Processor modelling using queueing theory

Erik Leonards

April 19, 2023

Bachelor thesis Mathematics and Computer Science

Supervisor: prof. dr. (Rude)sindo Núñez Queija, prof. dr. Anuj Pathania

Informatics Institute
Korteweg-de Vries Institute for Mathematics
Faculty of Sciences
University of Amsterdam

Abstract

Schrijf een samenvatting van hoogstens een halve bladzijde waarin je kort uitlegt wat je hebt gedaan. De samenvatting schrijf je als laatste. Je mag er vanuit gaan dat je docent de lezer is.

Title: Processor modelling using queueing theory
Authors: Erik Leonards, erik.leonards@student.uva.nl, 13170007
Supervisors: prof. dr. (Rude)sindo Núñez Queija, prof. dr. Anuj Pathania
Second grader: dr. Florian Speelman, prof. dr.
End date: April 19, 2023

Informatics Institute
University of Amsterdam
Science Park 904, 1098 XH Amsterdam
`http://www.ivi.uva.nl`

Korteweg-de Vries Institute for Mathematics
University of Amsterdam
Science Park 904, 1098 XH Amsterdam
`http://www.kdvi.uva.nl`

Contents

1. Introduction

Vertel iets over je artikel, noem je vraagstelling. Richt je tekst op medestudenten. (Richtlijn 2 bladzijden).

2. The processor

2.1. CPU architecture

2.2. Task simulation software

3. Queueing theory and Markov chains

3.1. Continuous Markov chains

3.2. A queueing network

3.2.1. The customers

A closed queueing network comsists of a finite number of service centers, a finite number of different classes of customers and a finite number of customers of each class. Let N be the number of service centers and let R be the number of different classes of customers. Moreover, for $1 \leq r \leq R$ let N_r be the number of customers of class r.

The customers travel through the network according to transition probabilities. Those transition probabilities are equal for all customers within the same class. Let $p_{i,j}^{(r)}$ denote the transition probability of a customer of class r to travel from service center i to j. Notice how the sequence of service centers being visited by a customer forms a Markov chain, because the next service center only depends on the current service center. This means that the path a customer takes is a Markov chain. Let

$$\phi := \{(i, r) \ : \ 1 \leq i \leq N, 1 \leq r \leq R\},$$

where i denotes the service center and r denotes the class of the customer.

3.2.2. The service centers

The service center is a queue with the First In First Out (FIFO) queueing policy. Each service center is determined by its capacity and service rate. The capacity denotes the number of customers a service center can serve at once. A service rate of $\mu \in \mathbb{R}_{>0}$ means that the service times of the customers being served are $E(\mu)$ (exponentially with parameter μ) distributed. Notice how the service times distributions are equal for all classes. For $1 \leq i \leq N$ let $C_i \in \mathbb{Z}_{>0}$ and μ_i be respectively the capacity and service rate of service center i. Now define

$$\phi_i(k) := \min\{C_i, k\}\, \mu_i,$$

to be the total rate at which customers leave service center i.

3.2.3. The state of the model

For a network with N service centers the state of the network is denoted by $x = (x_1, x_2, \ldots, x_N)$, where x_i denotes the state of service center i. The state x_i of service center i is denoted with $x_i = (x_{i,1}, x_{i,2}, \dots, x_{i,n_i})$ where n_i is the number of customers at service center i and $x_{i,k}$ is the class of the customer at the k-th position in the queue. Formally, the state space is given by

$$\psi \coloneqq \left\{ (x_1, \dots, x_N) \ : \ x_i \in \{1, \dots, R\}^{n_i}, \ \sum_{i=1}^{N} \sum_{k=1}^{n_i} \delta_{r, x_{i,k}} = N_r \right\},$$

where the second condition ensures that only N_r number of class r customers are in the system and

$$\delta_{i,j} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases},$$

denotes the Dirac delta function.

It is important to observe that the queue contains both the waiting and served customers. To be more precise, the customers $x_{i,1}, x_{i,2}, \dots, x_{i,\min\{C_i, n_i\}}$ are being served by service center i with rate μ_i. Furthermore, the next service center of a customer is determined by the transition probabilities $p_{i,j}^{(r)}$. This means that the state space and transition probabilities induce transition rates between the states.

