Properties of Pair of Coupled Maps

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1. Introduction

The natural world and engineered systems are filled with phenomena governed by nonlinear interactions. Unlike linear systems where outputs are proportional to inputs and superposition holds, nonlinear systems exhibit a rich tapestry of behaviors, including multiple stable states, periodic oscillations, quasi-periodicity, and the hallmark of deterministic chaos – extreme sensitivity to initial conditions alongside bounded, aperiodic long-term evolution. Understanding and predicting the behavior of these nonlinear systems is a central challenge across diverse scientific and engineering disciplines. Coupled map systems consist of networks of relatively simple dynamical units (maps) that interact according to specific coupling rules, giving rise to remarkably rich collective behaviors. Originally introduced as mathematical abstractions, coupled maps have evolved into powerful modeling tools with applications spanning physics, biology, neuroscience, economics, and numerous other fields.

In reality, few systems exist in complete isolation. More often, individual units or subsystems interact with each other, forming coupled systems. The study of coupled dynamical systems investigates how the behavior of individual components is modified by their mutual influence, and critically, what new collective or emergent behaviors arise from these interactions. This coupling can take many forms, from diffusive coupling representing the exchange of mass or energy, to all-to-all coupling in certain network structures, or more complex, specific interaction topologies.

What makes coupled map systems particularly intriguing is their ability to generate complex global behaviors from simple local dynamics and interactions. A single map, such as the logistic map $x_{n+1} = rx_n(1 - x_n)$, can already exhibit period-doubling bifurcations leading to chaos. When multiple such maps are coupled together, new phenomena emerge: synchronization, where initially different units begin to behave identically; pattern formation, where spatial structures spontaneously develop; and various forms of intermittency and turbulence that can model real-world phenomena across scales.

$$x_{n+1} = f(x)$$

If an initial condition x_0 is specified at time level n = 0, the system state at time level n = 1 is $x_l = f(x_0)$,and so on. The function f can be any linear or non-linear map, for instance, the exponential map,

$$f(x) = xe^{r(1-x)} \tag{1}$$

as per Ricker to model the population dynamics of a prey species In this paper, explores the properties of a pair of coupled logistic maps given by the following equations:

$$\begin{aligned} x_{n+1} &= df(x_n) + (1-d)f(y_n) \\ y_{n+1} &= (1-d)f(x_n) + df(y_n) \end{aligned} \tag{2}$$

where d is the coupling strength, which is a number between 0 and 1 and represents the fraction of the total population that interacts with the other population. This form of coupling often arises in physical systems, such as in the study of coupled oscillators. The equation can be interpreted as follows: at each time step n, the population x_n is updated by taking a weighted average of its own value and the value of the other population y_n with weights d and $(1 - d)$ respectively. The weights are chosen such that the total population remains constant. For example, if $d = 0.5$, the population x_n is updated by taking the average of its own value and the value of y_n. This form of coupling is often used to model the interactions between two species, such as predator-prey systems or competing species. The authors investigate several "global, generic properties" that emerge from this coupling structure, independent of the specific functional form of f (though their numerical examples use specific maps). We shall denote the mapping for brevity, as

$$(x_{n+1}, y_{n+1}) = M(d) \circ (x_n, y_n)$$

The paper have revealed that coupled maps has a great number of phenomena, including synchronization, chaos, and the emergence of complex patterns. These systems can exhibit a wide range of behaviors depending on the coupling strength, the nature of the maps, and the initial conditions. For example, weak coupling can lead to synchronization, where the maps converge to a common trajectory, while strong coupling can induce chaotic behavior or even lead to the formation of stable patterns.

This report aims to provide a comprehensive examination of coupled map systems, beginning with fundamental theoretical concepts and progressing through dynamical properties to diverse applications. We will explore established results, highlighting the universal principles that govern these systems while acknowledging the domain-specific insights they have yielded. Throughout, we emphasize the mathematical elegance of coupled maps and their practical utility as models of complex, real-world phenomena.

