Module 3

Constitutive Equations

Learning Objectives

- Understand basic stress-strain response of engineering materials.
- Quantify the linear elastic stress-strain response in terms of tensorial quantities and in particular the fourth-order elasticity or stiffness tensor describing Hooke's Law.
- Understand the relation between internal material symmetries and macroscopic anisotropy, as well as the implications on the structure of the stiffness tensor.
- Quantify the response of anisotropic materials to loadings aligned as well as rotated with respect to the material principal axes with emphasis on orthotropic and transversely-isotropic materials.
- Understand the nature of temperature effects as a source of thermal expansion strains.
- Quantify the linear elastic stress and strain tensors from experimental strain-gauge measurements.
- Quantify the linear elastic stress and strain tensors resulting from special material loading conditions.

3.1 Linear elasticity and Hooke's Law

Readings: Reddy 3.4.1 3.4.2
BC 2.6

Consider the stress strain curve $\sigma = f(\epsilon)$ of a linear elastic material subjected to uni-axial stress loading conditions (Figure 3.1).

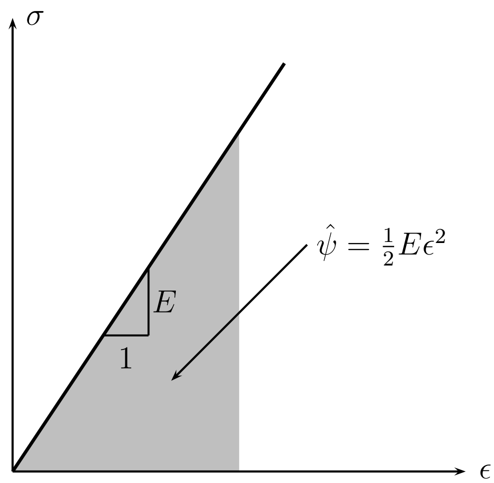


Figure 3.1: Stress-strain curve for a linear elastic material subject to uni-axial stress σ (Note that this is not uni-axial strain due to Poisson effect)

In this expression, E is Young's modulus.

Strain Energy Density
For a given value of the strain ϵ, the *strain energy density (per unit volume)* $\psi = \hat{\psi}(\epsilon)$, is defined as the area under the curve. In this case,

$$\psi(\epsilon) = \frac{1}{2}E\epsilon^2$$

We note, that according to this definition,

$$\sigma = \frac{\partial \hat{\psi}}{\partial \epsilon} = E\epsilon$$

In general, for (possibly non-linear) elastic materials:

$$\sigma_{ij} = \sigma_{ij}(\epsilon) = \frac{\partial \hat{\psi}}{\partial \epsilon_{ij}} \tag{3.1}$$

Generalized Hooke's Law
Defines the most general linear relation among all the components of the stress and strain tensor

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{3.2}$$

In this expression: C_{ijkl} are the components of the fourth-order *stiffness* tensor of material properties or *Elastic moduli*. The fourth-order stiffness tensor has 81 and 16 components for three-dimensional and two-dimensional problems, respectively. The strain energy density in this case is a quadratic function of the strain:

$$\hat{\psi}(\epsilon) = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl} \tag{3.3}$$

Concept Question 3.1.1. *Derivation of Hooke's law.*

Derive the Hooke's law from quadratic strain energy function Starting from the quadratic strain energy function and the definition for the stress components given in the notes,

1. derive the Generalized Hooke's law $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$. ■ **Solution:** We start by computing:

   $$\frac{\partial \epsilon_{ij}}{\partial \epsilon_{kl}} = \delta_{ik}\delta_{jl}$$

   However, we have lost the symmetry, i.e. the lhs is symmetric with respect to ij and with respect to kl, but the rhs is only symmetric with respect to ik and jl. But we can easily recover the same symmetries as follows:

   $$\frac{\partial \epsilon_{ij}}{\partial \epsilon_{kl}} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

   Now we use this to compute σ_{ij} as follows:

   $$\begin{aligned} \sigma_{ij}(\epsilon) &= \frac{\partial \hat{\psi}}{\partial \epsilon_{ij}} \\ &= \frac{\partial}{\partial \epsilon_{ij}} \left(\frac{1}{2} C_{klmn} \epsilon_{kl} \epsilon_{mn} \right) \\ &= \frac{1}{2} C_{klmn} \left(\frac{\partial \epsilon_{kl}}{\partial \epsilon_{ij}} \epsilon_{mn} + \epsilon_{kl} \frac{\partial \epsilon_{mn}}{\partial \epsilon_{ij}} \right) \\ & \text{using the first equation:} \\ &= \frac{1}{2} C_{klmn} \left(\delta_{ki}\delta_{lj}\epsilon_{kl} + \epsilon_{kl}\delta_{mi}\delta_{nj} \right) \\ &= \frac{1}{2} \left(C_{ijmn}\epsilon_{mn} + C_{klij}\epsilon_{kl} \right) \end{aligned}$$

   if $C_{klij} = C_{ijkl}$, we obtain:

   $$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

   ■

3.2 Transformation of basis for the elasticity tensor components

Readings: BC 2.6.2, Reddy 3.4.2

The stiffness tensor can be written in two different orthonormal basis as:

$$\begin{aligned}\mathbf{C} &= C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \\ &= \tilde{C}_{pqrs}\tilde{\mathbf{e}}_p \otimes \tilde{\mathbf{e}}_q \otimes \tilde{\mathbf{e}}_r \otimes \tilde{\mathbf{e}}_s\end{aligned} \tag{3.4}$$

As we've done for first and second order tensors, in order to transform the components from the $\mathbf{e}_i$ to the $\tilde{\mathbf{e}}_j$ basis, we take dot products with the basis vectors $\tilde{\mathbf{e}}_j$ using repeatedly the fact that $(\mathbf{e_m} \otimes \mathbf{e}_n) \cdot \tilde{\mathbf{e}}_k = (\mathbf{e}_n \cdot \tilde{\mathbf{e}}_k)\mathbf{e_m}$ and obtain:

$$\boxed{\tilde{C}_{ijkl} = C_{pqrs}(\mathbf{e}_p \cdot \tilde{\mathbf{e}}_i)(\mathbf{e}_q \cdot \tilde{\mathbf{e}}_j)(\mathbf{e}_r \cdot \tilde{\mathbf{e}}_k)(\mathbf{e}_s \cdot \tilde{\mathbf{e}}_l)} \tag{3.5}$$

3.3 Symmetries of the stiffness tensor

Readings: BC 2.1.1

The stiffness tensor has the following *minor symmetries* which result from the symmetry of the stress and strain tensors:

$$\sigma_{ij} = \sigma_{ji} \;\Rightarrow C_{jikl} = C_{ijkl} \tag{3.6}$$

Proof by (generalizable) example:

$$\begin{gathered}\text{From Hooke's law we have } \sigma_{21} = C_{21kl}\epsilon_{kl},\; \sigma_{12} = C_{12kl}\epsilon_{kl} \\ \text{and from the symmetry of the stress tensor we have } \sigma_{21} = \sigma_{12} \\ \Rightarrow \quad \text{Hence } C_{21kl}\epsilon_{kl} = C_{12kl}\epsilon_{kl} \\ \text{Also, we have } \big(C_{21kl} - C_{12kl}\big)\epsilon_{kl} = 0 \;\Rightarrow\; \text{Hence } C_{21kl} = C_{12kl}\end{gathered}$$

This reduces the number of material constants from $81 = 3 \times 3 \times 3 \times 3 \to 54 = 6 \times 3 \times 3$. In a similar fashion we can make use of the symmetry of the strain tensor

$$\epsilon_{ij} = \epsilon_{ji} \;\Rightarrow C_{ijlk} = C_{ijkl} \tag{3.7}$$

This further reduces the number of material constants to $36 = 6 \times 6$. To further reduce the number of material constants consider equation (3.1), (3.1):

$$\sigma_{ij} = \frac{\partial \hat{\psi}}{\partial \epsilon_{ij}} = C_{ijkl}\epsilon_{kl} \tag{3.8}$$

$$\frac{\partial^2 \hat{\psi}}{\partial \epsilon_{mn}\partial \epsilon_{ij}} = \frac{\partial}{\partial \epsilon_{mn}}\big(C_{ijkl}\epsilon_{kl}\big) \tag{3.9}$$

$$C_{ijkl}\delta_{km}\delta_{ln} = \frac{\partial^2 \hat{\psi}}{\partial \epsilon_{mn}\partial \epsilon_{ij}} \tag{3.10}$$

$$C_{ijmn} = \frac{\partial^2 \hat{\psi}}{\partial \epsilon_{mn}\partial \epsilon_{ij}} \tag{3.11}$$

Assuming equivalence of the mixed partials:

$$C_{ijkl} = \frac{\partial^2 \hat{\psi}}{\partial \epsilon_{kl} \partial \epsilon_{ij}} = \frac{\partial^2 \hat{\psi}}{\partial \epsilon_{ij} \partial \epsilon_{kl}} = C_{klij} \tag{3.12}$$

This further reduces the number of material constants to 21. The most general anisotropic linear elastic material therefore has 21 material constants. We can write the stress-strain relations for a linear elastic material exploiting these symmetries as follows:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \tag{3.13}$$

3.4 Engineering or Voigt notation

Since the tensor notation is already lost in the matrix notation, we might as well give indices to all the components that make more sense for matrix operation:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & symm & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} \tag{3.14}$$

We have: 1) combined pairs of indices as follows: $()_{11} \to ()_1, ()_{22} \to ()_2, ()_{33} \to ()_3, ()_{23} \to ()_4, ()_{13} \to ()_5, ()_{12} \to ()_6$, and, 2) defined the *engineering shear strains* as the sum of symmetric components, e.g. $\epsilon_4 = 2\epsilon_{23} = \epsilon_{23} + \epsilon_{32}$, etc.

