# Applied math exercise 3

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Exercise 3.5. In some communication system bits like 0 and 1 are transmitted in several steps. In every step the probability that the bit stays correct, is equal to 0.8.

(a) Describe this communication system as a Markov process, and give its state diagram and transition matrix.

Answer: S_1 **means the bit is transmitted correctly.** S_2 **means the bit is transmitted incorrectly.**
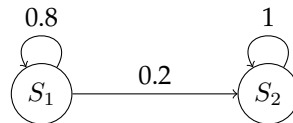
$$P = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ 0 & 1 \end{pmatrix} \tag{1}$$

(b) Determine the probability that the bit 0 gets received as 0 after 4 steps.

Answer: Since the state S_2 is absorbing that means the only way to receive 0 after 4 steps is $S_1 S_1 S_1 S_1$ the probability following this path is $(0.8)^4 = \frac{256}{625} = 0.4096$

Exercise 3.6. Some mathematician travels every morning to the university and every afternoon to his house. He has one umbrella. If it is dry, he doesn't bring his umbrella with him. Otherwise he does take hise umbrella with him if possible (if for example the umbrella is at the university when he leaves home, he obviously cannot use his umbrella). The probability that it rains in the morning is 0.2, while it rains in the afternoon with probability 0.1. This situation can be described by a Markov chain. We distinguish the following four states:

S_1 : Mathematician at home, umbrella at home.
S_2 : Mathematician at home, umbrella at university.
S_3 : Mathematician at university, umbrella at home.
S_4 : Mathematician at university, umbrella at university.

(2)

Every trip of the mathematician corresponds to a transition between two of these states.

(a) Give the state diagram of this Markov model. Put S1 and S4 next to each other.

Answer:
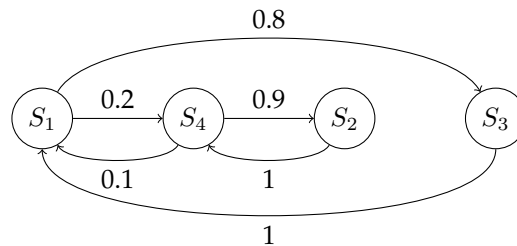
(b) Determine the transition matrix P.

Answer:

$$P = \begin{pmatrix} 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 \end{pmatrix}$$

(c) Suppose that on some Monday evening (t = 1) both the mathematician and his umbrella are at home. Compute the probability that the mathematician and his umbrella are at home on Wednesday evening (t = 5).

Answer:

$$P(X_5 = S_1 | X_1 = S_1) = P_{11}^{(4)}$$

$$P^4 = \begin{pmatrix} 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 \end{pmatrix}^4 = \begin{pmatrix} 0.69 & 0.31 & 0 & 0 \\ 0.17 & 0.83 & 0 & 0 \\ 0 & 0 & 0.66 & 0.34 \\ 0 & 0 & 0.14 & 0.86 \end{pmatrix} \qquad (3)$$
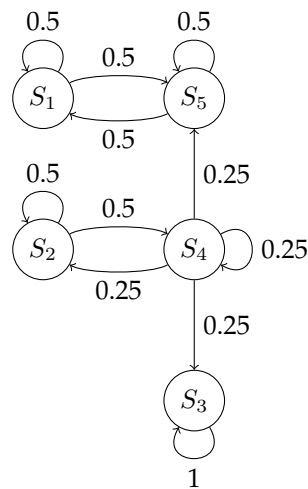
$$P_{11}^{(4)} = 0.69$$

Exercise 3.9.

(a) Identify the communicating classes of the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{4}$$

Answer: Based on the diagram of the Markov chain the communicating classes are $\{S_1, S_5\}, \{S_2, S_4\}, \{S_3\}$

(b) Specify in each case which classes are recurrent and which are transient. Are there any absorbing states?

Answer: Since the class $\{S_3\}$ has no outward edges, then the class is closed. From that, we can directly conclude that $\{S_3\}$ is also recurrent and S_3 is absorbing. Analogously For the class $\{S_1, S_5\}$ it holds that it is closed/recurrent and both S_1 and S_5 are absorbing states. For the last class $\{S_2, S_4\}$ it is neither closed nor recurrent since there are edges that flow into closed classes, and that makes S_2, S_4 transient states as a result.