The investigation of these global properties is significant because it provides a framework for understanding how interaction shapes collective behavior. Discovering that simple coupling can lead to:

- predictable symmetries (reducing analytical effort),
- the emergence of stable periodic states from chaotic components (offering mechanisms for control or explaining observed stability in nature), and
- robust synchronized states (with implications for communication and coordinated activity)

The paper's [1] findings highlight that even in seemingly simple two-unit coupled systems, a rich and structured array of dynamical behaviors can emerge, driven by the interplay between the intrinsic dynamics of the individual units and the nature of their coupling. The "seven-zone dynamics" characterized by distinct bifurcations (tangent, Sacker/Hopf-like) further underscores the structured complexity arising from these interactions.

2. Analytical Results

Some analytical results are discussed below, which will be useful in understanding the properties of coupled maps. The results are derived from the equations Eq. 2 the coupled maps and are presented in a concise manner.

1. **Result 1**: For each orbit $\{(\hat{x}_n, \hat{y}_n) \mid n = 0, 1, 2....\}$ corresponding to the parameter value d, there corresponds an orbit $\{(\hat{y}_n, \hat{x}_n) \mid n = 0, 1, 2....\}$.

   Corollary 1: For each n-periodic orbit of the map described by $\{(\hat{x}_n, \hat{y}_n) \mid n = 0, 1, 2....\}$ corresponding to the parameter d, there exists another n-periodic orbit described by $\{(\hat{y}_n, \hat{x}_n) \mid n = 0, 1, 2....\}$.

2. **Result 2**: Consider the orbit $\{(\hat{x}_n, \hat{y}_n) \mid n = 0, 1, 2, ...\}$ corresponding to a certain value of the parameter $d = ½ - d_o, 0 < d_o < ½$. For each such orbit, there corresponds an orbit given by $\{(\widehat{x_0}, \hat{y_0}), (\hat{y}_1, \hat{x}_1), (\hat{x}_2, \hat{y}_2), (\hat{y}_3, \hat{x}_3)....\}$, corresponding to the parameter $d = ½ + d_0$, i.e., each

alternate point of the second orbit has its x- and y-coordinate switched with respect to the corresponding point of the first orbit.

Corollary 2: If the map $M(½ - d_0), 0 \leq d_0 \leq ½$, has a $2n$-period orbit starting from some $(x_0, y_0), n = 1, 2.....$ then the map $M(½ + d_0)$ must have a $2n$- period orbit starting from the same point $(x_0, y_0), n = 1, 2....$

Corollary 3: If the map $M(½ - d_0), 0 \leq d_0 \leq ½$, has a $2n - 1$-period orbit starting from some $(x_0, y_0), n = 1, 2.....$ with $x_0 \neq y_0$ then the map $M(½ + d_0)$ must have a $2(2n - 1)$- period orbit starting from the same point $(x_0, y_0), n = 1, 2....$

3. **Result 3**:The orbit of the map starting with (x_0, x_0) will always consist of points of the form (x_n, x_n). In other words, orbits which begin on the diagonal in the (x, y) phase space lie entirely on the diagonal.
4. **Result 4**: For $d = ½$ the orbit of the map starting with (x_0, y_0) will consist of points of the form (x_n, x_n) after the first iteration.
5. **Result 5**: Each Lyapunov exponent of the map $M(½ - d_0), 0 \leq d_0 \leq ½$, starting from some (x_0, y_0) is the same as that of the map $M(½ + d_0)$ starting from the same point (x_0, y_0).
6. **Result 6**: When $d = l$, one Lyapunov exponent tends and to $-\infty$.

These analytical results provide a robust framework for understanding the behavior of the coupled map system. They reveal inherent symmetries related to variable interchange and the coupling parameter d, explain the special invariant nature of the diagonal subspace, connect the coupled system's dynamics directly to the underlying 1D map f along the diagonal and at $d = 0.5$, demonstrate the mirrored stability properties around $d = 0.5$ via Lyapunov exponents, and analytically predict the boundaries of the stable synchronization regime. These findings hold generally for the given coupling structure and significantly simplify the interpretation of complex numerical observations.

3. Numerical Results

The numerical results are obtained by simulating the coupled map system Eq. 2 using the equations governing the dynamics. The simulations are performed for different values of the coupling strength d and the initial conditions. The results are presented in the form of plots and tables, which illustrate the behavior of the system under different conditions.