When the material has symmetries within its structure the number of material constants is reduced even further. We now turn to a brief discussion of material symmetries and anisotropy.

3.5 Material Symmetries and anisotropy

Anisotropy refers to the directional dependence of material properties (mechanical or otherwise). It plays an important role in Aerospace Materials due to the wide use of engineered composites.

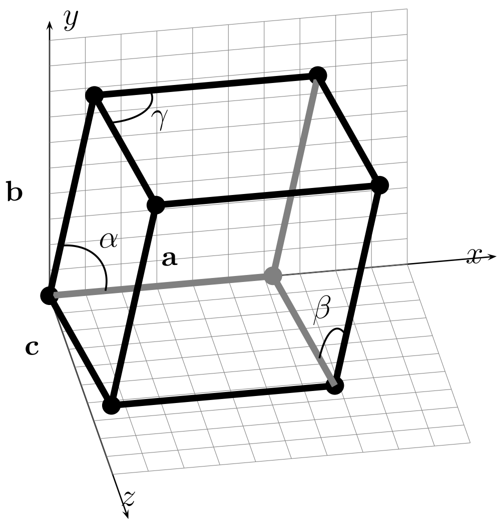
The different types of material anisotropy are determined by the existence of symmetries in the internal structure of the material. The more the internal symmetries, the simpler the structure of the stiffness tensor. Each type of symmetry results in the invariance of the stiffness tensor to a specific *symmetry transformations* (rotations about specific axes and reflections about specific planes). These symmetry transformations can be represented by an orthogonal second order tensor, i.e. $\mathbf{Q} = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, such that $\mathbf{Q}^{-1} = \mathbf{Q}^T$ and:

$$det(Q_{ij}) = \begin{cases} +1 & \text{rotation} \\ -1 & \text{reflection} \end{cases}$$

The invariance of the stiffness tensor under these transformations is expressed as follows:

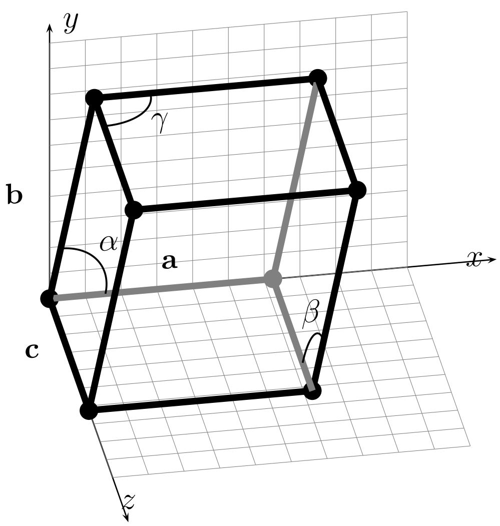
$$C_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs} \tag{3.15}$$

Let's take a brief look at various **classes of material symmetry**, corresponding **symmetry transformations**, implications on the **anisotropy of the material**, and the **structure of the stiffness tensor**:



Triclinic: no symmetry planes, fully anisotropic.
$\alpha, \beta, \gamma < 90$
Number of independent coefficients: 21
Symmetry transformation: None

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix}$$



Monoclinic: one symmetry plane (xy).
$a \neq b \neq c,\ \beta = \gamma = 90, \alpha < 90$
Number of independent coefficients: 13
Symmetry transformation: reflection about z-axis

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2313} & 0 \\ & symm & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

Concept Question 3.5.1. *Monoclinic symmetry.*

Let's consider a monoclinic material.

1. Derive the structure of the stiffness tensor for such a material and show that the tensor has 13 independent components.

■ **Solution:** The symmetry transformations can be represented by an orthogonal second order tensor, i.e. $\mathbf{Q} = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, such that$\mathbf{Q}^{-1} = \mathbf{Q}^T$ and:

$$det(Q_{ij}) = \begin{cases} +1 & \text{rotation} \\ -1 & \text{reflection} \end{cases}$$

The invariance of the stiffness tensor under these transformations is expressed as follows:

$$C_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs} \tag{3.16}$$

Herein, in the case of a monoclinic material, there is one symmetry plane (xy). Hence the second order tensor $\mathbf{Q}$ is written as follows:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.17}$$

Applying the corresponding symmetry transformation to the stiffness tensor:

$$\begin{aligned} C_{11} = C_{1111} &= Q_{1p}Q_{1q}Q_{1r}Q_{1s}C_{pqrs} \\ &= \delta_{1p}\delta_{1q}\delta_{1r}\delta_{1s}C_{pqrs} \end{aligned}$$

the term $\delta_{1p}\delta_{1q}\delta_{1r}\delta_{1s}$ is $\simeq 0$ if only $p = 1$ and $q = 1$ and $r = 1$ and $s = 1$:

$$C_{1111} = C_{1111}$$

In the similar manner we obtain:

$$\begin{aligned} C_{12} = C_{1122} &= Q_{1p}Q_{1q}Q_{2r}Q_{2s}C_{pqrs} \\ &= \delta_{1p}\delta_{1q}\delta_{2r}\delta_{2s}C_{pqrs} \\ &= C_{1122} \end{aligned}$$

$$\begin{aligned} C_{13} = C_{1133} &= Q_{1p}Q_{1q}Q_{3r}Q_{3s}C_{pqrs} \\ &= \delta_{1p}\delta_{1q}(-\delta_{3r})(-\delta_{3s})C_{pqrs} \\ &= C_{1133} \end{aligned}$$

$$\begin{aligned} C_{14} = C_{1123} &= Q_{1p}Q_{1q}Q_{2r}Q_{3s}C_{pqrs} \\ &= \delta_{1p}\delta_{1q}\delta_{2r}(-\delta_{3s})C_{pqrs} \\ &= -C_{1123} = 0 \end{aligned}$$

$$
\begin{aligned}
C_{15} = C_{1113} &= Q_{1p}Q_{1q}Q_{1r}Q_{3s}C_{pqrs} \\
&= \delta_{1p}\delta_{1q}\delta_{1r}(-\delta_{3s})C_{pqrs} \\
&= -C_{1113} = 0
\end{aligned}
$$

$$
\begin{aligned}
C_{16} = C_{1112} &= Q_{1p}Q_{1q}Q_{1r}Q_{2s}C_{pqrs} \\
&= \delta_{1p}\delta_{1q}\delta_{1r}\delta_{2s}C_{pqrs} \\
&= C_{1112}
\end{aligned}
$$

$$
\begin{aligned}
C_{22} = C_{2222} &= Q_{2p}Q_{2q}Q_{2r}Q_{2s}C_{pqrs} \\
&= \delta_{2p}\delta_{2q}\delta_{2r}\delta_{2s}C_{pqrs} \\
&= C_{2222}
\end{aligned}
$$

$$
\begin{aligned}
C_{23} = C_{2233} &= Q_{2p}Q_{2q}Q_{3r}Q_{3s}C_{pqrs} \\
&= \delta_{2p}\delta_{2q}(-\delta_{3r})(-\delta_{3s})C_{pqrs} \\
&= C_{2233}
\end{aligned}
$$

$$
\begin{aligned}
C_{24} = C_{2223} &= Q_{2p}Q_{2q}Q_{2r}Q_{3s}C_{pqrs} \\
&= \delta_{2p}\delta_{2q}\delta_{2r}(-\delta_{3s})C_{pqrs} \\
&= -C_{2223} = 0
\end{aligned}
$$

