Calculus Assignment 4

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1 Compute the following local linear approximations:

$$
\begin{aligned}
&x(t) = t^2 \text{ for } t_0 = 1 \\
&x'(t) = 2t \\
&\widehat{x}(t) = x'(t_0)(t - t_0) + x(t_0) = 2t_0t - 2{t_0}^2 + {t_0}^2 = 2t - 1 \\
\\
&x(t) = t^2e^t \text{ for } t_0 = 1 \\
&x'(t) = (2t + t^2)e^t \\
&\widehat{x}(t) = x'(t_0)(t - t_0) + x(t_0) = (2t_0 + {t_0}^2)e^{t_0}t- \\
&\qquad - (2t_0 + {t_0}^2)e^{t_0}t_0 + {t_0}^2e^{t_0} = 3et - 2e \\
\\
&x(t) = 2te^t \text{ for } t_0 = 0 \\
&x'(t) = (t + 1)2e^t \\
&\widehat{x}(t) = x'(t_0)(t - t_0) + x(t_0) = (t_0 + 1)2e^{t_0}t + {t_0}^2e^{t_0} = 2t
\end{aligned}
\tag{1}
$$

2 Compute the second derivatives of the following functions:

$$x(t) = 5t$$
$$x''(t) = 0$$

$$x(t) = t^2 + 2t$$
$$x''(t) = 2$$

$$x(t) = te^t + 2t^2$$
$$x''(t) = (e^t + te^t + 4t)' = 2e^t + te^t + 4 = (t+2)e^t + 4$$

$$\begin{aligned} x(t) &= t^3 3^t \\ x''(t) &= (3t^2 3^t + ln(3)t^3 3^t)' = 6t3^t + 3ln(3)t^2 3^t + 3ln(3)t^2 3^t + ln(27)t^3 3^t = \\ &= (6 + 6ln(3)t + ln(27)t^2)t3^t \end{aligned} \tag{2}$$

3 Use the derivative and the second derivative to compute the local quadratic approximation of the following function:

$$x(t) = t^3 e^t \tag{3}$$

for $t_0 = -1$. Plot the function, its local linear approximation and its local quadratic approximation use Geogebra.

Answer:

$$\begin{aligned}
x'(t) &= 3t^2e^t + t^3e^t \\
x''(t) &= (6 + 6t + t^2)te^t \\
\widehat{x}(t) &= \frac{1}{2}x''(t_0)(t - t_0)^2 + x'(t_0)(t - t_0) + x(t_0) = \\
&= \frac{1}{2}(6 + 6t_0 + {t_0}^2)t_0e^{t_0}(t^2 - 2tt_0 + {t_0}^2) + (3{t_0}^2e^{t_0} + {t_0}^3e^{t_0})(t - t_0) + {t_0}^3e^{t_0} = \\
&= -\frac{t^2 + 2t + 1}{2e} + \frac{3t + 3}{e} - \frac{t + 1}{e} - \frac{1}{e} = \\
&= -\frac{t^2 + 2t + 1 - 6t - 6 + 2t + 2 + 2}{2e} = \\
&= -\frac{t^2 - 2t - 1}{2e} = -\frac{1}{2e}t^2 + \frac{1}{e}t + \frac{1}{2e}
\end{aligned} \tag{4}$$

4 Determine the range of input values where the following function is decreasing.

$$x(t) = t^3 - t \tag{5}$$

Answer:

$$\begin{aligned}
x'(t) = 3t^2 - 1 &< 0 \\
t^2 - \frac{1}{3} &< 0 \\
(t - \frac{1}{\sqrt{3}})(t + \frac{1}{\sqrt{3}}) &< 0 \\
t \in (-\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}})&
\end{aligned} \tag{6}$$

5 Find all the critical points of the following function:

Answer:

$$\begin{aligned}
&x(t) = t - t^3 \\
&x'(t) = 1 - 3t^2 = 0 \\
&\frac{1}{\sqrt{3}} - t^2 = 0 \\
&(t - \frac{1}{\sqrt{3}})(t + \frac{1}{\sqrt{3}}) = 0 \\
&\text{Critical points: } A = (-\frac{1}{\sqrt{3}}, -\frac{2\sqrt{3}}{9}), B = (\frac{1}{\sqrt{3}}, \frac{2\sqrt{3}}{9})
\end{aligned} \tag{7}$$

$$\begin{aligned}
&x(t) = t^2 - t^4 \\
&x'(t) = 2t - 4t^3 = 0 \\
&-(2t^2 - 1)2t = 0 \\
&(t - \frac{1}{\sqrt{2}})(t + \frac{1}{\sqrt{2}})2t = 0 \\
&\text{Critical points: } A = (-\frac{1}{\sqrt{2}}, \frac{1}{4}), B = (0,0), C = (\frac{1}{\sqrt{2}}, \frac{1}{4})
\end{aligned} \tag{8}$$

$$\begin{aligned}
&x(t) = t^2 e^t \\
&x'(t) = (t+2)te^t = 0 \\
&\text{Critical points: } A = (-2, \frac{4}{e^2}), B = (0,0)
\end{aligned} \tag{9}$$

6 Use the second derivative to check if the critical points you found in the previous exercise are points of minimum or points of maximum.

Answer:

$$\begin{aligned}
&x(t) = t - t^3 \\
&x'(t) = 1 - 3t^2 \\
&x''(t) = -6t \\
&x''(-\frac{1}{\sqrt{3}}) = 2\sqrt{3} > 0 \text{ meaning point A is local minimum} \\
&x''(\frac{1}{\sqrt{3}}) = -2\sqrt{3} < 0 \text{ meaning point C is local maximum}
\end{aligned} \tag{10}$$

$$
\begin{aligned}
&x(t) = t^2 - t^4 \\
&x'(t) = 2t - 4t^3 \\
&x''(t) = 2 - 12t^2 \\
&x''(-\frac{1}{\sqrt{2}}) = -4 < 0 \text{ meaning point A is local maximum} \\
&x''(0) = 2 > 0 \text{ meaning point B is local minimum} \\
&x''(\frac{1}{\sqrt{2}}) = -4 < 0 \text{ meaning point C is local maximum}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&x(t) = t^2 e^t \\
&x'(t) = (t+2)te^t \\
&x''(t) = (t+1)2e^t + (t+2)te^t = (t^2 + 4t + 2)e^t \\
&x''(-2) = -\frac{2}{e^2} < 0 \text{ meaning point A is local maximum} \\
&x''(0) = 2 > 0 \text{ meaning point B is local minimum}
\end{aligned} \tag{12}
$$