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Theory and Methodology

Rule and utility based evidential reasoning approach for multiattribute decision analysis under uncertainties

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Abstract

In this paper generic decision models and both rule and utility based techniques for transforming assessment information are developed to enhance an evidential reasoning (ER) approach for dealing with multiple attribute decision analysis (MADA) problems of both a quantitative and qualitative nature under uncertainties. In the existing ER approach, a modelling framework is established for representing subjective assessments under uncertainty, in which a set of evaluation grades for a qualitative attribute is defined. The attribute may then be assessed to one or more of these grades with certain degrees of belief. Using such a distributed assessment framework, the features of a range of evidence can be catered for whilst the assessor is not forced to pre-aggregate various types of evidence into a single numerical value. Both complete and incomplete assessments can be accommodated in a unified manner within the framework. For assessing different qualitative attributes, however, different sets of evaluation grades may need to be defined to facilitate data collection. Moreover, some attributes are quantitative and may be assessed using certain or random numbers. This increases complexity in attribute aggregation. In this paper, a generalised and extended decision matrix is constructed and rule and utility based techniques are developed for transforming various types of information within the matrix for aggregating attributes via ER. The transformation processes are characterised by a group of matrix equations. The techniques can be used in a hybrid way at different levels of an attribute hierarchy. It is proved in this paper that such transformations are equivalent with regard to underlying utility and rational in terms of preserving the features of original assessments. Complementary to distributed descriptions, utility intervals are introduced to describe and analyse incomplete and imprecise information. Two numerical examples are provided to demonstrate the implementation procedures of the new techniques and the potential and scope of the rule and utility based ER approach in supporting decision analysis under uncertainties.

Keywords: Multiple attribute decision analysis; Uncertainty modelling; Evidential reasoning; Utility; Hybrid OR/AI method; Quantitative and qualitative assessment

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1. Multiattribute decision analysis under uncertainties

1.1. Quantitative and qualitative MADA

A multiple attribute decision analysis (MADA) problem can be generally modelled using a decision matrix (Hwang and Yoon, 1981). Several approaches have been proposed to deal with MADA problems of both a quantitative and qualitative nature. Multiattribute utility (value) function approaches are among the simplest and most commonly used (Keeney and Raiffa, 1976; Hwang and Yoon, 1981; Belton, 1986; Winston, 1994; Yang, 1996). If a MADA problem involves a large number of attributes and alternatives, however, estimating utilities at all alternatives on each attribute will become a tedious procedure, though certain regression methods could be used to facilitate utility estimation (Jacquet-Lagrèze and Siskos, 1982; Yang and Sen, 1996).

Pairwise comparisons were primarily used to estimate relative weights of attributes in several approaches including the eigenvector method (Saaty, 1988), the geometric least square method (Islei and Lockett, 1988) and the geometric mean method (Barzilai and Golany, 1990; Barzilai, 1997). Pairwise comparison matrices have also been used to assess alternatives with respect to a particular attribute such as in AHP (Saaty, 1988) and in judgmental modelling based on the geometric least square method (Proudlove, 1999). However, using pairwise comparisons to assess alternatives may lead to problems like rank reversal as within the AHP framework. As such there has been a long debate on how quantitative and qualitative assessments should be modelled and aggregated (Johnson et al., 1979; Belton and Gears, 1981; Belton, 1986; Islei and Lockett, 1988; Stewart, 1992; Barzilai, 1997).

A MADA problem may also be modelled using a generalised decision matrix, where an attribute is assessed using a belief structure represented by expectations (Yang and Singh, 1994; Yang and Sen, 1994). An expectation was originally designed to model qualitative assessments with uncertainty in the evidential reasoning (ER) approach developed on the basis of decision theory and the Dempster–Shafer theory of evidence (Yang and Singh, 1994; Yang and Sen, 1994). The ER approach has been used to deal with MADA problems in engineering and management, for example motorcycle assessment (Yang and Sen, 1994), general cargo ship design (Sen and Yang, 1995), system safety analysis and synthesis (Wang et al., 1995, 1996), software safety synthesis (Wang, 1997), retro-fit ferry design (Yang and Sen, 1997), and car assessment (Yang and Xu, 1998).

In the following sections, the generalised decision matrix will be first discussed and further extended to accommodate various types of numerical data and qualitative assessments with uncertainties. Rule and utility based techniques will then be developed to provide a systematic procedure to transform various types of information into a unified format, so that both quantitative and qualitative information with uncertainties can be handled in a consistent manner. This leads to the development of a new rule and utility based ER approach. Different from traditional MADA approaches, the new ER approach employs expectations and utility intervals to handle both certain and uncertain information, whether it is complete or incomplete, precise or imprecise. The new ER approach will be illustrated using numerical examples in Sections 4 and 5.

1.2. Generalised decision matrix for MADA modelling

Most decision problems involve multiple attributes (criteria). In assessment of the quality of a motor engine, for example, attributes such as responsiveness, fuel economy, quietness, vibration and starting could be used. Weights may also be assigned to attributes to represent their relative importance. To assess an alternative (e.g. engine) on attributes, numerical values or subjective judgements can be used to differentiate one alternative from another. In this section, we use a belief structure to describe subjective assessment information. This structure will be extended in the next sections to accommodate a wider range of information.

A belief structure is represented by an expectation that was originally designed to model a subjective assessment with uncertainty (Yang and Singh, 1994; Yang and Sen, 1994). To evaluate the quietness of an engine (e.g., Honda engine), for example, an expert may state that he is 50% sure it is good and 30% sure it is excellent (Isitt, 1990). In the statement, good and excellent denote distinctive evaluation grades (standards), and the percentage values of 50 and 30 are referred to as the degrees of belief, which indicate the extents that the corresponding grades are assessed to. The above assessment can be expressed as the following expectation:

$$S(\text{quietness}) = \{(\text{good}, 0.5), (\text{excellent}, 0.3)\}, \tag{1}$$

where S(quietness) stands for the state of the engine's quietness and the real numbers 0.5 and 0.3 denote the degrees of belief of 50% and 30%, respectively.

To assess the engine on other attributes or the quality of different engines, other evaluation grades may also be used such as 'poor', 'indifferent' and 'average' (Isitt, 1990). To assess the above engine on the other four attributes, for example, the following expectations can be acquired (Isitt, 1990):

$$S(\text{responsiveness}) = \{(\text{good}, 1.0)\}, \tag{2}$$

$$S(\text{fuel economy}) = \{(\text{indifferent}, 0.5), (\text{average}, 0.5)\}, \tag{3}$$

$$S(\text{vibration}) = \{(\text{good}, 0.5), (\text{excellent}, 0.5)\}, \tag{4}$$

$$S(\text{starting}) = \{(\text{good}, 1.0)\}. \tag{5}$$

Note that expectation (1) describes an incomplete assessment as its total degree of belief is $0.5 + 0.3 < 1$ (or $50\% + 30\% < 100\%$) whilst expectations (2)–(5) describe complete assessments. Incomplete assessments are likely to acquire in real life decision problems and may result from the lack of data and evidence (incompleteness) or the inability of the assessor to provide precise judgements (imprecision) due to the novelty and complexity of the problem in question.

In the above example, engine quality may be referred to as a general (upper level) attribute and quietness etc as basic (associated lower level) attributes. In general, suppose a MADA problem has L basic attributes e_i $(i = 1, \ldots, L)$ and M alternatives a_l $(l = 1, \ldots, M)$. Then a generalised decision matrix can be constructed as follows.

First, define a set of L basic attributes as follows:

$$E = \{e_i, i = 1, \ldots, L\}. \tag{6}$$

Suppose the L basic attributes include all the factors influencing the assessment of the associated general attribute. Use an appropriate method such as an eigenvector or a geometric mean method to estimate the relative weights of the L attributes $\omega = \{\omega_1 \cdots \omega_i \cdots \omega_L\}$, where ω_i is the relative weight for basic attribute i and is normalised, so that

$$0 \leqslant \omega_i \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{L} \omega_i = 1. \tag{7}$$

Secondly, define N distinctive (mutually exclusive) evaluation grades H_n $(n = 1, \ldots, N)$ as a complete (collectively exhaustive) set of standards for assessing each alternative on all attributes, or

$$H = \{H_n,\ n = 1, \ldots, N\}. \tag{8}$$

A multiattribute decision problem may then be modelled using the following expectations for alternatives a_l $(l = 1, \ldots, M)$ on attributes e_i $(i = 1, \ldots, L)$:

$$S(e_i(a_l)) = \{(H_n, \beta_{n,i}(a_l)),\ n = 1, \ldots, N\}, \quad i = 1, \ldots L, \quad l = 1, \ldots, M, \tag{9}$$

where $\beta_{n,i}(a_l) \geqslant 0$ and $\sum_{n=1}^{N} \beta_{n,i}(a_l) \leqslant 1$. $\beta_{n,i}(a_l)$ denotes a degree of belief. An expectation for e_i and a_l reads that an attribute e_i at an alternative a_l is assessed to a grade H_n with a degree of belief of $\beta_{n,i}(a_l)$ $(n = 1, \ldots, N)$. A generalised decision matrix D_g with $S(e_i(a_l))$ as its element is constructed as follows:

$$D_{\mathrm{g}} = (S(e_i(a_l)))_{L \times M}. \tag{10}$$

Note that the generalised decision matrix consists of an elementary model to represent more general multi-level decision problems (Yang and Sen, 1994; Yang, 1999).

1.3. Attribute aggregation under uncertainty using the ER algorithm

Suppose the five basic attributes in the engine quality assessment example provide a complete set of criteria for assessing the quality of an engine. Then it is true that the quality of an engine should be just good if its responsiveness, fuel economy, quietness, vibration and starting are all assessed to be absolutely good. However, such consensus assessments are rare and attributes are often assessed to different evaluation grades, as shown by Eqs. (1)–(5). Then a problem arises as to how to generate an overall assessment about the quality of the engine by aggregating the various judgements as given in Eqs. (1)–(5). The ER algorithm provides a systematic and rational way of dealing with the aggregation problem, as presented in this subsection.

Developed on the basis of an evaluation analysis model and the evidence combination rule of the Dempster–Shafer theory of evidence, the ER algorithm uses the concepts in set theory and probability theory for aggregating multiple attributes (Yang and Singh, 1994; Yang and Sen, 1994). Suppose there is a simple two level evaluation hierarchy with a general attribute y at the upper level and several basic attributes at the lower level. The above engine assessment problem constitutes such a hierarchy with 'engine quality' as a general attribute and the five assessment criteria as basic attributes.

The details about the ER algorithm can be found in the references (Yang and Singh, 1994; Yang and Sen, 1994; Yang, 1999). The assessments represented in Eqs. (9) and (10) for an alternative a_l can be aggregated using the following ER algorithm:

$$m_{n,i} = \omega_i \beta_{n,i}(a_l), \quad n = 1, \ldots, N, \quad i = 1, \ldots, L, \tag{11}$$

$$m_{H,i} = 1 - \sum_{n=1}^{N} m_{n,i} = 1 - \omega_i \sum_{n=1}^{N} \beta_{n,i}(a_l), \quad i = 1, \ldots, L, \tag{12}$$

$$\{H_n\} : m_{n,I(i+1)} = K_{I(i+1)}(m_{n,I(i)} m_{n,i+1} + m_{n,I(i)} m_{H,i+1} + m_{H,I(i)} m_{n,i+1}), \quad n = 1, \ldots, N, \tag{13}$$

$$\{H\} : m_{H,I(i+1)} = K_{I(i+1)} m_{H,I(i)} m_{H,i+1}, \tag{14}$$

$$K_{I(i+1)} = \left[1 - \sum_{t=1}^{N} \sum_{\substack{j=1 \\ j \neq t}}^{N} m_{t,I(i)} m_{j,i+1}\right]^{-1}, \quad i = 1, \ldots, L-1, \tag{15}$$

$$\beta_H(a_l) = C_2 = \sum_{i=1}^{L} \omega_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}(a_l)\right), \tag{16a}$$

$$\beta_n(a_l) = \frac{1 - C_2}{1 - m_{H,I(L)}} m_{n,I(L)}, \quad n = 1, \ldots, N, \tag{16b}$$

where $m_{n,I(1)} = m_{n,1}(n = 1, \ldots, N)$ and $m_{H,I(1)} = m_{H,1}$. It has been proved that $\sum_{n=1}^{N} \beta_n(a_l) + \beta_H(a_l) = 1$ (Yang and Singh, 1994; Yang and Sen, 1994; Yang, 1999).