Exercise 4.8. A flea moves around the vertices of a triangle in the following manner: whenever it is at vertex $S_i (i = 1, 2, 3)$ it moves to its clockwise neighbour vertex with probability $p_i = \frac{1}{2^i}$ and to the counterclockwise neighbour with probability $q_i = 1 - p_i$. Find the proportion of time that the flea is at each of the vertices.

Answer: From the definition of this markov chain it is evident that it is irreducible as from every state we can reach every other state.
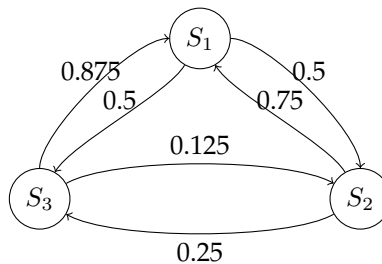
since X is an irreducible markov chain that means it has a stationary distribution $\pi P = \pi$

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.75 & 0 & 0.25 \\ 0.875 & 0.125 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} * \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.75 & 0 & 0.25 \\ 0.875 & 0.125 & 0 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

$$\begin{pmatrix} 0.75\pi_2 + 0.875\pi_3 \\ 0.5\pi_1 + 0.125\pi_3 \\ 0.5\pi_1 + 0.25\pi_2 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

$$\begin{pmatrix} -\pi_1 + 0.75\pi_2 + 0.875\pi_3 \\ 0.5\pi_1 - \pi_2 + 0.125\pi_3 \\ 0.5\pi_1 + 0.25\pi_2 - \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\pi_1 + 0.75\pi_2 + 0.875\pi_3 \\ \pi_1 - 2\pi_2 + 0.25\pi_3 \\ \pi_1 + 0.5\pi_2 - 2\pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\pi_1 + 0.75\pi_2 + 0.875\pi_3 \\ -1.25\pi_2 + 1.125\pi_3 \\ 1.25\pi_2 - 1.125\pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\pi_1 + 0.75\pi_2 + 0.875\pi_3 \\ -1.25\pi_2 + 1.125\pi_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

but $\pi_1 + \pi_2 + \pi_3 = 1$

$$\begin{pmatrix} -\pi_1 + 0.75\pi_2 + 0.875\pi_3 \\ -1.25\pi_2 + 1.125\pi_3 \\ 0 \\ \pi_1 + \pi_2 + \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -\pi_1 + 0.75\pi_2 + 0.875\pi_3 \\ -1.25\pi_2 + 1.125\pi_3 \\ 0 \\ 1.75\pi_2 + 1.875\pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -\pi_1 + 0.75\pi_2 + 0.875\pi_3 \\ -8.75\pi_2 + 7.875\pi_3 \\ 0 \\ 8.75\pi_2 + 9.375\pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -\pi_1 + 0.75\pi_2 + 0.875\pi_3 \\ -1.25\pi_2 + 1.125\pi_3 \\ 0 \\ 17.25\pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

$$\pi_3 = \frac{5}{17.25} = \frac{20}{69}$$

$$-\frac{5}{4}\pi_2 + \frac{9}{8}\frac{20}{69} = 0$$

$$\pi_2 = \frac{20*9*4}{69*8*5} = \frac{6}{23}$$

$$-\pi_1 + \frac{3}{4}\frac{6}{23} + \frac{7}{8}\frac{20}{69} = 0 \tag{6}$$

$$\pi_1 = \frac{9}{46} + \frac{35}{138}$$

$$\pi_1 = \frac{27}{138} + \frac{35}{138} = \frac{62}{138} = \frac{31}{69}$$

$$\pi = (\frac{31}{69}, \frac{6}{23}, \frac{20}{69})$$

π **is the Markov chain's stationary distribution meaning it represents the proportion of time the flea spends at every vertex.**

Exercise 4.14. Three people have a discussion about the upcoming elections. Everyone prefers one of two political parties, which we will abbreviate by V and W. We describe this by a Markov chain with four states $(S_0, S_1, S_2$ and $S_3)$. In state S_i, W has i followers. If a faction has only one follower, then he has probability $\frac{1}{3}$ to convince another person to vote for that faction, while with probability $\frac{2}{3}$ he himself gets persuaded to vote for the other political party. As soon as everyone has the same opinion, the discussion ends. At time t = 0 the political party W has one follower.

