# CHAPTER 4

# NUMERICAL INTEGRATION METHODS TO EVALUATE TRIPLE INTEGRALS USING GENERALIZED GAUSSIAN QUADRATURE[†]

## 4.1 Introduction

Many applications in science and engineering require the solution of three dimensional integrals. In physics, triple integral arises in the computation of mass, volume, moment of inertia and force on a three dimensional object. In the field of FEM, triple integrals need to be evaluated while finding the stiffness matrix, mass matrix, body force vector, etc. Usually these integrals cannot be solved analytically; thus numerical methods have to be applied.

Cubature formulae over different three-dimensional regions are derived by solving many systems of non-linear equations which is a very tedious procedure. Due to this difficulty, derivation of cubature rules becomes a challenge as the order increases. Also, these cubature rules are restricted to particular domains. Eg. there are cubature rules over a zero-one cube, a standard tetrahedron, a prism, a sphere, a pyramid etc. No extensive research has taken place to derive quadrature or cubature rules over regions with curved faces.

In this chapter, a general numerical integration formula is derived to evaluate triple integrals of the form,

$$I = \iiint_{\Omega} f(x,y,z)\, d\Omega = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dzdydx. \tag{4.1}$$

Consider the integration domain in Eq. (4.1),

$$\Omega = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), h_1(x,y) \le z \le h_2(x,y)\}$$

[†] The work in Chapter 4 is communicated to an international journal (see page 205).

If $g_1(x)$, $g_2(x)$, $h_1(x,y)$ and $h_2(x,y)$ are linear functions, then $\Omega$ is either a 4 face region, a 5-face region or a 6-face region like a tetrahedron, prism, pyramid, cube, cuboid etc. Each face or boundary of these regions will be either triangles or trapeziums. Few such regions are shown in Fig. 4.1.

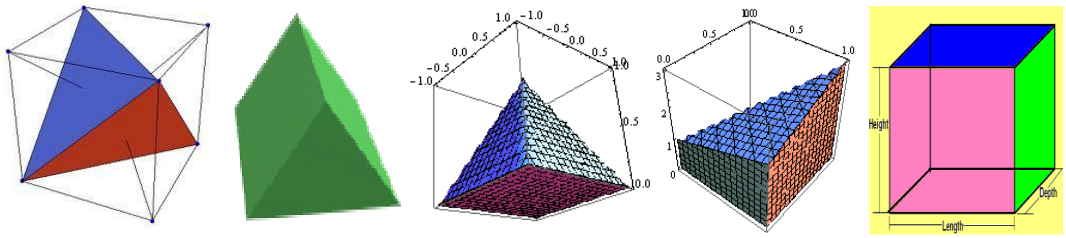


**Fig.4. 1: Three-dimensional regions with planar faces**

Whereas, if $g_1(x)$, $g_2(x)$, $h_1(x,y)$ and $h_2(x,y)$ are any functions (linear, quadratic, cubic, exponential, trigonometric etc.), then the bounded region $\Omega$ will be n-faced (n = 1, 2, 3, 4, 5 and 6), like sphere, cone, cylinder, or any other region having planar and non-planar faces. Few such regions with curved boundaries that are found in some engineering applications are plotted in Fig. 4.2.

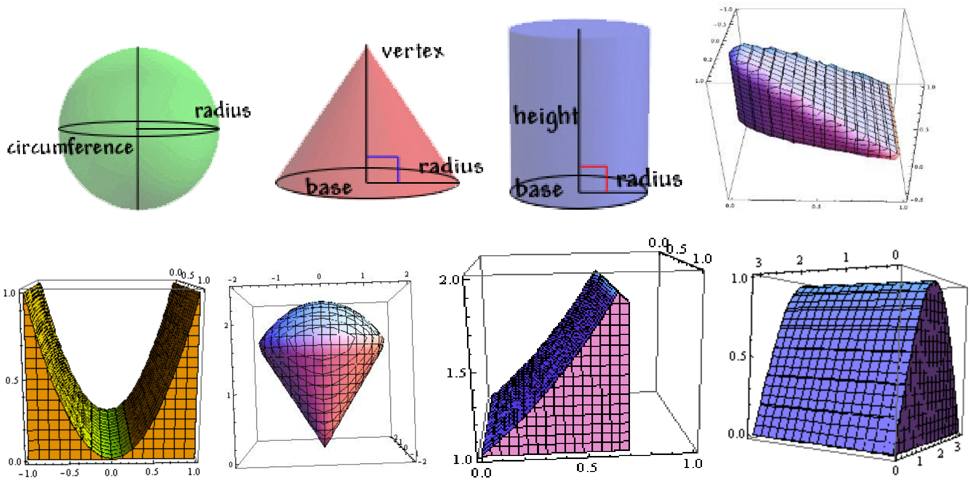


**Fig.4. 2: Bounded regions with curved or / and planar boundaries with n-faces where n=1, 2, 3, 4, 5 or 6**

Any other bounded three dimensional regions which cannot be written as $\Omega$ can be discretized into finite number of regions each of which can be written as $\Omega$.

The derivation of the integration formula to evaluate the integral in Eq. (4.1) is specified in section 4.2. In section 4.3, the integration rules and the formulae to evaluate the nodes and weights for integration over different three-dimensional regions are provided. The results of integration are also tabulated.

The quadrature / cubature rules over regions with circular boundaries like cones, cylinders and paraboloids are seldom found. The integration rule derived in section 4.2 gives reasonable accuracy for these regions with circular boundaries. Hence, a new transformation is planned to derive efficient integration rules for integration over such regions, which is presented in section 4.4. Section 4.5 describes another integration formula to evaluate triple integrals over a sphere and section 4.6 explains an integration method to integrate functions over an irregular three dimensional domain.