$$
\begin{aligned}
C_{25} = C_{2213} &= Q_{2p}Q_{2q}Q_{1r}Q_{3s}C_{pqrs} \\
&= \delta_{2p}\delta_{2q}\delta_{1r}(-\delta_{3s})C_{pqrs} \\
&= -C_{2213} = 0
\end{aligned}
$$

$$
\begin{aligned}
C_{26} = C_{2212} &= Q_{2p}Q_{2q}Q_{1r}Q_{2s}C_{pqrs} \\
&= \delta_{2p}\delta_{2q}\delta_{1r}\delta_{2s}C_{pqrs} \\
&= C_{2212}
\end{aligned}
$$

$$
\begin{aligned}
C_{33} = C_{3333} &= Q_{3p}Q_{3q}Q_{3r}Q_{3s}C_{pqrs} \\
&= (-\delta_{3p})(-\delta_{3q})(-\delta_{3r})(-\delta_{3s})C_{pqrs} \\
&= C_{3333}
\end{aligned}
$$

$$
\begin{aligned}
C_{34} = C_{3323} &= Q_{3p}Q_{3q}Q_{2r}Q_{3s}C_{pqrs} \\
&= (-\delta_{3p})(-\delta_{3q})\delta_{2r}(-\delta_{3s})C_{pqrs} \\
&= -C_{3323} = 0
\end{aligned}
$$

$$\begin{aligned} C_{35} = C_{3313} &= Q_{3p}Q_{3q}Q_{1r}Q_{3s}C_{pqrs} \\ &= (-\delta_{3p})(-\delta_{3q})\delta_{1r}(-\delta_{3s})C_{pqrs} \\ &= -C_{3313} = 0 \end{aligned}$$

$$\begin{aligned} C_{36} = C_{3312} &= Q_{3p}Q_{3q}Q_{1r}Q_{2s}C_{pqrs} \\ &= (-\delta_{3p})(-\delta_{3q})\delta_{1r}\delta_{2s}C_{pqrs} \\ &= C_{3312} \end{aligned}$$

$$\begin{aligned} C_{44} = C_{2323} &= Q_{2p}Q_{3q}Q_{2r}Q_{3s}C_{pqrs} \\ &= \delta_{2p}(-\delta_{3q})\delta_{2r}(-\delta_{3s})C_{pqrs} \\ &= C_{2323} \end{aligned}$$

$$\begin{aligned} C_{45} = C_{2313} &= Q_{2p}Q_{3q}Q_{1r}Q_{3s}C_{pqrs} \\ &= \delta_{2p}(-\delta_{3q})\delta_{1r}(-\delta_{3s})C_{pqrs} \\ &= C_{2313} \end{aligned}$$

$$\begin{aligned} C_{46} = C_{2312} &= Q_{2p}Q_{3q}Q_{1r}Q_{2s}C_{pqrs} \\ &= \delta_{2p}(-\delta_{3q})\delta_{1r}\delta_{2s}C_{pqrs} \\ &= -C_{2312} = 0 \end{aligned}$$

$$\begin{aligned} C_{55} = C_{1313} &= Q_{1p}Q_{3q}Q_{1r}Q_{3s}C_{pqrs} \\ &= \delta_{1p}(-\delta_{3q})\delta_{1r}(-\delta_{3s})C_{pqrs} \\ &= C_{1313} \end{aligned}$$

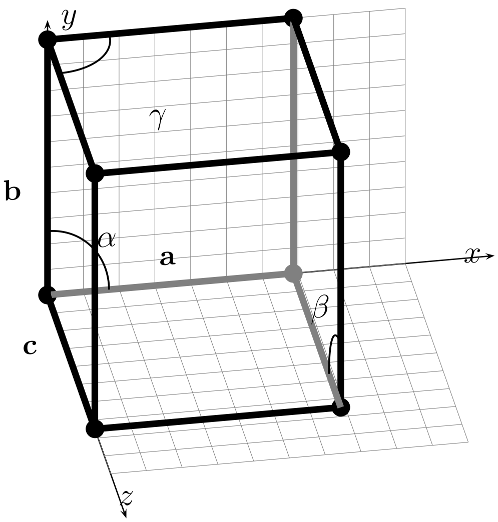
$$\begin{aligned} C_{56} = C_{1312} &= Q_{1p}Q_{3q}Q_{1r}Q_{2s}C_{pqrs} \\ &= \delta_{1p}(-\delta_{3q})\delta_{1r}\delta_{2s}C_{pqrs} \\ &= -C_{1312} = 0 \end{aligned}$$

$$\begin{aligned} C_{66} = C_{1212} &= Q_{1p}Q_{2q}Q_{1r}Q_{2s}C_{pqrs} \\ &= \delta_{1p}\delta_{2q}\delta_{1r}\delta_{2s}C_{pqrs} \\ &= C_{1212} \end{aligned}$$

Hence, the elastic tensor has 13 independant components:

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2313} & 0 \\ & symm & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

■



Orthotropic: three mutually orthogonal planes of reflection symmetry. $a \neq b \neq c$, $\alpha = \beta = \gamma = 90$
Number of independent coefficients: 9
Symmetry transformations: reflections about all three orthogonal planes

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{2323} & 0 & 0 \\ & symm & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

Concept Question 3.5.2. *Orthotropic elastic tensor.*
Consider an orthotropic linear elastic material where **1**, **2** and **3** are the orthotropic axes.

1. Use the symmetry transformations corresponding to this material shown in the notes to derive the structure of the elastic tensor.

2. In particular, show that the elastic tensor has 9 independent components.

■ **Solution:** For the reflection about the plane (**1**,**2**), the stress after reflection σ^* is expressed as a function of the stress before the reflection and the reflection transformation R:

$$\sigma^* = \mathbf{R}^T \sigma \mathbf{R} \tag{3.18}$$

with (1,2)

$$R = R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.19}$$

$$\sigma^* = \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{13} \\ \sigma_{12} & \sigma_{22} & -\sigma_{23} \\ -\sigma_{13} & -\sigma_{23} & \sigma_{33} \end{bmatrix}. \tag{3.20}$$

and

$$\varepsilon^* = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & -\varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & -\varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & \varepsilon_{33} \end{bmatrix}. \tag{3.21}$$

or you can write:

$$\epsilon^*_{ij} = Q_{ip} Q_{jp} \epsilon_{pq} \tag{3.22}$$

and

$$\sigma^*_{ij} = Q_{ip} Q_{jp} \sigma_{pq} \tag{3.23}$$

with $\mathbf{Q} = \mathbf{R}$, which leads to:

$$\begin{aligned} \epsilon^*_{11} &= \delta_{1p}\delta_{1q}\epsilon_{pq} \\ &= \epsilon_{11} \end{aligned}$$

$$\begin{aligned} \epsilon^*_{22} &= \delta_{2p}\delta_{2q}\epsilon_{pq} \\ &= \epsilon_{22} \end{aligned}$$

$$\begin{aligned} \epsilon^*_{33} &= (-\delta_{3p})(-\delta_{3q})\epsilon_{pq} \\ &= \epsilon_{33} \end{aligned}$$

$$\begin{aligned} \epsilon^*_{12} &= \delta_{1p}\delta_{2q}\epsilon_{pq} \\ &= \epsilon_{12} \end{aligned}$$

$$\begin{aligned} \epsilon^*_{23} &= \delta_{2p}(-\delta_{3q})\epsilon_{pq} \\ &= -\epsilon_{23} \end{aligned}$$

$$\begin{aligned} \epsilon^*_{13} &= \delta_{1p}(-\delta_{3q})\epsilon_{pq} \\ &= -\epsilon_{13} \end{aligned}$$

and in the similar manner:

$$\begin{aligned} \sigma^*_{11} &= \sigma_{11} \\ \sigma^*_{22} &= \sigma_{22} \\ \sigma^*_{33} &= \sigma_{33} \\ \sigma^*_{12} &= \sigma_{12} \\ \sigma^*_{23} &= -\sigma_{23} \\ \sigma^*_{13} &= -\sigma_{13} \end{aligned}$$