Figure 1 show the the behavior of the system for different values of the coupling strength d. The bifurcation diagram shows the periodic behavior of the system for certain ranges of d, while the Lyapunov exponent plot Figure 4 shows the stability of the system for different initial conditions. The results indicate that the coupled map system exhibits a rich variety of behaviors, including periodic orbits, chaotic behavior, and synchronization. Thus coupling two chaotic systems can stabilize both of them. In fact, as Figure 1 shows, for 0.03 < d < 0.13 we observe two one-period orbits, i.e., by coupling two chaotic units we can arrive at complete equilibrium. This stabilizing phenomenon may very well explain why there is so much stability in this physical world despite all the reported chaos.

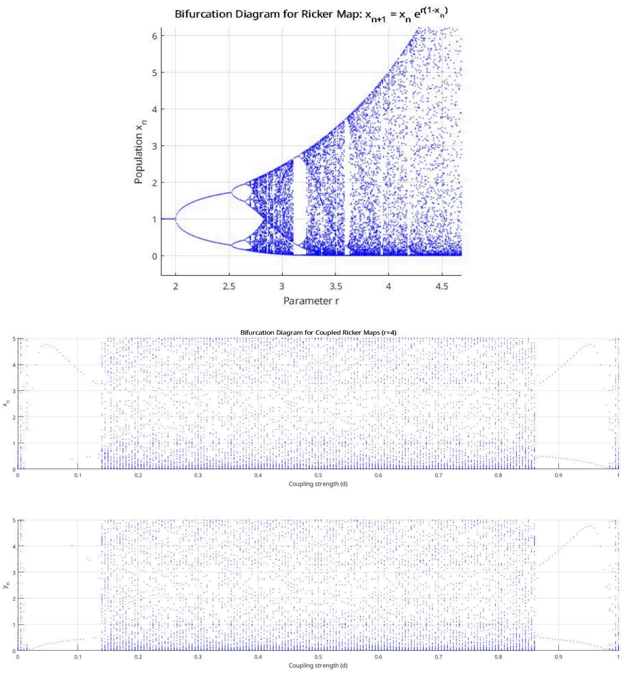

Figure 1: Bifurcation diagram of the exponential map plotted against the growth parameter r. For r say, r = 4, the map is chaotic. By coupling two such maps in the form described by Eq. 2, we can obtain periodic behavior for certain ranges of the coupling parameter d.

When synchronization is observed, the response of the coupled system, after a large number of iterations, is limited to the line x = y. Thus, instead of spanning the two- dimensional surface, the dynamics is confined to a one-dimensional line. This one-dimensional dynamics may be viewed as a step towards stabilization.

When the two units evolve synchronously, both populations are identical, i.e., $x = y$. Therefore, the difference between the two populations $x - y$ should be identically zero. Synchronicity variation with the coupling parameter d, could be understood from Figure 2.[2] It can be seen that the range of d for which the dynamics is synchronous is significant. coupled exponential map displays a variety of behavior over the range of d values of interest. The overall dynamics of the coupled exponential map system can broadly be divided into seven zones as in Figure 2(c). The transition from one zone to another is marked by a bifurcation with respect to the parameter d. Either a tangent bifurcation(Zone II to Zone III) or a Hopf-like bifurcation (Zone II to Zone III) may mark the transition from one zone to another.

Some features of dynamics in the seven zones are as follows:

- Zone I: The system is primarily chaotic and desynchronized. The two maps follow different chaotic paths, and the largest Lyapunov exponent is positive.
- Zone II: The system is stable and periodic, with the two maps converging to a finite set of points. Both Lyapunov exponents are negative, confirming stability.
- Zone III: The system is chaotic and desynchronized, with trajectories filling a complex region similar to Zone I. The largest Lyapunov exponent is positive.
- Zone IV: The system is chaotic and synchronized, with trajectories starting off the diagonal x=y being attracted to it. The tangential Lyapunov exponent is positive, while the transverse Lyapunov exponent is negative.
- Zone V: The system is chaotic and desynchronized, similar to Zone III. The largest Lyapunov exponent is positive.

- Zone VI: The system is stable and periodic, with trajectories converging to a finite set of points. Both Lyapunov exponents are negative.
- Zone VII: The system is primarily chaotic and desynchronized, similar to Zone I. The largest Lyapunov exponent is generally positive.