The aggregated assessment for a_l can be described by the following expectation:

$$S(y(a_l)) = \{(H_n, \beta_n(a_l)),\ n = 1, \ldots, N\}. \tag{17}$$

Suppose the utility of an evaluation grade H_n is denoted by $u(H_n)$. The expected utility of $S(y(a_l))$ is defined as follows:

$$u(S(y(a_l))) = \sum_{n=1}^{N} u(H_n)\beta_n(a_l). \tag{18}$$

Note that $\beta_n(a_l)$ denotes the lower bound of the likelihood that a_l is assessed to H_n. The upper bound of the likelihood is given by $(\beta_n(a_l) + \beta_H(a_l))$.

Complementary to the distributed assessment (Eqs. (17) and (18)), a utility interval can be established if the assessment is incomplete or imprecise. Without loss of generality, suppose the least preferred grade having the lowest utility is H_1 and the most preferred grade having the highest utility is H_N. Then the maximum, minimum and average utilities of a_l are given by:

$$u_{\max}(a_l) = \sum_{n=1}^{N-1} \beta_n(a_l)u(H_n) + (\beta_N(a_l) + \beta_H(a_l))u(H_N), \tag{19}$$

$$u_{\min}(a_l) = (\beta_1(a_l) + \beta_H(a_l))u(H_1) + \sum_{n=2}^{N} \beta_n(a_l)u(H_n), \tag{20}$$

$$u_{\text{aver}}(a_l) = \frac{u_{\max}(a_l) + u_{\min}(a_l)}{2}. \tag{21}$$

Note that if $u(H_1) = 0$ then $u(S(y(a_l))) = u_{\min}(a_l)$. Also note that if all original assessments $S(e_i(a_l))$ in the generalised decision matrix are complete, then $\beta_H(a_l) = 0$ and $u(S(y(a_l))) = u_{\max}(a_l) = u_{\min}(a_l) = u_{\text{aver}}(a_l)$. It has to be made clear that the above utilities are only used for characterising an assessment but not used in attribute aggregation.

The ranking of two alternatives a_l and a_k is based on their utility intervals. a_l is said to be preferred to a_k if and only if $u_{\min}(a_l) > u_{\max}(a_k)$; a_l is said to be indifferent to a_k if and only if $u_{\min}(a_l) = u_{\min}(a_k)$ and $u_{\max}(a_l) = u_{\max}(a_k)$. Otherwise, average utility may be used to generate a ranking, though such a ranking may be inconclusive. For instance, if $u_{\text{aver}}(a_l) > u_{\text{aver}}(a_k)$ but $u_{\max}(a_k) > u_{\min}(a_l)$, then one could say that a_l is preferred to a_k on an average basis. However, this ranking is not reliable, as there is a chance that a_k has higher utility than a_l. To generate a reliable ranking, the quality of original assessments must be improved by reducing imprecision and/or incompleteness present in the original information associated with a_l and a_k. Note that to clarify the relation between a_l and a_k there is no need to improve the quality of information related to other alternatives.

1.4. Extending the generalised decision matrix for generic MADA modelling

In the generalised decision matrix discussed in Section 1.2, a single set of evaluation grades was used, for example

$$H = \{\text{poor, indifferent, average, good, excellent}\}. \tag{22}$$

It was used to assess the general attribute (quality of engine) as well as each of the five basic attributes, i.e., responsiveness, fuel economy, quietness, vibration and starting. This group of grades is assumed to collectively provide a complete set of meaningful and distinctive standards for assessing a general attribute as well as all basic attributes. In order to facilitate the aggregation of basic attributes, it is desirable to assess them based on the same standards as defined for their associated general attribute. To facilitate data collection, however, it is more natural and acceptable to acquire assessment information in a manner appropriate to a particular attribute.

On one hand, a basic attribute may be quantitative and can be measured using precise numbers. For instance, fuel economy of a motorcycle engine may be assessed by fuel consumption that can be measured using a quantity such as how many miles a motorcycle can travel per gallon of fuel (mpg). The fuel consumption of an engine could be 37 mpg. The fuel economy of this engine is then precisely assessed by

$$S(\text{fuel economy}) = 37 \text{ mpg}. \tag{23}$$

Furthermore, fuel consumption could change under different weather and road conditions. For instance, the fuel consumption of a motorcycle engine in winter could be 31 mpg in urban areas and 39 mpg in suburb; in summer it could be 35 mpg in urban areas and 43 mpg in suburb. If the motorcycle is to be used equally frequently in those conditions, then its fuel economy could be more realistically assessed using the following expectation:

$$S(\text{fuel economy}) = \{(31, 0.25), (35, 0.25), (39, 0.25), (43, 0.25)\}, \tag{24}$$

where in (31, 0.25) the decimal number 0.25 denotes the perceived frequency (25% of the times) that the motorcycle may be used in urban areas in winter. The assessment given in Eq. (24) has an average value of 37 mpg, which is the same as that given in Eq. (23). However, Eq. (24) provides a more comprehensive and complete picture about the fuel consumption. This description is useful and necessary in situations where the frequencies are not equal or are not summed to one.

Note that Eq. (23) can be written as the following equivalent expectation:

$$S(\text{fuel economy}) = \{(37, 1)\}. \tag{25}$$

In general, let $h_{n,i}$ be the nth possible value of the ith attribute ($n = 1, \ldots, N_i$) and $\gamma_{n,i}(a_l)$ the likelihood that a_l may take this value. Then, a quantitative assessment may be represented using the following expectation:

$$S(e_i(a_l)) = \{(h_{n,i,}\gamma_{n,i}(a_l)), n = 1, \ldots, N_i\}, \tag{26}$$

where $\gamma_{n,i}(a_l) \geqslant 0$ and $\sum_{n=1}^{N_i} \gamma_{n,i}(a_l) \leqslant 1$.

On the other hand, a qualitative basic attribute may be assessed using a set of grades appropriate for this attribute but different from those defined in Eq. (22). In terms of quietness, for example, it is natural to judge that an engine is very quiet, quiet, normal, noisy, or very noisy. Thus, the basic attribute quietness may be assessed using the following set of evaluation grades:

$$H^{\text{quietness}} = \{\text{very noisy, noisy, normal, quiet, very quiet}\}. \tag{27}$$

To assess the vibration of an engine, one may use another set of grades, for example

$$H^{\text{vibration}} = \{\text{heavy, above average, below average, light}\}. \tag{28}$$

In general, let $H_{n,i}$ denote the nth evaluation grade for the ith attribute and $\gamma_{n,i}(a_l)$ the degree of belief that a_l is assessed to $H_{n,i}$. Then, a set of evaluation grades defined for attribute i is denoted by

$$H^i = \{H_{n,i}, n = 1, \ldots, N_i\}. \tag{29}$$

A subjective assessment may then be represented using the following expectation:

$$S(e_i(a_l)) = \{(H_{n,i}, \gamma_{n,i}(a_l)),\ n = 1, \ldots, N_i\}, \tag{30}$$

where $\gamma_{n,i}(a_l) \geqslant 0$ and $\sum_{n=1}^{N_i} \gamma_{n,i}(a_l) \leqslant 1$.

A quantitative and qualitative MADA problem with uncertainties may be modelled using the following generalised and extended decision matrix D_{ge}:

$$D_{\text{ge}} = S(e_i(a_l))_{L\times M}, \tag{31}$$

where $S(e_i(a_l))$ is given by Eq. (26) or (30).

Obviously, matrix (10) is a special case of matrix (31) in that in the former all attributes are assumed to be qualitative and employ the same set of evaluation grades. To solve problem (31) using either the ER approach or other methods, it is fundamental to transform the various sets of evaluation standards to a unified set so that all attributes can be assessed in a consistent and compatible manner. This to some extent mirrors the normalisation of multiple quantitative attributes with different units in traditional MADA techniques. The key difference is how to preserve the original features of incomplete assessments without having to make further assumptions or to use average values in aggregation.

One way of conducting the transformation is to define a set of grades for a general attribute and then equivalently and rationally transform other sets of grades defined for basic attributes to this general set. In the following sections, two different types of techniques are investigated for accomplishing the above task, including a rule based technique and a utility based technique.

2. Rule based information transformation techniques

The purpose of defining a unique set of evaluation grades for a particular basic attribute is to facilitate raw data collection. The grades thus defined need to be interpreted and transformed for assessment of a general attribute. Such transformation can be conducted using the decision maker's knowledge and experience described as rules. Both qualitative assessments and quantitative data can be transformed in this way.

2.1. Qualitative assessment transformation technique

In assessment, different words may be used to describe equivalent standards. Such equivalence can be established using equivalence rules. For instance, if a 'very noisy' engine means that the quality of the engine is 'poor' as far as quietness is concerned, then an evaluation grade 'very noisy' in quietness assessment is said to be equivalent to a grade 'poor' in quality assessment.

Similarly, if 'noisy' is equivalent to 'indifferent', 'normal' to 'average', 'quiet' to 'good', and 'very quiet' to 'excellent', then we could say that the set of grades {very noisy, noisy, normal, quiet, very quiet} in quietness assessment is equivalent to the set {poor, indifferent, average, good, excellent} in quality assessment. Suppose each grade $H_{n,i}$ of a basic set H^i defined in Eq. (29) means a grade H_n of a general set H defined in Eq. (8), or

$$H_{n,i} \text{ means } H_n, \quad n = 1, \ldots, N. \tag{32}$$

Then with $N = N_i$ the basic set H^i is said to be equivalent to the general set H.

Suppose H^i is equivalent to H and $N = N_i$. Then, a general assessment

$$S(e_i) = \{(H_n, \beta_{n,i}),\ n = 1, \ldots, N\} \tag{33}$$

is said to be equivalent to a basic assessment

$$S^i(e_i) = \{(H_{n,i}, \gamma_{n,i}),\ n = 1, \ldots, N_i\} \tag{34}$$

if and only if

$$\beta_{n,i} = \gamma_{n,i}, \quad n = 1, \ldots, N. \tag{35}$$

In general, it may not always be the case that $N = N_i$. It is also common that $H_{n,i}$ in H^i may not exactly mean any single grade in H but a number of grades in H to certain degrees. For instance, a 'heavily' vibrating engine might mean that the quality of the engine is between 'poor' and 'indifferent' as far as vibration is concerned. Generally, if a grade $H_{n,i}$ in H^i means a grade H_l in H to a degree of $\alpha_{l,n}$ $(l = 1, \ldots, N)$ with $0 \leqslant \alpha_{l,n} \leqslant 1$ and $\sum_{l=1}^{N} \alpha_{l,n} = 1$, then we say that

$$H_{n,i} \text{ is equivalent to } \{(H_l, \alpha_{l,n}),\ l = 1, \ldots, N\}. \tag{36}$$

Equivalence rules described by Eq. (36) or (32) needs to be provided by the decision maker. By equivalence, it is meant that the underlying utility of $H_{n,i}$ is equal to the expected utility of the expectation $\{(H_l, \alpha_{l,n}),\ l = 1, \ldots, N\}$ or $u(H_{n,i}) = \sum_{l=1}^{N} \alpha_{l,n} u(H_l)$. Eq. (32) is a special case of Eq. (36) with $\alpha_{n,n} = 1$ and $\alpha_{l,n} = 0$ for all $l = 1, \ldots, N$ and $l \neq n$.

Given Eq. (36), a basic assessment $S^i(e_i)$ defined in Eq. (34) is said to be equivalent to a general assessment $S(e_i)$ defined in Eq. (33) if and only if

$$\beta_{l,i} = \sum_{n=1}^{N_i} \alpha_{l,n} \gamma_{n,i}, \quad l = 1, \ldots, N. \tag{37}$$

We shall prove in Section 2.3 that given Eq. (36) the expected utility of $S(e_i)$ is equal to the expected utility of $S^i(e_i)$ if $\beta_{l,i}$ is calculated by Eq. (37).