Let's write the consitutive equation $\sigma^* = C : \varepsilon^*$

$$\begin{bmatrix} \sigma^*_{11} \\ \sigma^*_{22} \\ \sigma^*_{33} \\ \sigma^*_{12} \\ \sigma^*_{23} \\ \sigma^*_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ sym & & & & & C_{66} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon^*_{11} \\ \varepsilon^*_{22} \\ \varepsilon^*_{33} \\ 2\varepsilon^*_{12} \\ 2\varepsilon^*_{23} \\ 2\varepsilon^*_{31} \end{bmatrix}. \tag{3.24}$$

$$\begin{aligned}
\sigma^*_{11} &= C_{11}\varepsilon^*_{11} + C_{12}\varepsilon^*_{22} + C_{13}\varepsilon^*_{33} + 2C_{14}\varepsilon^*_{12} + 2C_{15}\varepsilon^*_{23} + 2C_{16}\varepsilon^*_{31} \\
\sigma^*_{22} &= C_{12}\varepsilon^*_{11} + C_{22}\varepsilon^*_{22} + C_{23}\varepsilon^*_{33} + 2C_{24}\varepsilon^*_{12} + 2C_{25}\varepsilon^*_{23} + 2C_{26}\varepsilon^*_{31} \\
\sigma^*_{33} &= C_{13}\varepsilon^*_{11} + C_{23}\varepsilon^*_{22} + C_{33}\varepsilon^*_{33} + 2C_{34}\varepsilon^*_{12} + 2C_{35}\varepsilon^*_{23} + 2C_{36}\varepsilon^*_{31} \\
\sigma^*_{12} &= C_{14}\varepsilon^*_{11} + C_{24}\varepsilon^*_{22} + C_{34}\varepsilon^*_{33} + 2C_{44}\varepsilon^*_{12} + 2C_{45}\varepsilon^*_{23} + 2C_{46}\varepsilon^*_{31} \\
\sigma^*_{23} &= C_{15}\varepsilon^*_{11} + C_{25}\varepsilon^*_{22} + C_{35}\varepsilon^*_{33} + 2C_{45}\varepsilon^*_{12} + 2C_{55}\varepsilon^*_{23} + 2C_{56}\varepsilon^*_{31} \\
\sigma^*_{31} &= C_{16}\varepsilon^*_{11} + C_{26}\varepsilon^*_{22} + C_{36}\varepsilon^*_{33} + 2C_{46}\varepsilon^*_{12} + 2C_{56}\varepsilon^*_{23} + 2C_{66}\varepsilon^*_{31} \qquad (3.25)
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + 2C_{14}\varepsilon_{12} + 2C_{15}\varepsilon_{23} + 2C_{16}\varepsilon_{31} \\
\sigma_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\varepsilon_{33} + 2C_{24}\varepsilon_{12} + 2C_{25}\varepsilon_{23} + 2C_{26}\varepsilon_{31} \\
\sigma_{33} &= C_{13}\varepsilon_{11} + C_{23}\varepsilon_{22} + C_{33}\varepsilon_{33} + 2C_{34}\varepsilon_{12} + 2C_{35}\varepsilon_{23} + 2C_{36}\varepsilon_{31} \\
\sigma_{12} &= C_{14}\varepsilon_{11} + C_{24}\varepsilon_{22} + C_{34}\varepsilon_{33} + 2C_{44}\varepsilon_{12} + 2C_{45}\varepsilon_{23} + 2C_{46}\varepsilon_{31} \\
\sigma_{23} &= C_{15}\varepsilon_{11} + C_{25}\varepsilon_{22} + C_{35}\varepsilon_{33} + 2C_{45}\varepsilon_{12} + 2C_{55}\varepsilon_{23} + 2C_{56}\varepsilon_{31} \\
\sigma_{31} &= C_{16}\varepsilon_{11} + C_{26}\varepsilon_{22} + C_{36}\varepsilon_{33} + 2C_{46}\varepsilon_{12} + 2C_{56}\varepsilon_{23} + 2C_{66}\varepsilon_{31} \qquad (3.26)
\end{aligned}$$

Expressing the components of the stress σ^* as a function of the components of the strain ε:

$$\begin{aligned}
\sigma^*_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + 2C_{14}\varepsilon_{12} - 2C_{15}\varepsilon_{23} - 2C_{16}\varepsilon_{31} \\
\sigma^*_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\varepsilon_{33} + 2C_{24}\varepsilon_{12} - 2C_{25}\varepsilon_{23} - 2C_{26}\varepsilon_{31} \\
\sigma^*_{33} &= C_{13}\varepsilon_{11} + C_{23}\varepsilon_{22} + C_{33}\varepsilon_{33} + 2C_{34}\varepsilon_{12} - 2C_{35}\varepsilon_{23} - 2C_{36}\varepsilon_{31} \\
\sigma^*_{12} &= C_{14}\varepsilon_{11} + C_{24}\varepsilon_{22} + C_{34}\varepsilon_{33} + 2C_{44}\varepsilon_{12} - 2C_{45}\varepsilon_{23} - 2C_{46}\varepsilon_{31} \\
\sigma^*_{23} &= C_{15}\varepsilon_{11} + C_{25}\varepsilon_{22} + C_{35}\varepsilon_{33} + 2C_{45}\varepsilon_{12} - 2C_{55}\varepsilon_{23} - 2C_{56}\varepsilon_{31} \\
\sigma^*_{31} &= C_{16}\varepsilon_{11} + C_{26}\varepsilon_{22} + C_{36}\varepsilon_{33} + 2C_{46}\varepsilon_{12} - 2C_{56}\varepsilon_{23} - 2C_{66}\varepsilon_{31}
\end{aligned}$$

and expressing the components of the stress σ^* as a function of the components of the stress σ (equation 3.30):

$$\begin{aligned}
\sigma^*_{11} &= \sigma_{11} \\
\sigma^*_{22} &= \sigma_{22} \\
\sigma^*_{33} &= \sigma_{33} \\
\sigma^*_{12} &= \sigma_{12} \\
\sigma^*_{23} &= -\sigma_{23} \\
\sigma^*_{31} &= -\sigma_{31}
\end{aligned}$$

to reach such an equality we need to have: $C_{15} = C_{16} = C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0$ hence the elastic tensor reads:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ sym & & & & & C_{66} \end{bmatrix}. \tag{3.27}$$

Now, let's consider the reflection about the plane (**2**,**3**), the stress after reflection σ^* is expressed as a function of the stress before the reflection and the reflection matrix R:

$$\sigma^* = R^T \sigma R \tag{3.28}$$

with (2,3)

$$R = R^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.29}$$

$$\sigma^* = \begin{bmatrix} \sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ -\sigma_{12} & \sigma_{22} & \sigma_{23} \\ -\sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}. \tag{3.30}$$

and

$$\varepsilon^* = \begin{bmatrix} \varepsilon_{11} & -\varepsilon_{12} & -\varepsilon_{13} \\ -\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ -\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}. \tag{3.31}$$

or you can write:

$$\epsilon^*_{ij} = Q_{ip} Q_{jp} \epsilon_{pq} \tag{3.32}$$

and

$$\sigma^*_{ij} = Q_{ip} Q_{jp} \sigma_{pq} \tag{3.33}$$

with $\mathbf{Q} = \mathbf{R}$, which leads to:

$$\begin{aligned} \epsilon^*_{11} &= (-\delta_{1p})(-\delta_{1q})\epsilon_{pq} \\ &= \epsilon_{11} \end{aligned}$$

$$\begin{aligned} \epsilon^*_{22} &= \delta_{2p}\delta_{2q}\epsilon_{pq} \\ &= \epsilon_{22} \end{aligned}$$

$$\begin{aligned} \epsilon^*_{33} &= \delta_{3p}\delta_{3q}\epsilon_{pq} \\ &= \epsilon_{33} \end{aligned}$$

$$\begin{aligned} \epsilon^*_{12} &= (-\delta_{1p})\delta_{2q}\epsilon_{pq} \\ &= -\epsilon_{12} \end{aligned}$$

$$
\begin{aligned}
\epsilon^*_{23} &= \delta_{2p}\delta_{3q}\epsilon_{pq} \\
&= \epsilon_{23}
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^*_{13} &= (-\delta_{1p})\delta_{3q}\epsilon_{pq} \\
&= -\epsilon_{13}
\end{aligned}
$$

and in the similar manner:

$$
\begin{aligned}
\sigma^*_{11} &= \sigma_{11} \\
\sigma^*_{22} &= \sigma_{22} \\
\sigma^*_{33} &= \sigma_{33} \\
\sigma^*_{12} &= -\sigma_{12} \\
\sigma^*_{23} &= \sigma_{23} \\
\sigma^*_{13} &= -\sigma_{13}
\end{aligned}
$$

