     1     1
     1     2     4
     1     3     9
     1     4    16
     1     5    25
```

The y vector is the observation vector for both models.

b) Determine the parameters of both models

```
X1tX1 =
     6    55
    55   979
>> X1ty=X1'*y
X1ty =
    62
   986
>> [X1tX1 X1ty]
ans =
     6    55    62
    55   979   986
>> rref(ans)
ans =
    1.0000         0    2.2703
         0    1.0000    0.8796
>> X2tX2=X2'*X2
X2tX2 =
     6    15    55
    15    55   225
    55   225   979
>> X2ty=X2'*y
X2ty =
    62
   234
   986
>> [X2tX2 X2ty]
ans =
     6    15    55    62
    15    55   225   234
    55   225   979   986
>> rref(ans)
ans =
    1.0000         0         0    1.0714
         0    1.0000         0    1.4786
         0         0    1.0000    0.6071
```

```
>> X1tX1 =
     6    55
    55   979
>> X1ty=X1'*y
X1ty =
    62
   986
>> [X1tX1 X1ty]
ans =
     6    55    62
    55   979   986
>> rref(ans)
ans =
    1.0000         0    2.2703
         0    1.0000    0.8796
>> X2tX2=X2'*X2
X2tX2 =
     6    15    55
    15    55   225
    55   225   979
>> X2ty=X2'*y
X2ty =
    62
   234
   986
>> [X2tX2 X2ty]
ans =
     6    15    55    62
    15    55   225   234
    55   225   979   986
>> rref(ans)
ans =

    1.0000         0         0    1.0714
         0    1.0000         0    1.4786
         0         0    1.0000    0.6071
```

$y_1(t) = 2.27 + 0.88t^2$
$y_2(t) = 1.07 + 1.48t + 0.61t^2$

c) Supply a substantiated answer to which model is the best fit for the measured data
In order to find the best fitted model, we need to compute the function values of models and then find the magnitude of the deviations/error. We can find the functional values by multiplying the design matrices with the vector that constitute the parameter.

```
>> x1= [2.2703 0.8796]'

x1 =

    2.2703
    0.8796

>> y1=X1*x1

y1 =

    2.2703
    3.1499
    5.7887
   10.1867
   16.3439
   24.2603

>> norm(y-y1)

ans =

    1.9877

>> x2=[1.0714 1.4786 0.6071]'

x2 =

    1.0714
    1.4786
    0.6071

>> y2=X2*x2

y2 =

    1.0714
    3.1571
    6.4570
   10.9711
   16.6994
   23.6419

>> norm(y-y2)

ans =
```

```
    0.9710
```

Since the second model yields the smallest error, it follows that this is the best fitted model of the two based on the provided data.

Assignment 6 (20%)

Let the matrix A be given by

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$$

a) Compute a full singular value decomposition of the matrix A

```
>> A=[1 2 2;1 2 2]'

A =

     1     1
     2     2
     2     2

>> AtA=A'*A

AtA =

     9     9
     9     9

>> eig(AtA)

ans =

     0
    18

>> AtA-18*eye(2)

ans =

    -9     9
     9    -9
```

```
>> rref(ans)

ans =

     1    -1
     0     0

>> v1=[1 1]'

v1 =

     1
     1

>> v1=v1/norm(v1)

v1 =

    0.7071
    0.7071

>> AtA-0*eye(2)

ans =

     9     9
     9     9

>> rref(ans)

ans =

     1     1
     0     0

>> v2=[-1 1]'

v2 =

    -1
     1

>> v2=v2/norm(v2)

v2 =
```

```
  -0.7071
   0.7071

>> u1=(1/sqrt(18))*A*v1

u1 =

    0.3333
    0.6667
    0.6667

>> u1'

ans =

    0.3333    0.6667    0.6667

>> rref(ans)

ans =

     1     2     2

>> u2=[-2 1 0]'

u2 =

    -2
     1
     0

>> u3=[-2 0 1]'

u3 =

    -2
     0
     1

>> u2=[-2 1 0]'

u2 =

    -2
     1
     0
```

```
>> u3=[-2 0 1]'

u3 =

    -2
     0
     1

>> u3=u3-proj(u3,u1)-proj(u3,u2)

u3 =

   -0.4000
   -0.8000
    1.0000

>> u2=u2/norm(u2)

u2 =

   -0.8944
    0.4472
    0.0000

>> u3=u3/norm(u3)

u3 =

   -0.2981
   -0.5963
    0.7454

>> V=[v1 v2]

V =

    0.7071   -0.7071
    0.7071    0.7071

>> S=[sqrt(18) 0;0 0;0 0]

S =

    4.2426         0
         0         0
```

```
         0         0

>> U=[u1 u2 u3]

U =

    0.3333   -0.8944   -0.2981
    0.6667    0.4472   -0.5963
    0.6667    0.0000    0.7454

>> U*S*V'

ans =

    1.0000    1.0000
    2.0000    2.0000
    2.0000    2.0000
```

b) How do the SVDs of A and A^T relate to each other (it is allowed to calculate the SVDs in Matlab in order to answer this question)

U and V are switched.