Optimización
Tarea 11

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1. Usa la ecuacion de Euler-Lagrange para buscar los extermos de las siguientes funcionales

   - $J[y] = \int_a^b \left(xy' + (y')^2\right) dx$
     Solución
     Sabemos que la ecuación de Euler-Lagrange está dada por
     $$\partial_y L + \frac{d}{dx}(\partial_{y'} L) = 0.$$
     Sea el Lagrangiano $L(x,y,y') = xy' + (y')^2$. Entonces, tenemos que
     $$\begin{aligned} &\partial_y L = 0, \\ &\partial_{y'} L = x + 2y' \quad \text{y} \\ &\frac{d}{dx}(\partial_{y'} L) = 1 + 2y''. \end{aligned}$$
     Luego, como $\frac{d}{dx}(x+2y') = 0$ se tiene que
     $$x + 2y' = c_1.$$
     Entonces,
     $$y' = \frac{c_1 - x}{2}$$
     $$\Rightarrow \boxed{y = \frac{c_1 x}{2} - \frac{x^2}{4} + c_2}$$

   - $J[y] = \int_a^b (1+x)(y')^2 \, dx$
     Solución
     Sea el Lagrangiano $L(x,y,y') = (1+x)(y')^2$. Entonces, tenemos que
     $$\begin{aligned} &\partial_y L = 0, \\ &\partial_{y'} L = 2(1+x)y' \quad \text{y} \\ &\frac{d}{dx}(\partial_{y'} L) = 2((1+x)y'' + y'). \end{aligned}$$
     Luego, por la ecuación de Euler-Lagrange sabemos que
     $$\frac{d}{dx}(2(1+x)y') = 0.$$
     Entonces,
     $$\begin{aligned} &2(1+x)y' = c_1 \\ \Rightarrow\ &y' = \frac{c_1}{2(1+x)}. \end{aligned}$$
     Por lo tanto,
     $$\boxed{y = \frac{c_1}{2} \ln(1+x) + c_2}$$

2. Derivar las ecuaciones de Euler Lagrange usando el Método de Lagrange de

$$\begin{array}{ll}\int_x \int_y & F(x,y,f,f_x,f_y)\,dxdy \\ \int_x \int_y & F(x,y,u,v,u_x,v_x,u_y,v_y)\,dxdy\end{array}$$

donde $f,u,v:\mathbb{R}^2 \to \mathbb{R}y$

$$f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}$$
$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}$$
$$v_x = \frac{\partial v}{\partial x}, \quad v_y = \frac{\partial v}{\partial y}$$

- $\int_x \int_y \quad F(x,y,f,f_x,f_y)\,dxdy$
  Solución
  Supongamos que $f = \hat{f}(x,y)$ resuelve el problema. Sea $h(x,y)$ una pequeña variación tal que

  $$f(x,y) = \hat{f}(x,y) + h(x,y),$$

  donde $h(x,y)$ en los extremos es igual a cero.
  Sabemos que las asunciones de Lagrange consideran pequeñas variaciones débiles, es decir

  $$h(x,y) = \varepsilon\eta(x,y),$$

  donde $h(x,y)$ y $h'(x,y)$ son del mismo orden y $\eta(x,y)$ es independiente de ε, con $\varepsilon \approx 0$. Luego,

  $$J(\varepsilon) := J[\hat{y} + \varepsilon\eta] = \int_y \int_x F(x,y,\hat{f} + \varepsilon\eta, \hat{f}_x + \varepsilon\eta_x, \hat{f}_y + \varepsilon\eta_y).$$

  La variación total es

  $$\begin{aligned}\Delta J &= J(\varepsilon) - J(0)\\ &= \int_y \int_x F(x,y,\hat{f} + \varepsilon\eta, \hat{f}_x + \varepsilon\eta_x, \hat{f}_y + \varepsilon\eta_y)dxdy - \int_y \int_x F(x,y,f,f_x,f)dxdy\\ &= \int_y \int_x \left[F(x,y,\hat{f} + \varepsilon\eta, \hat{f}_x + \varepsilon\eta_x, \hat{f}_y + \varepsilon\eta_y) - F(x,y,f,f_x,f)\right] dxdy.\end{aligned}$$

  Ahora, usando la expansión de Taylor alrededor de $\varepsilon = 0$ tenemos que

  $$J(\varepsilon) = J(0) + \left(\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}\right)\varepsilon + \frac{1}{2}\left(\left.\frac{d^2J}{d\varepsilon^2}\right|_{\varepsilon=0}\right)\varepsilon^2 + O(\varepsilon^3).$$

  Luego, de $\Delta J = J(\varepsilon) - J(0)$ se tiene que

  $$\Delta J = \left(\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}\right)\varepsilon + \frac{1}{2}\left(\left.\frac{d^2J}{d\varepsilon^2}\right|_{\varepsilon=0}\right)\varepsilon^2 + O(\varepsilon^3).$$