From the constitutive equation (equations 3.24, 3.25 and 3.26) Expressing the components of the stress σ^* as a function of the components of the strain ε:

$$
\begin{aligned}
\sigma^*_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} - 2C_{14}\varepsilon_{12} + 2C_{15}\varepsilon_{23} - 2C_{16}\varepsilon_{31} \\
\sigma^*_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\varepsilon_{33} - 2C_{24}\varepsilon_{12} + 2C_{25}\varepsilon_{23} - 2C_{26}\varepsilon_{31} \\
\sigma^*_{33} &= C_{13}\varepsilon_{11} + C_{23}\varepsilon_{22} + C_{33}\varepsilon_{33} - 2C_{34}\varepsilon_{12} + 2C_{35}\varepsilon_{23} - 2C_{36}\varepsilon_{31} \\
\sigma^*_{12} &= C_{14}\varepsilon_{11} + C_{24}\varepsilon_{22} + C_{34}\varepsilon_{33} - 2C_{44}\varepsilon_{12} + 2C_{45}\varepsilon_{23} - 2C_{46}\varepsilon_{31} \\
\sigma^*_{23} &= C_{15}\varepsilon_{11} + C_{25}\varepsilon_{22} + C_{35}\varepsilon_{33} - 2C_{45}\varepsilon_{12} + 2C_{55}\varepsilon_{23} - 2C_{56}\varepsilon_{31} \\
\sigma^*_{31} &= C_{16}\varepsilon_{11} + C_{26}\varepsilon_{22} + C_{36}\varepsilon_{33} - 2C_{46}\varepsilon_{12} + 2C_{56}\varepsilon_{23} - 2C_{66}\varepsilon_{31}
\end{aligned}
$$

using the previous results on the elastic tensor we have:

$$
\begin{aligned}
\sigma^*_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} - 2C_{14}\varepsilon_{12} \\
\sigma^*_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\varepsilon_{33} - 2C_{24}\varepsilon_{12} \\
\sigma^*_{33} &= C_{13}\varepsilon_{11} + C_{23}\varepsilon_{22} + C_{33}\varepsilon_{33} - 2C_{34}\varepsilon_{12} \\
\sigma^*_{12} &= C_{14}\varepsilon_{11} + C_{24}\varepsilon_{22} + C_{34}\varepsilon_{33} - 2C_{44}\varepsilon_{12} \\
\sigma^*_{23} &= 2C_{55}\varepsilon_{23} \\
\sigma^*_{31} &= -2C_{66}\varepsilon_{31}
\end{aligned}
$$

and expressing the components of the stress σ^* as a function of the components of the stress σ (equation 3.30):

$$
\begin{aligned}
\sigma^*_{11} &= \sigma_{11} \\
\sigma^*_{22} &= \sigma_{22} \\
\sigma^*_{33} &= \sigma_{33} \\
\sigma^*_{12} &= -\sigma_{12} \\
\sigma^*_{23} &= \sigma_{23} \\
\sigma^*_{31} &= -\sigma_{31}
\end{aligned}
$$

to reach such an equality we need to have: $C_{14} = C_{24} = C_{34} = 0$ hence the elastic tensor reads:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ sym & & & & & C_{66} \end{bmatrix}. \tag{3.34}$$

Considering the reflection about the plane (1,3) would be redundant. To conclude we have 9 independent components in the elastic tensor. ■

Concept Question 3.5.3. *Orthotropic elasticity in plane stress.*

Let's consider a two-dimensional orthotropic material based on the solution of the previous exercise.

1. Determine (in tensor notation) the constitutive relation $\varepsilon = f(\sigma)$ for two-dimensional orthotropic material in plane stress as a function of the engineering constants (i.e., Young's modulus, shear modulus and Poisson ratio).

2. Deduce the fourth-rank elastic tensor within the constitutive relation $\sigma = f(\varepsilon)$. Express the components of the stress tensor as a function of the components of both, the elastic tensor and the strain tensor.

■ **Solution:**

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ & 1/E_2 & 0 \\ & & 1/G_{12} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}. \tag{3.35}$$

Using the matrix rule (with A and B are both square matrix):

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \tag{3.36}$$

Hence, herein:

$$\begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 \\ -\nu_{12}/E_1 & 1/E_2 \end{bmatrix}^{-1} = \frac{E_1 E_2}{(1-\nu_{12}\nu_{21})} \begin{bmatrix} 1/E_2 & -\nu_{21}/E_2 \\ \nu_{12}/E_1 & 1/E_1 \end{bmatrix} \tag{3.37}$$

$$= \frac{1}{(1-\nu_{12}\nu_{21})} \begin{bmatrix} E_1 & \nu_{21}E_1 \\ \nu_{12}E_2 & E_2 \end{bmatrix} \tag{3.38}$$

Therefore:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} E_1/(1-\nu_{12}\nu_{21}) & \nu_{21}E_1/(1-\nu_{12}\nu_{21}) & 0 \\ \nu_{12}E_2/(1-\nu_{12}\nu_{21}) & E_2/(1-\nu_{12}\nu_{21}) & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} \tag{3.39}$$

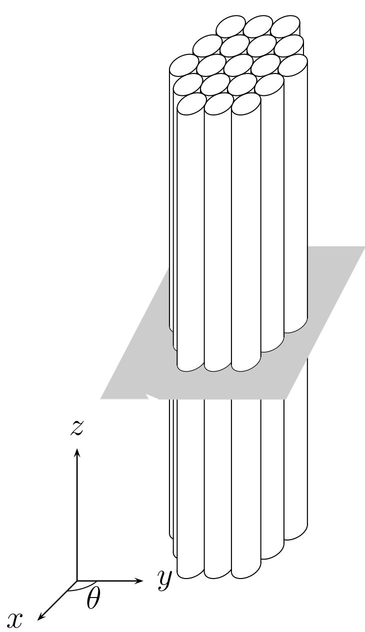
So:

$$\sigma_{11} = \frac{E_1}{(1-\nu_{12}\nu_{21})}(\varepsilon_{11}+\nu_{21}\varepsilon_{22}) \tag{3.40}$$

$$\sigma_{22} = \frac{E_2}{(1-\nu_{12}\nu_{21})}(\nu_{12}\varepsilon_{11}+\varepsilon_{22}) \tag{3.41}$$

$$\sigma_{12} = G_{12}(2\varepsilon_{12}) \tag{3.42}$$

■

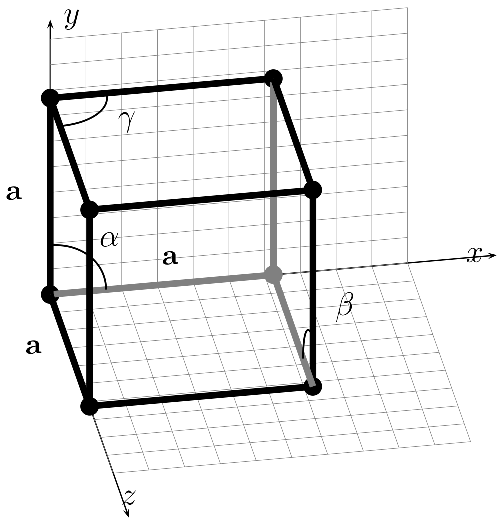


Transversely isotropic: The physical properties are symmetric about an axis that is normal to a plane of isotropy (xy-plane in the figure). Three mutually orthogonal planes of reflection symmetry and axial symmetry with respect to z-axis.
Number of independent coefficients: 5
Symmetry transformations: reflections about all three orthogonal planes plus all rotations about z-axis.

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0 \leq \theta \leq 2\pi$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{1111} & C_{1133} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{2323} & 0 & 0 \\ & symm & & & C_{2323} & 0 \\ & & & & & \frac{1}{2}(C_{1111}-C_{1122}) \end{bmatrix}$$



Cubic: three mutually orthogonal planes of reflection symmetry plus 90° rotation symmetry with respect to those planes. $a = b = c$, $\alpha = \beta = \gamma = 90$
Number of independent coefficients: 3
Symmetry transformations: reflections and 90° rotations about all three orthogonal planes

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & symm & & & C_{1212} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

3.6 Isotropic linear elastic materials

Most general isotropic 4th order isotropic tensor:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \big(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \big) \tag{3.43}$$

Replacing in:

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \tag{3.44}$$

gives:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + \mu \big(\epsilon_{ij} + \epsilon_{ji} \big) \tag{3.45}$$

$$\boxed{\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + \mu \big(\epsilon_{ij} + \epsilon_{ji} \big)} \tag{3.46}$$

Examples

$$\sigma_{11} = \lambda \delta_{11} \big(\epsilon_{11} + \epsilon_{22} + \epsilon_{33} \big) + \mu \big(\epsilon_{11} + \epsilon_{11} \big) = \big(\lambda + 2\mu \big) \epsilon_{11} + \lambda \epsilon_{22} + \lambda \epsilon_{33} \tag{3.47}$$

$$\sigma_{12} = 2\mu \epsilon_{12} \tag{3.48}$$

Concept Question 3.6.1. *Isotropic linear elastic tensor.*

Consider an isotropic linear elastic material.

