

# An Evidential Reasoning Approach for Multiple-Attribute Decision Making with Uncertainty

Jian-Bo Yang and Madan G. Singh, *Fellow, IEEE*

**Abstract**—A new evidential reasoning based approach is proposed that may be used to deal with uncertain decision knowledge in multiple-attribute decision making (MADM) problems with both quantitative and qualitative attributes. This approach is based on an evaluation analysis model and the evidence combination rule of the Dempster-Shafer theory. It is akin to a preference modeling approach, comprising an evidential reasoning framework for evaluation and quantification of qualitative attributes. Two operational algorithms have been developed within this approach for combining multiple uncertain subjective judgments. Based on this approach and a traditional MADM method, a decision making procedure is proposed to rank alternatives in MADM problems with uncertainty. A numerical example is discussed to demonstrate the implementation of the proposed approach. A multiple-attribute motor cycle evaluation problem is then presented to illustrate the hybrid decision making procedure.

## I. INTRODUCTION

**M**ULTIPLE-ATTRIBUTE decision making problems with both quantitative and qualitative attributes are common in practice [6], which we simply call hybrid MADM problems in this paper. At the concept design stage in engineering design, for example, alternative designs for a large engineering product need to be ranked or sorted by taking into account many technical and economical performances which are usually measured or evaluated using either numerical values with certain units or subjective judgments with uncertainty based on *a priori* experience. The aim of design is then to select from the existing alternative designs the best compromise alternative which attains these performances as closely as possible.

To solve a hybrid MADM problem, the first step is to evaluate and quantify the state of a qualitative attribute at each alternative. One of the simplest ways is to define a few evaluation grades for the attribute, which are quantified using a certain scale. The state of the attribute at an alternative may be evaluated to one of the grades. The scale of the confirmed grade may then be used as a numerical value for measuring the state of the attribute at the

alternative [6]. This approach is conceptually clear and easy to understand. However, it may be difficult to apply in practice because of two main reasons.

First of all, in a MADM problem, a qualitative attribute often represents an abstract concept representing an aggregated technical or economical performance comparable with other attributes. Such a qualitative attribute is generally difficult to assess directly, but may be possible to evaluate indirectly through a number of factors, which detail the attribute and are easier to assess directly. Secondly, it is improper to assume that subjective judgments for such evaluations might always be deterministic, even for the assessments of a factor. In other words, the decision maker may not always be 100% sure that the state of a factor is exactly confirmed to one of the evaluation grades. In fact, one or more grades may be confirmed at the same time with total confidence of exact or smaller than 100%. In the new approach to be reported in this paper, uncertain subjective judgments for the evaluation of qualitative attributes through multiple relevant factors will be accommodated within a framework based on the concept of preference degree and an evaluation analysis model [35].

Several tools are available for reasoning with uncertain decision knowledge. The Dempster-Shafer theory (simply D-S theory) is selected for the development of the new approach because of: 1) its powerful evidence combination rule, and 2) its reasonable requirement for the basic probability assignments that given a piece of evidence, the commitment of belief in a hypothesis does not necessarily mean that the remaining belief must be assigned to the complement of the hypothesis, but to the whole sample space [1], [19].

The second advantage of the D-S theory [i.e., 2)] indicates that the theory is well suited for handling incomplete uncertainty. This is particularly important and useful for dealing with uncertain subjective judgments when multiple factors need to be considered simultaneously. This is because even though each uncertain subjective judgment for the evaluation of a single factor provides a complete commitment to the evaluation grades (i.e., with the total confidence of exactly 100%), the total support from the factor for evaluation of its associated attribute may still be incomplete as each factor may have a different relative importance or a different role in evaluation of the attribute, as will be shown in the application examples in Section V. In other words, the total support from a

Manuscript received June 18, 1990; revised April 21, 1991, November 9, 1992, and February 17, 1993. This work was supported in part by the U.K. Science and Engineering Research Council under Grant GR/F 95306.

J.-B. Yang is with the Engineering Design Centre, University of Newcastle upon Tyne, Newcastle upon Tyne NE1 7RU, England.

M. G. Singh is with the Decision Technologies Group, Computation Department, University of Manchester Institute of Science and Technology, Manchester M60 1QD, England.

IEEE Log Number 9212928.

single factor could only be 100% if the factor absolutely dominated all other factors. However, this is not always the case. Otherwise, a single-factor analysis should be enough, which actually means that the attribute could be assessed directly. The D-S theory is so great that it can deal with such incomplete uncertainty in a more rational way than other tools in that given a piece of evidence, the unassigned belief in a hypothesis is just supposed to denote the unknown uncertainty, which instead of being necessarily assigned to the complement of the hypothesis, may eventually be assigned to any hyperthesis in the sample space when more evidence is gathered. This is the main reason why we have chosen the D-S theory to handle uncertainty for multiple factor analysis. Some concepts and the evidence combination rule of the D-S theory are introduced to develop evidence combination algorithms for combining uncertain decision knowledge.

In this paper, we will focus on developing the evidential reasoning approach. A decision making procedure for ranking alternatives in a hybrid MADM problem with uncertainty is also proposed, which is composed of the new approach and a traditional MADM method. In Section II, necessary basics are briefly discussed about hybrid MADM with uncertainty, the concept of preference degree, the evaluation analysis model, and the evidence combination rule. The new evidential reasoning approach is then explored in detail. In Section IV, the procedure for alternative ranking is proposed. Section V first presents a numerical example to demonstrate the implementation process of the new evidential reasoning approach. A multiple-attribute motorcycle evaluation problem is then presented to illustrate how to use the approach to deal with a real-world hybrid MADM problem with uncertainty.

## II. BASICS ABOUT HYBRID MADM WITH UNCERTAINTY

### A. Hybrid MADM Problems with Uncertainty

A hybrid MADM problem may be expressed using the following formula (1) or by an extended decision matrix, such as Table I.

$$\text{optimize } y(a) = [y_1(a) \cdots y_k(a) \cdots y_{k_1+k_2}(a)], \quad a \in \Omega \quad (1)$$

In (1),  $\Omega$  is a discrete set of alternatives. In Table I,  $y_{ij}$  is a numerical value of  $y_j$  at  $a_i$  ( $i = 1, \dots, l; j = 1, \dots, k_1$ ) and  $S_{ij}$  are subjective judgments with uncertainty for evaluation of the states of  $y_{k_1+j}$  at  $a_i$  ( $i = 1, \dots, l; j = 1, \dots, k_2$ ). The problem is to rank these alternatives or to select the best compromise alternative, with both quantitative and qualitative attributes being simultaneously satisfied as much as possible.

It is therefore fundamental to evaluate and quantify qualitative attributes so that the extended decision matrix can be transformed into an ordinary decision matrix, and then a traditional MADM method may be used for ranking alternatives. A simple method for the evaluation and quantification is to define a few evaluation grades such

that the state of an attribute at an alternative could be evaluated to one of the grades. Then, these grades may be quantified using certain scales [6]. This method may be practical if the decision maker is able to evaluate qualitative attributes synthetically and deterministically using only a few discrete evaluation grades.

In a hybrid MADM problem, however, a qualitative attribute may represent an aggregated technical and economical concept so that it is comparable with other attributes. Such an attribute may only be evaluated through a number of relevant factors which detail the attribute and are easier to evaluate directly. In addition to this, the evaluations of a factor may not always be deterministic. Rather, uncertain subjective judgments may often be provided by the decision maker.

In a problem of evaluating different types of motorcycles, for example, the following type of uncertain subjective judgments was frequently used [7].

*Statement i >* The *responsiveness* of the *engine* of "Yamaha" is evaluated to be *good* with a confidence degree of 0.3 and to be *excellent* with a confidence degree of 0.6.

In the statement, "Yamaha" is an alternative motor cycle, *engine* a qualitative attribute comparable with other attributes such as *price*, *responsiveness* a factor for the evaluation of *engine*, *good* and *excellent* are evaluation grades representing distinct states of *engine*, and the confidence degrees 0.3 and 0.6 represent the uncertainty in the evaluation. Note that the total confidence degree in Statement i > is 0.9, smaller than one.

To evaluate *engine* of a motorcycle, other factors such as *fuel economy* and *quietness* may need to be considered as well. In this case, similar statements may also be used to evaluate the *fuel economy* and *quietness* of the *engine* of "Yamaha." It is then essential to combine these multiple uncertain judgments to produce an aggregated evaluation for *engine*. The following sections are therefore focused on the development of an approach, comprising multiple-factor analysis and evidential reasoning, so that qualitative attributes may be evaluated and quantified. As a result, a decision making procedure will be proposed for ranking alternatives in a hybrid MADM problem with uncertainty.

### B. Evaluation Analysis Model

In [27], [35], an evaluation analysis model was proposed to represent uncertain subjective judgments, such as Statement i >. The model is shown in Fig. 1.

In the attribute level of the model, the state of an attribute (such as *engine*) at each alternative  $a$  (such as "Yamaha") is required to be evaluated. In the evaluation grade level,  $H_n$  is called an evaluation grade (such as *good*) ( $n = 1, \dots, N$ ). A set of evaluation grades for an attribute  $y_k$  is denoted by

$$H = \{H_1 \ H_2 \ \cdots \ H_n \ \cdots \ H_N\} \quad (2)$$

where  $N$  is the number of evaluation grades.  $H_n$  represents a grade to which the state of  $y_k$  may be evaluated.  $H_1$  and

TABLE I  
AN EXTENDED DECISION MATRIX

Alternatives ( $a_i$ )	Quantitative Attributes ( $y_k$ )				Qualitative Attributes ( $y_k$ )			
	$y_1$	$y_2$	$\dots$	$y_{k_1}$	$y_{k_1+1}$	$y_{k_1+2}$	$\dots$	$y_{k_1+k_2}$
$a_1$	$y_{11}$	$y_{12}$	$\dots$	$y_{1k_1}$	$SJ_{11}$	$SJ_{12}$	$\dots$	$SJ_{1k_2}$
$a_2$	$y_{21}$	$y_{22}$	$\dots$	$y_{2k_1}$	$SJ_{21}$	$SJ_{22}$	$\dots$	$SJ_{2k_2}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$a_i$	$y_{i1}$	$y_{i2}$	$\dots$	$y_{ik_1}$	$SJ_{i1}$	$SJ_{i2}$	$\dots$	$SJ_{ik_2}$

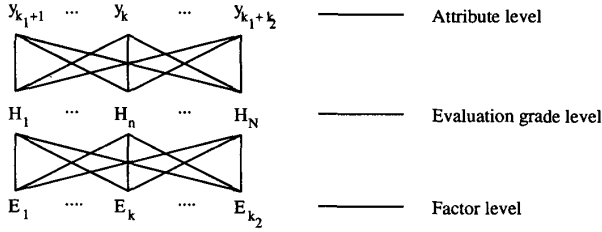


Fig. 1. An evaluation analysis model.

$H_N$  are set to be the worst and the best grades, respectively, and  $H_{n+1}$  is supposed to be preferred to  $H_n$ . It should be kept in mind that an attribute may have its own set of evaluation grades different from those of other attributes, although Fig. 1 only lists one set and  $H$  of (2) is not defined as  $H^k$  in order to simplify the following discussion.

In the model, the concept of preference degree was introduced, which may be used to quantify these evaluation grades and eventually to quantify subjective judgments with uncertainty. A preference degree takes values from the close interval  $[-1, 1]$ , which may be called the preference degree space. The set of evaluation grades may thus be quantified by

$$p\{H\} = [p(H_1) \dots p(H_n) \dots p(H_N)]^T \quad (3)$$

where  $p(H_n)$  is the scale of  $H_n$  and satisfies the following basic conditions:

$$\begin{aligned} p(H_1) &= -1, \quad p(H_N) = 1; \\ p(H_{n+1}) &> p(H_n), \quad n = 1, \dots, N-1. \end{aligned} \quad (4)$$

Besides,  $p(H_n)$  ( $n = 2, \dots, N-1$ ) should be so assigned that an additional consistence condition, defined by (21) in the next section, can be satisfied.

Suppose  $N = 7$ , for example,  $H$  may be defined as follows:

$$\begin{aligned} H &= \{H_1 \ H_2 \ H_3 \ H_4 \ H_5 \ H_6 \ H_7\} \\ &= \{\text{the most unsatisfactory, very unsatisfactory,} \\ &\quad \text{unsatisfactory, indifferent,} \\ &\quad \text{satisfactory, very satisfactory, the most satisfactory}\}. \end{aligned} \quad (5)$$

Without loss of generality, we may scale  $H_n$  ( $n = 1, \dots, 7$ ) by using real numbers in  $[-1, 1]$ ; for example,

$$\begin{aligned} p\{H\} &= [p(H_1) \ p(H_2) \ p(H_3) \ p(H_4) \ p(H_5) \\ &\quad p(H_6) \ p(H_7)]^T \\ &= [-1 \ -0.8 \ -0.4 \ 0 \ 0.4 \ 0.8 \ 1]^T. \end{aligned} \quad (6)$$

In the factor level,  $E_k$  represents a set of factors which are associated with the evaluation of the attribute  $y_k(a)$  and denoted by

$$E_k = \{e_k^1 \ e_k^2 \ \dots \ e_k^{L_k}\} \quad k = k_1 + 1, \dots, k_1 + k_2 \quad (7)$$

where  $e_k^i$  ( $i = 1, \dots, L_k$ ) are factors (such as *responsiveness*) influencing the evaluation of  $y_k(a)$ . The state of  $e_k^i$  can be directly evaluated at an alternative  $a$ , that is,  $e_k^i = e_k^i(a)$ .

