Investments TA Session

Duration and Convexity

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Introduction

Questions

Any questions before we start?

Plan for today

- Yield-to-Maturity
- Duration
- Convexity
- Return Attribution

Yield-to-Maturity

Yield-to-Maturity: Definition

A bond's **yield-to-maturity**, or **yield**, is the single rate that when used to discount a bond's cash flows produces the bond's market price.

Solving for Yield-to-Maturity

Consider a 10-year U.S. Treasury Note with price P and coupon C.

Yield-to-maturity y solves the equation

$$P = \sum_{n=1}^{19} \frac{C/2}{(1+y/2)^{2n}} + \frac{100 + C/2}{(1+y/2)^{20}}.$$

In R, we can search for the yield that equates the price function to the market price.

```
price <- function(yield, coupon, maturity) {
  t <- 1:(2 * maturity)
  df <- 1/(1+yield/2)^t
  cf <- c(rep(coupon / 2, 2*maturity - 1), 100 + coupon / 2)
  sum(df * cf)
}
```

Example: A 10-year U.S. Treasury Note

TREASURY NEWS

Department of the Treasury • Bureau of the Fiscal Service

For Immediate Release
February 08, 2017

CONTACT: Treasury Securities Services
202-504-3550

TREASURY AUCTION RESULTS

Term and Type of Security	10-Year Note
CUSIP Number	912828V98
Series	B-2027
Interest Rate	2-1/4%
High Yield [1]	2.333%
Allotted at High	5.23%
Price	99.263516
Accrued Interest per $1,000	None
Median Yield [2]	2.260%
Low Yield [3]	2.201%
Issue Date	February 15, 2017
Maturity Date	February 15, 2027
Original Issue Date	February 15, 2017
Dated Date	February 15, 2017

Example: Caculating T-Note's Price and Yield

For the auction results, we are told the price is 99.263516 and the yield is 2.333%.

```
options(digits = 8)
price <- function(yield, coupon, maturity) {
  t <- 1:(2 * maturity)
  df <- 1/(1+yield/2)^t
  cf <- c(rep(coupon / 2, 2 * maturity - 1), 100 + coupon / 2)
  sum(df * cf)
}
price(0.02333, 2.25, 10)
```

```
## [1] 99.263516
```

```
f <- function(yield, coupon, maturity){
 price(yield, coupon, maturity) - 99.263516
}
uniroot(f,c(0,10),2.25, 10, tol=10^-9)$root
```

```
## [1] 0.02333
```

What are Yields used for?

A bond's yield is a convenient way to talk about its price.

- Sometimes bonds are traded on yield, or yield spreads, rather than prices.

Yields can be used for **return attribution**.

- If a bond's yield-to-maturity remains unchanged over a *short period* of time, that bond's realized total return equals it's yield.

Yields can be Misleading

A yield is a useful summary of bond pricing, but you need to know exactly how to calculate the bond's price.

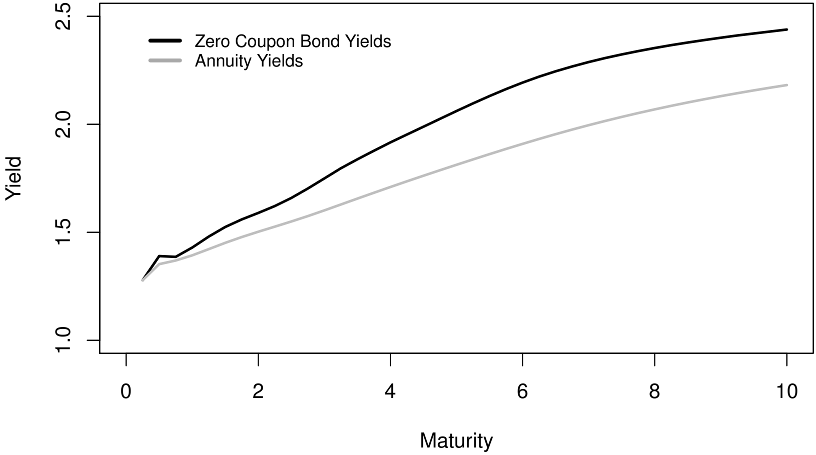
- There are many different yield and day count conventions.

Yield is **not** a measure of relative value or realized return to maturity.

- If one bond has a higher yield than another, it is not a better value.
- A bond purchased at a particular yield and held to maturity will not necessarily earn that yield.

Yields depend on the Timing of Cash Flows

With an upward-sloping yield curve, annuities have lower yields than zero-coupon bonds, but this does not mean that zero coupon bonds are cheap to annuities.



