Factoring

Contents

Special Product / Factoring Formulas

If A and B are any real numbers or algebraic expressions, then

- $(A + B)(A - B) = A^2 - B^2$
- $(A + B)^2 = A^2 + 2AB + B^2$
- $(A - B)^2 = A^2 - 2AB + B^2$
- $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$
- $A^3 - B^3 = (A - B)(A^2 - AB + B^2)$
- $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
- $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

To understand how to factor, we must first understand the Distributive Property,

$$a(b + c) = ab + ac$$

.

For example, we can find the product of $3x^2$ and $4x + 3$:

$$3x^2(4x + 3) = 3x^2(4x) + 3x^2(3) = 12x^3 + 9x^2$$

Because the distributive property is an equality, the reverse of this is also true!

$$12x^3 + 9x^2 = 3x^2(4x) + 3x^2(3) = 3x^2(4x + 3)$$

The resulting expression is in **factored form** because it is written as a product of two polynomials, whereas the original expression is a two - termed sum.

$$\begin{gathered} \longrightarrow Factoring \longrightarrow \\ 12x^3 + 9x^2 = 3x^2(4x + 3) \\ \longleftarrow Expanding \longleftarrow \end{gathered}$$

1 Greatest Common Factors

To factor the GCF out of a polynomial, we do the following:

1. Find the GCF of all of the terms in the polynomial

2. Express each as a product of the GCF and another factor
3. Use the distributive property to factor out the GCF

Example 1

Factor the GCF out of $2x^3 - 6x^2$
Solution:

Step 1: Find the GCF

- $2x^3 = 2 \cdot x \cdot x \cdot x$
- $6x^2 = 2 \cdot 3 \cdot x \cdot x$

Therefore, the GCF of $2x^3 - 6x^2$ is $2 \cdot x \cdot x = 2x^2$
Step 2: Express each term as a product of $2x^2$ and another factor

- $2x^3 = (2x^2)(x)$
- $6x^2 = (2x^2)(3)$

Therefore, the polynomial can be written $2x^3 - 6x^2 = (2x^2)(x) - (2x^2)(3)$
Step 3: Factor out the GCF
Now we can apply the distributive property to factor out $2x^2$.
$(2x^2)(x) - (2x^2)(3) = 2x^2(x-3)$
We get our factors to be $2x^2$ and $x - 3$.

Exercise 1

Factor each expression:

a) $3x^2 - 6x$

b) $8x^4y^2 + 6x^3y^3 - 2xy^4$

c) $(2x+4)(x-3) - 5(x-3)$

Solution:

a) The greatest common factor of $3x^2$ and $-6x$ is $3x$, so we have:

$$3x^2 - 6x = 3x(x) - 3x(2) = 3x(x-2)$$

b) The greatest common factor of $8x^4y^2$, $6x^3y^3$, and $-2xy^4$ is $2xy^2$, so we have:

$$8x^4y^2 + 6x^3y^3 - 2xy^4 = (2xy^2)(4x^3) + (2xy^2)(3x^2y) + (2xy^2)(-y^2) = 2xy^2(4x^3 + 3x^2y - y^2)$$

c) The greatest common factor of $(2x+4)(x-3)$ and $-5(x-3)$ is $x-3$, so we have:

$$(2x+4)(x-3)-5(x-3) = (x-3)(2x+4)+(x-3)(-5) = (x-3)(2x+4-5) = (x-3)(2x-1)$$

2 Quadratic of the form $x^2 + bx + c$

To factor a quadratic of the form $x^2 + bx + c$, we note that

$$(x+r)(x+s) = x^2 + rx + sx + rs = x^2 + (r+s)x + rs$$

Therefore, we need to find numbers r and s such that $r + s = b$ and $rs = c$. There is no set way to find these two values, so we find them by trial and error. Note, not all quadratic equations will have r and s as integers so it may not be realistic doing this process and you should use the quadratic formula instead.

Example 2

Factor each equation:

a) $x^2 + 7x + 12$ b) $x^2 - 5x - 6$

Solution:

a) We need to find two numbers whose product equals 12 and whose sum is 7. Let's list all pairs of numbers that have a product of 12 and then write the sum of them.

		Sum
1	12	13
2	6	8
3	4	7

The last row of the table on the left shows that the pair of numbers 3 and 4 have a sum of 7, along with the product of 12. Therefore, r and s are the two integers 3 and 4. Thus, the factorization is

$$x^2 + 7x + 12 = (x+3)(x+4)$$

b) We need to find two numbers whose product equals -6 and whose sum is -5. Let's list all pairs of numbers that have a product of -6 and then write the sum of them. Note that since our product is negative, one of our factors will be negative while the other is positive.