3.2.4. Aggregate states

In research it might only be interesting to know the number of customers of a certain class at a certain service center. This is what aggregate states describe. An aggregate state is denoted by $x^{(A)} \in \mathbb{N}^{N \times R}$ where $x_{i,r}^{(A)}$ denotes the number of class r customers at service center i. The aggregate state space is then formally defined by

$$\psi^{(A)} \coloneqq \left\{ x^{(A)} \in \mathbb{N}^{N \times R} \ : \ \sum_{i=1}^{N} x_{i,r}^{(A)} = N_r, 1 \leq i \leq N \right\}.$$

For a state in $\psi^{(A)}$ we will use n_i to denote the total number of customers at service center i, i.e. we have

$$n_i = \sum_{r=1}^{N} x_{i,r}^{(A)}.$$

3.2.5. The model as continuous Markov chain

As observed in Section 3.2.3, the state space, transition probabilities $p_{i,j}^{(r)}$, capacities C_i and rates μ_i induce transition rates between the states. To be more precise, a customer from class r being served by service center i induces a rate of $p_{i,j}^{(r)} \mu_i$ between the current state and the state in which this customer is removed from queue x_i and added at the end of queue x_j. Furthermore, note how the transition rates between states in ψ is zero if two or more customers travel at once between service centers. This is due to the fact that two or more customers travelling at once has zero probability of occurring. Let $Q = (q_{xy} : x, y \in \psi)$ be the transition rate matrix.

3.2.6. The invariant distribution

Let $\pi = (\pi_x : x \in \psi)$ be the invariant distribution. This distribution can be found by solving the balance equations

$$\pi_x \sum_{y \in \psi} q_{xy} = \sum_{y \in \psi} \pi_y q_{yx}, \quad x \in \psi.$$

Informally, these equations equate the total rate of outflow to the total rate of inflow.

Before presenting the solution to the balance equations, we must first introduce the *one customer invariant measure terms.*

Definitie 3.2.1. *The one customer invariant measure terms are given by*

$$e = (e_{i,r} : 1 \leq i \leq N, 1 \leq r \leq R),$$

which satisfy the equations

$$\sum_{j=1}^{N} e_{j,r} p_{j,i}^{(r)} = e_{i,r}. \tag{3.1}$$

Notice how every class of customers has its own set of equations and there could be multiple solutions for each class. Also observe that for every class r normalising $e_{i,r}$ will give you an invariant distribution for the continuous Markov chain induced by the case of $N_r = 1$ and $N_s = 0$ for all $r \neq s$. This corresponds to the scenario of letting one class r customer travel over the network. The customer will always be served and will never be waiting, so this will give you a continuous Markov chain.

Theorem 3.2.2 gives the general solution to the balance equations. The theorem and its proof originates from [Baskett et al., 1975].

Theorem 3.2.2. *For a closed queueing network with service centers exhibiting the FIFO server discipline, it holds that the invariant distribution $\pi = (\pi_x : x \in \psi)$ is given by*

$$\pi_x = C \prod_{i=1}^{N} \prod_{k=1}^{n_i} \frac{e_{i,x_{i,k}}}{\min\{C_i, k\}\, \mu_i}, \tag{3.2}$$

where C is a normalising constant such that $\sum_{x \in \psi} \pi_x = 1$.

Proof. The theorem will be proved by checking the balance equations.

As mentioned in Section 3.2.5, it holds for a state $x \in \psi$ that the only non-zero transition rates q_{xy} correspond to the transition of exactly one customer moving to another service center. This gives reason for to define

$$F_x \coloneqq \left\{(k, i, j) \in \mathbb{N}^3 \ : \ 1 \leq k \leq \min\{C_i, n_i\}\right\}, \quad x \in \psi$$

the set of all non-zero probability transitions from state x. Here $(k, i, j) \in F_x$ corresponds to a customer at the k-th position of service center i to move to service center j. Notice how k must be less than both C_i and n_i, because those are the only customers that are served by service center i when the system is in state x.