Yields and Returns

Between coupon payments, if a bond's yield remains unchanged, then the total return of the bond over that period equals its yield-to-maturity.

$$
\begin{aligned}
P_0 &= \frac{C/2}{1+y/2} + \frac{C/2}{(1+y/2)^2} + \cdots + \frac{100+C/2}{(1+y/2)^{2T}} \\
P_{1/2} &= \frac{C}{2} + \frac{C/2}{1+y/2} + \cdots + \frac{100+C/2}{(1+y/2)^{2T-1}} \\
&\Rightarrow P_{1/2} = (1+y/2)\,P_0 \\
&\Rightarrow y = 2\left(\frac{P_{1/2}}{P_0} - 1\right)
\end{aligned}
$$

Calculating Yields between Coupon Payments

Calculating Yields between Coupon Payments

The **Street Convention** and the **Treasury Convention** are two methods to calculate the yield of a bond between coupon payment dates.

- Yields using the Treasury Convention are shown on the Treasury auction results.
- The Street Convention is generally used for pricing and it is shown on Bloomberg.

Calculating Yields between Coupon Payments (Con't)

First, we need to determine the fraction w of time between the settlement date and the next coupon date.

Using the appropriate day count convention, this is determined as follows:

$$w = \frac{\text{Days between settlement date and next coupon payment date}}{\text{Days in the coupon period}}$$

Consider a bond with N semi-annual coupon payments remaining.

Using the **Street Convention**, the yield solves

$$P_{\text{Full}} = \sum_{i=0}^{N-1} \frac{C/2}{(1+y/2)^{i+w-1}} + \frac{100}{(1+y/2)^{N+w-1}}$$

Calculating Yields between Coupon Payments (Con't)

The **Street Convention** assumes that the first discount factor is

$$D(w) = \frac{1}{\left(1 + \frac{y}{2}\right)^{w}}.$$

The **Treasury Convention** assumes simple interest in the first period, so that the first discount factor is

$$D(w) = \frac{1}{\left(1 + \frac{y}{2}w\right)}.$$

Therefore, in the **Treasury Convention** the yield solves

$$P_{\text{Full}} = \sum_{i=0}^{N-1} \frac{C/2}{\left(1 + \frac{y}{2}w\right)\left(1 + \frac{y}{2}\right)^{i-1}} + \frac{100}{\left(1 + \frac{y}{2}w\right)\left(1 + \frac{y}{2}\right)^{N-1}}$$

Treasury Convention Example

TREASURY NEWS

Department of the Treasury • Bureau of the Fiscal Service

For Immediate Release
October 11, 2017

CONTACT: Treasury Securities Services
202-504-3550

TREASURY AUCTION RESULTS

Term and Type of Security	9-Year 10-Month Note
CUSIP Number	9128282R0
Series	E-2027
Interest Rate	2-1/4%
High Yield [1]	2.346%
Allotted at High	17.96%
Price	99.158502
Accrued Interest per $1,000	$3.79076
Median Yield [2]	2.300%
Low Yield [3]	2.231%
Issue Date	October 16, 2017
Maturity Date	August 15, 2027
Original Issue Date	August 15, 2017
Dated Date	August 15, 2017

Treasury Convention Example in R

```r
priceTC <- function(yld, cpn, matdt, setdt) {
  pmtdts <- seq(from = matdt, to = setdt, by = "-6 months") %>% sort
  n <- pmtdts %>% length
  ncd <- pmtdts[1]
  lcd <- seq(from = matdt, len = n + 1, by = "-6 months")[n + 1]
  cf <- c(rep(cpn / 2, n-1), cpn / 2 + 100)
  basis <- (ncd - lcd) %>% as.numeric
  accrual <- (setdt - lcd) %>% as.numeric
  ncdays <- basis - accrual
  df <- c(1/(1 +yld/2*ncdays / basis), rep(1/(1 + yld/2), n-1))
  df <- cumprod(df)
  px <- sum(df * cf)
  ai <- accrual/basis * cpn / 2
  px - ai
}
priceTC(0.02346, 2.25, as.Date("2027-08-15"), as.Date("2017-10-16"))
```

```
## [1] 99.158502
```

Street Convention Example in R

```
priceSC <- function(yld, cpn, matdt, setdt) {
  pmtdts <- seq(from = matdt, to = setdt, by = "-6 months") %>% sort
  n <- pmtdts %>% length
  ncd <- pmtdts[1]
  lcd <- seq(from = matdt, len = n + 1, by = "-6 months")[n + 1]
  cf <- c(rep(cpn / 2, n-1), cpn / 2 + 100)
  basis <- (ncd - lcd) %>% as.numeric
  accrual <- (setdt - lcd) %>% as.numeric
  ncdays <- basis - accrual
  df <- c(1/(1 +yld/2)^(ncdays / basis), rep(1/(1 + yld/2), n-1))
  df <- cumprod(df)
  px <- sum(df * cf)
  ai <- accrual/basis * cpn / 2
  px - ai
}
priceSC(0.02346, 2.25, as.Date("2027-08-15"), as.Date("2017-10-16"))
```

```
## [1] 99.160012
```

Street Convention on Bloomberg

```r
# 98 + 20.5 / 32 = 98.640625
priceSC(0.02406444, 2.25, as.Date("2027-08-15"), as.Date("2017-10-30"))
```

```
## [1] 98.640625
```

Duration

Duration

Duration is a *linear* measure of risk of a fixed-income investment to changes in yields.

