Calculus Assignment 5

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1 Compute the following derivatives using the chain rule:

Answer:

$$\begin{aligned} z(t) &= e^{-\frac{t^2}{2}} \\ z'(t) &= -te^{-\frac{t^2}{2}} \end{aligned} \tag{1}$$

$$\begin{aligned} z(t) &= -(1-e^t)^2 \\ z'(t) &= 2(1-e^t)e^t \end{aligned} \tag{2}$$

$$\begin{aligned} z(t) &= e^{e^t} \\ z'(t) &= e^{e^t}e^t \end{aligned} \tag{3}$$

2 Consider the function $z(t) = 1/x(t)$. Using the chain rule, prove that:

$$z'(t) = (\frac{1}{x(t)})' = -\frac{x'(t)}{(x(t))^2} \tag{4}$$

Proof:

$$\begin{aligned} z(t) &= \frac{1}{x(t)} \\ z'(t) &= (\frac{1}{x(t)})' = ((x(t))^{-1})' = -(x(t))^{-2} * x'(t) = -\frac{x'(t)}{(x(t))^2} \end{aligned} \tag{5}$$

3 Using the result of the previous problem, derive the formula for the derivative of the ratio of two functions:

$$z'(t) = (\frac{x(t)}{y(t)})' = \frac{x'(t)y(t) - x(t)y'(t)}{(y(t))^2} \tag{6}$$

assuming $y(t)$ is not equal to zero

Proof:

$$\begin{aligned} z'(t) = (\frac{x(t)}{y(t)})' = (x(t)\frac{1}{y(t))})' = x'(t)\frac{1}{y(t)} + x(t)(\frac{1}{y(t)})' = \\ = \frac{x'(t)}{y(t)} - x(t)\frac{y'(t)}{(y(t))^2} = \frac{x'(t)y(t)}{(y(t))^2} - \frac{x(t)y'(t)}{(y(t))^2} = \frac{x'(t)y(t) - x(t)y'(t)}{(y(t))^2} \end{aligned} \tag{7}$$

4 Compute the derivative of the following functions:

$$\begin{aligned} z(t) &= \frac{e^t}{t^2} \\ z'(t) &= \frac{t^2e^t - 2te^t}{t^4} = \frac{(t-2)e^t}{t^3} \end{aligned} \tag{8}$$

$$\begin{aligned} z(t) &= \frac{ln(t+2)}{e^t} \\ z'(t) &= \frac{\frac{e^t}{t+2} - ln(t+2)e^t}{e^{2t}} = \frac{1-(t+2)ln(t+2)}{(t+2)e^t} \end{aligned} \tag{9}$$

$$\begin{aligned} z(t) &= \frac{3^{-t^2}}{(1+t^2)} \\ z'(t) &= \frac{-2ln(3)t3^{-t^2}(1+t^2) - 2t3^{-t^2}}{(1+t^2)^2} = \frac{-2(t^2ln(3)+ln(3)+1)3^{-t^2}t}{(1+t^2)^2} \end{aligned} \tag{10}$$

$$\begin{aligned} z(t) &= \frac{e^t}{(1+e^t)} \\ z'(t) &= \frac{(1+e^t)e^t - e^te^t}{(1+e^t)^2} = \frac{e^t}{(e^t+1)^2} \end{aligned} \tag{11}$$

5 Using the chain rule twice, show that:

$$(z \circ y \circ x)'(t) = z'(y(x(y))) * y'(x(t)) * x'(t) \tag{12}$$

Proof:

$$(z \circ y \circ x)'(t) = (z(y(x(t))))' = z'(y(x(t))) * (y(x(t)))' = z'(y(x(t))) * y'(x(t)) * x'(t) \tag{13}$$

6 Compute the derivatives of the following functions:

$$\begin{aligned} z(t) &= ln((e^t + 2)^2) \\ z'(t) &= \frac{1}{(e^t+2)^2} * 2(e^t+2) * e^t = \frac{2e^t}{e^t+2} \end{aligned} \tag{14}$$

$$\begin{aligned} z(t) &= \frac{1}{(e^{t^3+t} + 3t)^3} \\ z'(t) &= -3\frac{1}{(e^{t^3+t}+3t)^4} * ((3t^2+1)e^{t^3+t} + 3t)' = \\ &= -3\frac{(3t^2+1)e^{t^3+t}+3}{(e^{t^3+t}+3t)^4} \end{aligned} \tag{15}$$

7 Compute all the partial derivatives of the following functions:

$$\begin{aligned} f(x,y) &= xe^{xy} \\ \frac{\partial f}{\partial x} &= (xy+1)e^{xy} \\ \frac{\partial f}{\partial y} &= x^2 e^{xy} \end{aligned} \tag{16}$$

$$\begin{aligned} f(x,y,z) &= x^3 - y^2 + 3z \\ \frac{\partial f}{\partial x} &= 3x^2 \\ \frac{\partial f}{\partial y} &= 2y \\ \frac{\partial f}{\partial z} &= 3 \end{aligned} \tag{17}$$

$$\begin{aligned} f(x,y) &= x^y \\ \frac{\partial f}{\partial x} &= yx^{y-1} \\ \frac{\partial f}{\partial y} &= ln(x)x^y \end{aligned} \tag{18}$$

8 Consider an artificial neuron with two inputs:

$$f(x_1, x_2) = \sigma(w_1 x_1 + w_2 x_2) \tag{19}$$

Part I: Write down a loss function to train the neuron to give a value of $\frac{1}{3}$ when $x_1 = \frac{1}{2}$ and $x_2 = -3$.

$$L(w_1, w_2) =? \tag{20}$$

Answer: If we want to find a loss function that satisfies the constraint the only requirement is that when:

$$\begin{aligned}
&f(\frac{1}{2}, -3) = \frac{1}{3} \longrightarrow L(w_1, w_2) = 0. \\
&L(w_1, w_2) + \frac{1}{3} = \frac{1}{3} \\
&L(w_1, w_2) + \frac{1}{3} = f(\frac{1}{2}, -3) \\
&L(w_1, w_2) = f(\frac{1}{2}, -3) - \frac{1}{3} = \sigma(\frac{w_1}{2} - 3w_2) - \frac{1}{3}
\end{aligned} \tag{21}$$

To complete the loss we square it and divide by half:

$$L(w_1, w_2) = \frac{1}{2}(\sigma(\frac{w_1}{2} - 3w_2) - \frac{1}{3})^2 \tag{22}$$

Part II: Compute the partial derivatives of this loss function with respect to the weights.

$$\begin{aligned}
\frac{\partial L}{\partial w_1} &= (\sigma(\frac{w_1}{2} - 3w_2) - \frac{1}{3}) * \sigma'(\frac{w_1}{2} - 3w_2) * \frac{1}{2} \\
\frac{\partial L}{\partial w_2} &= (\sigma(\frac{w_1}{2} - 3w_2) - \frac{1}{3}) * \sigma'(\frac{w_1}{2} - 3w_2) * (-3)
\end{aligned} \tag{23}$$