1. Write the three-dimensional elastic/stiffness matrix in Voigt notation.

■ **Solution:**

Exploiting the material symmetries and the Voigt notation, the 21 constants of an anisotropic linear elastic material can be written in matrix form as

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & symm & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & symm & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}.$$

Considering that the most general 4th order **isotropic** tensor can be expressed as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big),$$

it is straightforward to write the corresponding stiffness matrix

$$\begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & symm & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}.$$

■

Compliance matrix for an isotropic elastic material

From experiments one finds:

$$\begin{aligned} \epsilon_{11} &= \frac{1}{E}\Big[\sigma_{11} - \nu\big(\sigma_{22} + \sigma_{33}\big)\Big] \\ \epsilon_{22} &= \frac{1}{E}\Big[\sigma_{22} - \nu\big(\sigma_{11} + \sigma_{33}\big)\Big] \\ \epsilon_{33} &= \frac{1}{E}\Big[\sigma_{33} - \nu\big(\sigma_{11} + \sigma_{22}\big)\Big] \\ 2\epsilon_{23} &= \frac{\sigma_{23}}{G},\quad 2\epsilon_{13} = \frac{\sigma_{13}}{G},\quad 2\epsilon_{12} = \frac{\sigma_{12}}{G} \end{aligned} \tag{3.49}$$

In these expressions, E is the Young's Modulus, ν the Poisson's ratio and G the shear modulus. They are referred to as the *engineering constants*, since they are obtained from experiments. The shear modulus G is related to the Young's modulus E and Poisson ratio ν by the expression $G = \frac{E}{2(1+\nu)}$. Equations (3.49) can be written in the following matrix form:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & symm & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{3.50}$$

Invert and compare with:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & symm & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \tag{3.51}$$

and conclude that:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)} \tag{3.52}$$

Concept Question 3.6.2. *Inverted Hooke's law.*

Let's consider a linear elastic material.

1. Verify that the compliance form of Hooke's law, Equation (3.50) can be written in index notation as:
$$\epsilon_{ij} = \frac{1}{E}\Big[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\Big]$$
2. Invert Equation (3.50) (e.g. using Mathematica or by hand) and verify Equation (3.51) using λ and μ given by (3.52).
3. Verify the expression:
$$\sigma_{ij} = \frac{E}{(1+\nu)}\left[\epsilon_{ij} + \frac{\nu}{(1-2\nu)}\epsilon_{kk}\delta_{ij}\right]$$

■ **Solution:** ■

Bulk Modulus

Establishes a relation between the *hydrostatic stress* or pressure: $p = \frac{1}{3}\sigma_{kk}$ and the volumetric strain $\theta = \epsilon_{kk}$.

$$p = K\theta \; ; \; K = \frac{E}{3(1-2\nu)} \tag{3.53}$$

Concept Question 3.6.3. *Bulk modulus derivation.* Let's consider a linear elastic material.

1. Derive the expression for the bulk modulus in Equation (3.53)

■ **Solution:** Add up the first three isotropic Hooke's constitutive equations in compliance form:

$$\begin{aligned}
\underbrace{\epsilon_{11}+\epsilon_{22}+\epsilon_{33}}_{\epsilon_{kk}=\theta} &= \frac{1}{E}\left[(\sigma_{11}+\sigma_{22}+\sigma_{33}) - \nu\left((\sigma_{22}+\sigma_{33})+(\sigma_{11}+\sigma_{33})+(\sigma_{11}+\sigma_{22})\right)\right] \\
&= \frac{1}{E}\left[(\sigma_{11}+\sigma_{22}+\sigma_{33}) - 2\nu\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)\right] \\
&= \frac{1}{E}\underbrace{(\sigma_{11}+\sigma_{22}+\sigma_{33})}_{\sigma_{kk}=3p}(1-2\nu) \\
\theta &= \underbrace{\frac{3(1-2\nu)}{E}}_{1/K}p
\end{aligned}$$

■

Concept Question 3.6.4. *Independent coefficients for linear elastic isotropic materials.*

For a linearly elastic, homogeneous, isotropic material, the constitutive laws involve three parameters: *Young's modulus*, E, *Poisson's ratio*, ν, and the *shear modulus*, G.

1. Write and explain the relation between stress and strain for this kind of material.

2. What is the physical meaning of the coefficients E, ν and G?

3. Are these three coefficients independent of each other? If not, derive the expressions that relate them. Indicate also the relationship with the Lame's constants.

4. Explain why the Poisson's ratio is constrained to the range $\nu \in (-1, 1/2)$. Hint: use the concept of *bulk modulus*. ■ **Solution:** In a homogeneous material the properties are the same at each point. Isotropic means that the physical properties are identical in all directions. Linear elastic makes reference to the relationship between strain and stress (the linear behaviour is exhibited in small deformations), which is represented

by the generalized Hooke's law (extensional and shear strains)

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right], \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right], \\
\varepsilon_{33} &= \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right], \\
\varepsilon_{12} &= \frac{1}{2G}\sigma_{12} \qquad \left(\gamma_{12} = \frac{1}{G}\sigma_{12}\right), \\
\varepsilon_{13} &= \frac{1}{2G}\sigma_{13} \qquad \left(\gamma_{13} = \frac{1}{G}\sigma_{13}\right), \\
\varepsilon_{23} &= \frac{1}{2G}\sigma_{23} \qquad \left(\gamma_{23} = \frac{1}{G}\sigma_{23}\right),
\end{aligned}$$

where $\gamma_{ij} = 2\varepsilon_{ij}$ is the engineering notation for the shearing strain.

The *Young's modulus* E is a measure of the stiffness of the material, i.e., the resistance of the material to be deformed when a stress (uniaxial stretching or compression) is applied. The concept of Young's modulus is considered only for the linear strain-stress behaviour, the value is positive ($E > 0$), and a larger value of E indicates a stiffer material.

The *Poisson's ratio* ν gives us information about the ratio between lateral and longitudinal strain in uniaxial tensile stress. Consider that we have a bar and we apply a stress in the axial direction: the Poisson's ratio measures the lateral contraction produced by the applied stress. When $\nu = 1/2$ the material is incompressible, which means that the volume remains constant. If $\nu = 0$, a stretching causes no lateral contraction.

The *shear modulus* G indicates the material response to shearing strains. It is positive and smaller than E.

The constitutive law for an isotropic, linear elastic and homogeneous material is given by the following equation

$$c_{ijkl} = \lambda\ \delta_{ij}\ \delta_{kl} + \mu\ (\delta_{ik}\ \delta_{jl} + \delta_{il}\ \delta_{jk}),$$

where δ_{ij} is the Kronecker-delta, and λ and μ are the Lamé constants.

Note that the constitutive equation is defined through two parameters. It means that the three coefficients E, ν and G are not independent to each other. The relationship between these coefficients and the Lamé constants is given by

$$G = \mu, \qquad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},$$

which allows us to find the expression that relates the coefficients

$$G = \frac{E}{2(1+\nu)}.$$

When the material is subject to hydrostatic pressure, the relationship between the pressure p and the volumetric strain $e = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ is linear

$$p = \kappa e,$$

where

$$\kappa = \frac{E}{3(1-2\nu)}$$

is the *bulk modulus.* This parameter describes the response of the material to an uniform pressure. Note that the value of the bulk modulus tends to infinity when $\nu \to 1/2$, which means that the volumetric strain vanishes under an applied pressure. This limit is known as *limit of incompressibility*, and the material is called *incompressible* when $\nu = 1/2$, as we have mentioned previously.

The coefficients E, ν, G and κ are related through the following expression

$$E = 2G(1+\nu) = 3\kappa(1-2\nu).$$

■

3.7 Thermoelastic effects

We are going to consider the strains produced by changes of temperature (ϵ^{θ}). These strains have inherently a dilatational nature (thermal expansion or contraction) and do not cause any shear. Thermal strains are proportional to temperature changes. For isotropic materials:

$$\epsilon^{\theta}_{ij} = \alpha \Delta\theta \delta_{ij} \tag{3.54}$$

The total strains (ϵ_{ij}) are then due to the (additive) contribution of the *mechanical strains* (ϵ^{M}_{ij}), i.e., those produced by the stresses and the thermal strains:

$$\begin{aligned} \epsilon_{ij} &= \epsilon^{M}_{ij} + \epsilon^{\theta}_{ij} \\ \sigma_{ij} &= C_{ijkl}\epsilon^{M}_{kl} = C_{ijkl}(\epsilon_{kl} - \epsilon^{\theta}_{kl}) \text{ or:} \end{aligned}$$

$$\boxed{\sigma_{ij} = C_{ijkl}(\epsilon_{kl} - \alpha\Delta\theta\delta_{kl})} \tag{3.55}$$

Concept Question 3.7.1. *Thermoelastic constitutive equation.*
Let's consider an isotropic elastic material.