There are many duration measures. The most important are:

- **Macaulay Duration** (D)
  - the time-weighted present value of cash flows
  - useful for intuition
- **Modified Duration** (D^*)
  - the percentage change in price for a unit change in yields
  - This is what we actually use in calculations

⚠ Both Macaulay and Modified duration are called "duration."

- This can create a lot of confusion!

Modified Duration

- Modified Duration and Macaulay Duration are related by

$$D^* = \frac{D}{1 + y_k/k}$$

where k is the compounding frequency.

  - e.g. $k = 2$ for a semiannual yield.

- Modified Duration is a measure of price sensitivity:

$$D^* = -\frac{1}{P}\frac{\partial P}{\partial y}$$

Convexity

Bond prices are a non-linear function of yields.

- Duration is a *linear* measure of interest rate risk.
- Duration is a good approximation of the change in price of a bond for small changes in yields.
- However, the error grows in with the size of the yield change.

We can improve the estimate by adding a second-order term, known as **convexity**.

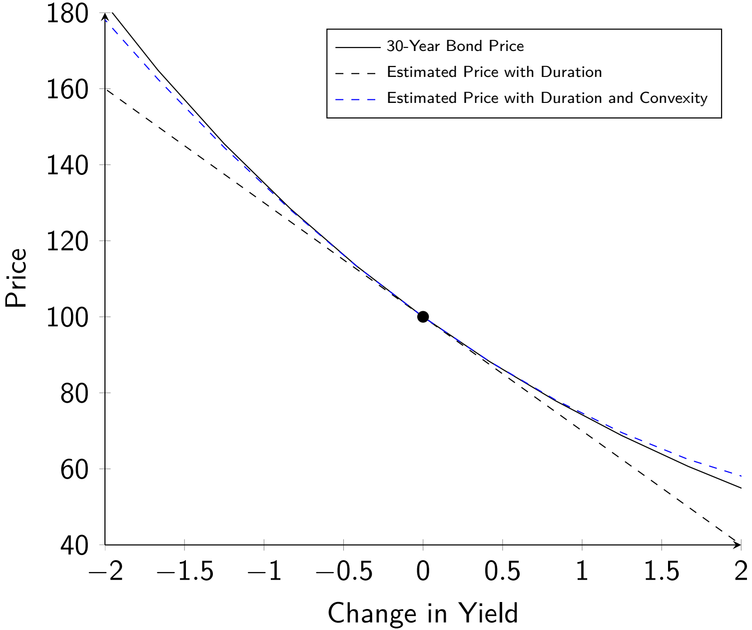
- Convexity measures how much the bond outperforms the linear estimate.

A lot of fixed income investing involves searching for cheap convexity.

- ie bonds that have a favorable price-yield relationship.

⚠ You need to be careful about the units convexity is quoted in.

30-Year Zero Coupon Bond Prices



Example: Zero Coupon Bond (Semiannual Compounding)

$$
\begin{aligned}
P &= \frac{100}{(1+y/2)^{2T}} \\
D &= T \\
D^* &= -\frac{1}{P}\frac{\partial P}{\partial y} = \frac{T}{1+y/2} \\
C &= \frac{1}{P}\frac{\partial^2 P}{\partial y^2} = \frac{T(T+1)}{(1+y/2)^2}
\end{aligned}
$$

Example: Zero Coupon Bond (Continuous Compounding)

$$
\begin{aligned}
P &= 100e^{-yT} \\
D &= T \\
D^* &= -\frac{1}{P}\frac{\partial P}{\partial y} = T \\
C &= \frac{1}{P}\frac{\partial^2 P}{\partial y^2} = T^2
\end{aligned}
$$

Understanding Price Changes

A second-order Taylor Approximation implies that

$$\begin{aligned} P(y+\Delta y) &\approx P(y) + \frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2P}{dy}(\Delta y)^2 \\ &= P(y) - D^* P \Delta y + \frac{1}{2} CP(\Delta y)^2 \end{aligned}$$

Over time (for zcb, continuous compounding):

$$\begin{aligned} \text{Because } \frac{dP}{dt} &= \frac{d}{dt} e^{-y(T-t)} = ye^{-y(T-t)} = yP, \\ \Delta P &\approx P(y) + \frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2P}{dy}(\Delta y)^2 + \frac{dP}{dt}\Delta t \\ &= P(y) - D^* P \Delta y + \frac{1}{2} CP(\Delta y)^2 + yP\Delta t \end{aligned}$$