## 4.2 Derivation of the generalized Gaussian quadrature rules to evaluate triple integrals

To derive a numerical integration method to integrate the integral in Eq. (4.1), the domain of integration,

$$\Omega = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x, y) \leq z \leq h_2(x, y)\}$$

is transformed to a zero-one cube in the $\xi - \eta - \gamma$ space,

$$C = \{(\xi, \eta, \gamma) \mid 0 \leq \xi \leq 1,\ 0 \leq \eta \leq 1, 0 \leq \gamma \leq 1\}$$

using the transformation,

$$x = (b - a)\xi + a$$

$$y = [g_2(x(\xi) - g_1(x(\xi))]\,\eta + g_1(x(\xi))$$

$$z = [h_2(\,x(\xi), y(\xi, \eta)) - h_1(x(\xi), y(\xi, \eta))]\,\gamma + h_1(x(\xi), y(\xi, \eta))$$

Here, $x$ is a function of $\xi$, $y$ is a function of $\xi, \eta$ and $z$ is a function of $\xi, \eta$ and $\gamma$. The Jacobian of the transformation is

$$|\,J\,| = (b - a)\ [g_2(x(\xi)) - g_1\big(x(\xi)\big)]\,[h_2(\,x(\xi), y(\xi, \eta)) - h_1(x(\xi), y(\xi, \eta))] \qquad (4.2)$$

After transforming the integral domain to a zero-one cube and then applying the generalized Gaussian quadrature formula for each integral, the proposed numerical integration rule for the integral in Eq. (4.1) can be derived in the following way:

$$
\begin{aligned}
I &= \iiint_{\Omega} f(x,y,z)\, d\Omega \\
&= \int_0^1 \int_0^1 \int_0^1 f(x(\xi), y(\xi,\eta), z(\xi,\eta,\gamma))\ |J|\ d\gamma\, d\eta\, d\xi \\
&\approx \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} w_1^i w_2^j w_3^k\ \ |J|\, f(x(\xi_i), y(\xi_i,\eta_j), z(\xi_i,\eta_j,\gamma_k)) \\
&\approx \sum_{m=1}^{N^3} c_m\, f(x_m, y_m, z_m)
\end{aligned}
\tag{4.3}
$$

where,

$$c_m = w_1^i w_2^j w_3^k (b-a)[g_2(x(\xi_i)) - g_1(x(\xi_i))]\left[h_2\left(x(\xi_i), y(\xi_i,\eta_j)\right) - h_1\left(x(\xi_i), y(\xi_i,\eta_j)\right)\right]; \tag{4.3a}$$

$$x_m = (b-a)\xi_i + a; \tag{4.3b}$$

$$y_m = [g_2(x(\xi_i) - g_1(x(\xi_i))]\eta_j + g_1(x(\xi_i)); \tag{4.3c}$$

$$z_m = \left[h_2\left(x(\xi_i), y(\xi_i,\eta_j)\right) - h_1\left(x(\xi_i), y(\xi_i,\eta_j)\right)\right]\gamma_k + h_1\left(x(\xi_i), y(\xi_i,\eta_j)\right) \tag{4.3d}$$

The formulae can also be written in a recursive style as,

$$x_m = (b-a)\xi_i + a; \tag{4.3e}$$

$$y_m = [g_2(x_m) - g_1(x_m)]\eta_j + g_1(x_m); \tag{4.3f}$$

$$z_m = [h_2(x_m, y_m) - h_1(x_m, y_m)]\gamma_k + h_1(x_m, y_m); \tag{4.3g}$$

$$c_m = w_1^i w_2^j w_3^k\, (b-a)[g_2(x_m) - g_1(x_m)][h_2(x_m, y_m) - h_1(x_m, y_m)] \tag{4.3h}$$

Here, $\xi_i, \eta_j$ *and* $\gamma_k$ are the node points between zero and one, and $w_1^i, w_2^j$ and $w_3^k$ are their corresponding weights, in one-dimension. Any of the Gaussian quadrature points and weights can be applied in this formula, like the Gauss-Legendre, Gauss-Jacobi etc. The generalized Gaussian quadrature nodes and weights, for product of polynomials and logarithmic function, given in Ma *et.al.* (1996) are used in the current work, as it gives optimum results in one and two dimensions.

Eq. (4.3a-d) or (4.3e-h) gives the weights and nodes $(c_m, x_m, y_m, z_m)$ for integrating any function over the region $\Omega$ and Eq. (4.3) is used to evaluate the integral numerically.

## 4.3 Integration formulae and numerical results over different three-dimensional regions

In this section, the integration rules, cubature points and weights for some commonly used three-dimensional domains are recorded as special cases of the general integration formulae derived in the previous section (Eq. (4.3)).

### 4.3.1 Triple Integrals with linear limits

If the integration limits in the integral in Eq. (4.1) are linear, then the integration domain will be a solid with planar faces like a cuboid, tetrahedron, prism, pyramid etc. Some domains which are commonly found in applications in science and engineering are considered here. The distribution of points in the domain of integration and the computed results for a variety of integrals over these domains are also tabulated.

## Cuboid

By substituting constant limits, $g_1 = c$, $g_2 = d$, $h_1 = e$ and $h_2 = f$ in Eqs. (4.3, 4.3a, b, c and d), one can get the weights and cubature points $(c_m, x_m, y_m, z_m)$ and the integral value of the function over a cuboid as,

$$I = \int_a^b \int_c^d \int_e^f f(x,y,z)\, dz\, dy\, dx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m) \tag{4.4}$$

where, $c_m = w_1^i w_2^j w_3^k (b-a)(d-c)(f-e)$; (4.4a)

$$x_m = (b-a)\xi_i + a; \quad y_m = (d-c)\eta_j + c; \quad z_m = (f-e)\,\gamma_k + e \tag{4.4b}$$

The quadrature points and their weights $(x_m, y_m, z_m, c_m)$ for N=5 for integration over the zero-one cube is given in Table 4.12. The distribution of these points $(x_m, y_m, z_m)$ in the cube is shown in Fig. 4.3.