Now for $x \in \psi$ and $(k, i, j) \in F_x$ let $y \in \psi$ correspond to the new state of the system when the k-th customer of service center i moves to service center i. Notice that we have

$$q_{xy} = p_{i,j}^{(x_{i,k})} \mu_i.$$

The left hand side of the balance equations can now be calculated by summing over all elements of F_x and for $x \in \psi$ we obtain

$$\begin{aligned}
\pi_x \sum_{y \in \psi} q_{xy} &= \pi_x \sum_{i=1}^{N} \sum_{k=1}^{\min\{C_i, n_i\}} \sum_{j=1}^{N} \mu_i p_{i,j}^{(x_{i,k})} \\
&= \pi_x \sum_{i=1}^{N} \mu_i \sum_{k=1}^{\min\{C_i, n_i\}} \sum_{j=1}^{N} p_{i,j}^{(x_{i,k})} \\
&= \pi_x \sum_{i=1}^{N} \mu_i \sum_{k=1}^{\min\{C_i, n_i\}} 1 \\
&= \pi_x \sum_{i=1}^{N} \mu_i \min\{C_i, n_i\},
\end{aligned}$$

which corresponds to the total flow out of center i.

Similarly to F_x, we can define

$$T_x \coloneqq \left\{(k, j, i) \in \mathbb{N}^3 \ : \ 1 \leq k \leq \min\{C_j, n_j + 1\}, n_i \geq 1\right\},$$

as the set of all non-zero probability transitions that result in state x. Here (k, j, i) corresponds to the k-th customer of service center j moving to service center i such that the resulting state is x. Notice how $n_i \geq 1$ is a necessary condition, because a customer cannot move from center j to center i in order to arrive at state x if there is no customer at center i in state x.

Now for $x \in \psi$ and $(k, j, i) \in T_x$ let $y \in \psi$ correspond to the state such that the k-th customer moving center j to center i results in state x. Then

$$q_{yx} = p_{j,i}^{(r)} \mu_j,$$

and notice that in state y there is one more customer in center j and one less in center i, thus

$$\pi_y = \pi_x \frac{e_{j,r}}{\min\{C_j, n_j + 1\} \mu_j} \frac{1}{\frac{e_{i,r}}{\min\{C_i, n_i\} \mu_i}} = \pi_x \frac{e_{j,r}}{\min\{C_j, n_j + 1\} \mu_j} \frac{\min\{C_i, n_i\} \mu_i}{e_i, r},$$

where n_i, n_j correspond to the number of customers in state x. The right hand side of the balance equations can now be calculated by summing over all elements of T_x and for

$x \in \psi$ we obtain

$$\begin{aligned}
\sum_{y\in\psi} \pi_y q_{yx} &= \pi_x \sum_{i=1,n_i\geq 1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{\min\{C_j,n_j+1\}} \frac{e_{j,x_{i,n_i}}}{\min\{C_j, n_j+1\}\mu_j} \frac{\min\{C_i,n_i\}\mu_i}{e_i, r} p_{j,i}^{(x_{i,n_i})} \mu_j \\
&= \pi_x \sum_{i=1}^{N} \frac{\min\{C_i,n_i\}\mu_i}{e_{i,x_{i,n_i}}} \sum_{j=1}^{N} \frac{e_{j,x_{i,n_i}}}{\min\{C_j,n_j+1\}} p_{j,i}^{(x_{i,n_i})} \sum_{k=1}^{\min\{C_j,n_j+1\}} 1 \\
&= \pi_x \sum_{i=1}^{N} \frac{\min\{C_i,n_i\}\mu_i}{e_{i,x_{i,n_i}}} \sum_{j=1}^{N} e_{j,x_{i,n_i}} p_{j,i}^{(x_{i,n_i})} \\
&= \pi_x \sum_{i=1}^{N} \frac{\min\{C_i,n_i\}\mu_i}{e_{i,x_{i,n_i}}} e_{i,x_{i,n_i}} \\
&= \pi_x \sum_{i=1}^{N} \min\{C_i,n_i\}\mu_i,
\end{aligned}$$

which is equal to the left hand side of the balance equations. This implies that $\pi = (\pi_x : x \in \psi)$ is a invariant distribution. □

The invariant distribution of Equation 3.2 enables us to calculate the equilibrium state probabilities of the aggregate states $\psi^{(A)}$. Theorem 3.2.3 gives these probabilities and originates from [Baskett et al., 1975].