To implement the transformation process, matrix equations are developed. The above transformation process can be represented by the following matrix equations:

$$\underline{\boldsymbol{b}}_i = \underline{\boldsymbol{A}}_i \times \underline{\boldsymbol{r}}_i, \tag{38}$$

$$\underline{\boldsymbol{u}}(H^i) = \underline{\boldsymbol{A}}_i^T \times \underline{\boldsymbol{u}}(H), \tag{39}$$

where

$$\underline{\boldsymbol{b}}_i = \begin{bmatrix} \beta_{1,i} \\ \beta_{2,i} \\ \vdots \\ \beta_{N,i} \end{bmatrix}, \quad \underline{\boldsymbol{A}}_i = \begin{array}{c} \\ H_1 \\ H_2 \\ \vdots \\ H_N \end{array} \begin{array}{c} \begin{array}{cccc} H_{1,i} & H_{2,i} & \cdots & H_{N_i,i} \end{array} \\ \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,N_i} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,N_i} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N,1} & \alpha_{N,2} & \cdots & \alpha_{N,N_i} \end{bmatrix} \end{array}, \quad \underline{\boldsymbol{r}}_i = \begin{bmatrix} \gamma_{1,i} \\ \gamma_{2,i} \\ \vdots \\ \gamma_{N_i,i} \end{bmatrix}, \tag{40}$$

$$\underline{\boldsymbol{u}}(H^i) = \begin{bmatrix} u(H_{1,i}) \\ u(H_{2,i}) \\ \vdots \\ u(H_{N_i,i}) \end{bmatrix}, \quad \underline{\boldsymbol{u}}(H) = \begin{bmatrix} u(H_1) \\ u(H_2) \\ \vdots \\ u(H_N) \end{bmatrix}, \tag{41}$$

$\underline{\boldsymbol{A}}_i$ may be referred to as transformation matrix.

One of the advantages of the above transformation is that the rules as represented by Eq. (36) are provided independently of the assessments of individual alternatives. This forms a basis for rational and consistent transformation of information from one form to another. Also note from Eq. (39) that given the utilities of the general grades $u(H_n)$ $(n = 1, \ldots, N)$, the utilities of the basic grades can be calculated using the equation. A transformation process would be irrational if an incomplete assessment is transformed to a complete assessment and visa versa. In Section 2.3, we shall prove that the features of an assessment are preserved in the transformation process proposed in this section.

2.2. *Quantitative data transformation technique*

A quantitative basic attribute can be assessed using numerical values as discussed in Section 1.4. In this case, equivalence rules also need to be extracted from the decision maker to transform a value to an equivalent expectation so that the quantitative attribute can be aggregated in conjunction with other qualitative attributes.

To carry out such a transformation, it is fundamental for the decision maker to provide rules relating each evaluation grade to a particular value. For instance, a fuel consumption of 50 mpg of a motor engine may mean that the quality of the engine is 'excellent' as far as fuel economy is concerned. In other words, 50 mpg fuel consumption is equivalent to 'excellent' fuel economy. Similarly, fuel consumption of 44, 38, 32 and 25 mpg may mean that fuel economy is 'good', 'average', 'indifferent' and 'poor', respectively. In general, suppose a value $h_{n,i}$ for an attribute e_i is judged to be equivalent to a grade H_n, or

$$h_{n,i} \text{ means } H_n \quad (n = 1, \ldots, N). \tag{42}$$

Without loss of generality, suppose e_i is a 'profit' attribute, that is a large value $h_{n+1,i}$ is preferred to a smaller value $h_{n,i}$. Let $h_{N,i}$ be the largest feasible value and $h_{1,i}$ the smallest. Then a value h_j on e_i may be represented using the following equivalent expectation:

$$S^i(h_j) = \{(h_{n.i}, \gamma_{n,j}),\ n = 1, \ldots, N\}, \tag{43}$$

where

$$\gamma_{n,j} = \frac{h_{n+1,i} - h_j}{h_{n+1,i} - h_{n,i}}, \quad \gamma_{n+1,j} = 1 - \gamma_{n,j} \quad \text{if } h_{n,i} \leqslant h_j \leqslant h_{n+1,i}, \tag{44}$$

$$\gamma_{k,j} = 0 \quad \text{for } k = 1, \ldots, N, \ \ k \neq n, n+1. \tag{45}$$

Note that representing h_j by Eq. (43) means that h_j is calculated by the expected value of $S^i(h_j)$ denoted by $E(S^i(h_j))$, or $h_j = E(S^i(h_j))$, and the utility of h_j is calculated by $u(S^i(h_j))$, or $u(h_j) = u(S^i(h_j))$. In the next section, we shall prove that the above equality relations are justified if the underlying utility function of the attribute e_i is assumed to be piecewise linear. The use of a piecewise linear utility function is of general significance as any nonlinear function may be approximated by a piecewise linear function if a sufficient number of values of the nonlinear function are estimated (Jacquet-Lagrèze and Siskos, 1982; Yang and Sen, 1996).

Given the equivalence rules described in Eq. (42), a value h_j can be represented by the following equivalent expectation:

$$S(h_j) = \{(H_n, \beta_{n,j}), \ n = 1, \ldots, N\}, \tag{46}$$

where

$$\beta_{n,j} = \gamma_{n,j}, \quad n = 1, \ldots, N. \tag{47}$$

Thus, given the equivalence rules between $h_{n,i}$ and H_n (Eq. (42)) and assuming a piecewise linear utility function for e_i, we are able to represent a numerical value by an equivalent expectation using Eqs. (44)–(47).

In real-world decision situations, some quantitative attribute may be a random variable and may not always take a single value but several values with different probabilities as discussed in Section 1.4. Another common example is that the future demand of a product may not be a single estimate. To assess such an attribute e_i, we may use the following distribution:

$$S^i(e_i) = \{(h_j, p_j), j = 1, \ldots, M_i\}, \tag{48}$$

where h_j $(j = 1, \ldots, M_i)$ are possible values that e_i may take and p_j is the probability that e_i takes a value h_j, where $\sum_{j=1}^{M_i} p_j \leqslant 1$. The above distribution reads that an attribute e_i takes a value h_j with a probability p_j $(j = 1, \ldots, M_i)$. Note that e_i taking a single value h_j for sure is a special case of Eq. (48) with $p_j = 1$ and $p_l = 0$ $(l = 1, \ldots, M_i, \ l \neq j)$.

Assuming a piecewise linear utility function for e_i, the distribution $S^i(e_i)$ given by Eq. (48) can be equivalently represented using $h_{n,i}$ by

$$\tilde{S}^i(e_i) = \{(h_{n,i}, \tilde{\gamma}_{n,i}), \ n = 1, \ldots, N\}, \tag{49}$$

where

$$\tilde{\gamma}_{n,i} = \begin{cases} \sum\limits_{j \in \pi_n} p_j \gamma_{n,j}, & \text{for } n = 1, \\ \sum\limits_{j \in \pi_{n-1}} p_j (1 - \gamma_{n-1,j}) + \sum\limits_{j \in \pi_n} p_j \gamma_{n,j}, & \text{for } \ 2 \leqslant n \leqslant N-1, \\ \sum\limits_{j \in \pi_{n-1}} p_j (1 - \gamma_{n-1,j}), & \text{for } n = N, \end{cases} \tag{50}$$

$$\pi_n = \begin{cases} \{j | h_{n,i} \leqslant h_j < h_{n+1,i}, \ j = 1, \ldots, M_i\}, & n = 1, \ldots, N-2, \\ \{j | h_{n,i} \leqslant h_j \leqslant h_{n+1,i}, \ j = 1, \ldots, M_i\}, & n = N-1, \end{cases} \tag{51}$$

and $\gamma_{n,j}$ is calculated by Eqs. (44) and (45). Note that $\pi_l \cap \pi_k = \phi(l, k = 1, \ldots, N-1; l \neq k)$ and $\bigcup_{n=1}^{N-1} \pi_n = \{1, 2, \ldots, M_i\}$.

Given the equivalence rule represented by Eq. (42), $S^i(e_i)$ (Eq. (48)) can be equivalently represented by the following expectation using the general grade set:

$$S(e_i) = \{(H_n, \beta_{n,i}),\ n = 1,\ldots,N\} \quad \text{with } \beta_{n,i} = \tilde{\gamma}_{n,i}. \tag{52}$$

The above equations will be used for proving several conclusions. To facilitate implementation, the transformation process can also be characterised using the following equivalent matrix equations:

$$\underline{\boldsymbol{b}}_i = \underline{\boldsymbol{A}}_i \times \underline{\boldsymbol{R}}_i \times \underline{\boldsymbol{P}}_i, \tag{53}$$

$$\underline{\boldsymbol{u}}(h^i) = \underline{\boldsymbol{A}}_i \times \underline{\boldsymbol{u}}(H) = \underline{\boldsymbol{u}}(H), \tag{54}$$

where $\underline{\boldsymbol{b}}_i$ and $\underline{\boldsymbol{u}}(H)$ are as defined in Eqs. (40) and (41) and

$$\underline{\boldsymbol{A}}_i = \begin{array}{c} \\ H_1 \\ H_2 \\ \vdots \\ H_N \end{array} \begin{array}{c} \begin{array}{cccc} h_{1,i} & h_{2,i} & \cdots & h_{N,i} \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \end{array}, \quad \underline{\boldsymbol{P}}_i = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{M_i} \end{bmatrix},$$

$$\underline{\boldsymbol{R}}_i = \begin{array}{c} \\ h_{1,i} \\ h_{2,i} \\ \vdots \\ h_{N,i} \end{array} \begin{array}{c} \begin{array}{cccc} h_1 & h_2 & \cdots & h_{M_i} \end{array} \\ \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,M_i} \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,M_i} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N,1} & \gamma_{N,2} & \cdots & \gamma_{N,M_i} \end{bmatrix} \end{array}, \quad \underline{\boldsymbol{u}}(h^i) = \begin{bmatrix} u(h_{1,i}) \\ u(h_{2,i}) \\ \vdots \\ u(h_{N,i}) \end{bmatrix}, \tag{55}$$

where $\underline{\boldsymbol{P}}_i$ is a probability vector and $\underline{\boldsymbol{R}}_i$ may be referred to as data conversion matrix whose elements ($\gamma_{n,j}$) are calculated using Eqs. (44) and (45). Note that the transformation matrix $\underline{\boldsymbol{A}}_i$ is an identity matrix consistent with the rules provided as in Eq. (42).

2.3. Properties of the rule based information transformation techniques

In the previous sections, qualitative assessments and quantitative data for basic attributes are transformed to a unified format having the same set of evaluation grades for a general attribute. The transformation must be equivalent in some sense. In this paper, by equivalent transformation it is meant that the underlying utility of an original assessment is equal to that of its transformed assessment. To conduct rational decision analysis, it is essential to preserve the features of original assessments. In particular, the completeness (or incompleteness) of an assessment must be preserved in the transformation process. In this section, we show that the techniques proposed in the last sections provide equivalent and rational transformations. The proofs of all the theorems of this section can be found in Appendix A.

Theorem 1 (Rule-based equivalent transformation of a qualitative assessment). *Define a general set of evaluation grades* $H = \{H_n,\ n = 1, \ldots, N\}$ *and a basic grade set* $H^i = \{H_{n,i},\ n = 1, \ldots, N_i\}$. *Suppose a qualitative attribute is originally assessed using* H^i *by Eq.* (34). *Suppose equivalence rules are provided as in Eq.* (32) or (36) *with* $u(H_{n,i}) = \sum_{l=1}^{N} \alpha_{l,n} u(H_l)$ *being implied. Then the attribute can be equivalently assessed using* H *by Eqs.* (33) *and* (37) *with* $u(S^i(e_i)) = u(S(e_i))$.

In the proof of the above conclusion, the equivalence between the original and the transformed assessments is established with respect to expected utility. This is sufficient if the original assessment $S^i(e_i)$ is complete, or $\sum_{n=1}^{N_i} \gamma_{n,i} = 1$. If $S^i(e_i)$ is incomplete, or $\sum_{n=1}^{N_i} \gamma_{n,i} < 1$, we have the following result.

Theorem 2 (Rule-based rational transformation of a qualitative assessment). *If an original assessment represented by Eq.* (34) *is transformed using Eq.* (37) *to a general assessment represented by Eq.* (33), *then the completeness* (*or incompleteness*) *of the assessment is preserved, that is*

$$\sum_{l=1}^{N} \beta_{l,i} = \sum_{n=1}^{N_i} \gamma_{n,i}.$$

The incompleteness of $S^i(e_i)$ is measured by $\left(1 - \sum_{n=1}^{N_i} \gamma_{n,i}\right)$. Theorem 2 shows that the incompleteness is not changed in the transformation process. This means that the utility interval of $S^i(e_i)$ is preserved and will be equal to that of $S(e_i)$ if the utilities of the worst grades in H^i and H are normalised to zero and those of the best grades to one. For the transformation of quantitative attributes, we have the following results.