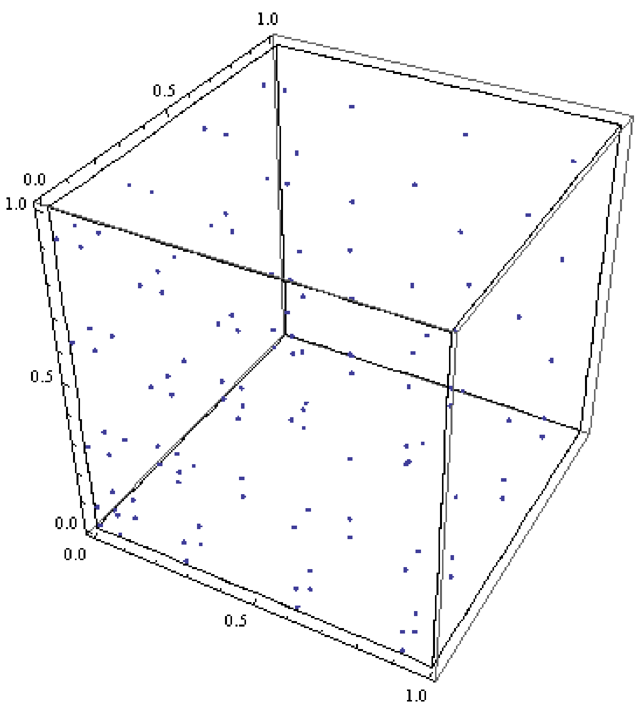


**Fig.4. 3: Distribution of the 125 nodal points ($x_m$, $y_m$, $z_m$) in the zero-one cube**

For numerical testing, some typical integrals with known exact values, few of which are given in Dooren and Ridder (1976) are pondered.

$$I_1 = \int_0^1 \int_0^1 \int_0^1 8(1+2(x+y+z))^{-1}\, dxdydz = 2.152142832595894$$

$$I_2 = \int_1^2 \int_3^4 \int_5^6 \sqrt{x+y+z}\ \ dzdydx = 3.23945017707172$$

$$I_3 = \int_0^1 \int_0^1 \int_0^1 (x+y+z)^{-2}\, dxdydz = 0.8630462173553432$$

$$I_4 = \int_0^{\pi} \int_0^{\pi} \int_0^{\pi/2} Cos(x+y+z)\, dzdydx = -4.00000000000000$$

$$I_5 = \int_0^1 \int_0^1 \int_0^1 \frac{1}{\sqrt{x+y+z}}\ \ dzdydx = 0.862877077142803$$

The computed value of these integrals using the current method and the absolute error is tabulated in Table 4.1.

It is clear from the table that the proposed integration method converges rapidly for almost all type of integrands including integrands with end point singularities and integrands with peak (*$I_3$ and $I_5$*). Also the functional evaluations required by the proposed method to obtain certain accuracy are less compared to the number of functional evaluations used by the method given in Dooren and Ridder (1976).

**Table 4. 1: Numerical integration results over different cuboids for different functions**

| Integral | | Computed value | Abs. Error |
|---|---|---|---|
| $I_1$ | **N=5** | 2.15214233328385 | 4.99E-07 |
| | **N=10** | 2.15214283259661 | 7.15E-13 |
| | **N=20** | 2.15214283259600 | 1.06E-13 |
| $I_2$ | **N=5** | 3.23945017706708 | 4.64E-12 |
| | **N=10** | 3.23945017707181 | 8.97E-14 |
| | **N=20** | 3.23945017707098 | 7.40E-13 |
| $I_3$ | **N=5** | 0.86862806889753 | 5.58E-03 |
| | **N=10** | 0.86356844367342 | 5.22E-04 |
| | **N=20** | 0.86308551222831 | 3.92E-05 |
| $I_4$ | **N=5** | - 4.00026561542668 | 2.65E-04 |
| | **N=10** | - 3.99999999998050 | 1.95E-11 |
| | **N=20** | - 3.99999999999989 | 1.10E-13 |
| $I_5$ | **N=5** | 0.86287769801611 | 6.20E-07 |
| | **N=10** | 0.86287707997018 | 2.83E-09 |
| | **N=20** | 0.86287707714824 | 5.43E-12 |

**Tetrahedron**

Corresponding to the triangles (2-simplex) in two-dimensions, the key element in FEM, in three-dimensional geometry is the tetrahedron (3-simplex). There are many publications which gives cubature points and weights to evaluate integrals over a tetrahedron (Rathod *et.al.* (2005), (2011)).

By substituting $a = 0$, $b = a$, $g_1 = 0$, $g_2 = a - x$, $h_1 = 0$ and $h_2 = a - x - y$ in Eq. (4.3a-d), the weights and nodes ($c_m, x_m, y_m, z_m$) required for integration over a tetrahedron $T = \{(x, y, z)|\ 0 \le x \le a, 0 \le y \le a - x, 0 \le z \le a - x - y\}$ by the present method can be acquired. The integral of a function can then be evaluated as,

$$I = \int_0^a \int_0^{a-x} \int_0^{a-x-y} f(x, y, z)\, dz\, dy\, dx \approx \sum_{m=1}^{N^3} c_m\, f(x_m\,, y_m\,, z_m\,) \tag{4.5}$$

where, $c_m = w_1^i w_2^j w_3^k a^3 (1 - \xi_i)\,(1 - \xi_i - \eta_j + \xi_i \eta_j);$ (4.5a)