Theorem 3.2.3. *For $x^{(A)} \in \psi^{(A)}$ let n_i be the total number of customers in service station i. Then the equilibrium state probabilities for $x^{(A)}$ are given by*

$$\pi_{x^{(A)}} = \left[\prod_{i=1}^{N} \binom{n_i}{x_{i,1}^{(A)}, \ldots, x_{i,R}^{(A)}}\right] \pi_x,$$

for any $\pi_x \in \psi$ that gives rise to the aggregate state $x^{(A)}$. The constant C is the constant from Theorem 3.2.2.

Proof. For $x^{(A)} \in \psi^{(A)}$ define $\psi_{x^{(A)}}$ as the set of states that give rise to the aggregate state $x^{(A)}$. Formally, this is defined as

$$\psi_{x^{(A)}} := \left\{ x \in \psi \ : \ \sum_{k=1}^{n_i} \delta_{r,x_{i,k}} = x_{i,r}^{(A)}, 1 \leq i \leq N, 1 \leq r \leq R \right\}.$$

Notice how

$$|\psi_{x^{(A)}}| = \prod_{i=1}^{N} \binom{n_i}{n_{i,1}, \ldots, n_{i,R}},$$

because only the number of customers of each class at service station i is known, which means that there are

$$\binom{n_i}{n_{i,1}, \ldots, n_{i,R}}$$

different queues possible at service station i. Also notice for $x \in \psi_{x^{(A)}}$ and service center i that

$$\prod_{k=1}^{n_i} e_{i,x_{i,k}} = \prod_{r=1}^{R} e_{i,x_{i,r}^{(A)}}$$

only depends on $x^{(A)}$ and not on the exact element within $\psi_{x^{(A)}}$. This implies that π_x is the same for all $x \in \psi_{x^{(A)}}$. We can use this to obtain an equilibrium state probability of

$$\pi_{x^{(A)}} = \sum_{x \in \psi_{x^{(A)}}} \pi_x = |\psi_{x^{(A)}}| \, \pi_x = \left[\prod_{i=1}^{N} \binom{n_i}{x_{i,1}^{(A)}, \ldots, x_{i,R}^{(A)}} \right] \pi_x$$

as desired. $\square$

4. Model description and analysis

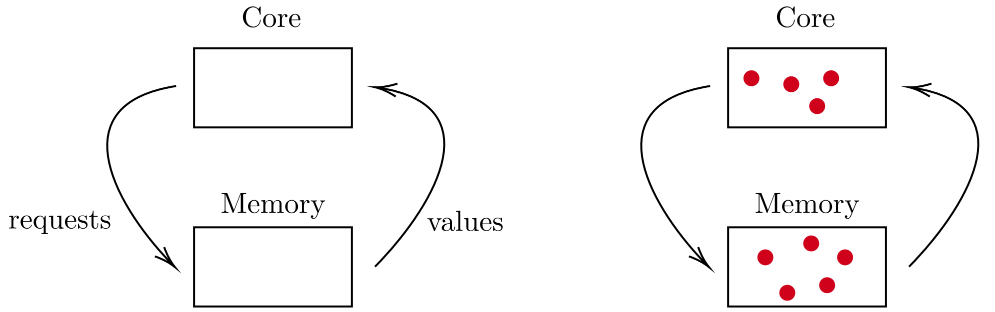
The queueing theory from Chapter 3 can now be used to create a model which imitates the dynamics of the processor as described in Chapter 2. Section 4.1.1 describes the model and the approach taken to arrive at the model, while Section 4.2 analyses the model and Section 4.3 presents derivations for the model parameters.

Note that the model is highly abstract and only aims at recreating the processor dynamics. This can give a non-trivial relation between the model parameters and the components and processes in the processor.