The new approach developed in this paper is then devoted to generating the preference degree for the state of an attribute  $y_k(a)$  at each alternative  $a$  through the direct evaluations of the relevant factors  $e_k^i$  ( $i = 1, \dots, L_k$ ). The generated preference degrees for the attribute have to satisfy certain rational assumptions such as the monotonicity of its marginal utilities.

If there is only one factor  $e_k^1$  associated with  $y_k(a)$  and its state at  $a$  is exactly confirmed to one of the evaluation grades in  $H$ , such as  $H_n$ , the procedure for evaluation of  $y_k(a)$  through  $e_k^1$  may be as simple as to use the scale of the confirmed evaluation grade as the preference degree of  $y_k(a)$ , denoted by  $p(y_k(a))$ , or  $p(y_k(a)) = p(e_k^1(a)) = p(H_n)$  where  $p(e_k^1(a))$  denotes the preference degree of  $e_k^1$  at  $a$ . A more general evaluation procedure is to be explored based on the evaluation analysis model and the evidence combination rule of the Dempster-Shafer theory.

### C. Evidence Combination Rule

The D-S theory is one of the powerful tools to deal with uncertainty. We do not attempt to discuss all of its details in this paper, but we will only use its evidence combination rule to develop evidence combination algorithms for the new approach.

In the D-S theory, a sample space is called a "frame of discernment," defined as  $\Theta$ . A basic hyperthesis (singleton) in  $\Theta$  is denoted by  $H_s$ , i.e.,  $H_s \subseteq \Theta$ . In  $\Theta$ , all basic hypertheses are required to be mutually exclusive and exhaustive. A probability mass to every subset  $\Psi$  of

$\Theta (\Psi \subseteq \Theta)$  can be assigned, denoted by  $m(\Psi)$ . The probability mass is called the basic probability assignment, which is a number in the interval  $[0, 1]$  to indicate belief in a hypothesis given a piece of evidence, or the degree to which the evidence supports the hypothesis.

A basic probability assignment satisfies the following condition [19]:

$$\sum_{\Psi \subseteq \Theta} m(\Psi) = 1, \quad m(\emptyset) = 0; \quad (8)$$

$$0 \leq m(\Psi) \leq 1, \text{ for all } \Psi \subseteq \Theta.$$

$m(\Psi)$  indicates that portion of the total belief exactly committed to hypothesis  $\Psi$  given a piece of evidence. In other words,  $m(\Psi)$  represents the direct support of evidence on  $\Psi$ . This portion of belief cannot be further subdivided among the subsets of  $\Psi$ , and does not include the portion of belief committed to subsets of  $\Psi$ .

The quantity  $m(\Theta)$  is a measure of that portion of the total belief that remains unassigned after commitment of belief to all subsets of  $\Theta$ . If  $m(\Psi) = s (\Psi \subseteq \Theta)$  and no belief is assigned to other subsets of  $\Theta$ , for example, then  $m(\Theta) = 1 - s$ . Thus, the remaining belief is assigned to  $\Theta$ , but not to the negation of the hypothesis  $\Psi$  (the complement of  $\Psi$ ).

Suppose there exist two pieces of evidence in  $\Theta$ , and that they provide two basic probability assignments to a subset  $\Psi$  of  $\Theta$ , i.e.,  $m_1(\Psi)$  and  $m_2(\Psi)$ . The problem is to obtain a combined probability assignment  $m_{12}(\Psi) = m_1(\Psi) \oplus m_2(\Psi)$ . The D-S theory provides an evidence combination rule defined below [19]:

$$m_{12}(\emptyset) = 0, \quad m_{12}(\Psi) = \sum_{A \cap B = \Psi} \frac{m_1(A)m_2(B)}{1 - K} \quad (9)$$

$$K = \sum_{A \cap B = \emptyset} m_1(A)m_2(B). \quad (10)$$

In the rule,  $m_{12}(\Psi)$  for hypothesis  $\Psi (\subseteq \Theta)$  is computed from  $m_1$  and  $m_2$  by adding all products of the form  $m_1(A)m_2(B)$  where  $A$  and  $B$  are selected from the subsets of  $\Theta$  in all possible ways such that their intersection is  $\Psi$ .  $K$  reflects the conflicting situations where both  $m_1(A)$  and  $m_2(B)$  are not zero, but the intersection  $A \cap B$  is empty. The commutativity of multiplication in the rule ensures that the rule yields the same value regardless of the order in which the two pieces of evidence are combined.

It is easy to show that the direct use of the combination rule will result in an exponential increase in computational complexity [1]. This is due to the need to enumerate all subsets or supersets of a given subset  $\Psi$  of  $\Theta$ . The following section is therefore intended to develop operational algorithms for evidence combination which reduce the computational complexity to linear time by utilizing the characteristics of the evidence combination process based on the evaluation analysis model.

### III. EVIDENTIAL REASONING APPROACH

#### A. An Evidential Reasoning Framework

In the evaluation analysis mode, an evaluation grade  $H_n$  may be considered as a basic hypothesis (singleton) in the D-S theory, a factor  $e_k^i$  as a piece of evidence, and a basic

probability assignment may be obtained from a confidence degree. To apply the evidence combination rule, however, the mutual exclusiveness and exhaustiveness of all basic hyperthesis have to be satisfied. It is therefore necessary that all the evaluation grades in  $H$  be defined as distinct grades. In other words, if one of the evaluation grades is absolutely confirmed, that is, the confidence degree is one, all the other grades must not be confirmed at all; if more than one grade is confirmed simultaneously, the total confidence degree must be one or smaller than one. In addition to this requirement, the evaluation grades defined in  $H$  must cover all possible grades the decision maker may use for evaluation of an attribute at all alternatives. Then, the frame of discernment may be defined by

$$\Theta = H = \{H_1 \cdots H_n \cdots H_N\}. \quad (11)$$

Let  $m(H_n/e_k^i(a))$  express a basic probability assignment to which  $e_k^i$  supports a hypothesis that the state of  $y_k$  at an alternative  $a$  is confirmed to  $H_n$ . Let  $\beta_{H_n}(e_k^i(a))$  be a confidence degree to which the decision maker considers that the state of  $e_k^i$  at an alternative  $a$  is confirmed to  $H_n$ .  $m(H_n/e_k^i(a))$  may be obtained from  $\beta_{H_n}(e_k^i(a))$ . For a rational decision maker, we assume that he only provides uncertain subjective judgments satisfying the following rationality assumption.

**Rationality Assumption:** If a decision maker recognizes that the state of a factor  $e_k^i$  has to be confirmed to an evaluation grade  $H_n$  to some extent, then he may express his uncertain subjective judgments only in one of the following three manners.

1)  $e_k^i$  is only confirmed to  $H_n$  to the extent of  $\beta_{H_n}(e_k^i(a))$  while  $0 < \beta_{H_n}(e_k^i(a)) \leq 1$ .

2)  $e_k^i$  may be confirmed to  $H_n$  and to  $H_{n+1}$  at the same time to the extents of  $\beta_{H_n}(e_k^i(a))$  and  $\beta_{H_{n+1}}(e_k^i(a))$ , respectively, while  $0 < \beta_{H_n}(e_k^i(a)), \beta_{H_{n+1}}(e_k^i(a)) \leq 1$ , and  $\beta_{H_n}(e_k^i(a)) + \beta_{H_{n+1}}(e_k^i(a)) \leq 1$ .

3)  $e_k^i$  may be confirmed to  $H_{n-1}$  and to  $H_n$  at the same time to the extents of  $\beta_{H_{n-1}}(e_k^i(a))$  and  $\beta_{H_n}(e_k^i(a))$ , respectively, while  $0 < \beta_{H_{n-1}}(e_k^i(a)), \beta_{H_n}(e_k^i(a)) \leq 1$ , and  $\beta_{H_{n-1}}(e_k^i(a)) + \beta_{H_n}(e_k^i(a)) \leq 1$ .

This assumption, however, is only made based upon our experience and may not be universally satisfied. In the extensions of the approach reported in [31], [32], this assumption has actually been abandoned, although the newly developed approaches need more computational effort. Obviously, Statement i) satisfies the rationality assumption. From the rationality assumption, we can classify the set of factors  $E_k = [e_k^1, e_k^2, \dots, e_k^{L_k}]$  into  $N - 1$  subsets  $S_n$ , defined by

$$S_n = \{e_{n,n+1}^1, \dots, e_{n,n+1}^i, \dots, e_{n,n+1}^{R_n}\} \quad (12)$$

$$n = 1, \dots, N - 1$$

where  $e_{n,n+1}^i$  is a factor in  $E_k$ , the state of which is confirmed to  $H_n$  and/or to  $H_{n+1}$ , and  $L_k = R_1 + R_2 + \dots + R_{N-1}$ . Because the evidence combination rule defined in (9) and (10) is independent of the order in which factors

are gathered [19], such classification will be useful to develop operational algorithms for evidence combination.

We are now in a position to summarize an evidential reasoning framework. The basic probability assignments of all hypothesis (subsets of  $\Theta$ ) are first generated from the confidence degrees. Suppose the basic probability assignment of  $H_n$  with respect to  $e_{n,n+1}^i$  is denoted by  $m(H_n/e_{n,n+1}^i)$ , simply  $m_{n,n+1}^{n,i}$ , that of  $H_{n+1}$  by  $m(H_{n+1}/e_{n,n+1}^i)$ , simply  $m_{n+1}^{n,i}$ , and that of  $\Theta$  by  $m(\Theta/e_{n,n+1}^i)$ , simply  $m_{\Theta}^{n,i}$ .

Secondly, combine the  $R_n$  factors which confirm  $H_n$  and/or  $H_{n+1}$ . At this step, let

$$e_{n,n+1}^{(i)} = \{e_{n,n+1}^1, e_{n,n+1}^2, \dots, e_{n,n+1}^i\} \quad (13)$$

$$m_n^{(i)} = m(H_n/e_{n,n+1}^{(i)}), \quad m_{n+1}^{(i)} = m(H_{n+1}/e_{n,n+1}^{(i)}),$$

$$m_{\Theta}^{(i)} = m(\Theta/e_{n,n+1}^{(i)}). \quad (14)$$

Therefore,  $e_{n,n+1}^{(R_n)} = \{e_{n,n+1}^1, \dots, e_{n,n+1}^{R_n}\} = S_n$ , and  $m_n^{(R_n)}$ ,  $m_{n+1}^{(R_n)}$ , and  $m_{\Theta}^{(R_n)}$  ( $n = 1, \dots, N-1$ ) are called the local probability assignments partially combined from the  $R_n$  factors.

Then, combine all factors in  $E_k$ . At this step, define

$$e_{1,j+1}^{C(j)} = \{e_{1,2}^{(R_1)}, e_{2,3}^{(R_2)}, \dots, e_{j,j+1}^{(R_j)}\} \quad (15)$$

$$b_n^{C(j)} = m(H_n/e_{1,j+1}^{C(j)}), \quad n = 1, \dots, N. \quad (16)$$

These symbols defined in (13)–(16) will be used to develop our partial and overall evidence combination algorithms, and they are expected to make it clearer to describe the computational procedures of the algorithms separately.

From (15), it is obvious that  $e_{1,N}^{C(N-1)} = \{e_1^1, e_2^2, \dots, e_N^N\} = E_k$  and  $b_n^{C(N-1)} = m(H_n/E_k)$ , where  $m(H_n/E_k)$  is the overall probability assignment to which the state of an attribute  $y_k(a)$  at an alternative  $a$  is confirmed to  $H_n$ . Suppose  $\Psi$  is a subset of  $\Theta$ . Then,  $m(\Psi/E_k)$  is defined as the overall probability assignment to which the state of  $y_k(a)$  at  $a$  is confirmed to  $\Psi$ . Let  $p(\Psi)$  stand for the scale of  $\Psi$ , which is defined as the average of the scales of the singletons involved in  $\Psi$ .

It is possible that the state of  $y_k$  at  $a_r$  may be confirmed by the factor set  $E_k$  to any subset  $\Psi$  of  $\Theta$  to an extent of  $m(\Psi/E_k(a_r))$ . The state of  $y_k(a_r)$  may therefore be denoted by the following expectation:

$$S(y_k(a_r)) = \{(m(\Psi/E_k(a_r)), \Psi), \quad \text{for all } \Psi \subseteq H\}. \quad (17)$$

In (17), each subset  $\Psi$  of  $H$  actually represents a possible state (single evaluation grades or their combinations) into which the state of an attribute may possibly fall at a particular alternative and  $m(\Psi/E_k(a_r))$  represents the total support by all the factors to the hypothesis that the state of  $y_k$  at  $a_r$  is confirmed to  $\Psi$ . Thus, (17) actually describes the distribution of the state of  $y_k$  at  $a_r$  among all possible states. If the distribution of the state of  $y_k$  at  $a_r$  favors good subsets  $\Psi$  in  $H$  more than the distribution at  $a_h$ ,  $S(y_k(a_r))$  should be better than  $S(y_k(a_h))$ . As  $\Sigma_{\Psi \subseteq H}$

$m(\Psi/E_k(a_r)) = 1$ ,  $m(\Psi/E_k(a_r))$  may thus be explained as a plausible probability that the state of  $y_k$  at  $a_r$  falls into  $\Psi$ . In this context, let us define  $p(\Psi)$  as a numerical value to express the relative intensity of the state  $\Psi$  compared with all other possible states and let a larger value represent a better state. Then, the intensity of the state of  $y_k$  at  $a_r$  may be defined as the expected value of the intensities of each state  $\Psi$  confirmed at  $a_r$  with a plausible probability of  $m(\Psi/E_k(a_r))$  for all  $\Psi \subseteq H$ . Let a preference degree express such an expected value, and the preference degree of  $y_k(a_r)$  be denoted by  $p_{rk} = p(y_k(a_r))$ , quantifying  $S(y_k(a_r))$ .  $p_{rk}$  is then calculated by

$$p_{rk} = \sum_{\Psi \subseteq H} m(\Psi/E_k(a_r)) p(\Psi). \quad (18)$$

It is therefore rational to state that if at an alternative  $a_r$  an attribute  $y_k$  has a larger preference degree than at another alternative  $a_h$ , then the state of  $y_k$  at  $a_r$  ought to be better than the state of  $y_k$  at  $a_h$ . In other words, for two alternatives  $a_r$  and  $a_h$ ,  $S(y_k(a_r))$  is preferred to  $S(y_k(a_h))$  if and only if  $p_{rk} > p_{hk}$ . A qualitative attribute  $y_k$  can thus be quantified with its marginal utilities being monotonous, which forms a rational basis for further decision analysis.