		Sum
-1	6	5
-2	3	1
1	-6	-5
2	-3	-1

The second to last row of the table on the left shows that the pair of numbers 1 and -6 have a sum of -5, along with the product of -6. Therefore, r and s are the two integers 1 and -6. Thus, the factorization is

$$x^2 - 5x - 6 = (x+1)(x-6)$$

□

3 Grouping

Polynomials with at least four terms can sometimes be factored by grouping terms. The following example illustrates this idea:

Example 3

Factor each polynomial:

a) $x^3 + x^2 + 4x + 4$

b) $x^3 - 2x^2 - 9x + 18$

Solution:

a)
$$\begin{aligned} x^3 + x^2 + 4x + 4 &= (x^3 + x^2) + (4x + 4) \\ &= x^2(x+1) + 4(x+1) \\ &= (x^2+4)(x+1) \end{aligned}$$

b)
$$\begin{aligned} x^3 - 2x^2 - 9x + 18 &= (x^3 - 2x^2) - (9x + 18) \\ &= x^2(x-2) - 9(x-2) \\ &= (x^2-9)(x-2) \\ &= (x+3)(x-3)(x-2) \end{aligned}$$

4 Quadratic of the form $ax^2 + bx + c$

Before trying to factor your equation, you want to make sure you factor the GCF out of each term. Sometimes this will make $a = 1$ instead which is a lot easier.

4.1 Trial and Error

To factor a quadratic of the form $ax^2 + bx + c$, with $a \neq 1$ we note that

$$(px + r)(qx + s) = pqx^2 + qrx + psx + rs = pqx^2 + (qr + ps)x + rs$$

Therefore, we need to find numbers p, q, r and s such that $pq = a$, $qr + ps = b$ and $rs = c$.

Example 4

Factor $6x^2 + 7x - 5$

Solution: We can factor 6 as $1 \cdot 6$ or $2 \cdot 3$ and we can factor -5 as $1 \cdot -5$ or $-1 \cdot 5$.

p	q	r	s	$qr + ps$
1	6	1	-5	1
1	6	-1	5	-1
6	1	1	-5	-29
6	1	-1	5	29
2	3	-1	5	7
2	3	1	-5	-7
3	2	-1	5	13
3	2	1	-5	-13

We have found that when $p = 2$, $q = 3$, $r = -1$ and $s = 5$, we get the specifications we were looking for. Thus the factorization is ☐

$$6x^2 + 7x - 5 = (2x - 1)(3x + 5)$$

4.2 Grouping (easier)

As a shourtcut/ tool to make factoring $ax^2 + bx + c$ a little bit easier, we use grouping.

1. Multiply $a \cdot c$, then list out the factors of your value.

2. Find the pair of numbers m and n that satisfy these two requirements:

- $m \cdot n = a \cdot c$
- $m + n = b$

3. Rewrite (split) the middle term bx using your factors found in step 2.

$$ax^2 + bx + c =$$

$$ax^2 + mx + nx + c$$

4. Now that there is four terms, we can now use grouping:

$$(ax^2 + mx) + (nx + c)$$

5. Factor out the GCF from each term. once you do this, the terms should share a common binomial factor.

6. Factor out the common binomial factor to write the factorization

7. Check your result by multiplying

Example 5

Factor: $8x^2 - 34x + 30$
Solution: We can first factor the GCF (2) from each term:

$$8x^2 - 34x + 30 = 2(4x^2 - 17x + 15)$$

Since $a \neq 1$ still, lets use our shortcut:
$a \cdot c = 4 \cdot 15 = 60 = m \cdot n$ and $b = -17 = m + n$. Next lets write out some factors of 60 to help us identify the two numbers m and n we are looking for.
Note: Since the product is positive but the sum is negative, both our numbers will be negative.

		Sum
-1	-60	-61
-2	-30	-32
-3	-20	-23
-4	-15	-19
-5	-12	-17
-6	-10	-16

The second to last row of the table on the left shows that the pair of numbers -5 and -12 have a sum of -17, along with the product of 60. Therefore, m and n are the two integers -5 and -12. So now lets rewrite our function and use grouping to factor:

$$\begin{aligned} 8x^2 - 34x + 30 = 2(4x^2 - 17x + 15) &= 2(4x^2 - 5x - 12x + 15) \\ &= 2(4x^2 - 5x) + (-12x + 15) \\ &= 2[x(4x - 5) - 3(4x - 5)] \\ &= 2(4x - 5)(x - 3) \end{aligned}$$