4.1. Model description and explanation

4.1.1. The model dynamics overview

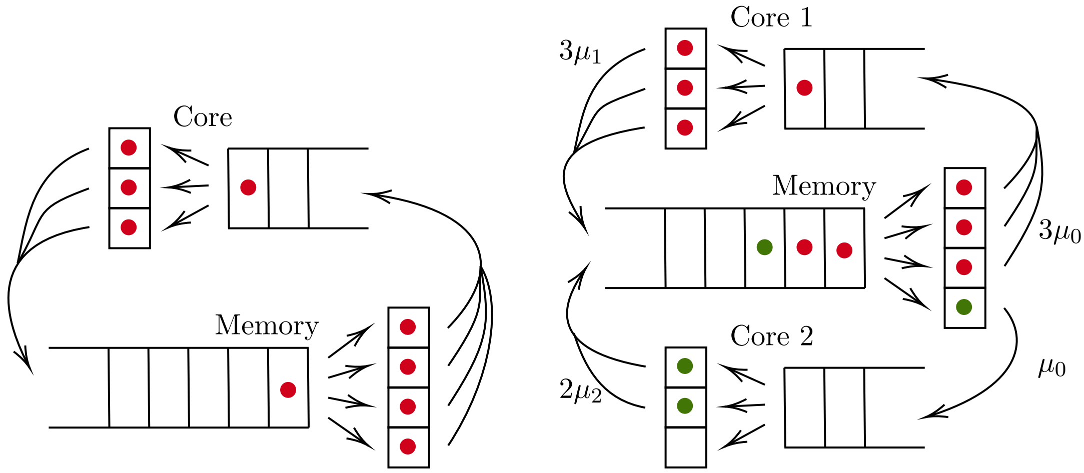
The most relevant interaction in the processor is the sending of memory requests by the core to the memory. There are two types of memory requests: local and shared requests. The local requests require the memory inside the core, and thus won't get effected by an increase in the number of tasks run in parallel. Therefore, those tasks won't be considered in the model. The shared requests require the shared memory between the cores. The service rate of these requests is effected by the other tasks due to the finite memory bandwidth. This gives reason to consider those requests in the model. The dynamic induced by those requests is visualised in Figure 4.1a.



(a) The core-memory interaction. (b) Resource requesting tokens in the system.

Figure 4.1.: The system dynamics modelled with resource requesting tokens travelling through the system, where the number of tokens correspond to the degree of parallelizability of a task.

However, there is a much more useful way of perceiving the core-memory interaction. This is done by viewing a task in execution as a task that requests resources from the different components of the system. The amount of attention a task wants at once is determined by the parallelizability of a task. A task with a high degree of parallelizability is able to perform many different computations and requests many different resources at once. This degree of parallelizability can be modelled with resource requesting tokens travelling through the network. This is visualised in Figure 4.1b and it gives an abstract framework of reasoning about task execution and its interference with the memory.



(a) Single core.

(b) Two cores.

Figure 4.2.: The core and memory depicted as queues serving tokens in a system with one and two cores.

The model can be further concretized by realising that both the core and memory can only serve a finite number of tokens at once. This can be incorporated into the model by replacing the core and memory rectangles in Figure 4.1b with a queue to obtain Figure 4.2a. The resulting network is a closed queueing network and multiple cores can easily added to the network. The case of two cores is given in Figure 4.2b. Since all cores in the system have equal capacity we can assume that the queueing capacities C_i are equal for all cores. The formal definition of the model is given in Definition 4.1.1, where service center 0 corresponds to the memory and centers $1 \leq i \leq N$ correspond to the cores.