Theorem 3 (Equivalent representation of a numerical value). *Let a numerical value* h_j *on a quantitative attribute* e_i *be represented by Eq.* (43). *Then* h_j *is equal to the expected value of the expectation* $S^i(h_j)$, *or* $h_j = E(S^i(h_j))$. *If a piecewise linear utility function is assumed for* e_i, *then the utility of* h_j *is equal to the expected utility of the expectation* $S^i(h_j)$, *or* $u(h_j) = u(S^i(h_j))$.

The first conclusion is the direct result of Eqs. (43)–(45). If a piecewise linear utility function is assumed for the attribute e_i, then the utility $u(h_j)$ of a value h_j can be calculated by

$$u(h_j) = u(h_{n+1,i}) - \frac{h_{n+1,i} - h_j}{h_{n+1,i} - h_{n,i}} (u(h_{n+1,i}) - u(h_{n,i})) \quad \text{if } h_{n,i} \leqslant h_j \leqslant h_{n+1,i}. \tag{56}$$

Note that the utility $u(h_{n,i})$ is calculated using Eq. (54).

Theorem 4 (Rule based equivalent transformation of quantitative data). *Define a general set of evaluation grades* $H = \{H_n, n = 1, \ldots, N\}$. *Suppose a quantitative attribute is originally assessed by Eq.* (48). *Acquire equivalence rules as described in Eq.* (42) *to identify a set of values* $h^i = \{h_{n,i},\ n = 1, \ldots, N\}$ *with* $u(h_{n,i}) = u(H_n)$ *being implied. Then the attribute can be equivalently assessed using* H *by Eq.* (52) *with* $u(S^i(e_i)) = u(S(e_i))$.

Theorem 5 (Rule-based rational transformation of quantitative data). *If an original assessment represented by Eq.* (48) *is transformed to a general assessment represented by Eq.* (52), *then the completeness* (*or incompleteness*) *of the assessment is preserved, that is*

$$\sum_{l=1}^{N} \beta_{l,i} = \sum_{j=1}^{M_i} p_j.$$

3. Utility based information transformation technique

In Section 2, both qualitative assessments and quantitative data were transformed to a unified format using equivalence rules as defined in Eqs. (32), (36) and (42). Assuming the equivalence of underlying utilities in those rules, we established equivalent transformation, though the explicit estimation of utilities was not required. This rule based technique is developed to provide a pragmatic means for extracting equivalence information from the decision maker for transforming information from one form to another.

If utilities are estimated explicitly, utility-based information transformation process can be established. In this section, techniques will be discussed for estimating utilities for both quantitative and qualitative attributes in the context of the ER framework. A utility-based information transformation technique will then be developed.

3.1. Utility estimation in the ER framework

Suppose an alternative a_l is assessed on a qualitative attribute e_i using an expectation $S(e_i(a_l))$ as shown in Eq. (9). Then the utility of the assessment is given by the expected utility of the expectation, as defined by Eq. (18), or

$$u(S(e_i(a_l))) = \sum_{n=1}^{N} u(H_n)\beta_{n,i}(a_l). \tag{57}$$

A utility interval for $S(e_i(a_l))$ can also be estimated by Eqs. (19) and (20). $u(H_n)$ can be estimated using the decision maker's preferences. If no preference information is available, it could be assumed that the utilities of evaluation grades are equidistantly distributed in the normalised utility space, so that $u(H_n) = (n-1)/(N-1)$ $(n = 1, \ldots, N)$. Otherwise, a probability assignment approach could be employed for utility estimation (Keeney and Raiffa, 1976; Farguhar, 1984; Winston, 1994).

An alternative a_l may also be assessed on a quantitative attribute e_i using a single numerical value or more generally a range of values as shown in Eq. (48). It is desirable to estimate the utility of each given value. However, this may require too much work if many values are involved. On the other hand, if a piecewise linear utility function is assumed for e_i, then one only needs to estimate the utilities of selected values. If no preference information is available, one could assume a linear marginal utility function for e_i in the normalised utility space, or $u(h_j) = u(h_{1,i}) + (u(h_{N,i}) - u(h_{1,i}))(h_j - h_{1,i})/(h_{N,i} - h_{1,i})$. Otherwise, the certain monetary equivalent (CME) approach (Keeney and Raiffa, 1976; Farguhar, 1984; Winston, 1994) may be used for this purpose.

3.2. Utility-based information transformation technique

From the discussions of the previous sections, a MADA problem can be represented by

$$S^i(e_i) = \{(H_{n,i}, \gamma_{n,i}),\ n = 1, \ldots, N_i\} \quad \text{(for qualitative attributes)} \tag{58a}$$

or

$$S^i(e_i) = \{(h_{n,i}, \gamma_{n,i}),\ n = 1, \ldots, N_i\} \quad \text{(for quantitative attributes)}\,. \tag{58b}$$

Having estimated the utilities of $H_{n,i}$ or $h_{n,i}$, we can establish a piecewise linear utility function for e_i (Jacquet-Lagrèze and Siskos, 1982; Yang and Sen, 1994). In traditional MADA approaches such as the multiattribute utility function approach, the average utility of an alternative on e_i would be calculated and aggregated in an additive, a multiplicative or other fashion. In doing so, one has to assume that assessments on e_i are all precise and complete and it is acceptable to use average utility for attribute aggregation and alternative ranking. If an incomplete or imprecise assessment is present, then the use of such a method will be inappropriate without making further assumptions or improving the quality of information.

In this section, we propose a utility-based technique to deal with both complete and incomplete information in a unified format. Suppose the utilities of the evaluation grade H_j $(j = 1, \ldots, N)$ of a general attribute y have been estimated and denoted by $u(H_j)$ $(j = 1, \ldots, N)$. Note that in a multi-level attribute hierarchy the utilities of the top level attribute must always be estimated. If y is an intermediate attribute, its utilities can be either estimated using the probability assignment or CME approach or calculated using Eq. (39) or (54). Nevertheless, given $u(H_j)$ $(j = 1, \ldots, N)$ and $u(H_{n,i})$ or $u(h_{n,i})$ $(n = 1, \ldots, N_i)$, an original assessment $S^i(e_i)$ can be transformed to an equivalent expectation $S(e_i)$ as follows:

$$S(e_i) = \{(H_j, \beta_{j,i}),\ j = 1, \ldots, N\}, \tag{59}$$

where

$$\beta_{j,i} = \begin{cases} \sum\limits_{n \in \pi_j} \gamma_{n,i} \tau_{j,n}, & \text{for } j = 1, \\ \sum\limits_{n \in \pi_{j-1}} \gamma_{n,i}(1 - \tau_{j-1,n}) + \sum\limits_{n \in \pi_j} \gamma_{n,i} \tau_{j,n}, & \text{for } 2 \leqslant j \leqslant N-1, \\ \sum\limits_{n \in \pi_{j-1}} \gamma_{n,i}(1 - \tau_{j-1,n}), & \text{for } j = N, \end{cases} \tag{60}$$

for a qualitative attribute

$$\tau_{j,n} = \frac{u(H_{j+1}) - u(H_{n,i})}{u(H_{j+1}) - u(H_j)} \quad \text{if } u(H_j) \leqslant u(H_{n,i}) \leqslant u(H_{j+1}), \tag{61a}$$

for a quantitative attribute

$$\tau_{j,n} = \frac{u(H_{j+1}) - u(h_{n,i})}{u(H_{j+1}) - u(H_j)} \quad \text{if } u(H_j) \leqslant u(h_{n,i}) \leqslant u(H_{j+1}) \tag{61b}$$

and

$$\pi_j = \begin{cases} \{n | u(H_j) \leqslant u(H_{n,i}) < u(H_{j+1}), n = 1, \ldots, N_i\}, & j = 1, \ldots, N-2, \\ \{n | u(H_j) \leqslant u(H_{n,i}) \leqslant u(H_{j+1}), n = 1, \ldots, N_i\}, & j = N-1. \end{cases} \tag{62}$$

Note that $\pi_l \cap \pi_k = \phi(l, k = 1, \ldots, N-1; l \neq k)$ and $\bigcup_{j=1}^{N-1} \pi_j = \{1, 2, \ldots, N_i\}$.

The above equations will be used for proving several conclusions. To facilitate implementation, the equivalent matrix form for a qualitative attribute e_i is given as follows:

$$\underline{\boldsymbol{b}}_i = \underline{\boldsymbol{A}}_i \times \underline{\boldsymbol{r}}_i, \tag{63}$$

where $\underline{\boldsymbol{b}}_i$ and $\underline{\boldsymbol{r}}_i$ are as defined in (40) and $\underline{\boldsymbol{A}}_i$ is given by

$$\underline{\boldsymbol{A}}_i = \begin{array}{c} \\ H_1 \\ H_2 \\ \vdots \\ H_N \end{array} \begin{array}{c} \begin{array}{cccc} H_{1,i} & H_{2,i} & \cdots & H_{N_i,i} \end{array} \\ \begin{bmatrix} \tau_{1,1} & \tau_{1,2} & \cdots & \tau_{1,N_i} \\ \tau_{2,1} & \tau_{2,2} & \cdots & \tau_{2,N_i} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{N,1} & \tau_{N,2} & \cdots & \tau_{N,N_i} \end{bmatrix} \end{array}. \tag{64}$$

In Eq. (64) $\tau_{j,n}$ is calculated by Eq. (61a).

A matrix form for transforming the assessments of a quantitative attribute is given by

$$\underline{\boldsymbol{b}}_i = \underline{\boldsymbol{A}}_i \times \underline{\boldsymbol{R}}_i \times \underline{\boldsymbol{P}}_i, \tag{65}$$

where $\underline{\boldsymbol{b}}_i$ is as defined in Eq. (40), $\underline{\boldsymbol{R}}_i$ and $\underline{\boldsymbol{P}}_i$ as in Eq. (55), and $\underline{\boldsymbol{A}}_i$ as in Eq. (64). However, the element of $\underline{\boldsymbol{A}}_i$ in Eq. (64) is calculated using Eq. (61b) and in $\underline{\boldsymbol{R}}_i$ of Eq. (55) N should be replaced by N_i. The degrees of belief $\beta_{j,i}$ are then aggregated using the ER algorithm.

3.3. Properties of the utility based information transformation technique

Similar to the rule based techniques, the above utility based technique also leads to an equivalent and rational transformation process. In this section, we will prove that assessment Eq. (58a) or (58b) is equivalent to assessment (59) in terms of underlying utility. It will also be shown that the completeness (or incompleteness) of an assessment is preserved in the transformation process. The proofs of the theorems of this section can be found in Appendix A.

Theorem 6 (Utility based equivalent transformation). *Suppose an original assessment $S^i(e_i)$ as represented by Eq.* (58a) *or* (58b) *is transformed to a general assessment $S(e_i)$ given by Eq.* (59). *If $\beta_{j,i}$ in Eq.* (59) *is calculated by s.* (60)–(62), *then there must be $u(S^i(e_i)) = u(S(e_i))$.*

The above conclusion ensures that the expected utilities of the original and the transformed assessments are the same. The following theorem also ensures that the completeness or incompleteness of an original assessment is preserved during the transformation process.

Theorem 7 (Utility based rational transformation). *Suppose an original assessment $S^i(e_i)$ represented by Eq.* (58a) or *Eq.* (58b) *is transformed to a general assessment $S(e_i)$ given by Eq.* (59). *If $\beta_{j,i}$ in Eq.* (59) *is calculated by* (60)–(62), *then there must be*

$$\sum_{j=1}^{N} \beta_{j,i} = \sum_{n=1}^{N_i} \gamma_{n,i}.$$

In summary, information transformation could be conducted at three levels. If no preference information is available, it could be assumed that the utilities of evaluation grades for a qualitative attribute are equidistantly distributed in the normalised utility space and a linear utility function might be assumed for a quantitative attribute. At this basic level, there is no participation of the decision maker in information transformation. If the decision maker has sufficient expertise in analysing an assessment problem but is not confident in estimating utilities, the rule based technique could be used for information transformation. If the decision maker is capable of estimating utilities, information transformation could be conducted through utility estimation.

4. Numerical example 1 – car performance assessment

The rule and utility based ER approach has been applied to solving decision problems in management and engineering. In the following sections, two examples are examined to demonstrate how the rule and the utility based transformation techniques can be implemented to support decision analysis under uncertainties in conjunction with the ER algorithm. The first car performance assessment is conducted using the rule based transformation technique, and the second motorcycle selection problem is dealt with using the utility based transformation technique.