$x_m = a\xi_i;\ \ y_m = a(1 - \xi_i)\eta_j;\quad z_m = a(1 - \xi_i - \eta_j + \xi_i\eta_j)\gamma_k$ (4.5b)

Recently in 2011, Rathod and his team suggested a product formula based on Gauss Legendre-Gauss Jacobi quadrature rules which according to them, has higher precision than the one in Rathod *et.al.* (2005). All five integrals evaluated in Rathod *et.al.* (2011) are assessed using Eq. (4.5) and the results attained are tabulated in Table 4.2 for comparison. Those five integrals with their exact solutions are:

$$I_6 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sqrt{x + y + z}\ dzdydx = 0.142857142857143$$

$$I_7 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{\sqrt{x + y + z}}\ dzdydx = 0.2$$

$$I_8 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} [(1 - x - y)^2 + z^2]^{-1/2}\ dzdydx = 0.440686793509772$$

$$I_9 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sin(x + 2y + 4z)\ dzdydx = 0.131902326890181$$

$$I_{10} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1 + x + y + z)^{-4}\ dzdydx = 0.020833333333333$$

**Table 4. 2: Comparative results of integration over a tetrahedron**

| | | Computed value using proposed method | Abs. Error | Computed value given in Rathod *et.al.* (2011) | Abs. Error |
|---|---|---|---|---|---|
| $I_6$ | N=5 | 0.142857116529783 | 2.63E-08 | 0.142857974957635 | 8.32E-07 |
| | N=10 | 0.142857142852557 | 4.58E-12 | 0.142488864056452 | 3.68E-04 |
| $I_7$ | N=5 | 0.200000275082717 | 2.75E-07 | 0.199948586087860 | 5.14E-05 |
| | N=10 | 0.200000002496587 | 2.49E-09 | 0.199607363698843 | 3.93E-04 |
| $I_8$ | N=5 | 0.440685900461523 | 8.93E-07 | 0.422997550101168 | 1.77E-02 |
| | N=10 | 0.440686793510626 | 8.54E-13 | 0.433942502667705 | 6.74E-03 |
| $I_9$ | N=5 | 0.131901021445645 | 1.31E-06 | 0.131902327536958 | 6.47E-10 |
| | N=10 | 0.131902326890124 | 5.70E-14 | 0.131643059515503 | 2.59E-04 |
| $I_{10}$ | N=5 | 0.0208332537441863 | 7.96E-08 | 0.020833289256266 | 4.41E-08 |
| | N=10 | 0.0208333333334694 | 1.36E-13 | 0.020806898469793 | 2.64E-05 |

**Note:** In the above table, we considered N=5 and N=10 only for comparison, as Rathod *et.al.* (2011) presents results only up to N=10.

Table 4.2 reveals that the proposed method gives more precise results than the method in Rathod *et.al.* (2011) for the same integrals $I_6$ to $I_{10}$. Moreover, the proposed integration formula (in Eq. 4.3) is a general one, of which tetrahedron is only a special case, whereas the quadrature rule in Rathod *et.al.* (2011) was only for a specific tetrahedron.

The quadrature points and their corresponding weights over the standard tetrahedron $\{(x, y, z)|\ 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$ for N=5 are tabulated in Table 4.13.

**Prism**

The nodes and weights $(x_m, y_m, z_m, c_m)$ required for integration over a prism, $P = \{(x, y, z)|\ 0 \le x \le h, 0 \le y \le a, 0 \le z \le a - y\}$ can be obtained by substituting $a = 0$, $b = h$, $g_1 = 0$, $g_2 = a$, $h_1 = 0$ and $h_2 = a - y$ in Eq. (4.3).

$$I = \int_0^h \int_0^a \int_0^{a-y} f(x, y, z)\, dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m) \tag{4.6}$$

where, $c_m = w_1^i w_2^j w_3^k a^2 h\left(1 - \eta_j\right);$ (4.6a)

$x_m = h\xi_i\,;\quad y_m = a\eta_j;\quad z_m = a(1 - \eta_j)\gamma_k$ (4.6b)

Integrals of some complicated functions over some prisms are given below, the numerical results of which are given in Table 4.3.

$$I_{11} = \int_0^3 \int_0^1 \int_0^{1-y} \sqrt{x + y + z}\ dzdydx = 2.15355213747502$$

$$I_{12} = \int_0^9 \int_0^3 \int_0^{3-y} \frac{1}{\sqrt{x + y + z}}\ dzdydx = 17.3631076695368$$

$$I_{13} = \int_0^1 \int_0^3 \int_0^{3-y} (y^2 + z^2) ln(3x) dzdydx = 1.33126589701948$$

Many other integrals are also tested and it is found that the method depicted here gives an accuracy of at least ten decimal places for more or less all functions including functions with end point singularities.

**Table 4. 3: Integration results over different prisms for three different functions**

| Integral | | Computed value | Abs. Error |
|---|---|---|---|
| $I_{11}$ | **N=5** | 2.15355139980635 | 7.37E-07 |
| | **N=10** | 2.15355213745917 | 1.58E-11 |
| | **N=20** | 2.15355213747495 | 7.01E-14 |
| $I_{12}$ | **N=5** | 17.3630049641611 | 1.02E-04 |
| | **N=10** | 17.3631075141767 | 1.55E-07 |
| | **N=20** | 17.3631076693077 | 2.29E-10 |
| $I_{13}$ | **N=5** | 1.33126589701942 | 5.99E-14 |

## Few other commonly found integrals over regions with linear boundaries

Listed below are the integration rules and results of integration over three more regions with planar faces. The first region is a pyramid given in Ethan J. Kubatko *et.al.* (2013). Fig. 4.4 shows the three regions and Table 4.4 provides the numerical results of integration over these regions.