#### B. Acquisition and Representation of Uncertain Decision Knowledge

An uncertain subjective judgment may be acquired using a statement such as Statement i>. It is used to evaluate the state of a factor or an attribute at an alternative, indicating to which evaluation grades the state is confirmed and to what extents these evaluation grades are confirmed.

It is assumed that such uncertain subjective judgments satisfy the rationality assumption. Suppose a judgment states that the state of a factor  $e_k^i$  at  $a_r$  is confirmed to  $H_n$  to the extent of  $\beta_{H_n}(e_k^i(a_r))$  and to  $H_{n+1}$  to the extent of  $\beta_{H_{n+1}}(e_k^i(a_r))$ . The state of  $e_k^i(a_r)$  may be represented by the following expectation:

$$S(e_k^i(a_r)) = \{(\beta_{H_n}(e_k^i(a_r)), H_n); (\beta_{H_{n+1}}(e_k^i(a_r)), H_{n+1})\} \quad (19)$$

where  $\beta_{H_n}(e_k^i(a_r)) + \beta_{H_{n+1}}(e_k^i(a_r)) \leq 1$ . Compared with (17) and (18), the state of  $e_k^i(a_r)$  may then be quantified using the preference degree, defined as the following expected scale:

$$p_{ri} = p(e_k^i(a_r)) = \beta_{H_n}(e_k^i(a_r)) p(H_n) + \beta_{H_{n+1}}(e_k^i(a_r)) p(H_{n+1}). \quad (20)$$

The scales of  $H_n$ , i.e.,  $p(H_n)$ , ( $n = 2, \dots, N-1$ ) therefore need to be defined so that in addition to the basic conditions defined by (4), the following consistence condition is also satisfied, that is, for two alternatives  $a_r$  and  $a_h$ ,

$$S(e_k^i(a_r)) \text{ is preferred to } S(e_k^i(a_h)) \text{ if and only if } p_{ri} > p_{hi}. \quad (21)$$

A preference comparison of the state of one factor with that of another, such as  $S(e_k^i(a_r))$  with  $S(e_k^i(a_h))$ , may be provided by the decision maker. If sufficient number of such preference comparisons are obtained,  $p(H_n)$  ( $n = 1, \dots, N$ ) may then be assessed by satisfying the constraints defined by (4) and (21).

By definition, the confidence degree  $\beta_{H_n}(e_k^i(a))$  expresses the intensity to which the state of a single factor  $e_k^i$  at  $a$  is confirmed to an evaluation grade  $H_n$ . On the other hand, the basic probability assignment  $m(H_n/e_k^i(a))$  represents the degree to which  $e_k^i$  supports a hypothesis that the state of the attribute  $y_k$  at  $a$  is confirmed to  $H_n$ . If there is only one factor  $e_k^i$  in  $E_k$ ,  $m(H_n/e_k^i(a))$  should be equal to  $\beta_{H_n}(e_k^i(a))$ ; if there are multiple factors in  $E_k$ , however, they may play different roles in evaluation of  $y_k$ , depending upon their relative importance. Therefore, the weighted confidence degree may be used as the basic probability assignment. Suppose  $\lambda_k^i$  is the normalized relative weight of  $e_k^i$  in  $E_k$  and  $\lambda_k = [\lambda_k^1 \dots \lambda_k^{L_k}]^T$ . Then,  $m(H_n/e_k^i(a))$  may be determined by

$$m(H_n/e_k^i(a)) = \lambda_k^i \beta_{H_n}(e_k^i(a)). \quad (22)$$

$\lambda_k$  may be obtained as follows. Suppose  $\zeta_k = [\zeta_k^1 \dots \zeta_k^{L_k}]^T$  is a uniform weight vector, where  $\zeta_k^i$  expresses the relative importance of  $e_k^i$  and

$$\sum_{i=1}^{L_k} \zeta_k^i = 1 \quad 0 \leq \zeta_k^i \leq 1. \quad (23)$$

Let  $e_k^l$  be the most important factor in  $E_k$ , called the key factor, that is,  $\zeta_k^l = \max_i \{\zeta_k^1, \dots, \zeta_k^i, \dots, \zeta_k^{L_k}\}$ . Normalize  $\zeta_k$  as follows:

$$\bar{\zeta}_k^i = \zeta_k^i / \zeta_k^l \quad i = 1, \dots, L_k. \quad (24)$$

If for the key factor the following relation is true

$$m(H_n/e_k^l) = \alpha_k \beta_{H_n}(e_k^l) \quad 0 < \alpha_k \leq 1 \quad (25)$$

then  $\lambda_k^i$  ( $i = 1, \dots, L_k$ ) may be determined by

$$\lambda_k^i = \alpha_k \bar{\zeta}_k^i \quad i = 1, \dots, L_k. \quad (26)$$

$\alpha_k$  may be referred to as a priority coefficient representing the importance of the role the most importance factor plays for evaluation of the attribute  $y_k$ .

In this way, the basic probability assignments required in the combination rule can be generated from uncertain subjective judgments. The overall probability assignment can then be obtained by combining all the basic probability assignments using the operational algorithms explored in the following subsections.

### C. Partial Combination Algorithm

As mentioned before, the set of factors can be classified into  $N - 1$  subsets denoted by  $S_n$  ( $n = 1, \dots, N - 1$ ). In  $S_n$ , there are  $R_n$  factors as defined by formula (12), the states of which may be confirmed to  $H_n$  and/or to  $H_{n+1}$ . In this subsection, an algorithm will be developed to generate the local probability assignments to  $H_n$  and  $H_{n+1}$  by combining these  $R_n$  factors. For computational purposes,

the "intersection tableau" [1] with values of probability assignments along the rows and columns, respectively, is adopted to develop the algorithm.

At first, combine two factors,  $e_{n,n+1}^1$  and  $e_{n,n+1}^2$ . Suppose the basic probability assignments  $m_n^{n,i}$  and  $m_{n+1}^{n,i}$  ( $i = 1, \dots, R_n$ ) are obtained using formula [22]; then  $m_\Theta^{n,i} = 1 - (m_n^{n,i} + m_{n+1}^{n,i})$  ( $i = 1, \dots, R_n$ ). All these basic probability assignments to  $H_n$ ,  $H_{n+1}$ , and  $\Theta$  with respect to  $e_{n,n+1}^i$  ( $i = 1, \dots, R_n$ ;  $n = 1, \dots, N - 1$ ) may then be expressed by the following basic probability assignment matrices  $M^n$ :

$$M^n = \begin{bmatrix} m_n^{n,1} & m_{n+1}^{n,1} & m_\Theta^{n,1} \\ m_n^{n,2} & m_{n+1}^{n,2} & m_\Theta^{n,2} \\ \dots & \dots & \dots \\ m_n^{n,R_n} & m_{n+1}^{n,R_n} & m_\Theta^{n,R_n} \end{bmatrix} \begin{Bmatrix} \{e_{n,n+1}^1\} \\ \{e_{n,n+1}^2\} \\ \dots \\ \{e_{n,n+1}^{R_n}\} \end{Bmatrix} \quad (n = 1, \dots, N - 1). \quad (27)$$

If  $R_n = 0$ , then  $m_n^{n,i} = 0$ ,  $m_{n+1}^{n,i} = 0$  and  $m_\Theta^{n,i} = 1$ . Then, construct intersection tableau 1. (See Table II.)

From the combination rule shown in formulas (9) and (10), we have

$$\{H_n\}: m_n^{l(2)} = K^{l(2)}(m_n^{n,1} m_n^{n,2} + m_{n+1}^{n,1} m_\Theta^{n,2} + m_\Theta^{n,1} m_n^{n,2})$$

$$\{H_{n+1}\}: m_{n+1}^{l(2)} = K^{l(2)}(m_{n+1}^{n,1} m_{n+1}^{n,2} + m_n^{n,1} m_\Theta^{n,2} + m_\Theta^{n,1} m_{n+1}^{n,2})$$

$$\{\Theta\}: m_\Theta^{l(2)} = K^{l(2)} m_\Theta^{n,1} m_\Theta^{n,2}$$

where

$$K^{l(2)} = [1 - (m_n^{n,1} m_{n+1}^{n,2} + m_{n+1}^{n,1} m_n^{n,2})]^{-1}.$$

As to  $e_{n,n+1}^{l(2)}$ , the partially combined probability assignments to other hypothesis in  $\Theta$  are all zero.

Now, let us combine  $e_{n,n+1}^{l(3)} = \{e_{n,n+1}^1, e_{n,n+1}^2, e_{n,n+1}^3\}$ . Similarly, construct intersection tableau 2 (Table III). From the combination rule, we can obtain

$$\{H_n\}: m_n^{l(3)} = K^{l(3)}(m_n^{l(2)} m_n^{n,3} + m_{n+1}^{l(2)} m_\Theta^{n,3} + m_\Theta^{l(2)} m_n^{n,3})$$

$$\{H_{n+1}\}: m_{n+1}^{l(3)} = K^{l(3)}(m_{n+1}^{l(2)} m_{n+1}^{n,3} + m_n^{l(2)} m_\Theta^{n,3} + m_\Theta^{l(2)} m_{n+1}^{n,3})$$

$$\{\Theta\}: m_\Theta^{l(3)} = K^{l(3)} m_\Theta^{l(2)} m_\Theta^{n,3}$$

where

$$K^{l(3)} = [1 - (m_n^{l(2)} m_{n+1}^{n,3} + m_{n+1}^{l(2)} m_n^{n,3})]^{-1}.$$

Since  $m_n^{l(1)} = m_n^{n,1}$ ,  $m_{n+1}^{l(1)} = m_{n+1}^{n,1}$  and  $m_\Theta^{l(1)} = m_\Theta^{n,1}$ , then it is natural that by combining  $e_{n,n+1}^{l(r+1)} = \{e_{n,n+1}^1, \dots, e_{n,n+1}^{l(r)}\}$ , we can obtain the following recursive formulas:

$$\{H_n\}: m_n^{l(r+1)} = K^{l(r+1)}(m_n^{l(r)} m_n^{n,r+1} + m_{n+1}^{l(r)} m_\Theta^{n,r+1} + m_\Theta^{l(r)} m_n^{n,r+1}) \quad (28.a)$$

TABLE II  
INTERSECTION TABLEAU 1

		$e_{n,n+1}^2$		
$e_{n,n+1}^{l(2)}$		$\{H_n\} (m_n^{n,2})$	$\{H_{n+1}\} (m_{n+1}^{n,2})$	$\{\Theta\} (m_\Theta^{n,2})$
$e_{n,n+1}^1$	$\{H_n\} (m_n^{n,1})$	$\{H_n\} (m_n^{n,1} m_n^{n,2})$	$\{\emptyset\} (m_n^{n,1} m_{n+1}^{n,2})$	$\{H_n\} (m_n^{n,1} m_\Theta^{n,2})$
	$\{H_{n+1}\} (m_{n+1}^{n,1})$	$\{\emptyset\} (m_{n+1}^{n,1} m_n^{n,2})$	$\{H_{n+1}\} (m_{n+1}^{n,1} m_{n+1}^{n,2})$	$\{H_{n+1}\} (m_{n+1}^{n,1} m_\Theta^{n,2})$
	$\{\Theta\} (m_\Theta^{n,1})$	$\{H_n\} (m_\Theta^{n,1} m_n^{n,2})$	$\{H_{n+1}\} (m_\Theta^{n,1} m_{n+1}^{n,2})$	$\{\Theta\} (m_\Theta^{n,1} m_\Theta^{n,2})$