A window-based and graphically designed intelligent decision system (IDS [1]) via ER has been developed on the basis of the ER approach. IDS provides a flexible and easy to use interface for modelling, analysis and reporting. It is capable of solving large MADA problems having both quantitative and qualitative attributes with uncertainties. All the analyses of this and next sections are conducted using the software. Some intermediate results are also reported for illustrative purpose.

4.1. Illustration of the rule-based ER approach

A performance assessment problem of executive cars is as shown in Table 1. Seven performance attributes are taken into account including four quantitative attributes: Acceleration (seconds from 0 to 60 mph), Braking (feet from 60 mph to 0 mph), Horsepower (hp) and Fuel economy (mpg), and three qualitative ones: Handling, Ride quality and Powertrain. The relative weight of the ith attribute is denoted by ω_i. Six different cars are chosen for this analysis. Note that e_1 and e_2 are attributes to be minimised and e_3, e_4, e_5, e_6 and e_7 are attributes to be maximised.

Most of the attributes in Table 1 are related to the technical performances of a car. As such, it may not make much sense to estimate marginal utilities of these attributes. In general, at a purely technical level the decision maker may not be able or confident to provide meaningful indifference judgements required for utility estimation. In such circumstances, an alternative way for conducting information transformation is to use equivalence rules.

Suppose the performance of a car is classified into several categories (grades) like 'Top' ($h_{6,1}$), 'Excellent' ($h_{5,1}$), 'Good' ($h_{4,1}$), 'Average' ($h_{3,1}$), 'Poor' ($h_{2,1}$) and 'Worst' ($h_{1,1}$). Define the following set of grades to assess car performance:

$$H = \{H_j, j = 1, \dots, 6\} = \{\text{Worst, Poor, Average, Good, Excellent, Top}\}. \tag{66}$$

All attributes may then be assessed with reference to this set of grades using the rule based information transformation technique.

To assess 'acceleration', for example, it is assumed that equivalence rules can be acquired. The first equivalence rule reads as follows:

1. If an executive car can accelerate from stand still to 60 mph in 7.4 s, then as far as acceleration is concerned the car's performance is at top level among the range of cars in question. This rule may be represented by a simple statement "If accelerating time is 7.4 s, then performance is top (or $h_{6,1} = 7.4$)".

Other equivalence rules could be acquired in a similar way as follows:

2. If accelerating time is 7.8 s, then performance is excellent (or $h_{5,1} = 7.8$).
3. If accelerating time is 8.2 s, then performance is good (or $h_{4,1} = 8.2$).
4. If accelerating time is 8.7 s, then performance is average (or $h_{3,1} = 8.7$).

[1] The free demo version of IDS can be obtained from the author: jian-bo.yang@umist.ac.uk. All the analysis results reported in this article can be generated using the demo.

Table 1
Original performance assessment of executive cars

Performance	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Acceleration (e_1)	8.8	8.0	7.7	8.4	8.0	7.9
Braking (e_2)	128	124	127	134	135	126
Handling (e_3)	B	A	B	B−	B+	A
Horsepower (e_4)	196	152	182	183	138	171
Ride quality (e_5)	A−	B−	B	B+	B+	A−
Powertrain (e_6)	B	B+	A	B	A−	A
Fuel economy (e_7)	20	20	21	20	19	20

5. If accelerating time is 9.2 s, then performance is poor (or $h_{2,1} = 9.2$).
6. If accelerating time is 10 s, then performance is worst (or $h_{1,1} = 10$).

The above equivalence rules are established for a range of executive cars in question. Other types of cars such as sport cars are not considered. Based on the rules, a set of evaluation grades for 'acceleration' equivalent to H results as follows:

$$H^1 = H^{\text{acceleration}} = \{h_{1,1}, h_{2,1}, h_{3,1}, h_{4,1}, h_{5,1}, h_{6,1}\} = \{10, 9.2, 8.7, 8.2, 7.8, 7.4\}. \tag{67}$$

Note that by equivalent it is implied that the underlying utility of $h_{i,1}$ is equal to that of H_i or $u(h_{i,1}) = u(H_i)$ $(i = 1, \ldots, 6)$. The transformation matrix for acceleration is given by

$$\underline{\boldsymbol{A}}_{acceleration} = \begin{array}{r} \\ worst \\ poor \\ average \\ good \\ excellent \\ top \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 9.2 & 8.7 & 8.2 & 7.8 & 7.4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}. \tag{68}$$

Given matrix (68), the underlying piecewise linear utility function of acceleration is as shown in Fig. 1, which is determined by the utilities of the evaluation grades, u(top), u(excellent), u(good), u(average), u(poor) and u(worst).

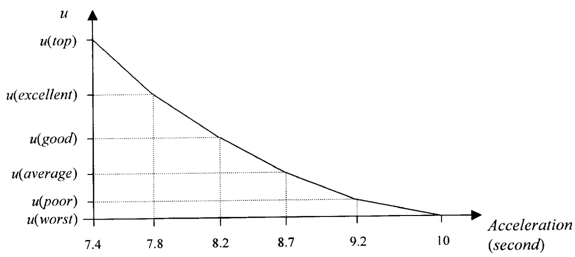

Fig. 1. Underlying piecewise linear utility function of acceleration.

Similarly, other sets of evaluation grades equivalent to H can be established for other quantitative attributes as follows:

$$H^2 = H^{\text{braking}} = \{140, 135, 131, 128, 126, 123\}, \tag{69}$$

$$H^4 = H^{\text{horse power}} = \{130, 145, 160, 175, 188, 200\}, \tag{70}$$

$$H^7 = H^{\text{fuel economy}} = \{17, 18, 19, 20, 21, 22\}. \tag{71}$$

The three qualitative attributes are all assessed using the following grades:

$$H^3 = H^5 = H^6 = \{C^-, C, C^+, B^-, B, B^+, A^-, A, A^+\}. \tag{72}$$

Suppose the following equivalence rules are acquired for relating the grades in H^3 (or H^5, H^6) to those in H:

$$\begin{aligned}
&C^- \Longleftrightarrow \{(\text{worst}, 1.0)\},\\
&C \Longleftrightarrow \{(\text{worst}, 0.6), (\text{poor}, 0.4)\},\\
&C^+ \Longleftrightarrow \{(\text{poor}, 0.6),\ (\text{average}, 0.4)\},\\
&B^- \Longleftrightarrow \{(\text{average}, 1.0)\},\\
&B \Longleftrightarrow \{(\text{average}, 0.4),\ (\text{good}, 0.6)\},\\
&B^+ \Longleftrightarrow \{\text{good}, 1.0)\},\\
&A^- \Longleftrightarrow \{(\text{good}, 0.6), (\text{excellent}, 0.4)\},\\
&A \Longleftrightarrow \{(\text{excellent}, 0.6), (\text{top}, 0.4)\},\\
&A^+ \Longleftrightarrow \{\text{top}, 1.0)\},
\end{aligned} \tag{73}$$

where ‘ $\Longleftrightarrow$ ’ reads ‘is equivalent to’. The transformation matrix for handling is then given by

$$\underline{\boldsymbol{A}}_{handling} = \begin{array}{r} \\ worst \\ poor \\ average \\ good \\ excellent \\ top \end{array} \begin{array}{c} \begin{array}{ccccccccc} C- & C & C+ & B- & B & B+ & A- & A & A+ \end{array} \\ \begin{bmatrix} 1 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 1 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 1 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 1 \end{bmatrix} \end{array}. \tag{74}$$

Note that in the above matrix the sum of the elements in any column is one. This means that the transformation from grade set (72) to grade set (66) is complete.

Given the equivalence rules as shown in Eqs. (67)–(74), the data shown in Table 1 can be automatically transformed using IDS, so that all attributes are assessed in a unified format using the same set of evaluation grades as given by Eq. (66). Take the accelerating time ($h_1 = 8.8$ s) of car 1 for example. Since $h_{2,1} = 9.2$ s and $h_{3,1} = 8.7$ s and $h_{3,1} < h_1 < h_{2,1}$, we can use Eq. (43) to describe h_1 as follows:

$$S^1(e_1(\text{car } 1)) = \{(h_{2,1}, \gamma_{2,1}), (h_{3,1}, \gamma_{3,1})\}, \tag{75}$$

where $\gamma_{2,1}$ and $\gamma_{3,1}$ are calculated using Eq. (44) as follows:

$$\gamma_{2,1} = \frac{h_{3,1} - h_1}{h_{3,1} - h_{2,1}} = \frac{8.7 - 8.8}{8.7 - 9.2} = 0.2, \quad \gamma_{3,1} = 1 - \gamma_{2,1} = 0.8. \tag{76}$$

Given that H^1 is equivalent to H, $S^1(e_1(car\ 1))$ can be equivalently represented using Eq. (46) where $\beta_{n,1} = 0$ ($n = 1, 4, 5, 6$), $\beta_{2,1} = \gamma_{2,1} = 0.2$ and $\beta_{3,1} = \gamma_{3,1} = 0.8$, or

$$S(e_1(\text{car } 1)) = \{(H_2, 0.2), (H_3, 0.8)\} = \{(\text{Poor}, 0.2), (\text{Average}, 0.8)\} = \{(P, 0.2), (A, 0.8)\}, \tag{77}$$

where P and A are abbreviated from 'Poor' and 'Average', respectively. The above assessment is as shown in row 2 and column 2 of Table 2. Abbreviate 'Worst', 'Good', 'Excellent' and 'Top' by W, G, E and T, respectively. Then all the numerical values for the four quantitative attributes of Table 1 can be transformed in a way similar to the above, as shown in row 2, 3, 5 and 8 of Table 2.

To transform qualitative attributes, take the handling of car 1 for example. Since the handling of car 1 is assessed to B as shown in column 2 and row 4 of Table 1, we can transform the assessment using Eq. (73) as follows:

$$S(e_3(\text{car1})) = \{(\text{Average}, 0.4), (\text{Good}, 0.6)\} = \{(A, 0.4), (G, 0.6)\}. \tag{78}$$

The above assessment is as shown in row 4 and column 2 of Table 2. Other qualitative assessments can be transformed in the same way, as shown in row 4, 6 and 7 of Table 2.

The assessment problem shown in Table 2 is in the same format as that defined in Eq. (9) or (10). They can then be aggregated using the ER algorithm. Intuitively, if an attribute of a car is assessed to a grade, then the overall performance of the car should also be assessed to the grade to certain degree. Take car 1 for example. Five attributes (e_2, e_3, e_5, e_6, e_7) for car 1 are all assessed to the grade 'good' to various degrees. Therefore, the performance of car 1 should be assessed to 'good' to a large degree. Similarly, the performance of car 1 should be assessed to 'Average', 'Excellent', 'Top' and 'Poor' to certain degrees but not to 'Worst' at all. The ER algorithm can be employed to calculate the overall degrees of belief.

Suppose the seven attributes are of equal importance, or $\omega_i = 1/7$ $(i = 1, \ldots, 7)$. The assessments of individual attributes for each car are aggregated into an overall assessment using the IDS software, as shown in Table 3. From the overall assessments of the six cars shown in Table 3. Some partial rankings of the cars could be generated without estimating utilities. For instance, one can say with confidence that the performances of these cars may be classified into three classes. Car 3 and car 6 belong to the first class, car 1

Table 2
Transformed distributed assessment of executive cars

Performance	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Acceleration	{(P, 0.2), (A, 0.8)}	{(G, 0.5), (E, 0.5)}	{(E, 0.75), (T, 0.25)}	{(A, 0.4), (G, 0.6)}	{(G, 0.5), (E, 0.5)}	{(G, 0.25), (E, 0.75)}
Braking	{(G, 1.0)}	{(E, 0.3333), (T, 0.6667)}	{(G, 0.5), (E, 0.5)}	{(P, 0.75), (A, 0.25)}	{(P, 1.0)}	{(E, 1.0)}
Handling	{(A, 0.4), (G, 0.6)}	{(E, 0.6), (T, 0.4)}	{(A, 0.4), (G, 0.6)}	{(A, 1.0)}	{(G, 1.0)}	{(E, 0.6), (T, 0.4)}
Horsepower	{(E, 0.3333), (T, 0.6667)}	{(P, 0.5333), (A, 0.4667)}	{(G, 0.4615), (E, 0.5385)}	{(G, 0.3846), (E, 0.6154)}	{(W, 0.4667), (P, 0.5333)}	{(A, 0.2667), (G, 0.7333)}
Ride quality	{(G, 0.6), (E, 0.4)}	{(A, 1.0)}	{(A, 0.4), (G, 0.6)}	{(G, 1.0)}	{(G, 1.0)}	{(G, 0.6), (E, 0.4)}
Powertrain	{(A, 0.4), (G, 0.6)}	{(G, 1.0)}	{(E, 0.6), (T, 0.4)}	{(A, 0.4), (G, 0.6)}	{(G, 0.6), (E, 0.4)}	{(E, 0.6), (T, 0.4)}
Fuel economy	{(G, 1.0)}	{(G, 1.0)}	{(E, 1.0)}	{(G, 1.0)}	{(A, 1.0)}	{(G, 1.0)}

Table 3
Overall distributed assessment of executive cars

	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Performance	{(P, 0.0243), (A, 0.211), (G, 0.5918), (E, 0.0918), (T, 0.081)}	{(P, 0.0703), (A, 0.2037), (G, 0.3755), (E, 0.204), (T, 0.1465)}	{(A, 0.1009), (G, 0.3016), (E, 0.5161), (T, 0.0814)}	{(P, 0.0917), (A, 0.2808), (G, 0.5522), (E, 0.0753)}	{(W, 0.06), (P, 0.2092), (A, 0.129), (G, 0.4813), (E, 0.1204)}	{(A, 0.0323), (G, 0.3634), (E, 0.5042), (T, 0.1001)}

and car 2 to the second class, and car 4 and car 5 to the third class. It may also be fair to say that in terms of performance car 5 is slightly worse than car 4 and car 2 is marginally better than car 1. However, it is not straightforward to differentiate between car 6 and car 3.