1. Write the relationship between stresses and strains for an isotropic elastic material whose Lamé constants are λ and μ and whose coefficient of thermal expansion is α.

■ **Solution:** When the thermal effects are considered, the relationship between stress and strain is given by

$$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha\Delta\theta\delta_{kl}\right),$$

where α is the coefficient of thermal expansion and $\Delta\theta$ the change in temperature.

By using the constitutive equation for linear elastic isotropic materials

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big),$$

and the fact that

$$C_{ijkl}\varepsilon_{kl} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij},$$

we can write

$$\begin{aligned}
\sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - C_{ijkl}\alpha\Delta\theta\delta_{kl} \\
&= \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - \big[\lambda\delta_{ij}\delta_{kl} + \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big)\big]\,\alpha\Delta\theta\delta_{kl} \\
&= \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - \lambda\alpha\Delta\theta\delta_{ij}\underbrace{\delta_{kl}\delta_{kl}}_{\delta_{kk}=\delta_{ll}=3} - \mu\alpha\Delta\theta\underbrace{\delta_{ik}\delta_{jl}\delta_{kl}}_{\delta_{ij}} - \mu\alpha\Delta\theta\underbrace{\delta_{il}\delta_{jk}\delta_{kl}}_{\delta_{ij}} \\
&= \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - 3\lambda\alpha\Delta\theta\delta_{ij} - 2\mu\alpha\Delta\theta\delta_{ij} \\
&= \lambda(\epsilon_{kk} - 3\alpha\Delta\theta)\delta_{ij} + 2\mu(\epsilon_{ij} - \alpha\Delta\theta\delta_{ij}).
\end{aligned}$$

■

Concept Question 3.7.2. *Thermoelasticity in a fully constrained specimen.* Let's consider a specimen which deformations are fully constrained (see Figure 3.2). The material behavior is considered isotropic linear elastic with E and ν the elastic constants, the Young's modulus and Poisson's ratio, respectively. A temperature gradient $\Delta\theta$ is prescribed on the specimen.

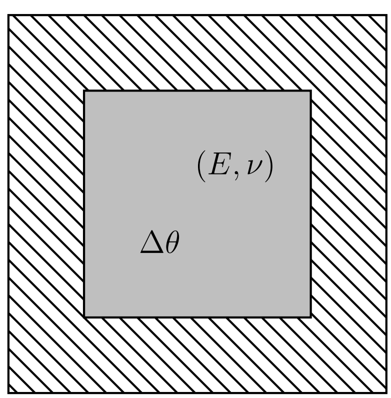


Figure 3.2: Specimen fully constrained.

1. Determine the internal stress state within the specimen.

■ **Solution:**

Herein, there is no mechanical strain components. Hence, using the following relation:

$$\sigma_{ij} = \lambda(\epsilon_{kk} - 3\alpha\Delta\theta)\delta_{ij} + 2\mu(\epsilon_{ij} - \alpha\Delta\theta\delta_{ij})$$

with $\epsilon_{kk} = 0$ and $\epsilon_{ij=0}$ we obtain:

$$\begin{aligned} \sigma_{ij} &= \lambda(0 - 3\alpha\Delta\theta)\delta_{ij} + 2\mu(0 - \alpha\Delta\theta\delta_{ij}) \\ &= -\left(3\lambda + 2\mu\right)\alpha\Delta\theta\delta_{ij} \end{aligned}$$

with ϵ^{θ}_{ij} the thermal strain, a diagonal matrix:

$$\epsilon^{\theta}_{ij} = \alpha\Delta\theta\delta_{ij}$$

hence, the stress matrix is also diagonal and the pressure describing the state of stress within the specimen is defined as follows:

$$\begin{aligned} p &= \frac{1}{3}\sigma_{ii} \\ &= -\frac{1}{3}\left(3\lambda + 2\mu\right)\alpha\Delta\theta\delta_{ii} \\ &= -\frac{3\lambda + 2\mu}{3}\epsilon^{\theta}_{ii} \\ &= -K\epsilon^{\theta}_{ii} \end{aligned}$$

with K the bulk modulus.

■

3.8 Particular states of stress and strain of interest

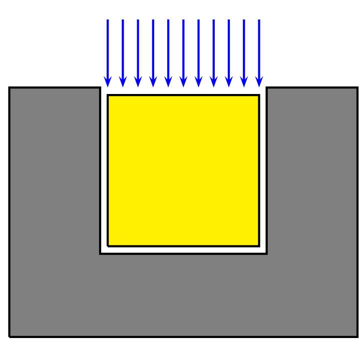
3.8.1 Uniaxial stress

$$\sigma_{11} = \sigma, \text{ all stress components vanish}$$

From (3.50):

$$\epsilon_{11} = \frac{\sigma}{E},\ \epsilon_{22} = -\frac{\nu}{E}\sigma,\ \epsilon_{33} = -\frac{\nu}{E}\sigma,\ \text{ all shear strain components vanish}$$

3.8.2 Uniaxial strain



Material	Mass density $[Mg \cdot m^{-3}]$	Young's Modulus $[GPa]$	Poisson Ratio	Thermal Expansion Coefficient $[10^{-6}K^{-1}$
Tungsten	13.4	410	0.30	5
CFRP	1.5-1.6	70 200	0.20	2
Low alloy steels	7.8	200 - 210	0.30	15
Stainless steel	7.5-7.7	190 - 200	0.30	11
Mild steel	7.8	196	0.30	15
Copper	8.9	124	0.34	16
Titanium	4.5	116	0.30	9
Silicon	2.5-3.2	107	0.22	5
Silica glass	2.6	94	0.16	0.5
Aluminum alloys	2.6-2.9	69-79	0.35	22
GFRP	1.4-2.2	7-45		10
Wood, parallel grain	0.4-0.8	9-16	0.2	40
PMMA	1.2	3.4	0.35-0.4	50
Polycarbonate	1.2 1.3	2.6	0.36	65
Natural Rubbers	0.83-0.91	0.01-0.1	0.49	200
PVC	1.3-1.6	0.003-0.01	0.41	70

Table 3.1: Representative isotropic properties of some materials

$$\epsilon_{11} = \epsilon, \text{ all other strain components vanish}$$

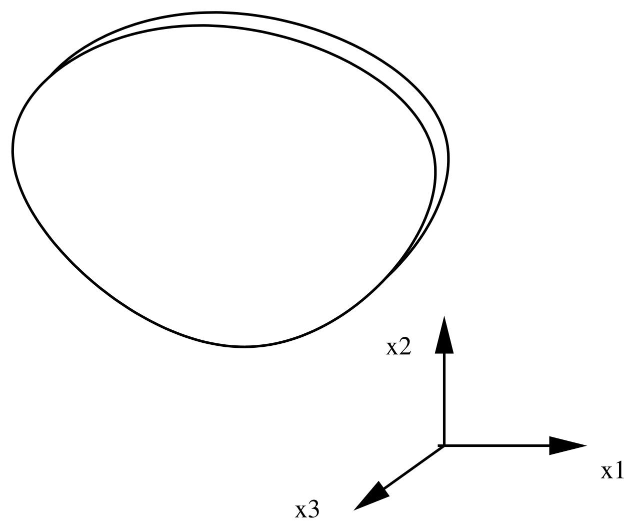
From (3.51):

$$\sigma_{11} = (\lambda + 2\mu)\epsilon_{11} = \frac{(1-\nu)}{(1+\nu)(1-2\nu)} E\epsilon_{11}$$

3.8.3 Plane stress

Consider situations in which:

$$\sigma_{i3} = 0 \tag{3.56}$$



.

Then:

$$\epsilon_{11} = \frac{1}{E}\big(\sigma_{11} - \nu\sigma_{22}\big) \tag{3.57}$$

$$\epsilon_{22} = \frac{1}{E}\big(\sigma_{22} - \nu\sigma_{11}\big) \tag{3.58}$$

$$\epsilon_{33} = \frac{-\nu}{E}\big(\sigma_{11} + \sigma_{22}\big) \neq 0\,!!! \tag{3.59}$$

$$\epsilon_{23} = \epsilon_{13} = 0 \tag{3.60}$$

$$\epsilon_{12} = \frac{\sigma_{12}}{2G} = \frac{(1+\nu)\sigma_{12}}{E} \tag{3.61}$$

In matrix form:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{3.62}$$

Inverting gives the:

Relations among stresses and strains for *plane stress*:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} \tag{3.63}$$

Concept Question 3.8.1. *Plane stress*

Let's consider an isotropic elastic material for a plate in plane stress state.