Definitie 4.1.1. *The N-core single memory model is defined as the tuple (μ, C), where*

$$\mu = (\mu_i \in \mathbb{R}_{>0} : 0 \leq i \leq N),$$

are the service rates of the service centers and

$$C = (C_i \in \mathbb{N} : 0 \leq i \leq N)$$

are the capacities, where $C_i = C_j$ for all $1 \leq i, j \leq N$. There are N classes of customers and

$$p_{i,0}^{(i)} = p_{0,i}^{(i)} = 1.$$

4.2. Model analysis

The theorems from Chapter 3 can now be used to get an expression for the terms $e_{i,r}$ and the invariant distribution. For the terms $e_{i,r}$ and $1 \le r \le N$ it must hold that

$$e_{i,r} = \sum_{j=0}^{N} e_{j,r} p_{j,i}^{(r)} = \begin{cases} \sum_{j=1}^{N} e_{j,r} & : i = 0 \\ e_{0,r} & : i \neq 0 \end{cases},$$

which means that we can simply let $e_{0,i} = 1$ and $e_{i,i} = 1$ for all $1 \le i \le N$, and all other terms are equal to 0. This directly implies that any $x \in \psi$ with $x_{i,k} \neq i$ for $1 \le i \le N$ has 0 probability of occurring in equilibrium, because then $e_{i,x_{i,k}} = 0$. In other words, in equilibrium there are only class i customers in core i, which should be the case. We can use this to obtain for $x \in \psi$ with $x_{i,k} = i$ for all $1 \le i \le N$ that

$$\begin{aligned} \pi_x &= C \prod_{i=0}^{N} \prod_{k=1}^{n_i} \frac{e_{i,x_{i,k}}}{\min\{C_i, k\}\, \mu_i} \\ &= C \prod_{i=0}^{N} \prod_{k=1}^{n_i} \frac{1}{\min\{C_i, k\}\, \mu_i} \\ &= C \prod_{i=0}^{N} \frac{1}{\min\{C_i, n_i\}! C_i^{\min\{C_i - n_i, 0\}} \mu_i^{n_i}}. \end{aligned}$$

Furthermore, for the equilibrium state probabilities of the aggregate state $x^{(A)} \in \psi^{(A)}$ we obtain

$$\begin{aligned} \pi_{x^{(A)}} &= C \left[\prod_{i=1}^{N} \frac{n_i!}{\prod_{r=1}^{N} \left(x_{i,r}^{(A)}! \right)} \right] \prod_{i=0}^{N} \frac{1}{\min\{C_i, n_i\}! C_i^{\min\{C_i - n_i, 0\}} \mu_i^{n_i}} \\ &= C \frac{n_0!}{\prod_{r=1}^{N} \left(x_{0,r}^{(A)}! \right)} \prod_{i=0}^{N} \frac{1}{\min\{C_i, n_i\}! C_i^{\min\{C_i - n_i, 0\}} \mu_i^{n_i}} \\ &= C \frac{n_0!}{\prod_{r=1}^{N} \left((N_r - n_r)! \right)} \prod_{i=0}^{N} \frac{1}{\min\{C_i, n_i\}! C_i^{\min\{C_i - n_i, 0\}} \mu_i^{n_i}}, \end{aligned}$$

where we used the fact that only class r customers are in service center r for $1 \le r \le N$. This implies that

$$x_{0,r}^{(M)} = N_r - n_r, \quad 1 \le r \le N$$

The constant C can then be calculated by summing over all $x^{(A)} \in \psi^{(A)}$. Notice that those states are completely determined by $n_1, n_2, \ldots, n_N$. This means we have

$$C = \left[\sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \cdots \sum_{n_N=0}^{N_N} \frac{n_0!}{\prod_{r=1}^{N} \left((N_r - n_r)! \right)} \prod_{i=0}^{N} \frac{1}{\min\{C_i, n_i\}! C_i^{\min\{C_i - n_i, 0\}} \mu_i^{n_i}} \right]^{-1},$$

where

$$n_0 = \sum_{r=1}^{N} N_r - n_r$$

4.3. Model parameter derivation

5. Model accurateness and shortcomings

5.1. Task simulation results

5.2. Model parameters and parallel simulation predictions

5.3. Comparisons

6. Conclusion

Bibliography

[Baskett et al., 1975] Baskett, F., Chandy, K. M., Muntz, R. R., and Palacios, F. G. (1975). Open, closed, and mixed networks of queues with different classes of customers. *Journal of the ACM (JACM)*, 22(2):248–260.

Populair summary

A. Proofs