TABLE III  
INTERSECTION TABLEAU 2

		$e_{n,n+1}^3$		
$e_{n,n+1}^{l(3)}$		$\{H_n\} (m_n^{n,3})$	$\{H_{n+1}\} (m_{n+1}^{n,3})$	$\{\Theta\} (m_\Theta^{n,3})$
$e_{n,n+1}^{l(2)}$	$\{H_n\} (m_n^{l(2)})$	$\{H_n\} (m_n^{l(2)} m_n^{n,3})$	$\{\emptyset\} (m_n^{l(2)} m_{n+1}^{n,3})$	$\{H_n\} (m_n^{l(2)} m_\Theta^{n,3})$
	$\{H_{n+1}\} (m_{n+1}^{l(2)})$	$\{\emptyset\} (m_{n+1}^{l(2)} m_n^{n,3})$	$\{H_{n+1}\} (m_{n+1}^{l(2)} m_{n+1}^{n,3})$	$\{H_{n+1}\} (m_{n+1}^{l(2)} m_\Theta^{n,3})$
	$\{\Theta\} (m_\Theta^{l(2)})$	$\{H_n\} (m_\Theta^{l(2)} m_n^{n,3})$	$\{H_{n+1}\} (m_\Theta^{l(2)} m_{n+1}^{n,3})$	$\{\Theta\} (m_\Theta^{l(2)} m_\Theta^{n,3})$

$$\{H_{n+1}\}: m_{n+1}^{l(r+1)} = K^{l(r+1)} (m_{n+1}^{l(r)} m_{n+1}^{n,r+1} + m_{n+1}^{l(r)} m_\Theta^{n,r+1} + m_\Theta^{l(r)} m_{n+1}^{n,r+1}) \quad (28.b)$$

$$\{\Theta\}: m_\Theta^{l(r+1)} = K^{l(r+1)} m_\Theta^{l(r)} m_\Theta^{n,r+1} \quad (28.c)$$

where

$$K^{l(r+1)} = [1 - (m_n^{l(r)} m_{n+1}^{n,r+1} + m_{n+1}^{l(r)} m_n^{n,r+1})]^{-1} \\ r = 1, \dots, R_n - 1; \quad n = 1, \dots, N - 1. \quad (28.d)$$

The formulas (28) constitute a partial combination algorithm.  $m_n^{l(r+1)}$ ,  $m_{n+1}^{l(r+1)}$ , and  $m_\Theta^{l(r+1)}$  are the partially combined probability assignments to  $H_n$ ,  $H_{n+1}$ , and  $\Theta$ , respectively, with respect to  $e_{n,n+1}^{l(r+1)} = \{e_{n,n+1}^{l(r)}, \dots, e_{n,n+1}^{l(r+1)}\}$ . The local probability assignments to  $H_n$ ,  $H_{n+1}$ , and  $\Theta$  with respect to the subset of factors  $S_n$  can be represented as  $m_n^{l(R_n)}$ ,  $m_{n+1}^{l(R_n)}$ , and  $m_\Theta^{l(R_n)}$ . To represent the results of the partial combination of all subsets of factors, the following matrix is suggested, called the local probability assignment matrix:

$$M = \begin{bmatrix} m_1^{l(R_1)} & m_2^{l(R_1)} & m_\Theta^{l(R_1)} \\ \dots & \dots & \dots \\ m_n^{l(R_n)} & m_{n+1}^{l(R_n)} & m_\Theta^{l(R_n)} \\ \dots & \dots & \dots \\ m_{N-1}^{l(R_{N-1})} & m_N^{l(R_{N-1})} & m_\Theta^{l(R_{N-1})} \end{bmatrix} \begin{bmatrix} \{e_{1,2}^{l(R_1)}\} \\ \dots \\ \{e_{n,n+1}^{l(R_n)}\} \\ \dots \\ \{e_{N-1,N}^{l(R_{N-1})}\} \end{bmatrix}. \quad (29)$$

#### D. Overall Combination Algorithm

After the partial combination, the subset of factors  $S_n$  may be regarded as an aggregated factor, and  $m_n^{l(R_n)}$  as a new basic probability assignment to the hypothesis  $H_n$ , confirmed by  $S_n$ . The problem is then to combine all these integrated factors in order to obtain the overall probability assignments to all subsets  $\Psi$  of  $\Theta$ , including the single-

tons  $H_n$  ( $n = 1, \dots, N$ ). We still use the intersection tableau to develop such an overall combination algorithm.

First of all, combine  $e_{1,3}^{C(2)} = \{e_{1,2}^{l(R_1)}, e_{2,3}^{l(R_2)}\}$ . Construct intersection tableau 3. (See Table IV.)

From the combination rule, we have

$$\{H_1\}: b_1^{C(2)} = K^{C(2)} m_1^{l(R_1)} m_\Theta^{l(R_2)} \\ \{H_2\}: b_2^{C(2)} = K^{C(2)} (m_2^{l(R_1)} m_2^{l(R_2)} + m_2^{l(R_1)} m_\Theta^{l(R_2)} + m_\Theta^{l(R_1)} m_2^{l(R_2)}) \\ \{H_3\}: b_3^{C(2)} = K^{C(2)} m_\Theta^{l(R_1)} m_3^{l(R_2)} \\ \{\Theta\}: b_\Theta^{C(2)} = K^{C(2)} m_\Theta^{l(R_1)} m_\Theta^{l(R_2)}$$

where

$$K^{C(2)} = [1 - (m_1^{l(R_1)} m_2^{l(R_2)} + m_1^{l(R_1)} m_3^{l(R_2)} + m_2^{l(R_1)} m_3^{l(R_2)})]^{-1}.$$

Then, construct intersection tableau 4 (see Table V) to combine  $e_{1,4}^{C(3)} = \{e_{1,3}^{C(2)}, e_{3,4}^{l(R_3)}\} = \{e_{1,2}^{l(R_1)}, e_{2,3}^{l(R_2)}, e_{3,4}^{l(R_3)}\}$ . According to the combination rule, we obtain

$$\{H_1\}: b_1^{C(3)} = K^{C(3)} b_1^{C(2)} m_\Theta^{l(R_3)} \\ \{H_2\}: b_2^{C(3)} = K^{C(3)} b_2^{C(2)} m_\Theta^{l(R_3)} \\ \{H_3\}: b_3^{C(3)} = K^{C(3)} (b_3^{C(2)} m_3^{l(R_3)} + b_3^{C(2)} m_\Theta^{l(R_3)} + b_\Theta^{C(2)} m_3^{l(R_3)}) \\ \{H_4\}: b_4^{C(3)} = K^{C(3)} b_\Theta^{C(2)} m_4^{l(R_3)} \\ \{\Theta\}: b_\Theta^{C(3)} = K^{C(3)} b_\Theta^{C(2)} m_\Theta^{l(R_3)}$$

where

$$K^{C(3)} = [1 - (b_1^{C(2)} m_3^{l(R_3)} + b_1^{C(2)} m_4^{l(R_3)} + b_2^{C(2)} m_3^{l(R_3)} + b_2^{C(2)} m_4^{l(R_3)} + b_3^{C(2)} m_4^{l(R_3)})]^{-1}$$

TABLE IV  
INTERSECTION TABLEAU 3

		$e_{2,3}^{I(R_2)}$		
		$\{H_2\} (m_2^{I(R_2)})$	$\{H_3\} (m_3^{I(R_2)})$	$\{\Theta\} (m_\Theta^{I(R_2)})$
$e_{1,2}^{I(R_1)}$	$\{H_1\} (m_1^{I(R_1)})$	$\{\emptyset\} (m_1^{I(R_1)} m_2^{I(R_2)})$	$\{\emptyset\} (m_1^{I(R_1)} m_3^{I(R_2)})$	$\{H_1\} (m_1^{I(R_1)} m_\Theta^{I(R_2)})$
	$\{H_2\} (m_2^{I(R_1)})$	$\{H_2\} (m_2^{I(R_1)} m_2^{I(R_2)})$	$\{\emptyset\} (m_2^{I(R_1)} m_3^{I(R_2)})$	$\{H_2\} (m_2^{I(R_1)} m_\Theta^{I(R_2)})$
	$\{\Theta\} (m_\Theta^{I(R_1)})$	$\{H_2\} (m_\Theta^{I(R_1)} m_2^{I(R_2)})$	$\{H_3\} (m_\Theta^{I(R_1)} m_3^{I(R_2)})$	$\{\Theta\} (m_\Theta^{I(R_1)} m_\Theta^{I(R_2)})$

TABLE V  
INTERSECTION TABLEAU 4

		$e_{3,4}^{I(R_3)}$		
		$\{H_3\} (m_3^{I(R_3)})$	$\{H_4\} (m_4^{I(R_3)})$	$\{\Theta\} (m_\Theta^{I(R_3)})$
$e_{1,3}^{C(2)}$	$\{H_1\} (b_1^{C(2)})$	$\{\emptyset\} (b_1^{C(2)} m_3^{I(R_3)})$	$\{\emptyset\} (b_1^{C(2)} m_4^{I(R_3)})$	$\{H_1\} (b_1^{C(2)} m_\Theta^{I(R_3)})$
	$\{H_2\} (b_2^{C(2)})$	$\{\emptyset\} (b_2^{C(2)} m_3^{I(R_3)})$	$\{\emptyset\} (b_2^{C(2)} m_4^{I(R_3)})$	$\{H_2\} (b_2^{C(2)} m_\Theta^{I(R_3)})$
	$\{H_3\} (b_3^{C(2)})$	$\{H_3\} (b_3^{C(2)} m_3^{I(R_3)})$	$\{\emptyset\} (b_3^{C(2)} m_4^{I(R_3)})$	$\{H_3\} (b_3^{C(2)} m_\Theta^{I(R_3)})$
	$\{\Theta\} (b_\Theta^{C(2)})$	$\{H_3\} (b_\Theta^{C(2)} m_3^{I(R_3)})$	$\{H_4\} (b_\Theta^{C(2)} m_4^{I(R_3)})$	$\{\Theta\} (b_\Theta^{C(2)} m_\Theta^{I(R_3)})$

$$= \left[ 1 - \left( \sum_{t=1}^2 b_t^{C(2)} (m_3^{I(R_3)} + m_4^{I(R_3)}) + b_3^{C(2)} m_4^{I(R_3)} \right) \right]^{-1}.$$

Since  $b_1^{C(1)} = m_1^{I(R_1)}$ ,  $b_2^{C(1)} = m_2^{I(R_1)}$ , and  $b_\Theta^{C(1)} = m_\Theta^{I(R_1)}$ , then we can combine  $e_{1,j+2}^{C(j+1)} = \{e_{1,2}^{I(R_1)}, \dots, e_{j+1,j+2}^{I(R_j+1)}\}$  in a similar way and obtain the following recursive algorithm:

$$\{H_1\}: b_1^{C(j+1)} = K^{C(j+1)} b_1^{C(j)} m_\Theta^{I(R_j+1)} \quad (30.a)$$

...

$$\{H_j\}: b_j^{C(j+1)} = K^{C(j+1)} b_j^{C(j)} m_\Theta^{I(R_j+1)} \quad (30.b)$$

$$\begin{aligned} \{H_{j+1}\}: b_{j+1}^{C(j+1)} &= K^{C(j+1)} (b_{j+1}^{C(j)} m_{j+1}^{I(R_j+1)} \\ &\quad + b_{j+1}^{C(j)} m_\Theta^{I(R_j+1)} + b_\Theta^{C(j)} m_{j+1}^{I(R_j+1)}) \end{aligned} \quad (30.c)$$

$$\{H_{j+2}\}: b_{j+2}^{C(j+1)} = K^{C(j+1)} b_\Theta^{C(j)} m_{j+2}^{I(R_j+1)} \quad (30.d)$$

$$\{\Theta\}: b_\Theta^{C(j+1)} = K^{C(j+1)} b_\Theta^{C(j)} m_\Theta^{I(R_j+1)} \quad (30.e)$$

where

$$\begin{aligned} K^{C(j+1)} &= \left[ 1 - \left( \sum_{t=1}^j b_t^{C(j)} (m_{j+1}^{I(R_j+1)} + m_{j+2}^{I(R_j+1)}) \right. \right. \\ &\quad \left. \left. + b_{j+1}^{C(j)} m_{j+2}^{I(R_j+1)} \right) \right]^{-1} \\ &\quad j = 1, \dots, N-2. \end{aligned} \quad (30.f)$$

When  $j = N-2$ , the overall probability assignments are generated and can be expressed by the following vector, called the overall probability assignment vector:

$$G = [b_1^{C(N-1)}, \dots, b_n^{C(N-1)}, \dots, b_N^{C(N-1)}, b_\Theta^{C(N-1)}]^T. \quad (31)$$

Notice that  $G$  is obtained by combining  $e_{1,N}^{C(N-1)}$  while

$$\begin{aligned} e_{1,N}^{C(N-1)} &= \{e_{1,2}^{I(R_1)}, \dots, e_{n,n+1}^{I(R_n)}, \dots, e_{N-1,N}^{I(R_{N-1})}\} \\ &= \{S_1 \ S_2 \ \dots \ S_n \ \dots \ S_{N-1}\} \\ &= \{e_k^1 \ e_k^2 \ \dots \ e_k^{L_k}\} = E_k. \end{aligned} \quad (32)$$

In other words,  $b_n^{C(N-1)}$  is the overall probability assignment to which  $H_n$  is confirmed by all factors  $e_k^i$  ( $i = 1, \dots, L_k$ ).