(i) $\int_0^1 \int_{x-1}^{1-x} \int_{x-1}^{1-x} f(x, y, z)\, dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m)$ (4.7)

where $c_m = 4\,(1-\xi_i)^2 w_1^i w_2^j w_3^k;$

$$x_m = \xi_i\,;\ y_m = (1-\xi_i)\big(2\eta_j - 1\big);\ z_m = (1-\xi_i)(2\gamma_k - 1)$$

(4.7a)

(ii) $\int_0^2 \int_{z/2}^{2-z/2} \int_{z-2}^{2-z} f(x, y, z)\, dydxdz \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m)$ (4.8)

where $c_m = 16\,(1-\gamma_k)^2 w_1^i w_2^j w_3^k;$

$$z_m = 2\gamma_k\,;\ x_m = 2(1-\gamma_k)\xi_i + \gamma_k;\ \ y_m = (4\eta_j - 2)(1-\gamma_k)$$

(4.8a)

(iii) $\int_0^1 \int_0^x \int_0^{1+x+y} f(x,y,z)\, dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m)$ (4.9)

where $c_m = \xi_i(1+\xi_i+\xi_i\eta_j) w_1^i w_2^j w_3^k$;

$x_m = \xi_i; \ \ y_m = \xi_i\eta_j; \ \ z_m = (1+\xi_i+\xi_i\eta_j)\,\gamma_k$

(4.9a)

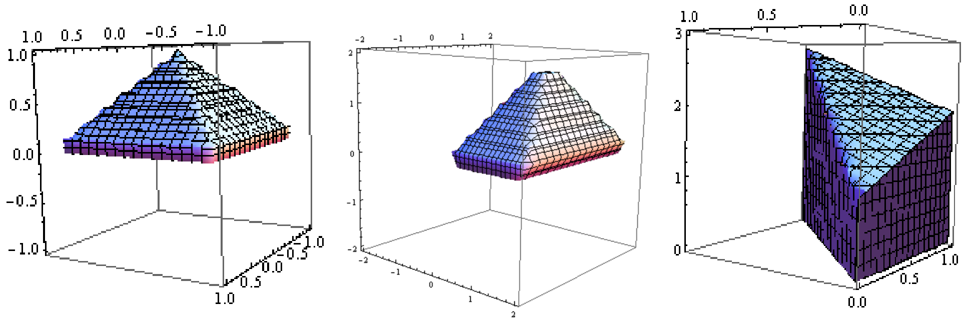

**Fig.4. 4: Domain of integration in (i), (ii) and (iii) in Section 4.3.1**

The following integrals are considered for numerical evaluation using the proposed method over the regions in Fig. 4.4

$$I_{14} = \int_0^1 \int_{x-1}^{1-x} \int_{x-1}^{1-x} \mathrm{Cos}(x)\,(y^2+z^2) dzdydx = \ 0.520809694372043$$

$$I_{15} = \int_0^1 \int_{x-1}^{1-x} \int_{x-1}^{1-x} xy^2z^4\ dzdydx = 0.00296296296296296$$

$$I_{16} = \int_0^2 \int_{z/2}^{2-z/2} \int_{z-2}^{2-z} e^{x/3}(y^2+z^2)\ dydxdz = \ 9.02681478076500$$

$$I_{17} = \int_0^2 \int_{z/2}^{2-z/2} \int_{z-2}^{2-z} xz\ \ dydxdz = 2.66666666666666$$

$$I_{18} = \int_0^1 \int_0^x \int_0^{1+x+y} \sqrt{x}(y^2+z^2)\ dzdydx = 1.48783068783069$$

$$I_{19} = \int_0^1 \int_0^x \int_0^{1+x+y} ln(3x)\,(y^2 + z^2)\ dzdydx = 1.377640890735786$$

**Table 4. 4: Integral values of some functions over domains given in Fig. 4.4**

| **Integral** | **Computed Value** | **Abs. Error** |
|---|---|---|
| $\boldsymbol{I_{14}}$ | | |
| **N=5** | 0.520846869551098 | 3.71E-05 |
| **N=10** | 0.520809694372009 | 3.41E-14 |
| $\boldsymbol{I_{15}}$ | | |
| **N=5** | 0.00295059511450889 | 1.23E-05 |
| **N=10** | 0.00296296296296282 | 1.40E-16 |
| $\boldsymbol{I_{16}}$ | | |
| **N=5** | 9.02675858042043 | 5.62E-05 |
| **N=10** | 9.02681478076513 | 1.31E-13 |
| $\boldsymbol{I_{17}}$ | | |
| **N=5** | 2.66666666666667 | 3.55E-15 |
| **N=10** | 2.66666666666661 | 5.63E-14 |
| $\boldsymbol{I_{18}}$ | | |
| **N=5** | 1.48782959157754 | 1.09E-06 |
| **N=10** | 1.48783068773415 | 9.65E-11 |
| **N=20** | 1.48783068783041 | 2.79E-13 |
| $\boldsymbol{I_{19}}$ | | |
| **N=5** | 1.37764089073579 | 8.65E-15 |

### 4.3.2 Triple Integrals with non-linear limits

Here, the integration rule and results for some triple integrals over regions with a curved face is given. The domains of integration of these integrals are shown in Fig. 4.5 and the numerical results tabulated in Table 4.5.