  Entonces,

  $$\begin{aligned}\left(\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}\right)\varepsilon &= \varepsilon \int_y \int_x \frac{d}{d\varepsilon} F(x,y,\hat{f} + \varepsilon\eta, \hat{f}_x + \varepsilon\eta_x, \hat{f}_y + \varepsilon\eta_y)dxdy\\ &= \varepsilon \int_y \int_x \left(\frac{\partial F}{\partial f}\frac{\partial f}{\partial \varepsilon} + \frac{\partial F}{\partial f_x}\frac{\partial f_x}{\partial \varepsilon} + \frac{\partial F}{\partial f_y}\frac{\partial f_y}{\partial \varepsilon}\right)dxdy\\ &= \varepsilon \int_y \int_x \left(\frac{\partial F}{\partial f}\eta + \frac{\partial F}{\partial f_x}\eta_x + \frac{\partial F}{\partial f_y}\eta_y\right)dxdy.\end{aligned}$$

  Utilizando la condición necesaria tenemos que

  $$\left(\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}\right)\varepsilon = 0.$$

  Entonces,

  $$\int_y \int_x \left(\frac{\partial F}{\partial f}\eta + \frac{\partial F}{\partial f_x}\eta_x + \frac{\partial F}{\partial f_y}\eta_y\right)dxdy = 0$$
  $$\int_y \int_x \frac{\partial F}{\partial f}\eta dxdy + \int_y \int_x \frac{\partial F}{\partial f_x}\eta_x dxdy + \int_y \int_x \frac{\partial F}{\partial f_y}\eta_y)dxdy = 0.$$

Por integración por partes, tomando a $u = \frac{\partial F}{\partial f_x}$, $du = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial f_x}\right)$ y $v = \eta$ tenemos

$$\int_x \frac{\partial F}{\partial f_x}\eta_x dx = \cancelto{0}{\frac{\partial F}{\partial f_x}\eta} - \int_x \eta \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial f_x}\right)dx.$$

Entonces,

$$\int_y \int_x \frac{\partial F}{\partial f_x}\eta_x dxdy = -\int_y \int_x \eta \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial f_x}\right)dxdy$$

Además, igualmente por integración por partes obtenemos

$$\begin{aligned}\int_y \int_x \frac{\partial F}{\partial f_y}\eta_y dxdy &= \int_x \int_y \frac{\partial F}{\partial f_y}\eta_y dydx \\ &= -\int_x \int_x \eta \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial f_y}\right)dydx.\end{aligned}$$

Por lo tanto, como

$$\begin{aligned}&\int_y \int_x \frac{\partial F}{\partial f}\eta dxdy + \int_y \int_x \frac{\partial F}{\partial f_x}\eta_x dxdy + \int_y \int_x \frac{\partial F}{\partial f_y}\eta_y)dxdy = 0 \\ \Rightarrow &\int_y \int_x \eta \left(\frac{\partial F}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial f_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial f_y}\right)\right)dxdy = 0\end{aligned}$$

Entonces, por el teorema fundamental de cálculo variacional obtenemos que

$$\boxed{\frac{\partial F}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial f_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial f_y}\right) = 0}$$

- $\int_x \int_y \quad F\left(x,y,u,v,u_x,v_x,u_y,v_y\right)dxdy$
  Solución
  Supongamos que $u = \hat{u}(x,y)$ y $v = \hat{v}(x,y)$ resuelven el problema. Sea la variación

$$\begin{aligned}u(x,y) &= \hat{u}(x,y) + h_1(x,y)\\ v(x,y) &= \hat{v}(x,y) + h_2(x,y),\end{aligned}$$

donde h_1, h_2 son cero en la frontera.
Las asunciones de Lagrange consideran pequeñas variaciones débiles, i.e.,

$$\begin{aligned}h_1(x,y) &= \varepsilon\eta_1(x,y),\\ h_2(x,y) &= \varepsilon\eta_2(x,y),\end{aligned}$$

tales que η_1 y η_2 son cero en la frontera y h_1, h_1' son del mismo orden, así como h_2, h_2'. η_1 y η_2 son independientes de $\varepsilon \approx 0$. Luego, definimos

$$\begin{aligned}J(\varepsilon) :&= J[\hat{u} + \varepsilon\eta_1, \hat{v} + \varepsilon\eta_2]\\ &= \int_y \int_x F\left(x,y,\hat{u}+\varepsilon\eta_1, \hat{v}+\varepsilon\eta_2, \hat{u_x}+\varepsilon\eta_{1x}, \hat{v_x}+\varepsilon\eta_{2x}, \hat{u_y}+\varepsilon\eta_{1y}, \hat{v_y}+\varepsilon\eta_{2y}\right)dxdy.\end{aligned}$$

Considerando que $\Delta J = J(\varepsilon) - J(0)$ y desarrollando la expansión de Taylor alrededor de ε al igual que en el inciso anterior obtenemos

$$\Delta J = \left(\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}\right)\varepsilon + \frac{1}{2}\left(\left.\frac{d^2 J}{d\varepsilon^2}\right|_{\varepsilon=0}\right)\varepsilon^2 + O(\varepsilon^3).$$