From the above discussion, it is obvious that the overall probability assignments are all zero for other hypothesis in  $\Theta$  except for the singletons  $H_n$  ( $n = 1, \dots, N$ ) and  $\Theta$ . So, it can be proved that the following equation is true:

$$\sum_{n=1}^N b_n^{C(N-1)} + b_\Theta^{C(N-1)} = 1. \quad (33)$$

Since  $m(H_n/E_k(a)) = b_n^{C(N-1)}$ , the preference degree  $p_{rk}$ , defined by (18), can then be calculated by

$$\begin{aligned} p_{rk} &= \sum_{n=1}^N m(H_n/E_k(a_r)) p(H_n) + m(\Theta/E_k(a_r)) p(\Theta) \\ &= \sum_{n=1}^N b_n^{C(N-1)} p(H_n) + b_\Theta^{C(N-1)} p(\Theta). \end{aligned} \quad (34)$$

#### IV. EVALUATION MATRIX AND ALTERNATIVE RANKING

##### A. Construction of An Evaluation Matrix

The evidential reasoning approach explored above is actually used to transform the uncertain subjective judgments about the state of a qualitative attribute  $y_k$  at an alternative  $a_r$  into the preference degree  $p_{rk} = p(y_k(a_r))$  for all  $k = k_1 + 1, \dots, k_1 + k_2$ ;  $r = 1, \dots, l$ . In this way, all qualitative attributes are evaluated and quantified using the values in the interval  $[-1 \ 1]$ .

The values of quantitative attributes which are generally incommensurate may also be transformed into the



preference degree space using the following formulas:

$$p_{rk} = p(y_{rk}) = \frac{2(y_{rk} - y_k^{\min})}{y_k^{\max} - y_k^{\min}} - 1, \quad (35)$$

$$k = 1, \dots, k_1; r = 1, \dots, l,$$

for benefit attributes

$$p_{rk} = p(y_{rk}) = \frac{2(y_k^{\max} - y_{rk})}{y_k^{\max} - y_k^{\min}} - 1; \quad (36)$$

$$k = 1, \dots, k_1; r = 1, \dots, l,$$

for cost attributes

$$y_k^{\max} = \max \{y_{1k} \dots y_{lk}\}, \quad (37)$$

$$y_k^{\min} = \min \{y_{1k} \dots y_{lk}\}.$$

The transformed attribute  $y_k$  may be denoted by a preference function  $p(y_k)$ . Thus, the original extended decision matrix defined by Table I is transformed into an evaluation matrix, an ordinary decision matrix defined by Table VI, in which the states of all attributes, either quantitative or qualitative, are represented in the preference degree space. The alternatives may then be ranked based on the evaluation matrix.

#### B. Alternative Ranking

At this stage, several traditional MADM methods can be selected to rank alternatives on the basis of the evaluation matrix. The CODASID method [21], [30], [33] may be one of them, which is based on a complete concordance and discordance analysis for information aggregation and the decision rule of the TOPSIS method for information synthesis (alternative ranking). The reasons that we have chosen CODASID for alternative ranking are not only that it is appropriate to address a MADM problem represented by an ordinary decision matrix, but also that we have developed software for CODASID so that the application examples can be readily tested [21], [33]. Obviously, the readers are not prohibited from adopting other proper MADM methods they prefer to deal with alternative ranking based on Table VI. The computational steps of CODASID are summarized below.

*Step 1:* Generate the weighted normalized evaluation matrix  $Z$  as follows:

$$Z = (r_{ij})_{l \times (k_1 + k_2)} \times \text{diag} \{\omega_1 \dots \omega_{k_1 + k_2}\} \quad (38)$$

$$= (z_{ij})_{l \times (k_1 + k_2)}$$

where  $\omega_k$  is the relative weight of  $y_k$  ( $k = 1, \dots, k_1 + k_2$ ) and

$$z_{ij} = \omega_j r_{ij}; \quad r_{ij} = \frac{p_{ij} - p_j^{\min}}{p_j^{\max} - p_j^{\min}}, \quad (39)$$

$$i = 1, \dots, l; j = 1, \dots, k_1 + k_2$$

$$p_j^{\max} = \max \{p_{1j} \dots p_{lj}\};$$

$$p_j^{\min} = \min \{p_{1j} \dots p_{lj}\}, \quad j = 1, \dots, k_1 + k_2. \quad (40)$$

TABLE VI  
THE EVALUATION MATRIX

Preference Degrees	$p(y_1)$	$\dots$	$p(y_{k_1})$	$p(y_{k_1+1})$	$\dots$	$p(y_{k_1+k_2})$
$a_1$	$p_{11}$	$\dots$	$p_{1k_1}$	$p_{1k_1+1}$	$\dots$	$p_{1k_1+k_2}$
$a_2$	$p_{21}$	$\dots$	$p_{2k_1}$	$p_{2k_1+1}$	$\dots$	$p_{2k_1+k_2}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$a_l$	$p_{l1}$	$\dots$	$p_{lk_1}$	$p_{lk_1+1}$	$\dots$	$p_{lk_1+k_2}$

*Step 2:* For each pair of alternatives  $(a_i, a_j)$  ( $i, j = 1, \dots, l; i \neq j$ ), construct the concordance set  $C_{ij}$  and the discordance set  $D_{ij}$ , based on the evaluation matrix:

$$C_{ij} = \{k | p_{ik} \geq p_{jk}, k = 1, \dots, k_1 + k_2\};$$

$$D_{ij} = \{k | p_{ik} < p_{jk}, k = 1, \dots, k_1 + k_2\}. \quad (41)$$

*Step 3:* Calculate the preference-evaluation discordance index  $d_{ij}$ , the preference concordance index  $pc_{ij}$ , and the evaluation concordance index  $ec_{ij}$ :

$$d_{ij} = \frac{\max_{k \in D_{ij}} |z_{ik} - z_{jk}|}{\max_{k \in J} |z_{ik} - z_{jk}|}; \quad pc_{ij} = \frac{\sum_{k \in C_{ij}} \omega_k}{\sum_{k=1}^{k_1+k_2} \omega_k};$$

$$ec_{ij} = \frac{\max_{k \in C_{ij}} |r_{ik} - r_{jk}|}{\max_{k \in J} |r_{ik} - r_{jk}|} \quad (42)$$

where  $0 \leq d_{ij}, pc_{ij}, ec_{ij} \leq 1$ , and  $J = \{1, \dots, k_1 + k_2\}$  is the index set of attributes.

*Step 4:* For all alternatives, calculate the net preference concordance dominance index  $pc(a_i)$ , the net evaluation concordance dominance index  $ec(a_i)$ , and the net preference-evaluation discordance dominance index  $d(a_i)$ . Then, construct the Judgment-Evaluation (J-E) matrix (Table VII):

$$pc(a_i) = \sum_{j=1}^l pc_{ij} - \sum_{j=1}^l pc_{ji};$$

$$ec(a_i) = \sum_{j=1}^l ec_{ij} - \sum_{j=1}^l ec_{ji};$$

$$d(a_i) = \sum_{j=1}^l d_{ij} - \sum_{j=1}^l d_{ji}, \quad i = 1, \dots, l. \quad (43)$$

*Step 5:*  $pc(a)$ ,  $ec(a)$ , and  $d(a)$  in Table VII are regarded as new composite attributes.  $pc(a)$  and  $ec(a)$  are for maximization and  $d(a)$  for minimization. Suppose  $\rho_1$ ,  $\rho_2$ , and  $\rho_3$  are tradeoff weights representing the relative importance of  $pc(a)$ ,  $ec(a)$ , and  $d(a)$ .  $\rho_i$  may be determined as follows [30]:

$$\rho_1 = \rho_2 = 0.25; \quad \rho_3 = 0.5. \quad (44)$$

*Step 6:* Normalize the three indexes  $pc(a)$ ,  $ec(a)$ , and

TABLE VII  
THE J-E MATRIX

$a$	$pc(a)$	$ec(a)$	$d(a)$
$a_1$	$pc_1$	$ec_1$	$d_1$
$a_2$	$pc_2$	$ec_2$	$d_2$
$\dots$	$\dots$	$\dots$	$\dots$
$a_l$	$pc_l$	$ec_l$	$d_l$

$d(a)$  as follows:

$$\begin{aligned}\bar{pc}(a_i) &= \frac{pc(a_i)}{\sqrt{\sum_{j=1}^l pc^2(a_j)}}; \quad \bar{ec}(a_i) = \frac{ec(a_i)}{\sqrt{\sum_{j=1}^l ec^2(a_j)}}; \\ \bar{d}(a_i) &= \frac{d(a_i)}{\sqrt{\sum_{j=1}^l d^2(a_j)}}, \quad i = 1, \dots, l\end{aligned}\quad (45)$$

yielding the normalized J-E matrix  $\bar{JE}$ :

$$\bar{JE} = \begin{bmatrix} \bar{pc}(a_1) & \bar{ec}(a_1) & \bar{d}(a_1) \\ \bar{pc}(a_2) & \bar{ec}(a_2) & \bar{d}(a_2) \\ \dots & \dots & \dots \\ \bar{pc}(a_l) & \bar{ec}(a_l) & \bar{d}(a_l) \end{bmatrix}. \quad (46)$$

Step 7:  $\bar{JE}$  is weighted by the tradeoff weight vector  $\rho = [\rho_1 \ \rho_2 \ \rho_3]^T$ , resulting in the weighted normalized J-E matrix  $JE$ :

$$JE = \begin{bmatrix} \bar{pc}(a_1) & \bar{ec}(a_1) & \bar{d}(a_1) \\ \bar{pc}(a_2) & \bar{ec}(a_2) & \bar{d}(a_2) \\ \dots & \dots & \dots \\ \bar{pc}(a_l) & \bar{ec}(a_l) & \bar{d}(a_l) \end{bmatrix} \quad (47)$$

where  $\bar{pc}(a_i) = \rho_1 \bar{pc}(a_i)$ ,  $\bar{ec}(a_i) = \rho_2 \bar{ec}(a_i)$ , and  $\bar{d}(a_i) = \rho_3 \bar{d}(a_i)$ ,  $i = 1, \dots, l$ .

Step 8: An ideal point  $a^*$  and a negative ideal point  $a^-$  in the J-E space can then be defined as follows:

$$\begin{aligned}\bar{pc}(a^*) &= \max \{ \bar{pc}(a_1) \dots \bar{pc}(a_l) \}, \\ \bar{ec}(a^*) &= \max \{ \bar{ec}(a_1) \dots \bar{ec}(a_l) \}, \\ \bar{d}(a^*) &= \min \{ \bar{d}(a_1) \dots \bar{d}(a_l) \}\end{aligned}\quad (48)$$

and

$$\begin{aligned}\bar{pc}(a^-) &= \min \{ \bar{pc}(a_1) \dots \bar{pc}(a_l) \}, \\ \bar{ec}(a^-) &= \min \{ \bar{ec}(a_1) \dots \bar{ec}(a_l) \}, \\ \bar{d}(a^-) &= \max \{ \bar{d}(a_1) \dots \bar{d}(a_l) \}.\end{aligned}\quad (49)$$

Step 9: The distance  $s_i^*$  between an alternative  $a_i$  and the ideal point  $a^*$  and the distance  $s_i^-$  between  $a_i$  and the negative ideal point  $a^-$  are defined as

$$\begin{aligned}s_i^* &= [(\bar{pc}(a_i) - \bar{pc}(a^*))^2 + (\bar{ec}(a_i) - \bar{ec}(a^*))^2 \\ &\quad + (\bar{d}(a_i) - \bar{d}(a^*))^2]^{1/2} \quad i = 1, \dots, l\end{aligned}\quad (50)$$

$$\begin{aligned}s_i^- &= [(\bar{pc}(a_i) - \bar{pc}(a^-))^2 + (\bar{ec}(a_i) - \bar{ec}(a^-))^2 \\ &\quad + (\bar{d}(a_i) - \bar{d}(a^-))^2]^{1/2} \quad i = 1, \dots, l.\end{aligned}\quad (51)$$

Step 10: The relative closeness index of  $a_i$  to the ideal point is finally defined as

$$\begin{aligned}u(a_i) &= \frac{s_i^-}{s_i^- + s_i^*}, \quad 0 \leq u(a_i) \leq 1, \quad i = 1, \dots, l; \\ u(a^-) &= 0, \quad u(a^*) = 1.\end{aligned}\quad (52)$$

A large value of  $u(a_i)$  indicates that  $a_i$  is more favorable since it is simultaneously closer to the ideal point and further from the negative ideal point.

### C. A Procedure for Hybrid MADM with Uncertainty

As a result of the discussion in the previous subsections, we are now in a position to formulate a procedure for dealing with a hybrid multiple-attribute decision making problem with uncertainty. This procedure is composed of the transformation, aggregation, and synthesis of information contained in the problem. The procedure may be summarized as the following steps.

Step 1: Define a hybrid MADM problem using the extended decision matrix as defined by Table I, where uncertain subjective judgments for evaluation of a qualitative attribute may be acquired using statements similar to Statement  $i >$  and represented by the evaluation analysis model.

Step 2: Transform the numerical value with a certain unit of a quantitative attribute at each alternative into the preference degree space using (35) or (36).

Step 3: Quantify the state of a qualitative attribute  $y_k$  at each alternative  $a_r$  using the evidential reasoning approach in order to obtain the preference degree  $p_{rk} = p(y_k(a_r))$ . Let  $k = 1, r = 1$ .

Step 4: First, calculate the basic probability assignments from the confidence degrees given in the uncertain subjective judgments by using (22), resulting in the basic probability assignment matrices  $M^n(y_k(a_r))$  ( $n = 1, \dots, N - 1$ ) defined by (27).

Step 5: Then, conduct partial combinations for the subsets of factors  $S_n$  ( $n = 1, \dots, N - 1$ ) in  $E_k$  using the algorithm shown in formulas (28), resulting in the local probability assignment matrix  $M(y_k(a_r))$  defined by (29).