(i) $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m)$ (4.10)

where, $c_m = 32\,(\xi_i - {\xi_i}^2)^2(1-\eta_j)w_1^i w_2^j w_3^k$;

$x_m = 2\xi_i - 1;\ y_m = 4(\xi_i - {\xi_i}^2)(\eta_j - 1) + 1;\ \ z_m = 4(\xi_i - {\xi_i}^2)(1-\eta_j)\gamma_k$

(4.10a)

(ii) $\int_{0}^{1}\int_{-1}^{0}\int_{0}^{y^2} f(x,y,z)\,dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m)$ (4.11)

where, $c_m = (\eta_j - 1)^2 w_1^i w_2^j w_3^k;\ x_m = \xi_i;\ y_m = \eta_j - 1,\ z_m = (\eta_j - 1)^2\gamma_k$

(4.11a)

(iii) $\int_{0}^{1}\int_{-1}^{1}\int_{0}^{y^2} f(x,y,z)\,dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m)$ (4.12)

where, $c_m = 2(2\eta_j - 1)^2\, w_1^i w_2^j w_3^k$;

$x_m = \xi_i\,;\ y_m = 2\eta_j - 1;\ z_m = (2\eta_j - 1)^2\gamma_k$ (4.12a)

(iv) $\int_{0}^{1}\int_{0}^{ln2}\int_{1}^{e^y} f(x,y,z)\,dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m)$ (4.13)

where, $c_m = ln2\left(e^{(ln2)\eta_j} - 1\right)w_1^i w_2^j w_3^k$;

$x_m = \xi_i;\ y_m = (ln2)\eta_j;\ z_m = \left(e^{(ln2)\eta_j} - 1\right)\gamma_k + 1$

(4.13a)

(v) $\int_{0}^{\pi}\int_{0}^{\pi}\int_{0}^{sinx} f(x,y,z)\,dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m)$ (4.14)

where, $c_m = \pi^2 \sin(\pi\xi_i)\, w_1^i w_2^j w_3^k$;

$x_m = \pi\xi_i;\ y_m = \pi\eta_j;\ z_m = \sin(\pi\xi_i)\gamma_k$ (4.14a)

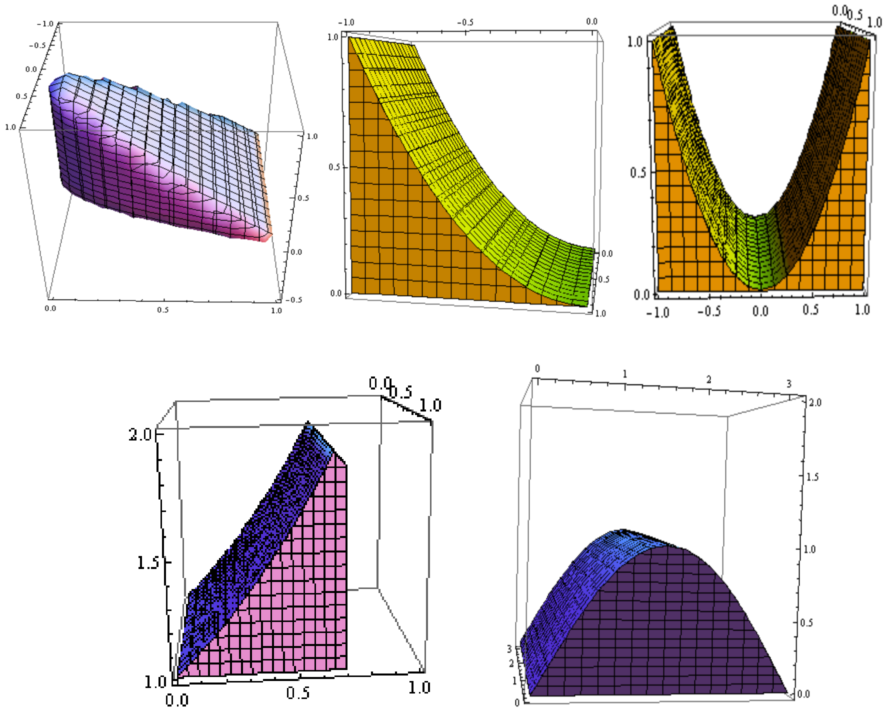

**Fig.4. 5: Domains of three-dimensional integrals with a curved face**

The following integrals are considered for integration tests over the curved domains in Fig. 4.5.

$$I_{20} = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} cos(x)\,(y^2+z^2)\,dzdydx = 0.178679271549711$$

$$I_{21} = \int_{0}^{1}\int_{-1}^{0}\int_{0}^{y^2} \sqrt{x}(y^2+z^2)\;dzdydx = 0.33015873015873$$

$$I_{22} = \int_{0}^{1}\int_{-1}^{1}\int_{0}^{y^2} x\sqrt{y^2+z^2}\,dzdydx = 0.137649738841096$$

$$I_{23} = \int_{0}^{1}\int_{0}^{ln2}\int_{1}^{e^y} (x^2+y^2+z^2)\,dzdydx = 0.726322079004793$$

$$I_{24} = \int_0^{\pi}\int_0^{\pi}\int_0^{sinx} x^2y^2z\,dzdydx = 22.6465432506750$$

**Table 4. 5: Integration of some functions over domains given in Fig. 4.5**

| Integrals | Computed Value | Abs. Error |
|---|---|---|
| $\boldsymbol{I_{20}}$ | | |
| **N=5** | 0.176882257002098 | 1.79E-03 |
| **N=10** | 0.178679200995834 | 7.05E-08 |
| **N=20** | 0.178679271549714 | 2.99E-15 |
| $\boldsymbol{I_{21}}$ | | |
| **N=5** | 0.329244988152129 | 9.13E-04 |
| **N=10** | 0.330158590060786 | 1.40E-07 |
| **N=20** | 0.330158727502470 | 2.65E-09 |
| $\boldsymbol{I_{22}}$ | | |
| **N=5** | 0.137650547523035 | 8.08E-07 |
| **N=10** | 0.137649738841164 | 6.80E-14 |
| **N=20** | 0.137649738841104 | 8.02E-15 |
| $\boldsymbol{I_{23}}$ | | |
| **N=5** | 0.726317302070989 | 4.77E-06 |
| **N=10** | 0.726322079004771 | 2.19E-14 |
| **N=20** | 0.726322079004786 | 6.99E-15 |
| $\boldsymbol{I_{24}}$ | | |
| **N=5** | 22.8572128398622 | 2.11E-01 |
| **N=10** | 22.6465428618721 | 3.88E-07 |
| **N=20** | 22.6465432506744 | 5.96E-13 |