1. Determine the out-of-plane ϵ_{33} strain component from the measurement of the in-plane normal strains $\epsilon_{11}, \epsilon_{22}$.

■ **Solution:** Solve for σ_{11} and σ_{22} in the plane stress stress-strain law in compliance form:

$$\epsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22})$$
$$\epsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11})$$

One obtains:

$$\sigma_{11} = \frac{E}{1-\nu^2}(\epsilon_{11} + \nu\epsilon_{22})$$
$$\sigma_{22} = \frac{E}{1-\nu^2}(\epsilon_{22} + \nu\epsilon_{11})$$

Insert in the third:

$$\begin{aligned}
\epsilon_{33} &= \frac{-\nu}{E}(\sigma_{11} + \sigma_{22}) \\
&= \frac{-\nu}{E}\left(\frac{E}{1-\nu^2}(\epsilon_{11} + \nu\epsilon_{22}) + \frac{E}{1-\nu^2}(\epsilon_{22} + \nu\epsilon_{11})\right) \\
&= \frac{-\nu(1+\nu)}{1-\nu^2}(\epsilon_{11} + \epsilon_{22}) = \frac{-\nu}{1-\nu}(\epsilon_{11} + \epsilon_{22})
\end{aligned}$$

■

3.8.4 Plane strain

In this case we consider situations in which:

$$\epsilon_{i3} = 0 \tag{3.64}$$

Then:

$$\epsilon_{33} = 0 = \frac{1}{E}\Big[\sigma_{33} - \nu\big(\sigma_{11} + \sigma_{22}\big)\Big], \text{ or:} \tag{3.65}$$

$$\sigma_{33} = \nu\big(\sigma_{11} + \sigma_{22}\big) \tag{3.66}$$

$$\begin{aligned}\epsilon_{11} &= \frac{1}{E}\Big\{\sigma_{11} - \nu\big[\sigma_{22} + \nu\big(\sigma_{11} + \sigma_{22}\big)\big]\Big\} \\ &= \frac{1}{E}\Big[\big(1-\nu^2\big)\sigma_{11} - \nu\big(1+\nu\big)\sigma_{22}\Big]\end{aligned} \tag{3.67}$$

$$\epsilon_{22} = \frac{1}{E}\Big[\big(1-\nu^2\big)\sigma_{22} - \nu\big(1+\nu\big)\sigma_{11}\Big] \tag{3.68}$$

In matrix form:

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1-\nu^2 & -\nu(1+\nu) & 0\\ -\nu(1+\nu) & 1-\nu^2 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} \tag{3.69}$$

Inverting gives the

Relations among stresses and strains for *plane strain*:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \dfrac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix} \tag{3.70}$$

Concept Question 3.8.2. *Plane strain.*

Using Mathematica:

1. Verify equations (3.63) and (3.70)

■ **Solution:** ■

Concept Question 3.8.3. *Comparison of plane-stress and plane-strain linear isotropic elasticity.*

Let's consider two linear elastic isotropic materials with the same Young's modulus E but different Poisson's ratio, $\nu = 0$ and $\nu = 1/3$. We are interested in comparing the behavior of these two materials for both, plane stress and plane strain models.

1. Express the relation between the stress components and the strain components in the case of both, plane stress and plane strain models. ■ **Solution:** In the

case of a linear elastic isotropic behavior, the fourth-order compliance tensor denoted **S**, relating the strain tensor to the stress tensor is given as follows:

$$\mathbf{S} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & symm & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix}$$

In the *plane stress* approach, the stress components out of the plane ($\mathbf{e}_1$,$\mathbf{e}_2$) are equal to 0:

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

Hence, the strain components read:

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{E}\left(\sigma_{11} - \nu\sigma_{22}\right), \\
\epsilon_{22} &= \frac{1}{E}\left(\sigma_{22} - \nu\sigma_{11}\right), \\
\epsilon_{33} &= -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right) \quad = \quad -\frac{\nu}{1-\nu}\left(\epsilon_{11} + \epsilon_{22}\right), \\
\epsilon_{12} &= \frac{1+\nu}{E}\sigma_{12}, \\
\epsilon_{13} &= 0 \\
\epsilon_{23} &= 0,
\end{aligned}$$

where E, ν and G are the Young's modulus, the Poisson's ratio, and the shear modulus, respectively.

In the case of a linear elastic isotropic behavior, the fourth-order elastic tensor denoted **C**, relating the stress tensor to the strain tensor is given as follows:

$$C_{ijkl} = \frac{\nu E}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

or

$$\mathbf{C} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & symm & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}$$

with:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu = \frac{E}{2(1+\nu)}$$

In the *plane strain* approach, the strain components out of the plane ($\mathbf{e}_1$,$\mathbf{e}_2$) are equal to 0:

$$\epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0$$

Hence, the stress components read:

$$\begin{aligned}
\sigma_{11} &= (\lambda + 2\mu)\epsilon_{11} + \lambda\epsilon_{22} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\epsilon_{11} + \frac{\nu E}{(1+\nu)(1-2\nu)}\epsilon_{22} \\
\sigma_{22} &= \lambda\epsilon_{11} + (\lambda + 2\mu)\epsilon_{22} = \frac{\nu E}{(1+\nu)(1-2\nu)}\epsilon_{11} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\epsilon_{22} \\
\sigma_{33} &= \lambda(\epsilon_{11} + \epsilon_{22}) = \frac{\lambda}{2(\lambda+\mu)}(\sigma_{11} + \sigma_{22}) = \nu(\sigma_{11} + \sigma_{22}) \\
\sigma_{12} &= 2\mu\epsilon_{12} = \frac{E}{1+\nu}\epsilon_{12} \\
\sigma_{23} &= 0 \\
\sigma_{13} &= 0
\end{aligned}$$

■

2. Under which conditions these two materials manifest the same elastic response for each hypothesis, plane strain and plane stress? ■ **Solution:** When the Poisson's ratios of both materials are equal to 0, the plane stress and plane strain approaches are equivalent; thus leading to the following stress-strain relations:

$$\begin{aligned}
\sigma_{11} &= E\epsilon_{11} \\
\sigma_{22} &= E\epsilon_{22} \\
\sigma_{33} &= 0 \\
\sigma_{12} &= E\epsilon_{12}
\end{aligned}$$

■

3. Derive the equation that relates ϵ_{11} and ϵ_{22} when $\sigma_{22} = 0$ for both, plane strain and plane stress models. For the material having a Poisson's ratio equals to $\nu = 1/3$, for which model (plane stress or plane strain) the deformation ϵ_{22} reaches the greatest value? ■ **Solution:** When $\sigma_{22} = 0$, the relation between ϵ_{11} and ϵ_{22} for the plane stress approach is given by

$$\epsilon_{22} = -\nu\epsilon_{11} = -\frac{1}{3}\epsilon_{11},$$

while for the plane strain approach is

$$\epsilon_{22} = -\frac{\nu}{1-\nu}\epsilon_{11} = -\frac{1}{2}\epsilon_{11}.$$

Herein, the strain component ϵ_{22} is larger for plane strain condition. ■

4. Let's consider a square specimen of each material, with a length equals to 1 m and the origin of the coordinate system is located at the left bottom corner of the specimen.

When a deformation of $\epsilon_{11} = 0.01$ is applied, calculate the displacement u_2 of the point with coordinates $(0.5, 1)$. ■ **Solution:** We know that the strain ϵ_{22} and the displacement u_2 are related through the equation

$$\epsilon_{22} = \frac{y_{final} - y_{initial}}{y_{initial}} = \frac{u_2}{L},$$

where $L = 1$ m is the length of the edge of the square. Thus, we can write

$$u_2 = \epsilon_{22} = \begin{cases} -\nu\epsilon_{11} & \text{plane stress} \\ -\dfrac{\nu}{1-\nu}\epsilon_{11} & \text{plane strain} \end{cases}$$

This last equation allows us to calculate the displacement u_2 at the point $(1/2, 1)$ for the different approaches and materials, as shown in the next table.

Approach / Material	$\nu = 0$	$\nu = 1/3$
Plane strain	$u_2 = 0$	$u_2 = -1/200 = -5.5$ mm
Plane stress	$u_2 = 0$	$u_2 = -1/300 \approx -3.3$ mm

■