To accurately rank these cars, the utilities of these six evaluation grades need to be estimated. The above partial rankings could be used to support utility estimation using a regression method. In the rest of the section, however, we show how to apply an alternative technique 'the probability assignment approach' (Winston, 1994).

First, normalise utility so that $u(\text{Worst}) = 0$ and $u(\text{Top}) = 1$. If no preference information is available, we could assume that the utilities of the six grades are equidistantly distributed in the utility space, so that $u(H_n) = (n-1)/(N-1)$ $(n = 1, \ldots, N)$, or $u(\text{Poor}) = 0.2$, $u(\text{Average}) = 0.4$, $u(\text{Good}) = 0.6$, and $u(\text{Excellent}) = 0.8$. If possible, preference information should always be acquired to more accurately estimate utilities. To estimate the utility of the grade 'Good', for example, the following tickets may be designed using the probability assignment approach (Fig. 2).

The decision maker is asked to identify a probability value p at which the two tickets are equivalent. Suppose the equivalence holds at $\hat{p} = 0.65$. Then $u(\text{Good}) = \hat{p}u(\text{Top}) + (1-\hat{p})u(\text{Worst}) = \hat{p} = 0.65$. The utilities of the other grades can be estimated similarly. Suppose $u(\text{Excellent}) = 0.85, u(\text{Poor}) = 0.2$ and $u(\text{Average}) = 0.4$.

Note that the above utility estimation is conducted at performance level, where the decision maker is expected to be in a better position to make indifference judgements on the tickets designed. Also note that utilities are estimated after attributes have been aggregated. This ensures that the features of original assessments are preserved during attribute aggregation, though in this example all original assessments are complete.

The utilities of the six cars can then be calculated using Eqs. (18) or (19)–(21), as shown in the second row of Table 4. Note that the expected, maximum, minimum and average utilities are the same because there is no incomplete assessment in Table 1. The ranking of the cars is finally generated on the basis of the magnitude of utility, as shown in the last row of Table 4. This utility based ranking coincides with the earlier observations about the classification of these cars as well as the superiority of car 2 over car 1 and car 4 over car 5. Moreover, the relation between car 3 and car 6 is now made clear.

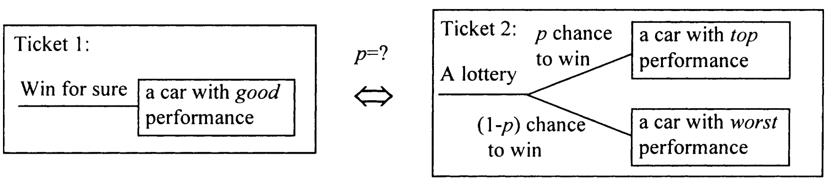

Fig. 2. Utility estimation by assigning probability.

Table 4
Utility and ranking of executive cars

	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Utility	0.633014	0.659525	0.756471	0.553569	0.508570	0.777803
Ranking	4	3	2	5	6	1

Table 5
Quantified assessment of executive cars

Performance	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Acceleration (e_1)	0.4615	0.7692	0.8846	0.6154	0.7692	0.8077
Braking (e_2)	0.7059	0.9412	0.7647	0.3529	0.2941	0.8235
Handling (e_3)	0.5	0.875	0.5	0.375	0.625	0.875
Horsepower (e_4)	0.9429	0.3143	0.7429	0.7571	0.1143	0.5857
Ride quality (e_5)	0.75	0.375	0.5	0.625	0.625	0.75
Powertrain (e_6)	0.5	0.625	0.875	0.5	0.75	0.875
Fuel economy (e_7)	0.6	0.6	0.8	0.6	0.4	0.6

4.2. Comparison with existing methods

The above example is simple in that there are only a handful of attributes and all original assessments are complete. If the marginal utility functions for all the seven attributes are estimated before aggregation, then several existing MADA methods could be used to analyse the problem. It is of interest to compare the results generated by the ER approach with those by existing methods. In this section, the multiattribute utility function (MAUF) method (Hwang and Yoon, 1981; Winston, 1994), Saaty's (left) eigenvector method (Saaty, 1988), Belton's normalised (left) eigenvector procedure (Belton and Gears, 1981) and Johnson's right eigenvector procedure (Johnson et al., 1979) are used for the comparative study.

To simplify the comparison and without loss of generality, let us assume that the evaluation grades defined in Eq. (72) are equidistantly distributed in the utility space with $u(C-) = 0$ and $u(A+) = 1$. Then the qualitative attributes could be quantified. Furthermore, suppose the marginal utility functions of all quantitative attributes are linear with the best and the worst values having the utilities of one and zero, respectively. Then Table 1 becomes Table 5.

A pairwise comparison matrix of the six cars on each of the seven attributes could also be constructed based on Table 5 (Hwang and Yoon, 1981). The above five methods have been applied to deal with the problem based on Table 5. The generated utilities and rankings are summarised in Table 6.

Table 6
Performance rankings of executive cars

Method		Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
The MAUF method	Utility	0.6374	0.643	0.7241	0.5466	0.5112	0.7598
	Ranking	4	3	2	5	6	1
Saaty's method	Utility	0.17	0.1667	0.189	0.1447	0.1313	0.1983
	Ranking	3	4	2	5	6	1
Belton's procedure	Utility	0.7378	0.7382	0.8341	0.6368	0.6011	0.8798
	Ranking	4	3	2	5	6	1
Johnson's procedure	Utility	0.8389	0.8398	0.8648	0.815	0.7653	0.8762
	Ranking	4	3	2	5	6	1
The ER approach	Utility	0.6319	0.6448	0.7308	0.5486	0.5178	0.7616
	Ranking	4	3	2	5	6	1

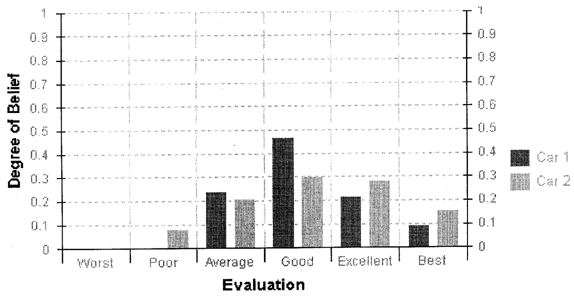

Fig. 3. Distributed assessments of cars 1 and 2.

It is clear from Table 6 that all the five rankings are consistent except that Saaty's method ranks car 1 ahead of car 2. This may be due to the fact that the difference between the two cars are marginal as measured using the average utilities. In fact the two cars are not clearly differentiated by Saaty's method, Belton's or Johnson's procedure.

In addition to average utilities, the ER approach can also provide a distributed assessment for each car. Fig. 3 shows the two distributed assessments for car 1 and car 2 generated using Table 5 by IDS. The visual difference between the two assessments is clear in Fig. 3. Car 2 has more excellent and best features than car 1. However, car 2 does have some poor features. This explains why the average assessments of the two cars are quite similar.

The above comparative study is preliminary. It only shows that the ER approach is capable of generating credible results as other well-known traditional MADA methods in a case where all of them are applicable. However, the ER approach has been developed not only to solve simple MADA problems such as that represented in Table 1 or Table 5 but also to deal with decision problems involving both complete and incomplete (or imprecise) information. A more complex problem is examined in the next section.

5. Example 2 – motorcycle assessment

To demonstrate the other features of the new ER approach, in this section we examine a motorcycle assessment problem by assuming incomplete and imprecise data of both a quantitative and qualitative nature. In the previous car ranking problem, qualitative attributes were assessed by simply selecting one of the nine defined evaluation grades (i.e. with 100% degree of belief). In this section, we use a belief structure to facilitate continuous and imprecise assessments for qualitative attributes. For quantitative attributes, both certain and random numbers are taken into account. IDS is used to support the following analysis.

The generalised and extended decision matrix shown in Table 7 describes a motorcycle assessment problem, which involves four candidate motorcycles for selection based on 29 attributes of a hierarchy. The relative weights of the same group of attributes are shown in the brackets. The assessment information includes certain and random numbers and subjective judgements with uncertainties. The qualitative attributes are all assessed on the basis of the same five evaluation grades, which are defined as poor, indifferent, average, good and excellent (see Eq. (22)) and abbreviated by P, I, A, G and E, respectively. The overall assessment of a motorcycle is also based on this set of grades.

Imprecise assessments, as lightly shaded, are involved in the decision matrix. Data absence is also assumed in Table 7, as shown by the shaded blank boxes. Some judgements and random numbers are incomplete in the sense that the total degree of belief in an assessment is not summed to one. For example, the assessment of the engine responsiveness of Yamaha is {(G, 0.3), (E, 0.6)} where the total belief degree is $(0.3 + 0.6) < 1$ (or $30\% + 60\% < 100\%$). The assessment for the fuel consumption of Yamaha is {(28, 0.25), (34, 0.25), (38, 0.25)} with the total belief degree of 0.75 (or 75%) as the fuel consumption in urban areas in winter is supposed to be not available.

Table 7
Assessment data for motorcycle selection problem

Attribute			Kawasaki	Yamaha	Honda	BMW
Price, pounds (9)			6499	5199	6199	8220
Displacement, cc (5)			1052	1188	998	987
Range, miles (7)			175	160	170	200
Top speed, mph (7)			160	155	160	145
Engine (14)	Responsiveness (0.2)		{(E, 0.8)}	{(G, 0.3), (E, 0.6)}	{(G, 1.0)}	{(I, 1.0)}
	Fuel consumption, mpg (0.4)		{(32, 0.25), (36, 0.25), (41, 0.25), (43, 0.25)}	{(28, 0.25), (34, 0.25), (38, 0.25)}	{(31, 0.25), {(35, 0.25), (39, 0.25), (43, 0.25)}	(35, 0.25), (39, 0.25), (46, 0.25), (48, 0.25)}
	Quietness (0.1)		{(I, 0.5), (A, 0.5)}	{(A, 1.0)}	{(G, 0.5), (E, 0.3)}	
	Vibration (0.1)		{(G, 1.0)}	{(I, 1.0)}	{(G, 0.5), (E, 0.5)}	{(P, 1.0)}
	Starting (0.2)		{(G, 1.0)}	{(A, 0.6), (G, 0.3)}	{(G, 1.0)}	{(A, 1.0)}
Operation (7)	Handling (0.5)	Steering (0.3)	{(E, 0.9)}	{(G, 1.0)}	{(A, 1.0)}	{(A, 0.6)}
		Bumpy bends (0.1)	{(A, 0.5), (G, 0.5)}	{(G, 1.0)}	{(G, 0.8), (E, 0.1)}	{(P, 0.5), (I, 0.5)}
		Manoeuvrability (0.4)	{(A, 1.0)}	{(E, 0.9)}		{(P, 1.0)}
		Top speed stability (0.3)	{(E, 1.0)}	{(G, 1.0)}	{(G, 1.0)}	{(G, 0.6), (E, 0.4)}
	Transmission (0.167)	Clutch operation (0.5)	{(A, 0.8)}	{(G, 1.0)}	{(E, 0.85)}	{(I, 0.2), (A, (0.8)}
		Gearbox operation (0.5)	{(A, 0.5), (G, 0.5)}	{(I, 0.5), (A, 0.5)}	{(E, 1.0)}	{(P, 1.0)}
	Brakes (0.333)	Stopping power (0.4)	{(G, 1.0)}	{(A, 0.3), (G, 0.6)}	{(G, 1.0)}	
		Braking stability (0.3)	{(G, 0.5), (E, 0.5)}	{(G, 1.0)}	{(A, 0.5), (G, 0.5)}	{(E, 1.0)}
		Feel at control (0.3)	{(P, 1.0)}	{(G, 0.5), (E, 0.5)}	{(G, 1.0)}	{(G, 0.5), (E, 0.5)}
General (14)	Quality of finish (0.4)		{(P, 0.5), (I, 0.5)}	{(G, 1.0)}	{(E, 1.0)}	{(G, 0.5), (E, 0.5)}
	Seat comfort (0.3)		{(G, 1.0)}	{(G, 0.5), (E, 0.5)}	{(G, 1.0)}	{(E, 1.0)}
	Headlight (0.1)		{(G, 1.0)}	{(A, 1.0)}	{(E, 1.0)}	{(G, 0.5), (E, 0.5)}
	Mirrors (0.1)		{(A, 0.5), (G, 0.5)}	{(G, 0.5), (E, 0.5)}	{(E, 1.0)}	{(G, 1.0)}
	Horn (0.1)			{(G, 1.0)}	{(G, 0.5), (E, 0.5)}	{(E, 1.0)}