## 4.4 Numerical integration over regions bounded by one or more circular edges

It has been found that for regions like cylinders, paraboloids and cones, where one or more circular edges appear, the integration formula derived in section 4.3 is not an apt choice. The accuracy was found to be poor for practically all functions, even while using more quadrature points. Hence, for these typical regions different integration rules are recommended using a new non-linear transformation to map such domains with circular edges, to a zero-one cube. This section explicates the derivation of these new quadrature rules over the above mentioned regions.

### Cylinder

Consider the integral of a function $f(x, y, z)$ over a circular cylinder with base $y^2 + z^2 = a^2$ and height *h* along the X direction,

$$I = \int_{0}^{h} \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x, y, z) dz dy dx \tag{4.15}$$

This integral can be transformed into a zero-one cube in the $\xi - \eta - \gamma$ space by using the transformation:

$$x = h\xi, \quad y = \mathrm{a}\eta cos(2\pi\gamma), \quad z = \mathrm{a}\eta sin(2\pi\gamma)$$

The Jacobian of the transformation is $|\, J \,| = \begin{vmatrix} h & 0 & 0 \\ a & \mathrm{a}cos(2\pi\gamma) & -2\pi\mathrm{a}\eta\, sin(2\pi\gamma) \\ 0 & \mathrm{a}sin(2\pi\gamma) & 2\pi\mathrm{a}\eta cos(2\pi\gamma) \end{vmatrix}$

$$= 2\pi a^2 h\eta$$

Now, the integral in Eq. (4.15) will be

$$I = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} f(x(\xi, \eta, \gamma), y(\xi, \eta, \gamma), z(\xi, \eta, \gamma))\ |J|\ d\gamma \mathrm{d}\eta \mathrm{d}\xi$$

$$\approx \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_1^i w_2^j w_3^k \; f(\,h\xi_i, a\eta_j cos(2\pi\,\gamma_k), a\eta_j sin(2\pi\,\gamma_k))2\pi a^2 h\eta_j$$

$$\approx \sum_{m=1}^{N^3} c_m\, f(x_m\,, y_m\,, z_m\,) \tag{4.16}$$

where, $c_m = \; w_1^i w_2^j w_3^k 2\pi a^2 h\eta_j \;$ ;

$x_m = h\xi_i\,;\;\; y_m = a\eta_j cos(2\pi\,\gamma_k)\,;\;\; z_m = a\eta_j sin(2\pi\,\gamma_k)$ (4.16a)

Eqs. (4.16) and (4.16a) should be used to evaluate the integral value of a function over the given circular cylinder.

Comparative results of integration of few functions over circular cylinders, using the integration rule given in Eq. (4.3, 4.3a-c) and the one given in Eq. (4.16, 4.16a) are tabulated in Table 4.6. The table clearly shows that the integration rule recently suggested in this section is the best choice for integrating any function over a circular cylinder.

The integrals evaluated over cylinders are given below:

$$I_{25} = \int_0^3 \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (y^2 + z^2) exp(x/3)\; dzdydx = 8.09721235362566$$

$$I_{26} = \int_0^5 \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} ln(3x)\; (y^2 + z^2) l\,n(3x)\; dzdydx = 13.4149949093632$$

$$I_{27} = \int_0^3 \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,\sqrt{y^2 + z^2}\; dzdydx = 9.42477796076938$$

$$I_{28} = \int_0^4 \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (y^2 + z^2)\sqrt{x} dzdydx = 8.37758040957278$$

$$I_{29} = \int_0^1 \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{\,x^2 + y^2 + z^2}\; dzdydx = 18.4023143690208$$

**Table 4. 6: Integration results over a cylinder using two methods**

| Integral | | Computed value using Eqs. (4.3, 4.3a-c) | Abs. Error | Computed value using Eqs. (4.16, 4.16a) | Abs. Error |
|---|---|---|---|---|---|
| $I_{25}$ | N=5 | 8.17718541519248 | 7.99E-02 | 8.09721165769299 | 6.95E-07 |
| | N=10 | 8.10746962489963 | 1.02E-02 | 8.09721235362570 | 4.08E-14 |
| | N=20 | 8.09857752205178 | 1.36E-03 | 8.09721235362566 | 1.01E-15 |
| $I_{26}$ | N=5 | 13.5474908345150 | 1.32E-01 | 13.4149949093633 | 9.94E-14 |
| $I_{27}$ | N=5 | 9.55038249036606 | 1.25E-01 | 9.42477796076933 | 4.97E-14 |
| $I_{28}$ | N=5 | 8.46015916261744 | 8.25E-02 | 8.37741789127231 | 1.62E-05 |
| | N=10 | 8.38818928209213 | 1.06E-02 | 8.37757685467096 | 3.55E-06 |
| | N=20 | 8.37899277989349 | 1.41E-03 | 8.37758034217137 | 6.74E-08 |
| $I_{29}$ | N=5 | 18.5673414058798 | 1.65E-01 | 18.4022794778556 | 3.48E-05 |
| | N=10 | 18.4117287791920 | 9.41E-03 | 18.4023143795737 | 1.05E-08 |
| | N=20 | 18.4043229412647 | 2.01E-03 | 18.4023143690217 | 8.98E-13 |

The betterment of results by using the new integration rule is due to the transformation which is used in the derivation. The distribution of nodal points over each circular cross section of the cylinder is in a circular fashion, if the new method (Eq. 4.16) is used, whereas it is along vertical lines when one uses the general integration rule in Eq. (4.3).