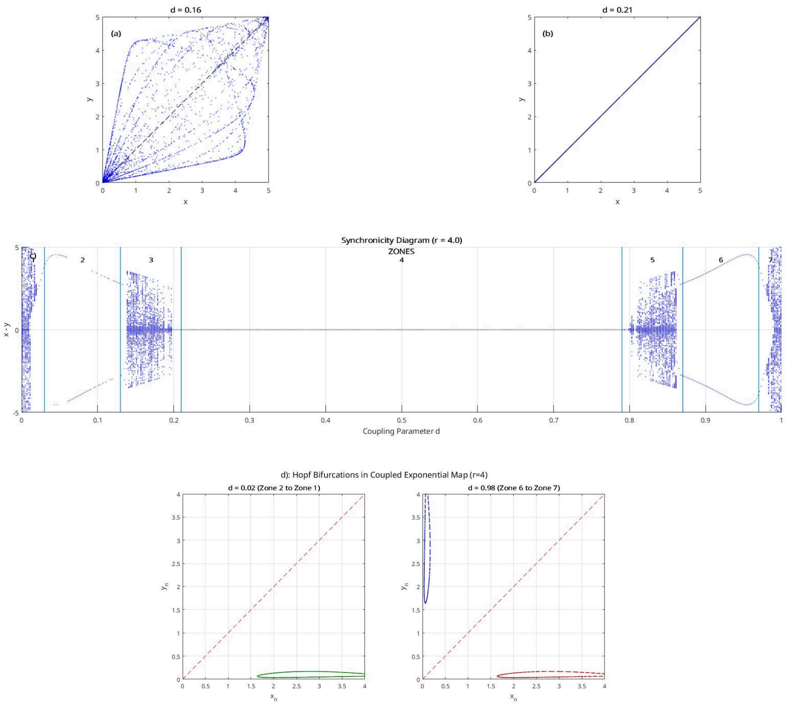

Figure 2: Plots depicting the diagonal attraction A coupled exponential map with $r = 4$ is used: $(a)d = 0.16$; $(b)d = 0.21$. For some values of d, as in (a), the response is over a region in the line $x = y$. (c) The difference $x - y$ plotted against d. Over a large range of d, the coupled response is synchronous as seen by the null values. (d) The transitions from zone 2 to zone 1, and from zone 6 to zone 7 are marked by Hopf bifurcations $(r = 4)$. The closed loop trajectory on the left is caused by a bifurcation of the one-period orbit in zone 2. The corresponding closed trajectory, caused by a Hopf bifuraction of the two-period orbit in zone 6, is shown on the right. This orbit alternates between the two closed loops. The loops are symmetric about the line x = y..

One of the major question would be what is the probability of the two maps being synchronized. The probability of synchronization is defined as the fraction of time that the two maps are synchronized. The probability of periodicity, synchronicity, and their total are plotted in Fig. 5 for cases where the underlying single map is chaotic. For every value of $\hat{r}$ chosen, the probabilities are obtained by investigating the orbits for 5000 values of d ranging from zero to one. The number of periodic orbits produced divided by 5000 yields the probability of periodic or- bits and so on. This procedure is repeated over three initial conditions to obtain an average probability.

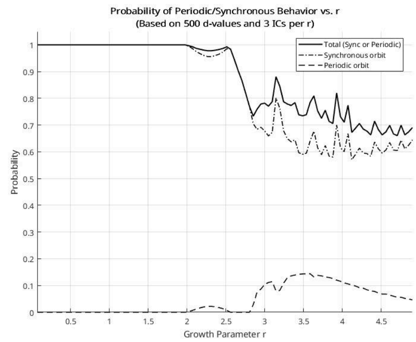

Figure 3: The probability of obtaining (i) periodic behavior (dashed lines), (ii) synchronous behavior (dashed dot) and (iii) their total, plotted against r.