Step 6: Conduct overall combination for all factors in  $E_k$  for  $y_k$  using the algorithm listed in formulas (30), yielding the overall probability assignment vector  $G(y_k(a_r))$  defined by (31).

Step 7: Using (34), calculate the preference degree of  $y_k(a_r)$ , i.e.,  $p_{rk}$ . If  $k \geq k_1 + k_2$  and  $r \geq l$ , continue. If  $k \leq k_1 + k_2$  and  $r < l$ , let  $r = r + 1$  and then go to Step 4. If  $k < k_1 + k_2$  and  $r \geq l$ , let  $k = k + 1$ ,  $r = 1$ , and then go to Step 4.

Step 8: Construct the evaluation matrix as shown in Table VI.

**Step 9:** Aggregate the information contained in the evaluation matrix using formulas (38)–(43), resulting in the J–E matrix defined in Table VII.

**Step 10:** Synthesize the information contained in the J–E matrix by formulas (44)–(52), generating the relative closeness indexes of all alternatives to the ideal point, that is,  $u(a_r)$ ,  $r = 1, \dots, l$ .

**Step 11:** Rank  $a_r$  based on  $u(a_r)$ ,  $r = 1, \dots, l$ . If  $u(a_{r_1}) \geq u(a_{r_2})$ , then  $a_{r_1}$  is preferred to  $a_{r_2}$ ,  $r_1, r_2 = 1, \dots, l$ ;  $r_1 \neq r_2$ .

## V. EXAMPLES

### A. A Numerical Example

A numerical example is discussed in this subsection to show how to implement the new evidential reasoning approach. The problem is to evaluate and quantify the state of an attribute  $y_k$  at an alternative  $a_r$  within an evidential reasoning framework.

First, define the set of evaluation grades for  $y_k$  as

$$H = \{H_1 \ H_2 \ H_3 \ H_4 \ H_5 \ H_6 \ H_7\} \quad (53)$$

which may be interpreted as in formula (5) and scaled as in (6) where  $p(H) = \sum_{n=1}^7 p(H_n)/7 = 0$ . In (53),  $H_n$  ( $n = 1, \dots, 7$ ) are supposed to be distinct grades. Suppose there are ten factors influencing the evaluation of the attribute  $y_k$ , denoted by

$$E_k = \{e_1 \ e_2 \ e_3 \ e_4 \ e_5 \ e_6 \ e_7 \ e_8 \ e_9 \ e_{10}\}. \quad (54)$$

The evaluation analysis model can then be depicted as in Fig. 2.

The uniform weights representing the relative importance of these factors are given by

$$\begin{aligned} \zeta_k &= [\zeta_k^1 \ \zeta_k^2 \ \zeta_k^3 \ \zeta_k^4 \ \zeta_k^5 \ \zeta_k^6 \ \zeta_k^7 \ \zeta_k^8 \ \zeta_k^9 \ \zeta_k^{10}]^T \\ &= [0.12, 0.085, 0.095, 0.09, 0.1, 0.14, 0.08, \\ &\quad 0.07, 0.13, 0.09]^T. \end{aligned} \quad (55)$$

Normalize  $\zeta_k$ , resulting in  $\bar{\zeta}_k$ , where

$$\begin{aligned} \bar{\zeta}_k &= \zeta_k / \zeta_k^6 = [0.86, 0.61, 0.68, 0.64, 0.71, 1.0, 0.57, \\ &\quad 0.5, 0.93, 0.64]^T. \end{aligned} \quad (56)$$

Note that  $e_6$  is the key factor. Suppose the decision maker considers that  $e_6$  has the absolute priority in evaluation of  $y_k(a)$ , that is,  $\alpha_k = 1$ . So,  $\lambda_k = \alpha_k \bar{\zeta}_k$ .

The uncertain subjective judgments for evaluation of the state of  $y_k$  at  $a_r$  are acquired and listed in Table VIII. These judgments may also be described using statements. For instance, it is stated that the state of the factor  $e_6$  at  $a_r$  is evaluated to be *indifferent* ( $H_4$ ) with a confidence degree of 0.7 and to be *satisfactory* ( $H_5$ ) with 0.2.

From Fig. 2 and Table VIII, we can then obtain the following notation:

$$\begin{aligned} R_1 &= 0; \\ e_{2,3}^{I(R_2)} &= \{e_{2,3}^1 = e_1\}, \quad R_1 = 1; \end{aligned}$$

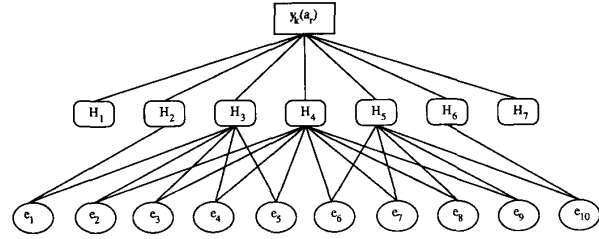


Fig. 2. The evaluation analysis submodel for the numerical example.

TABLE VIII  
UNCERTAIN SUBJECTIVE JUDGMENTS FOR EVALUATION OF  $y_k(a_r)$

Confidence Degrees ( $\beta$ )	Evaluation Grades						
	$H_1$	$H_2$	$H_3$	$H_4$	$H_5$	$H_6$	$H_7$
$e_1$		0.4	0.2				
$e_2$			0.5	0.4			
$e_3$			0.4	0.5			
$e_4$			0.3	0.6			
$e_5$			0.8				
$e_6$				0.7	0.2		
$e_7$				0.8			
$e_8$				0.5	0.5		
$e_9$				0.6	0.25		
$e_{10}$					0.5	0.5	

$$e_{3,4}^{I(R_3)} = \{e_{3,4}^1 = e_2, e_{3,4}^2 = e_3, e_{3,4}^3 = e_4, e_{3,4}^4 = e_5\}$$

$$R_3 = 4;$$

$$e_{4,5}^{I(R_4)} = \{e_{4,5}^1 = e_6, e_{4,5}^2 = e_7, e_{4,5}^3 = e_8, e_{4,5}^4 = e_9\}$$

$$R_4 = 4;$$

$$e_{5,6}^{I(R_5)} = \{e_{5,6}^1 = e_{10}\}, \quad R_5 = 1;$$

$$R_6 = 0.$$

The basic probability assignments can be calculated from the given confidence degrees by using formula (22). For instance,

$$\begin{aligned} m_{2,3}^{2,1} &= m(H_2/e_{2,3}^1) = m(H_2/e_1) = 0.4 \times \lambda_k^1 \\ &= 0.4 \times 0.86 = 0.344; \end{aligned}$$

$$m_{3,4}^{2,1} = m(H_3/e_{2,3}^1) = m(H_3/e_1) = 0.2 \times \lambda_k^1 = 0.172;$$

$$m_{\theta}^{2,1} = 1 - (m_{2,3}^{2,1} + m_{3,4}^{2,1}) = 0.484;$$

$$\begin{aligned} m_{3,1}^{3,1} &= m(H_3/e_{3,4}^1) = m(H_3/e_2) = 0.5 \times \lambda_k^2 \\ &= 0.5 \times 0.61 = 0.305; \end{aligned}$$

$$\begin{aligned} m_{3,2}^{3,2} &= m(H_3/e_{3,4}^2) = m(H_3/e_3) = 0.4 \times \lambda_k^3 \\ &= 0.4 \times 0.68 = 0.272; \end{aligned}$$

$$\begin{aligned} m_{3,3}^{3,3} &= m(H_3/e_{3,4}^3) = m(H_3/e_4) = 0.3 \times \lambda_k^4 \\ &= 0.3 \times 0.64 = 0.192; \end{aligned}$$

$$\begin{aligned} m_{3,4}^{3,4} &= m(H_3/e_{3,4}^4) = m(H_3/e_5) = 0.8 \times \lambda_k^5 \\ &= 0.8 \times 0.71 = 0.568. \end{aligned}$$

On the whole, the following basic probability assignment matrices are obtained:

$$\begin{aligned}
 M^1 &= [0 \ 0 \ 1] \quad R_1 = 0; \\
 M^2 &= [0.344 \ 0.172 \ 0.484] \quad R_2 = 1; \\
 M^3 &= \begin{bmatrix} 0.305 & 0.244 & 0.451 \\ 0.272 & 0.34 & 0.388 \\ 0.192 & 0.384 & 0.424 \\ 0.568 & 0.0 & 0.432 \end{bmatrix} \quad R_3 = 4; \\
 M^4 &= \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.456 & 0.0 & 0.544 \\ 0.25 & 0.25 & 0.5 \\ 0.558 & 0.233 & 0.209 \end{bmatrix} \quad R_4 = 4; \\
 M^5 &= [0.32 \ 0.32 \ 0.36] \quad R_5 = 1; \\
 M^6 &= [0 \ 0 \ 1] \quad R_6 = 0.
 \end{aligned}$$

These ten factors are then combined using the partial combination algorithm (28) and the overall combination algorithm (30). As a result of the partial combination, the following local probability assignment matrix can be produced:

$$M = \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 0.344 & 0.172 & 0.484 \\ 0.609 & 0.32 & 0.071 \\ 0.894 & 0.095 & 0.011 \\ 0.32 & 0.32 & 0.36 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}.$$

As a result of the overall combination, the following overall probability assignment vector can be obtained:

$$\begin{aligned}
 G &= [b_1^{C(6)} \ b_2^{C(6)} \ b_3^{C(6)} \ b_4^{C(6)} \ b_5^{C(6)} \ b_6^{C(6)} \\
 &\quad b_7^{C(6)} \ b_H^{C(6)}]^T \\
 &= [0.0 \ 0.001 \ 0.025 \ 0.934 \ 0.036 \ 0.002 \\
 &\quad 0.0 \ 0.002]^T.
 \end{aligned}$$

From the above distribution of the overall probability assignments, it is obvious that the state of  $y_k$  at  $a_r$  is confirmed by the whole set of factors to the grade  $H_4$  to a very high extent of 0.934, although the confidence degrees are almost uniformly distributed among the grades  $H_2$ ,  $H_3$ ,  $H_4$ ,  $H_5$ , and  $H_6$ . Such a result is quite reasonable because the states of eight factors at  $a_r$  are confirmed to  $H_4$  to different extents, including those of the two most important factors  $e_6$  and  $e_9$  at  $a_r$ . This result may demonstrate the property of the D-S theory that it can model the narrowing of the hypothesis set with the accumulation of evidence.

Finally, the preference degree of  $y_k(a_r)$ , i.e.,  $p(y_k(a_r))$ , can be generated by formula (34):

$$p(y_k(a_r)) = \sum_{n=1}^7 b_{H_n}^{C(6)} p(H_n) + b_H^{C(6)} p(H) = 0.0052 \quad (57)$$

which means that it is almost certain that the state of  $y_k$  at  $a_r$  is *indifferent*.

## B. A Motorcycle Evaluation Problem

A customer intends to buy a motorcycle. Four types of motorcycle are available for selection, that is, "Kawasaki," "Yamaha," "Honda," and "BMW." The technical and economical performances of the four types of motorcycle are also available [7]. These performances are represented by either numerical values with units or subjective judgments with uncertainty. The customer, however, only takes into account six of the performances (attributes), including both qualitative and quantitative attributes. These six attributes are described in Table IX, in which the numerical values of the quantitative attributes and the uncertain subjective judgments for evaluation of the qualitative attributes are discussed in depth in [7].

The uncertain subjective judgments listed in Table IX are represented in a compact form. In Table X, the uncertain subjective judgments for evaluation of the engine of "Kawasaki" are demonstrated. These judgments can be described using the following statements.

- 1) The *responsiveness* of the engine of "Kawasaki" is *excellent* with a confidence degree of 0.8,
- 2) the *fuel economy* of the engine of "Kawasaki" is *absolutely average*, and
- 3) The *quietness* of the engine of "Kawasaki" might be *half indifferent* and *half average*.