To apply traditional MADA methods to deal with the above problem, one has to try hard to find the missing information and eliminate the imprecision, if this is possible and cost effective; otherwise additional assumptions need to be made about these missing and imprecise assessments, or certain attributes have to be abandoned for further analysis. However, the ER approach is well suited to solving the problem using the very information of Table 7. The IDS software is used to support the following analysis. For illustrative purpose, the utility-based information transformation technique is employed in this study.

The certain monetary equivalent (CME) approach (Winston, 1994) is used to estimate the utilities of the quantitative attributes. Take price for example. Suppose for this range of motorcycles the highest acceptable price is '£9,000' and the lowest possible price is '£5,000'. Note that price is a cost attribute and therefore low price is preferred. Normalise the utility of price by assigning $u(9000) = 0$ and $u(5000) = 1$.

Following the procedure of the CME approach, we first identify a price value having the average utility of £9000 and £5000. Suppose this price value is £7500. Then we have $u(7500) = (u(9000) + u(5000))/2 = 0.5$. Furthermore, suppose £6500 has the average utility of £7500 and £5000, or $u(6500) = (u(7500) + u(5000))/2 = 0.75$, and £8500 has the average utility of £9000 and £7500, or $u(8500) = (u(9000) + u(7500))/2 = 0.25$. Let

$$h_{1,1} = 9000, \quad h_{2,1} = 8500, \quad h_{3,1} = 7500, \quad h_{4,1} = 6500, \quad h_{5,1} = 5000.$$

Note that if no preference information is available a linear marginal utility function could be assumed for price. Under this assumption we would have $\overline{h}_{1,1} = 9000$, $\overline{h}_{2,1} = 8000$, $\overline{h}_{3,1} = 7000$, $\overline{h}_{4,1} = 6000$, and $\overline{h}_{5,1} = 5000$.

Given the above discussion, a price value of Table 7 can be equivalently represented using Eq. (43). For example, the price of Kawasaki is $h_1 = 6499$. Since $h_{5,1} < h_1 < h_{4,1}$, we have

$$S^1(6499) = \{(h_{4,1}, \gamma_{4,1}),\ (h_{5,1}, \gamma_{5,1})\}, \tag{79}$$

where

$$\gamma_{4,1} = \frac{h_{5,1} - h_1}{h_{5,1} - h_{4,1}} = \frac{5000 - 6499}{5000 - 6500} \approx 0.9993, \quad \gamma_{5,1} = 1 - \gamma_{4,1} \approx 0.0007. \tag{80}$$

So

$$S^1(6499) = \{(h_{4,1}, 0.9993),\ (h_{5,1}, 0.0007)\}. \tag{81}$$

Similarly, the other price values can be represented as follows:

$$S^1(5199) = \{(h_{4,1}, 0.1337), (h_{5,1}, 0.8673)\}, \tag{82}$$

$$S^1(6199) = \{(h_{4,1}, 0.7993), (h_{5,1}, 0.2007)\}, \tag{83}$$

$$S^1(8220) = \{(h_{2,1}, 0.72), (h_{3,1}, 0.28)\}. \tag{84}$$

The probability assignment approach (Winston, 1994) could be used to estimate the utilities of the five evaluation grades for the qualitative attributes. To illustrate the following transformation process and simplify discussion, suppose the utilities of the five evaluation grades are equidistantly distributed in the normalised utility space, or

$$u(P) = 0, \quad u(I) = 0.25, \quad u(A) = 0.5, \quad u(G) = 0.75, \quad u(E) = 1. \tag{85}$$

It should be noted that utility values different from the above may be estimated for the grades if preference information is available.

The assessments given in Eqs. (81)–(84) can be further transformed using Eqs. (60)–(62). Let $H_1 = P$, $H_2 = I$, $H_3 = A$, $H_4 = G$ and $H_5 = E$. Since $u(h_{n,1}) = u(H_n)(n = 1, \ldots, 5)$, Eq. (62) becomes

$$\pi_1 = \{1\}, \quad \pi_2 = \{2\}, \quad \pi_3 = \{3\}, \quad \pi_4 = \{4, 5\}. \tag{86}$$

From Eq. (60), we have:

$$\begin{aligned}
\beta_{1,i} &= \sum_{n\in\pi_1} \gamma_{n,i}\tau_{1,n} = \gamma_{1,i}\tau_{1,1}, \\
\beta_{2,i} &= \sum_{n\in\pi_1} \gamma_{n,i}(1 - \tau_{1,n}) + \sum_{n\in\pi_2} \gamma_{n,i}\tau_{2,n} = \gamma_{1,i}(1 - \tau_{1,1}) + \gamma_{2,i}\tau_{2,2}, \\
\beta_{3,i} &= \sum_{n\in\pi_2} \gamma_{n,i}(1 - \tau_{2,n}) + \sum_{n\in\pi_3} \gamma_{n,i}\tau_{3,n} = \gamma_{2,i}(1 - \tau_{2,2}) + \gamma_{3,i}\tau_{3,3}, \\
\beta_{4,i} &= \sum_{n\in\pi_3} \gamma_{n,i}(1 - \tau_{3,n}) + \sum_{n\in\pi_4} \gamma_{n,i}\tau_{4,n} = \gamma_{3,i}(1 - \tau_{3,3}) + (\gamma_{4,i}\tau_{4,4} + \gamma_{5,i}\tau_{4,5}), \\
\beta_{5,i} &= \sum_{n\in\pi_4} \gamma_{n,i}(1 - \tau_{4,n}) = \gamma_{4,i}(1 - \tau_{4,4}) + \gamma_{5,i}(1 - \tau_{4,5}).
\end{aligned} \tag{87}$$

From Eq. (61b), $\tau_{n,i}$ can be calculated by

$$\tau_{n,n} = \frac{u(H_{n+1}) - u(h_{n,1})}{u(H_{n+1}) - u(H_n)} = 1, \quad n = 1, 2, 3, 4, \qquad \tau_{4,5} = \frac{u(H_5) - u(h_{5,1})}{u(H_5) - u(H_4)} = 0. \tag{88}$$

We therefore have

$$\beta_{n,1} = \gamma_{n,1}, \quad n = 1, \ldots, 5. \tag{89}$$

Eq. (89) can also be generated simply using Eq. (63) with $\underline{\boldsymbol{A}}_i$ of (64) calculated using Eqs. (61a) and (61b).

Using Eq. (89), the assessments shown in Eqs. (81)–(84) can be transformed to the set of evaluation grades $H = \{P, I, A, G, E\}$ as follows:

$$S(6499) = \{(H_4, 0.9993), (H_5, 0.0007)\}, \tag{90}$$

$$S(5199) = \{(H_{4,}, 0.1337), (H_5, 0.8673)\}, \tag{91}$$

$$S(6199) = \{(H_4, 0.7993), (H_5, 0.2007)\}, \tag{92}$$

$$S(8220) = \{(H_2, 0.72), (H_3, 0.28)\}. \tag{93}$$

The numerical data for the other quantitative attributes can be transformed in the same way. Suppose the utilities of the attributes are estimated as follows. For displacement

$$u(900) = 0, \quad u(950) = 0.25, \quad u(1000) = 0.5, \quad u(1100) = 0.75, \quad u(1200) = 1. \tag{94}$$

For range

$$u(150) = 0, \quad u(165) = 0.25, \quad u(175) = 0.5, \quad u(190) = 0.75, \quad u(200) = 1. \tag{95}$$

For top speed

$$u(140) = 0, \quad u(148) = 0.25, \quad u(155) = 0.5, \quad u(160) = 0.75, \quad u(165) = 1. \tag{96}$$

For fuel consumption

$$u(25) = 0, \quad u(32) = 0.25, \quad u(38) = 0.5, \quad u(44) = 0.75, \quad u(50) = 1. \tag{97}$$

In Table 7, the attributes are of a three-level hierarchy. In the ER approach, each group of the bottom level attributes associated with the same upper-level attribute are first aggregated to generate an assessment for the upper-level attribute using the ER algorithm. Once the assessments for a group of upper-level attributes associated with the same higher-level attribute are all generated, these assessments can be further aggregated in the same fashion to generate an assessment for the higher-level attribute. This hierarchical aggregation process based on the ER algorithm is readily accomplished using IDS. Table 8 shows the final assessments for the four motorcycles generated by aggregating all the attributes shown in Table 7.

The above results show that Honda is clearly the most recommended motorcycle as its minimum utility is larger than the maximum utilities of the other motorcycles. This is logical as it has the best engine quality, excellent general finish and relatively low price. Yamaha is ranked the second due to its low price followed by Kawasaki. BMW is ranked the last due to its high price and below average transmission and handling system. The above ranking is conclusive for the weights provided despite the imprecision and absence of some data. This shows that decision could be made on the basis of incomplete information. Note, however, that the above ranking is the personal choice of the decision maker who provided the weights of all the attributes and also estimated their marginal utilities. This means that given the same assessment data shown in Table 7, another decision maker may achieve a different ranking.

6. Concluding remarks

The complexity in dealing with real-world decision problems results from the fact that both quantitative and qualitative attributes with uncertainties need to be handled together in a way that is rational, systematic, reliable, flexible, and transparent. The rule and utility-based ER approach developed in this paper provides a rigorous yet pragmatic way to support multiple attribute decision analysis (MADA) under uncertainties. This approach is not only capable of generating credible results for simple MADA problems as other well-known methods can do, but also flexible to handle a wider range of complex MADA problems. In particular, the proposed modelling framework can consistently accommodate both numerical

Table 8
Overall assessment of motorcycle

Assesssment	Kawasaki	Yamaha	Honda	BMW
Distributed assessment	{(P, 0.0495), (I, 0.0835), (A, 0.2136), (G, 0.5622), (E, 0.0568)}	{(P, 0.0453), (I, 0.1148), (A, 0.2097), (G, 0.3218), (E, 0.2738)}	{(P, 0.003), (I, 0.0779), (A, 0.1619), (G, 0.5303), (E, 0.2004)}	{(P, 0.0868), (I, 0.2475), (A, 0.1864), (G, 0.1128), (E, 0.3234)}
Maximum utility	0.6405	0.6833	0.7251	0.6062
Minimum utility	0.606	0.6487	0.6985	0.5631
Average utility	0.6232	0.666	0.7118	0.5847
Ranking	3	2	1	4

data and subjective judgements of various formats. The rule and utility-based techniques provide systematic yet rational ways for transforming information from one form to another. The use of distributed assessments and utility intervals makes it possible to describe incomplete and imprecise information in an explicit manner, which forms a basis for sensitivity analysis. The two numerical examples demonstrated the implementation processes of the new rule and utility-based ER approach. The ER approach is applicable to general MADA problems and has been and is being applied to decision problems in management and engineering, including product and process design, risk and safety analysis and synthesis, project management, marketing strategy analysis, quality and environmental management.