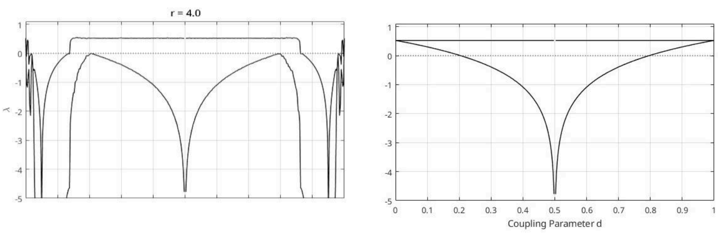

Figure 4: The spectrum of Lyapunov exponents. (a) The ICs do not fall on the diagonal $(x_0 \neq y_0)$ here are ranges of d for which both exponents are negative which indicates that the behavior in those ranges is non-chaotic. (b) The ICs fall on the diagonal $(x_0 = y_0)$. At least one exponent is positive indicating that the diagonal is chaotic.

4. Novel results

4.1. Largest Lyapunov Exponent map

2D parameter scan of Lyapunov exponents for the coupled exponential map. The largest Lyapunov exponent is plotted against the coupling strength d and map non-linearity r .Lines or sharp transitions in the color map corresponding to $\lambda_1 \approx 0$ indicate bifurcation curves in the (r, d) parameter plane.Figure 5 helps understand how the seven zones discussed in the paper for a fixed r (like r=4) evolve or change shape as r itself is varied.

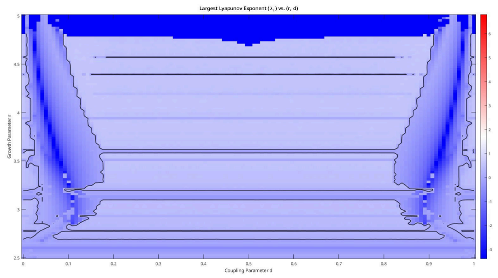

Figure 5: Contour map of the largest Lyapunov exponent for a coupled exponential map.

4.2. Synchronization Time vs. Coupling d

Figure 6, Shows how quickly the system synchronizes once the coupling d is within the stable synchronization zone (Zone IV). offers valuable insights into the dynamics and efficiency of the synchronization process itself. If rapid synchronization is a desired feature (e.g., in secure communication systems where quick lock-in is needed, or in neural systems for fast processing), this plot can help identify the range of coupling strengths d that minimize the time to synchronize.

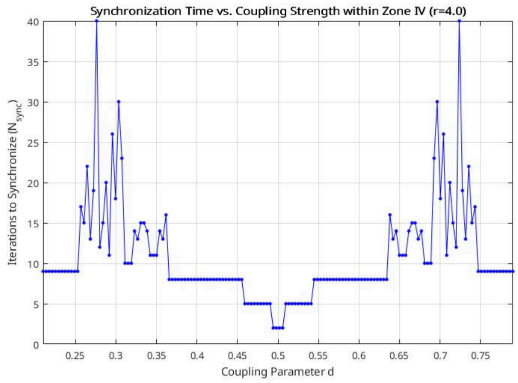

Figure 6: Show the iterations required for synchronization for various values of d.

5. Conclusion

[1] analytical derivations and numerical simulations, we confirmed the paper's findings on quasi-symmetry around d=0.5, the emergence of periodicity, and the significant phenomenon of chaotic synchronization within distinct zones of coupling strength. Lyapunov exponent calculations validated the stability and chaotic nature of these zones. Furthermore, additional visualizations, such as 2D parameter scans of the largest Lyapunov exponent and synchronization time analyses, were proposed and demonstrated to offer deeper global insights into the system's rich dynamics, highlighting the profound impact of coupling strength and intrinsic map nonlinearity on the collective behavior of even simple interacting systems. These results underscore the power of coupling to induce order or complex synchronized states from potentially chaotic components, with broad implications for understanding natural and engineered systems.

References

[1] F. Udwadia and N. Raju, "Some global properties of a pair of coupled maps: Quasi-symmetry, periodicity, and synchronicity," Physica D: Nonlinear Phenomena, vol. 111, no. 1, pp. 16–26, 1998, doi: https://doi.org/10.1016/S0167-2789(97)80002-4.

[2] F. Udwadia and N. Raju, "Dynamics of coupled nonlinear maps and its application to ecological modeling," Applied Mathematics and Computation, vol. 82, no. 2, pp. 137–179, 1997, doi: https://doi.org/10.1016/S0096-3003(96)00027-6.