Since no single motorcycle type dominates or is dominated by the other types from Table IX, the customer has to provide his preference information about the relative importance of the six attributes. He uses a ten-point scale to estimate the relative importance. The relative weights of the six attributes are thus estimated as follows:

$$\begin{aligned}
 \hat{W} &= [\hat{\omega}_1 \ \hat{\omega}_2 \ \hat{\omega}_3 \ \hat{\omega}_4 \ \hat{\omega}_5 \ \hat{\omega}_6]^T \\
 &= [9 \ 5 \ 7 \ 7 \ 7 \ 4]^T. \quad (58)
 \end{aligned}$$

$\hat{W}$  is then normalized by

$$\begin{aligned}
 W &= \hat{W}/39 = [\omega_1 \ \omega_2 \ \cdots \ \omega_6]^T \\
 &= [0.23 \ 0.127 \ 0.18 \ 0.18 \ 0.18 \ 0.103]^T. \quad (59)
 \end{aligned}$$

Three sets of factors for evaluation of the three qualitative attributes are defined by

$$\begin{aligned}
 E_4 &= \{e_4^1 \ e_4^2 \ e_4^3\} \\
 &= \{\text{responsiveness} \ \text{fuel economy} \ \text{quietness}\} \quad (60)
 \end{aligned}$$

TABLE IX  
AN EXTENDED DECISION MATRIX FOR EVALUATION OF FOUR TYPES OF MOTORCYCLE

Types of Attributes	Definition of Attributes	Units or Factors	Types of Motorcycle (Alternatives)			
			Kawasaki ( $a_1$ )	Yamaha ( $a_2$ )	Honda ( $a_3$ )	BMW ( $a_4$ )
Quantitative	Price ( $y_1$ )	lb	6499	5199	6199	8220
	Displacement ( $y_2$ )	cm <sup>3</sup>	1052	1188	998	987
	Top Speed ( $y_3$ )	mi/h	160	155	160	145
Qualitative	Engine ( $y_4$ )	Responsiveness ( $e_4^1$ )	$E(0.8)$	$G(0.3)$ $E(0.6)$	$G(1.0)$	$I(1.0)$
		Fuel Economy ( $e_4^2$ )	$A(1.0)$	$I(1.0)$	$I(0.5)$ $A(0.5)$	$E(1.0)$
		Quietness ( $e_4^3$ )	$I(0.5)$ $A(0.5)$	$A(1.0)$	$G(0.5)$ $E(0.3)$	$E(1.0)$
		Steering ( $e_4^4$ )	$E(0.9)$	$G(1.0)$	$A(1.0)$	$A(0.6)$
	Handling ( $y_5$ )	Bumping Bends ( $e_5^1$ )	$A(0.5)$ $G(0.5)$	$G(1.0)$	$G(0.8)$ $E(0.1)$	$P(0.5)$ $I(0.5)$
		Maneuverability ( $e_5^2$ )	$A(1.0)$	$E(0.9)$	$I(1.0)$	$P(1.0)$
		Top Speed Stability ( $e_5^3$ )	$E(1.0)$	$G(1.0)$	$G(1.0)$	$G(0.6)$ $E(0.4)$
		Quality of Finish ( $e_5^4$ )	$P(0.5)$ $I(0.5)$	$G(1.0)$	$E(1.0)$	$G(0.5)$ $E(0.5)$
	General ( $y_6$ )	Seat Comfort ( $e_6^1$ )	$G(1.0)$	$G(0.5)$ $E(0.5)$	$G(0.6)$	$E(1.0)$
		Headlight ( $e_6^2$ )	$G(1.0)$	$A(1.0)$	$E(1.0)$	$G(0.5)$ $E(0.5)$

The evaluation grades for the qualitative attributes are defined as  $P(\beta)$ —poor,  $I(\beta)$ —indifferent,  $A(\beta)$ —average,  $G(\beta)$ —good, and  $E(\beta)$ —excellent, where  $\beta$  represents confidence degree [7].

TABLE X  
UNCERTAIN SUBJECTIVE JUDGMENTS FOR  $y_4(a_1)$

Confidence Degrees ( $\beta$ )	Evaluation Grades				
	Poor	Indifferent	Average	Good	Excellent
Factors					
Responsiveness					0.8
Fuel economy			1.0		
Quietness		0.5	0.5		

$$E_5 = \{e_5^1 \ e_5^2 \ e_5^3 \ e_5^4\}$$

$$= \{\text{steering} \ \text{bumpy bends} \ \text{maneuverability} \ \text{top speed stability}\} \quad (61)$$

$$E_6 = \{e_6^1 \ e_6^2 \ e_6^3\}$$

$$= \{\text{quality of finish} \ \text{seat comfort} \ \text{headlight}\}. \quad (62)$$

The customer realizes that the factors in the same factor set have equal relative importance for evaluation of the

corresponding attribute, and that the priority coefficients  $\alpha_4, \alpha_5, \alpha_6$  are all equal to 0.9. This means that the state of the corresponding attribute is only regarded to be confirmed by 90% to the same evaluation grade as that confirmed absolutely by a key factor's state. Thus, the following weights for the factors are obtained:

$$\lambda_4 = [\lambda_4^1 \ \lambda_4^2 \ \lambda_4^3]^T = [0.9 \ 0.9 \ 0.9]^T \quad (63)$$

$$\lambda_5 = [\lambda_5^1 \ \lambda_5^2 \ \lambda_5^3 \ \lambda_5^4]^T = [0.9 \ 0.9 \ 0.9 \ 0.9]^T \quad (64)$$

$$\lambda_6 = [\lambda_6^1 \ \lambda_6^2 \ \lambda_6^3]^T = [0.9 \ 0.9 \ 0.9]^T. \quad (65)$$

In [7], the same set of evaluation grades is used for the three qualitative attributes, which includes five distinct evaluation grades and is defined by

$$H = \{H_1 \ H_2 \ H_3 \ H_4 \ H_5\} \\ = \{\text{poor} \ \text{indifferent} \ \text{average} \ \text{good} \ \text{excellent}\}. \quad (66)$$

$H$  is transformed into the preference degree space using the following scale:

$$p\{H\} = [p(H_1) \ p(H_2) \ p(H_3) \ p(H_4) \ p(H_5)]^T \\ = [-1 \ -0.4 \ 0 \ 0.4 \ 1]^T \quad (67)$$

where  $p(H_n)$  ( $n = 2, 3, 4$ ) are assigned by the customer so that they satisfy the basic conditions (4) and the consistency condition (21), and  $p(H) = \sum_{n=1}^5 p(H_n)/5 = 0$ .

The evaluation analysis model for evaluation of the three qualitative attributes may then be depicted as in Fig. 3.

Each of the preference degrees for quantifying the states of the qualitative attributes at all alternative motorcycles, i.e.,  $p_{rk} = p(y_k(a_r))$  ( $k = 4, 5, 6; r = 1, \dots, 4$ ), is generated following the same process demonstrated in the last subsection. The basic probability assignments are obtained from the confidence degrees given in Table IX and the relative weights  $\lambda_k$  ( $k = 4, 5, 6$ ) assigned by the customer. The overall probability assignments are generated using the evidence combination algorithms. The results are listed in Tables XI–XXII.

Then, the preference degrees of the three qualitative attributes at the four alternative motorcycle types are calculated using (34). For instance, from (67) and Table XI,

$$p_{14} = b_{H_1}^{C(4)} p(H_1) + b_{H_2}^{C(4)} p(H_2) + b_{H_3}^{C(4)} p(H_3) \\ + b_{H_4}^{C(4)} p(H_4) + b_{H_5}^{C(4)} p(H_5) + b_H^{C(4)} p(H) \\ = 0.0 \times (-1) + 0.072 \times (-0.4) + 0.87 \times 0 \\ + 0.0 \times 0.4 + 0.041 \times 1 + 0.017 \times 0 \\ = 0.012. \quad (68)$$

The values of the three quantitative attributes at each alternative are normalized using (35) for  $y_2$  and  $y_3$  and using (36) for  $y_1$ . Table XXIII shows the obtained evaluation matrix for the motorcycle evaluation problem.

The CODASID method is then used to rank the four motorcycle types, based on Table XXIII. Using formulas (38)–(43), we can obtain the following judgment and evaluation matrix (Table XXIV).

Finally, the relative closeness indexes of the four motorcycle types are generated by formulas (44)–(52), that is,

$$[u(a_1) \ u(a_2) \ u(a_3) \ u(a_4)]^T \\ = [0.281 \ 0.895 \ 0.942 \ 0.205]^T. \quad (69)$$

So the preference order is

$$a_3 > a_2 > a_1 > a_4. \quad (70)$$

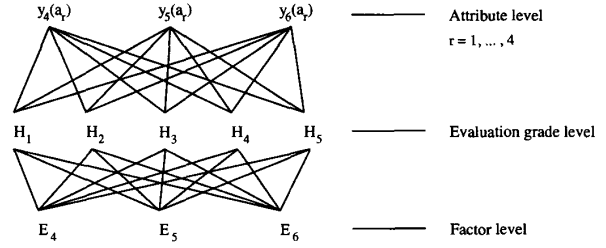


Fig. 3. The evaluation analysis model for the motorcycle evaluation problem.

TABLE XI  
PROBABILITY ASSIGNMENTS FOR  $y_4(a_1)$

Basic Probability Assignments ( $\beta \times \lambda_4$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_4^1$					0.72
$e_4^2$			0.9		
$e_4^3$		0.45	0.45		
Overall Probability Assignments $b_{H_k}^{C(4)}$	0.000	0.072	0.870	0.000	0.041

TABLE XII  
PROBABILITY ASSIGNMENTS FOR  $y_4(a_2)$

Basic Probability Assignments ( $\beta \times \lambda_4$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_4^1$				0.27	0.54
$e_4^2$		0.9			
$e_4^3$			0.9		
Overall Probability Assignments $b_{H_k}^{C(4)}$	0.000	0.387	0.387	0.061	0.122

TABLE XIII  
PROBABILITY ASSIGNMENTS FOR  $y_4(a_3)$

Basic Probability Assignments ( $\beta \times \lambda_4$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_4^1$				0.9	
$e_4^2$		0.45	0.45		
$e_4^3$				0.45	0.27
Overall Probability Assignments $b_{H_k}^{C(4)}$	0.000	0.125	0.125	0.696	0.027

Hence, "Honda" is regarded as the best compromise choice, "Yamaha" is quite competitive, but its *top speed* and *engine* are not as good as those of "Honda," which are supposed to be very important.

It may be noted that the preference order (70) partially depends on the customer's preference, that is, the relative

TABLE XIV  
PROBABILITY ASSIGNMENTS FOR  $y_4(a_4)$

Basic Probability Assignments ( $\beta \times \lambda_4$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_4^1$		0.9			
$e_4^2$					0.9
$e_4^3$					0.9
Overall Probability Assignments $b_{H_n}^{C(4)}$	0.000	0.083	0.000	0.000	0.908

TABLE XV  
PROBABILITY ASSIGNMENTS FOR  $y_5(a_1)$

Basic Probability Assignments ( $\beta \times \lambda_5$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_5^1$					0.81
$e_5^2$			0.45	0.45	
$e_5^3$			0.9		
$e_5^4$					0.9
Overall Probability Assignments $b_{H_n}^{C(4)}$	0.000	0.000	0.486	0.04	0.465

TABLE XVI  
PROBABILITY ASSIGNMENTS FOR  $y_5(a_2)$

Basic Probability Assignments ( $\beta \times \lambda_5$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_5^1$				0.9	
$e_5^2$				0.9	
$e_5^3$					0.81
$e_5^4$				0.9	
Overall Probability Assignments $b_{H_n}^{C(4)}$	0.000	0.000	0.000	0.995	0.004

TABLE XVII  
PROBABILITY ASSIGNMENTS FOR  $y_5(a_3)$

Basic Probability Assignments ( $\beta \times \lambda_5$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_5^1$			0.9		
$e_5^2$				0.72	0.09
$e_5^3$		0.9			
$e_5^4$				0.9	
Overall Probability Assignments $b_{H_n}^{C(4)}$	0.000	0.136	0.136	0.707	0.007

TABLE XVIII  
PROBABILITY ASSIGNMENTS FOR  $y_5(a_4)$

Basic Probability Assignments ( $\beta \times \lambda_5$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_5^1$			0.54		
$e_5^2$	0.45	0.45			
$e_5^3$	0.9				
$e_5^4$				0.54	0.36
Overall Probability Assignments $b_{H_n}^{C(4)}$	0.775	0.065	0.017	0.078	0.052

TABLE XIX  
PROBABILITY ASSIGNMENTS FOR  $y_6(a_1)$

Basic Probability Assignments ( $\beta \times \lambda_6$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_6^1$	0.45	0.45			
$e_6^2$				0.9	
$e_6^3$				0.9	
Overall Probability Assignments $b_{H_n}^{C(4)}$	0.041	0.041	0.000	0.908	0.000

TABLE XX  
PROBABILITY ASSIGNMENTS FOR  $y_6(a_2)$

Basic Probability Assignments ( $\beta \times \lambda_6$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_6^1$				0.9	
$e_6^2$				0.45	0.45
$e_6^3$			0.9		
Overall Probability Assignments $b_{H_n}^{C(4)}$	0.000	0.000	0.131	0.788	0.066

TABLE XXI  
PROBABILITY ASSIGNMENTS FOR  $y_6(a_3)$

Basic Probability Assignments ( $\beta \times \lambda_6$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_6^1$					0.9
$e_6^2$				0.54	
$e_6^3$					0.9
Overall Probability Assignments $b_{H_n}^{C(4)}$	0.000	0.000	0.000	0.012	0.979

weights of the six attributes and those of the factors. If he assigns different weights to the attributes or to the factors, different preference orders may be generated. For in-

stance, if he recognizes that *top speed* ( $y_3$ ) and *engine* ( $y_4$ ) are not as important as suggested by (58) and adopts the following devaluated weights for  $y_3$  and  $y_4$ ,

TABLE XXII  
PROBABILITY ASSIGNMENTS FOR  $y_6(a_4)$

Basic Probability Assignments ( $\beta \times \lambda_6$ )	Evaluation Grades				
	$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors $e_6^1$ $e_6^2$ $e_6^3$				0.45	0.45
					0.9
				0.45	0.45
Overall Probability Assignments $b_{H_n}^{C(a)}$	0.000	0.000	0.000	0.088	0.909

TABLE XXIII  
THE EVALUATION MATRIX

	$p(y_1)$	$p(y_2)$	$p(y_3)$	$p(y_4)$	$p(y_5)$	$p(y_6)$
$a_1$	0.139	-0.353	1.000	0.012	0.481	0.306
$a_2$	1.000	1.000	0.333	-0.057	0.402	0.381
$a_3$	0.338	-0.891	1.000	0.255	0.235	0.984
$a_4$	-1.000	-1.000	-1.000	0.875	-0.718	0.944

TABLE XXIV  
THE J-E MATRIX

	$pc(a)$	$ec(a)$	$d(a)$
$a_1$	0.308	-1.179	-0.964
$a_2$	0.436	0.566	0.867
$a_3$	0.769	1.006	0.728
$a_4$	-1.513	-0.394	-0.631

$$\hat{W} = [9 \ 5 \ 5 \ 5 \ 7 \ 4]^T \quad (71)$$

then the four motorcycle types will be ranked as follows:

$$[u(a_1) \ u(a_2) \ u(a_3) \ u(a_4)]^T \\ = [0.306 \ 0.916 \ 0.869 \ 0.144]^T \quad (72)$$

$$a_2 > a_3 > a_1 > a_4. \quad (73)$$

In this case, the cheapest "Yamaha" is ranked to be the best compromise choice.