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Appendix A. The proofs of the theorems

Proof of Theorem 1. Note that Eq. (32) is a special case of Eq. (36). From the definition of expected utility (Eq. (18)) and considering Eq. (37), we have

$$\begin{aligned} u(S^i(e_i)) &= \sum_{n=1}^{N_i}\gamma_{n,i}u(H_{n,i}) = \sum_{n=1}^{N_i}\gamma_{n,i}\sum_{l=1}^{N}\alpha_{l,n}u(H_l) = \sum_{l=1}^{N}\left(\sum_{n=1}^{N_i}\alpha_{l,n}\gamma_{n,i}\right)u(H_l) = \sum_{l=1}^{N}\beta_{l,i}u(H_l) \\ &= u(S(e_i)). \qquad \square \end{aligned}$$

Proof of Theorem 2. From Eq. (37) and considering $\sum_{l=1}^{N}\alpha_{n,l} = 1$, we have

$$\sum_{l=1}^{N}\beta_{l,i} = \sum_{l=1}^{N}\sum_{n=1}^{N_i}\alpha_{l,n}\gamma_{n,i} = \sum_{n=1}^{N_i}\left(\sum_{l=1}^{N}\alpha_{l,n}\right)\gamma_{n,i} = \sum_{n=1}^{N_i}\gamma_{n,i}. \qquad \square$$

Proof of Theorem 3. The expected value of the expectation as defined by Eqs. (43)–(45) is given by

$$E(S^i(h_j)) = \gamma_{n,j}h_{n,i} + (1-\gamma_{n,j})h_{n+1,i} = \frac{h_{n+1,i}-h_j}{h_{n+1,i}-h_{n,i}}h_{n,i} + \left(1 - \frac{h_{n+1,i}-h_j}{h_{n+1,i}-h_{n,i}}\right)h_{n+1,i} = h_j.$$

The expected utility of the expectation $S^i(h_j)$ is defined by

$$u(S^i(h_j)) = \gamma_{n,j}u(h_{n,i}) + \gamma_{n+1,j}u(h_{n+1,i}) = \gamma_{n,j}u(h_{n,i}) + (1-\gamma_{n,j})u(h_{n+1,i}).$$

Suppose the utility function of e_i is piecewise linear. If $h_{n,i} \leqslant h_j \leqslant h_{n+1,i}$, then the utility of h_j is calculated as follows (Yang and Sen, 1996):

$$\begin{aligned} u(h_j) &= u(h_{n+1,i}) - \frac{h_{n+1,i}-h_j}{h_{n+1,i}-h_{n,i}}(u(h_{n+1,i}) - u(h_{n,i})) \\ &= u(h_{n+1,i}) - \gamma_{n,j}(u(h_{n+1,i}) - u(h_{n,i})) \\ &= \gamma_{n,j}u(h_{n,i}) + (1-\gamma_{n,j})u(h_{n+1,i}) = u(S^i(h_j)). \qquad \square \end{aligned}$$

Proof of Theorem 4. From the definition of $u(S^i(e_i))$, Eqs. (43)–(45), and Theorem 3, we have

$$\begin{aligned}
u(S^i(e_i)) &= \sum_{j=1}^{M_i} p_j u(h_j) = \sum_{j=1}^{M_i} p_j u(S^i(h_j)) \\
&= \sum_{j=1}^{M_i} p_j \big(\gamma_{n,j} u(h_{n,i}) + (1-\gamma_{n,j}) u(h_{n+1,i})\big) \quad \text{if } h_{n,i} \le h_j \le h_{n+1,i},\ n = 1, \ldots, N-1 \\
&= \sum_{j=1}^{M_i} p_j \big(\gamma_{1,j} u(h_{1,i}) + (1-\gamma_{1,j}) u(h_{2,i})\big) \qquad \text{if } h_{1,i} \le h_j < h_{2,i} \\
&\quad + \sum_{j=1}^{M_i} p_j \big(\gamma_{2,j} u(h_{2,i}) + (1-\gamma_{2,j}) u(h_{3,i})\big) \quad \text{if } h_{2,i} \le h_j < h_{3,i} \\
&\quad + \ldots \\
&\quad + \sum_{j=1}^{M_i} p_j \big(\gamma_{N-2,j} u(h_{N-2,i}) + (1-\gamma_{N-2,j}) u(h_{N-1,i})\big) \qquad \text{if } h_{N-2,i} \le h_j < h_{N-1,i} \\
&\quad + \sum_{j=1}^{M_i} p_j \big(\gamma_{N-1,j} u(h_{N-1,i}) + (1-\gamma_{N-1,j}) u(h_{N,i})\ \big) \qquad \text{if } h_{N-1,i} \le h_j \le h_{N,i}.
\end{aligned}$$

Using Eq. (51), we then have

$$\begin{aligned}
u(S^i(e_i)) = {} & \sum_{j\in\pi_1} p_j \gamma_{1,j} u(h_{1,i}) + \sum_{n=2}^{N-1} \left(\sum_{j\in\pi_{n-1}} p_j (1-\gamma_{n-1,j}) + \sum_{j\in\pi_n} p_j \gamma_{n,j} \right) u(h_{n,i}) \\
& + \sum_{j\in\pi_{N-1}} p_j (1-\gamma_{N-1,j}) u(h_{N,i}).
\end{aligned}$$

From Eq. (52), the definition of $u(S(e_i))$ and Eq. (50), it is true that

$$\begin{aligned}
u(S(e_i)) &= \sum_{n=1}^{N} \beta_{n,i} u(H_n) \\
&= \sum_{j\in\pi_1} p_j \gamma_{1,j} u(H_1) + \sum_{n=2}^{N-1} \left(\sum_{j\in\pi_{n-1}} p_j (1-\gamma_{n-1,j}) + \sum_{j\in\pi_n} p_j \gamma_{n,j} \right) u(H_n) + \sum_{j\in\pi_{N-1}} p_j (1-\gamma_{N-1,j}) u(H_N).
\end{aligned}$$

Since the equivalence rules represented by Eq. (42) imply $u(h_{n,i}) = u(H_n)$, we have the conclusion. □

Proof of Theorem 5. From Eqs. (50) and (52), we have

$$\begin{aligned}
\sum_{n=1}^{N} \beta_{n,i} &= \sum_{j\in\pi_1} p_j \gamma_{1,j} + \sum_{n=2}^{N-1} \left(\sum_{j\in\pi_{n-1}} p_j (1-\gamma_{n-1,j}) + \sum_{j\in\pi_n} p_j \gamma_{n,j} \right) + \sum_{j\in\pi_{N-1}} p_j (1-\gamma_{N-1,j}) \\
&= \sum_{n=1}^{N-1} \left(\sum_{j\in\pi_n} p_j \gamma_{n,j} \right) + \sum_{n=2}^{N} \left(\sum_{j\in\pi_{n-1}} p_j (1-\gamma_{n-1,j}) \right).
\end{aligned}$$

Let $n = k + 1$ in the second term of the last equation. Then

$$\sum_{n=1}^{N} \beta_{n,i} = \sum_{n=1}^{N-1} \left(\sum_{j \in \pi_n} p_j \gamma_{n,j} \right) + \sum_{k=1}^{N-1} \left(\sum_{j \in \pi_k} p_j (1 - \gamma_{k,j}) \right) = \sum_{n=1}^{N-1} \sum_{j \in \pi_n} \big(\gamma_{n,j} + (1 - \gamma_{n,j}) \big) p_j = \sum_{n=1}^{N-1} \sum_{j \in \pi_n} p_j.$$

From Eq. (51), we have

$$\bigcup_{n=1}^{N-1} \pi_n = \{1, \ldots, M_i\}.$$

Finally

$$\sum_{n=1}^{N} \beta_{n,i} = \sum_{n=1}^{N-1} \sum_{j \in \pi_n} p_j = \sum_{j=1}^{M_i} p_j. \qquad \square$$

Proof of Theorem 6. From the definition of the expected utilities of $S^i(e_i)$ and $S(e_i)$, we have

$$u(S^i(e_i)) = \sum_{n=1}^{N_i} \gamma_{n,i} u(H_{n,i}) \quad \text{and } u(S(e_i)) = \sum_{j=1}^{N} \beta_{j,i} u(H_j).$$

From Eq. (60), $u(S(e_i))$ can be calculated by

$$\begin{aligned} u(S(e_i)) &= \sum_{n \in \pi_1} \gamma_{n,i} \tau_{1,n} u(H_1) + \sum_{j=2}^{N-1} \left(\sum_{n \in \pi_{j-1}} \gamma_{n,i} (1 - \tau_{j-1,n}) + \sum_{n \in \pi_j} \gamma_{n,i} \tau_{j,n} \right) u(H_j) \\ &\quad + \sum_{n \in \pi_{N-1}} \gamma_{n,i} (1 - \tau_{N-1,j}) u(H_N) \\ &= \sum_{j=1}^{N-1} \left(\sum_{n \in \pi_j} \gamma_{n,i} \tau_{j,n} \right) u(H_j) + \sum_{j=2}^{N} \left(\sum_{n \in \pi_{j-1}} \gamma_{n,i} (1 - \tau_{j-1,n}) \right) u(H_j). \end{aligned}$$

In the second term of the last equation, let $k = j - 1$. Then

$$\begin{aligned} u(S(e_i)) &= \sum_{j=1}^{N-1} \left(\sum_{n \in \pi_j} \gamma_{n,i} \tau_{j,n} \right) u(H_j) + \sum_{k=1}^{N-1} \left(\sum_{n \in \pi_k} \gamma_{n,i} (1 - \tau_{k,n}) \right) u(H_{k+1}) \\ &= \sum_{j=1}^{N-1} \sum_{n \in \pi_j} \gamma_{n,i} \big(\tau_{j,n} u(H_j) + (1 - \tau_{j,n}) u(H_{j+1}) \big). \end{aligned}$$

From Eq. (61a), we generate

$$u(H_{n,i}) = \tau_{j,n} u(H_j) + (1 - \tau_{j,n}) u(H_{j+1}).$$

From Eq. (62), there is

$$\bigcup_{j=1}^{N-1} \pi_j = (1, \ldots, N_i).$$

We therefore conclude that

$$u(S(e_i)) = \sum_{j=1}^{N-1} \sum_{n \in \pi_j} \gamma_{n,i} u(H_{n,i}) = \sum_{n=1}^{N_i} \gamma_{n,i} u(H_{n,i}) = u(S^i(e_i)). \qquad \square$$

Proof of Theorem 7. From Eq. (58a), the total degree of belief in an original assessment $S^i(e_i)$ is given by $\sum_{n=1}^{N_i} \gamma_{n,i}$. From Eqs. (59) and (60) the total degree of belief in a transformed assessment is calculated by

$$\begin{aligned}\sum_{j=1}^{N} \beta_{j,i} &= \sum_{n \in \pi_1} \gamma_{n,i} \tau_{1,n} + \sum_{j=2}^{N-1} \left(\sum_{n \in \pi_{j-1}} \gamma_{n,i}(1 - \tau_{j-1,n}) + \sum_{n \in \pi_j} \gamma_{n,i} \tau_{j,n} \right) + \sum_{n \in \pi_{N-1}} \gamma_{n,i}(1 - \tau_{N-1,j}) \\ &= \sum_{j=1}^{N-1} \sum_{n \in \pi_j} \gamma_{n,i} \tau_{j,n} + \sum_{j=2}^{N} \sum_{n \in \pi_{j-1}} \gamma_{n,i}(1 - \tau_{j-1,n}).\end{aligned}$$

In the second term of the last equation, let $k = j - 1$. Then

$$\sum_{j=1}^{N} \beta_{j,i} = \sum_{j=1}^{N-1} \sum_{n \in \pi_j} \gamma_{n,i} \tau_{j,n} + \sum_{k=1}^{N-1} \sum_{n \in \pi_k} \gamma_{n,i}(1 - \tau_{k,n}) = \sum_{j=1}^{N-1} \sum_{n \in \pi_j} \gamma_{n,i}(\tau_{j,n} + (1 - \tau_{j,n})).$$

Since from Eq. (62) there is

$$\bigcup_{j=1}^{N-1} \pi_j = (1, \ldots, N_i),$$

we finally have

$$\sum_{j=1}^{N} \beta_{j,i} = \sum_{j=1}^{N-1} \sum_{n \in \pi_j} \gamma_{n,i} = \sum_{n=1}^{N_i} \gamma_{n,i}. \qquad \square$$

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