(a) Draw the state diagram of this Markov chain, and give its transition matrix.
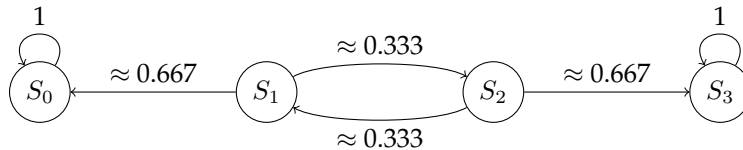
Answer:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

(b) Give the probability distribution at t = 3.

Answer: The probability distribution at time t = 3 is dependant on the initial distribution. Since the initial distribution isn't given the resulting distribution will be a function of the initial distribution.

	t = 0	t = 1	t = 2	t = 3
S_0	π_0	$\pi_0 + \frac{2}{3}\pi_1$	$\pi_0 + \frac{2}{3}\pi_1 + \frac{2}{9}\pi_2$	$\pi_0 + \frac{20}{27}\pi_1 + \frac{2}{9}\pi_2$
S_1	π_1	$\frac{1}{3}\pi_2$	$\frac{1}{9}\pi_1$	$\frac{1}{27}\pi_2$
S_2	π_2	$\frac{1}{3}\pi_1$	$\frac{1}{9}\pi_2$	$\frac{1}{27}\pi_1$
S_3	π_3	$\pi_3 + \frac{2}{3}\pi_2$	$\pi_3 + \frac{2}{3}\pi_2 + \frac{2}{9}\pi_1$	$\pi_3 + \frac{20}{27}\pi_2 + \frac{2}{9}\pi_1$

Table 1: Visualization of P(T | C).

(c) Which of the states are absorbing?

Answer: Absorbing state is a state which once entered cannot be left, in this case the state S_i has $P_{ii}^{(t)} = 1$, meaning once entered in can only stay in S_i. In this case those would be the states S_0, S_3.

(d) What is the probability to eventually end in state S_3?

Answer: Since $\{S_3\}$ is a closed class that means that once we enter we cannot leave so all we need to do is calculate the probability from each state that we enter S_3 once.

$$h_{0,3} = 0 \text{ since } S_0 \text{ is closed}$$

$$h_{1,3} = \frac{2}{3}h_{0,3} + \frac{1}{3}h_{2,3}$$

$$h_{2,3} = \frac{2}{3}h_{3,3} + \frac{1}{3}h_{1,3}$$

$$h_{3,3} = 1$$

$$h_{1,3} = \frac{1}{3}h_{2,3}$$

$$h_{2,3} = \frac{2}{3} + \frac{1}{3}h_{1,3}$$

$$h_{2,3} = \frac{2}{3} + \frac{1}{9}h_{2,3}$$

$$\frac{8}{9}h_{2,3} = \frac{2}{3} \tag{8}$$

$$h_{2,3} = \frac{3}{4}$$

$$h_{1,3} = \frac{1}{3}\frac{3}{4}$$

$$h_{1,3} = \frac{1}{4}$$

All of these probabilities assume we start at that state.

The total probability is the weighted sum with the initial distribution.

$$P(\text{ending in } S_3) = \pi_0 h_{0,3} + \pi_1 h_{1,3} + \pi_2 h_{2,3} + \pi_3 h_{3,3} =$$

$$= \frac{1}{4}\pi_1 + \frac{3}{4}\pi_2 + \pi_3$$

Something that should be noted is that this probability is entirely dependent on the initial distribution since the Markov chain is reducible. So without an initial distribution, an exact answer cannot be given.