## VI. CONCLUSION

The evidential reasoning approach proposed in this paper provides an alternative way to treat uncertain decision knowledge. The presented decision making procedure composed of this approach and the CODASID method can be used to deal with hybrid multiple-attribute decision making problems with uncertainty. The evidential reasoning framework involved in the approach is suitable for representation and quantification of subjective judgments with uncertainty. The obtained two evidence combination algorithms are computationally useful for combining multiple uncertain subjective judgments. The presented examples have demonstrated the implementation process of the proposed approach, and perhaps its potential to treat uncertainty in hybrid MADM problems through multiple-factor analysis and evidential reasoning.

However, the approach reported in this paper is only at the early stage of its development. More work needs to be done for evolution of the approach into better approaches for dealing with more general problems. For instance, one question may be that only the parallel combination of factors is considered in the approach. In real world problems, however, multiple factors associated with the evaluations of an attribute may constitute a hierarchical structure [35]. In this case, sequential propagation of the evaluations for these factors may occur, which needs to be explored in further research.

In addition to this question, it may be argued that some technical details presented in this paper need more proper justification or more formal definition by using universally accepted rules or laws. For instance, the following questions may be proposed as well. Is the rationality assumption of Section III-A always rational? If not, is it possible to extend the approach on a more general basis so that more general problems could be treated? Are there any common rules to follow for assignment of the priority coefficient  $\alpha_k$  defined in (25) as its value is important to conduct a rational transformation of the given confidence degrees and preference weights into the basic probability assignments? Some of these questions have actually been addressed to a large extent in the authors' current work [31], [32]. To answer all of these questions or similar ones, however, more effort should be placed on the work of other researchers such as Keefer *et al.* [13] and Miller *et al.* [18].

## ACKNOWLEDGMENT

The anonymous referees' comments have made important contributions to the improvement of this paper. A Robertson of the Engineering Design Centre, University of Newcastle upon Tyne, England, provided the motorcycle evaluation problem.



## REFERENCES

- [1] B. G. Buchanan and E. H. Shortliffe, *Rule-Based Expert Systems*. Reading, MA: Addison-Wesley, 1984.
- [2] V. Changkong and Y. Y. Haimes, *Multiobjective Decision Making—Theory and Methodology*. Amsterdam: North-Holland Elsevier Science, 1983.
- [3] T. Chen, *Decision Analysis*. Beijing: Publishing House for Science and Technology, 1987 (in Chinese).
- [4] J. L. Cohon, *Multiobjective Programming and Planning*. New York: Academic, 1983.
- [5] P. C. Fishburn, "Methods of estimating additive utilities," *Management Sci.*, vol. 13, no. 7, pp. 435–453, 1967.
- [6] C. L. Huang and K. Yong, *Multiple Attribute Decision Making Methods and Applications, A State-of-Art Survey*. Berlin: Springer-Verlag, 1981.
- [7] T. Isitt, "The sports tourers," *Motor Cycle Int.*, no. 64, pp. 18–27, Sept. 1990.
- [8] E. Jacquet-Lagrange and J. Siskos, "Assessing a set of additive utility functions for multicriteria decision making, the UTA method," *European J. Oper. Res.*, vol. 10, pp. 151–164, 1982.
- [9] E. Jacquet-Lagrange and M. F. Shkun, "Decision support systems for semi-structured buying decisions," *European J. Oper. Res.*, vol. 16, pp. 48–58, 1984.
- [10] E. Jacquet-Lagrange and R. Meziani, "MOLP with an interactive assessment of piecewise linear utility function," *European J. Oper. Res.*, vol. 31, pp. 350–357, 1987.
- [11] M. Jarke and F. J. Radermacher, "The AI potential of model management and its central role in decision support," *Decision Support Syst.*, vol. 4, no. 4, pp. 387–404, 1988.
- [12] M. Jarke, M. T. Jelassi, and M. F. Shkun, "MEDIATOR: Towards a negotiation support system," *European J. Oper. Res.*, vol. 31, pp. 314–334, 1987.
- [13] D. L. Keefer and S. E. Bodily, "Three point approximations for continuous random variables," *Management Sci.*, vol. 29, pp. 595–609, 1983.
- [14] S. G. Kochan, *Programming in C*. New York: Hayden, 1983.
- [15] G. Lilien and P. Kotler, *Marketing Decision Making: A Model Building Approach*. New York: Harper and Row, 1983.
- [16] B. Liu, "Analysis hierarchy process—A tool for programming and decisionmaking," *Syst. Eng.*, vol. 2, no. 2, 1984, (in Chinese).
- [17] B. Malakooti, "Theories and an exact interactive paired-comparison approach for discrete multiple-criteria problems," *IEEE Trans. Syst., Man, Cybern.*, vol. 19, no. 2, pp. 365–378, 1989.
- [18] A. C. Miller and T. R. Rice, "Discrete approximations of probability distributions," *Management Sci.*, vol. 29, pp. 352–362, 1983.
- [19] R. Lopez de Mantaras, *Approximate Reasoning Models*. Chichester, England: Ellis Horwood Limited, 1990.
- [20] T. L. Saaty, *The Analytic Hierarchy Process: Planning, Priority Setting, Resource Allocation*. New York: McGraw-Hill, 1980.
- [21] P. Sen and J. B. Yang, "A multiple criteria decision support environment for engineering design," in *Proc. 9th Int. Conf. Eng. Design*, The Hague, Aug. 1993.
- [22] P. Sen, J. B. Yang, and P. Meldrum, "Interactive trade-off analysis in multiple criteria preliminary design of a semi-submersible," in *Proc. 6th Int. Conf. Marine Eng. Syst.*, Univ. Hamburg, Germany, Sept. 1993.
- [23] M. G. Singh, J. B. Singh, and M. Corstjens, "Mark-Opt—A negotiation tool for manufacturers and retailers," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-15, no. 4, pp. 483–495, 1985.
- [24] M. G. Singh, R. Cook, and M. Corstjens, "A hybrid knowledge-based system for allocating retail space and for other allocation problems," *Interface*, vol. 18, no. 5, pp. 13–22, 1988.
- [25] C. C. White, A. P. Sage, and S. Dozono, "A model of multiattribute decisionmaking and trade-off weight determination under uncertainty," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-14, no. 2, pp. 223–229, 1984.
- [26] C. C. White, "A survey on the integration of decision analysis and expert systems for decision support," *IEEE Trans. Syst., Man, Cybern.*, vol. 20, no. 2, pp. 358–364, 1990.
- [27] J. B. Yang, C. Chen, and Z. J. Zhang, "A multiobjective fuzzy decisionmaking method for planning management expert systems," in *Proc. Int. Academic Conf. Economics and Management*, Shanghai, P.R. China, 1987.
- [28] —, "The interactive decomposition method for multiobjective linear programming and its applications," *Inform. Decision Technol.*, vol. 14, no. 3, pp. 275–288, 1988.
- [29] —, "The interactive step trade-off method (ISTM) for multiobjective optimization," *IEEE Trans. Syst., Man, Cybern.*, vol. 20, no. 3, pp. 688–695, 1990.
- [30] J. B. Yang, P. Sen, and P. F. Meldrum, "Multiattribute engineering design selection through concordance and discordance analysis by similarity to ideal designs," *J. Multi-Criteria Decision Anal.*, accepted for publication, 1993.
- [31] J. B. Yang and P. Sen, "A hierarchical evaluation process for multiple attribute design selection and uncertainty," in *Proc. 6th Int. Conf. Indust. Eng. Appl. Artificial Intell. Expert Syst.* (IEA/AIE-93), Edinburgh, Scotland, June 1993.
- [32] —, "A multi-level evaluation process for hybrid MADM with uncertainty," *IEEE Trans. Syst., Man, Cybern.*, accepted for publication, 1993.
- [33] J. B. Yang, "A hybrid multicriteria decision support system for engineering design," Res. Rep., EDCN/MCDM/PAPERS/3/2, Eng. Design Centre, Univ. Newcastle upon Tyne, England, 1992.
- [34] J. B. Yang and P. Sen, "An artificial neural network approach for nonlinear optimization with discrete design variables," in *Proc. 16th IFIP Conf. Syst. Modelling and Optimiz.*, Compiègne, France, July 1993.
- [35] Z. J. Zhang, J. B. Yang, and D. L. Xu, "A hierarchical analysis model for multiobjective decisionmaking," in *Proc. 4th IFAC/IFIP/IFORS/IEA Conf. ManMachine Syst.*, Xi'an, P.R. China, 1989.



**Jian-Bo Yang** was born in Hunan Province, People's Republic of China, in November 1961. He studied at the Northwestern Polytechnical University, Xian, P.R. China from 1978 to 1984, and at Shanghai Jiao Tong University, Shanghai, P.R. China, from 1984 to 1987. He received the B.Sc. and M.Sc. degrees in control engineering in 1981 and in 1984, respectively, and the Ph.D. degree in systems engineering in 1987.

He is currently a Senior Research Associate at the Engineering Design Centre, University of Newcastle upon Tyne, England. In 1987, he joined the Department of Automatic Control, Shanghai Jiao Tong University. In 1990, he joined the Computation Department, University of Manchester Institute of Science and Technology (UMIST), England, as a Visiting Postdoctoral Research Fellow. Since early 1991, he has been working at the University of Newcastle upon Tyne as a Research Associate at first, and then as a Senior Research Associate. His current research interests include multiple-criteria decision making (MCDM) with uncertainty, integrated MCDM-based decision support systems, nonlinear multiple objective optimization, intelligent MCDM techniques, multiple objective control theory, and nonlinear discrete optimization based on artificial neural network. He has published over 40 papers in these areas and a book on multiple criteria decision making methods and applications (in Chinese). He has developed a few large software packages for applications of MCDM and large-scale optimization techniques to industrial and engineering systems and for optimal controller design and system simulation for the propulsion systems of fighter aeroplanes.

Dr. Yang was granted in 1988, as the first young researcher (under 40 years old) in the China Society of Systems Engineering, the Science and Technology Award for Youth from the China Association for Science and Technology.



**Madan G. Singh** (SM'82-F'89) received the B.Sc. degree from Exeter, the M.Sc. degree from Manchester, the Ph.D. degree from Cambridge, and the Docteur es Sciences degree from Toulouse.

He was elected to the chair of Control Engineering at UMIST in 1979. He has been a Professor of Information Engineering in the Computation Department at UMIST since 1987. He was the Head of the Control Systems Centre from 1981 to 1983 and from 1985 to 1987. Before 1979, he had been an Associate Professor at the University of Toulouse, France (1976–78), a Researcher at the French CNRS (1978–79), a Fellow of St. John's College, Cambridge (1974–77), and an ICI European Postdoctoral

Fellow in Toulouse, France (1973–75). His research interests are on the application of complex systems and information engineering methodologies to decision making problems in industry and commerce. He was a part time Visiting Research Professor of Management Sciences at INSEAD (the European Institute of Business Management), Fontainebleau, France, over the period 1980–84. He has acted as a consultant to a large number of major companies (Midland Bank, Lloyds Bank, TSB, Shell UK Oil, BP Oil, Unilever, Audi-Volkswagen, Vauxhall, Mercedes Glaxo, GMT, British Airways, etc.) He is the Chairman and founder of Control Sciences Ltd. and its subsidiary, Knowledge Support Systems Ltd. These companies provide services concerned with decision technologies as applied to key tactical decision making problems for large companies based on products developed by Prof. Singh as a part of his research. Through marketing and franchise agreements, these products are currently being used by major corporations worldwide to improve their decision making.

Dr. Singh is the author or coauthor of eight books, the editor or coeditor of a further nine books, and an author or coauthor of over 140 scientific

articles. He is the Editor-in-Chief of the eight volume *Encyclopedia of Systems and Control* (Pergamon Books Ltd. 1987) and of the series, *Advances in Systems, Control and Information Engineering*, which aims to keep the main Encyclopedia up to date by producing supplementary volumes and concise subject encyclopedias. He was given an "Outstanding Contribution Award" by the IEEE Systems, Man, and Cybernetics Society for editing this major reference work. He is the Editor-in-Chief of the journal, *Information and Decision Technologies* (North-Holland) and he is on the Editorial Boards of a number of other international journals, including the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS. He is also the Coordinating Editor of three book series (with Pergamon, North-Holland, and Plenum). He is the editor of the *IEEE SMC Newsletter*. He is the Chairman of the IMACS Technical Committee on Managerial Decision Making, and was a Vice Chairman of the Systems Engineering Committee of the IFAC (1981–84). He was the Vice President (Publications) of the IEEE Systems, Man, and Cybernetics Society (1990–1992) and is currently Vice President for Technical Activities. He gave the 1988 Rank Xerox Lecture.