Entonces, por integración por partes obtenemos

$$
\begin{aligned}
\left(\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}\right)\varepsilon &= \varepsilon\int_y\int_x \frac{d}{d\varepsilon}F\left(x,y,\hat{u}+\varepsilon\eta_1,\hat{v}+\varepsilon\eta_2,\hat{u_x}+\varepsilon\eta_{1_x},\hat{v_x}+\varepsilon\eta_{2_x},\hat{u_y}+\varepsilon\eta_{1_y},\hat{v_y}+\varepsilon\eta_{2_y}\right)dxdy \\
&= \varepsilon\int_y\int_x\left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial\varepsilon}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial\varepsilon}+\frac{\partial F}{\partial u_x}\frac{\partial u_x}{\partial\varepsilon}+\frac{\partial F}{\partial v_x}\frac{\partial v_x}{\partial\varepsilon}+\frac{\partial F}{\partial u_y}\frac{\partial u_y}{\partial\varepsilon}+\frac{\partial F}{\partial v_y}\frac{\partial v_y}{\partial\varepsilon}\right)dxdy \\
&= \varepsilon\int_y\int_x\left(\frac{\partial F}{\partial u}\eta_1+\frac{\partial F}{\partial v}\eta_2+\frac{\partial F}{\partial u_x}\eta_{1_x}+\frac{\partial F}{\partial v_x}\eta_{2_x}+\frac{\partial F}{\partial u_y}\eta_{1_y}+\frac{\partial F}{\partial v_y}\eta_{2_y}\right)dxdy \\
&= \varepsilon\left[\int_y\int_x\frac{\partial F}{\partial u}\eta_1 dxdy+\int_y\int_x\frac{\partial F}{\partial v}\eta_2 dxdy+\int_y\int_x\frac{\partial F}{\partial u_x}\eta_{1_x}dxdy+\right. \\
&\left.+\int_y\int_x\frac{\partial F}{\partial v_x}\eta_{2_x}dxdy+\int_y\int_x\frac{\partial F}{\partial u_y}\eta_{1_y}dxdy+\int_y\int_x\frac{\partial F}{\partial v_y}\eta_{2_y}dxdy\right] \\
&= \varepsilon\left[\int_y\int_x\frac{\partial F}{\partial u}\eta_1 dxdy+\int_y\int_x\frac{\partial F}{\partial v}\eta_2 dxdy-\int_y\int_x\eta_1\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)dxdy-\right. \\
&\left.+\int_y\int_x\eta_2\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)dxdy-\int_y\int_x\eta_1\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)dxdy-\int_y\int_x\eta_2\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)dxdy\right].
\end{aligned}
$$

Usando la condición necesario de $\left(\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}\right)\varepsilon=0$ obtenemos

$$
\varepsilon\left[\int_y\int_x\eta_1\left(\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right)dxdy+\int_y\int_x\eta_2\left(\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)\right)dxdy\right]=0.
$$

Por lo tanto, por el teorema fundamental del cálculo variacional concluimos que

$$
\boxed{\left\{\begin{array}{rcl}
\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) &=& 0 \\
\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) &=& 0
\end{array}\right.}
$$

3. Obtener las ecuaciones de Euler-Lagrange de

$$
\int_x\int_y (f-g)^2+\lambda\|\nabla f\|^2\Big)\,dxdy
$$

$$
\int_x\int_y (p-q-p_xu-q_xv)^2+\lambda\left(\|\nabla u\|^2+\|\nabla v\|^2\right)dxdy
$$

donde $f,u,v:\mathbb{R}^2\to\mathbb{R}$ y $g,p,q:\mathbb{R}^2\to\mathbb{R}$ son funciones dadas.

- $\int_x\int_y\left((f-g)^2+\lambda(f_x^2+f_y^2)\right)dxdy$
  Solución
  Por el problema 2, tenemos que la ecuación de Euler-Lagrange es

$$
2f-2g-2\lambda\frac{\partial}{\partial x}(f_x)-2\lambda\frac{\partial}{\partial y}(f_y)=0
$$

$$
\Rightarrow \boxed{f-g-\lambda\left(\frac{\partial}{\partial x}(f_x)+\frac{\partial}{\partial y}(f_y)\right)=0}
$$

- $\int_x\int_y (p-q-p_xu-q_xv)^2+\lambda\left(u_x^2+u_y^2+v_x^2+v_y^2\right)dxdy$
  Solución
  Nuevamente, por el problema 2 encontramos que la ecuación de Euler-Lagrange es

$$
\boxed{\left\{\begin{array}{rcl}
-2p_x(p-q-p_xu-q_xv)-\frac{\partial}{\partial_x}(2\lambda u_x)-\frac{\partial}{\partial_y}(2\lambda u_y) &=& 0 \\
-2q_x(p-q-p_xu-q_xv)-\frac{\partial}{\partial_x}(2\lambda v_x)-\frac{\partial}{\partial_y}(2\lambda v_y) &=& 0
\end{array}\right.}
$$