In a similar manner, a numerical integration formula over an elliptic cylinder with the base $\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ and a height of $h$ along the X direction can be derived. The formula will be:

$$I = \int_{0}^{h} \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-y^2}}^{\frac{b}{a}\sqrt{a^2-y^2}} f(x,y,z)dzdydx \approx \sum_{m=1}^{N^3} c_m f(x_m, y_m, z_m) \tag{4.17}$$

where, $c_m = w_1^i w_2^j w_3^k 2\pi abh\eta_j$ ;

$$x_m = h\xi_i \,;\; y_m = a\eta_j cos(2\pi\,\gamma_k);\; z_m = b\eta_j sin(2\pi\,\gamma_k) \tag{4.17a}$$

Table 4.7 tabulates the results of integration of three functions over different elliptic cylinders.

**Table 4. 7: Integration results over elliptic cylinders**

| Integral with exact solution over elliptic cylinders | Computed value using Eqs. (4.17,4.17a) | Abs. Error |
|---|---|---|
| $I_{30} = \int_0^3 \int_{-1}^1 \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} dzdydx = 18.8495559215388$ | | |
| **N=5** | 18.8495559215388 | 4.26E-14 |
| $I_{31} = \int_0^1 \int_{-2}^2 \int_{-0.5\sqrt{4-y^2}}^{0.5\sqrt{4-y^2}} (x^2+y^2+z^2)dzdydx = 9.94837673636768$ | | |
| **N=5** | 9.77898860326025 | 1.69E-01 |
| **N=10** | 9.94845760521649 | 8.08E-05 |
| **N=20** | 9.94837673636754 | 1.38E-13 |
| $I_{32} = \int_0^4 \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} e^{x/3}(y^2+z^2)dzdydx = 9.32139816239476$ | | |
| **N=5** | 9.12036201012729 | 2.01E-01 |
| **N=10** | 9.32149414055879 | 9.59E-05 |
| **N=20** | 9.32139816239469 | 6.92E-14 |

**Cone**

Integral of a function $f(x,y,z)$ over a cone $z = \sqrt{x^2+y^2}$ , which lies between $z = 0$ and $z = a$ planes is given by,

$$I = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{\sqrt{x^2+y^2}}^{a} f(x,y,z)dzdydx$$

(4.18)

This region can be transformed into a zero-one cube using the transformation

$$x = \mathrm{a}\xi cos(2\pi\eta), \quad \mathrm{y} = \mathrm{a}\xi sin(2\pi\eta), \quad \mathrm{z} = \mathrm{a}(1-\xi)\gamma + \mathrm{a}\xi$$

The Jacobian of the transformation is,

$$|\,J\,| = \begin{vmatrix} \mathrm{a}cos(2\pi\eta) & -2\pi\mathrm{a}\xi sin(2\pi\eta) & 0 \\ \mathrm{a}sin(2\pi\eta) & 2\pi\mathrm{a}\xi cos(2\pi\eta) & 0 \\ a - a\gamma & 0 & a - a\xi \end{vmatrix} = 2\pi a^3\xi(1-\xi)$$

Integral in Eqn. (4.18) will now be

$$I = \int_0^1\int_0^1\int_0^1 f(\mathrm{a}\xi cos(2\pi\eta), \mathrm{a}\xi sin(2\pi\eta), \mathrm{a}(1-\xi)\gamma + \mathrm{a}\xi)\,|\,J\,|\,d\gamma\,d\eta\,d\xi$$

$$\approx \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_1^i w_2^j w_3^k\ |J| f(\,\mathrm{a}\xi_i cos(2\pi\eta_j), \mathrm{a}\xi_i sin(2\pi\eta_j), \mathrm{a}(1-\xi_i)\gamma_k + \mathrm{a}\xi_i)$$

$$\approx \sum_{m=1}^{N^3} c_m\, f(x_m\,, y_m\,, z_m\,)$$

(4.19)

where, $c_m = 2\pi a^3\xi_i(1-\xi_i)w_1^i w_2^j w_3^k$;

$x_m = \mathrm{a}\xi_i cos(2\pi\eta_j)$; $y_m = \mathrm{a}\xi_i sin(2\pi\eta_j)$; $z_m = \mathrm{a}(1-\xi_i)\gamma_k + \mathrm{a}\xi_i$ (4.19a)

By substituting the generalized Gaussian quadrature points $\xi_i$, $\eta_j$ and $\gamma_k$ along with their weights $w_1^i$ , $w_2^j$ and $w_3^k$ for one dimension in Eq. (4.19a) we can get the nodes $x_m$ , $y_m$ , $z_m$ and the weights $c_m$ required in the integration formula in Eq. (4.19). The integral values computed using Eq. (4.19) over different cones is tabulated in Table 4.8 below.

**Table 4. 8: Integration results over a cone**

| Integrals with exact values | Computed Value using Eqs. (4.19 and 4.19a) | Abs. Error |
|---|---|---|
| $I_{33} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} dzdydx = 1.04719755119660$ | | |
| **N=5** | 1.04719755119661 | 9.99E-15 |
| $I_{34} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2+z^2}\ dzdydx = 0.957362203787823$ | | |
| **N=5** | 0.957363549131513 | 1.34E-06 |
| **N=10** | 0.957362203740914 | 4.69E-11 |
| **N=20** | 0.957362203787829 | 5.99E-15 |
| $I_{35} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} \frac{1}{\sqrt{x^2+y^2+z^2}}\ dzdydx = 1.30129028456857$ | | |
| **N=5** | 1.30139452193870 | 1.04E-04 |
| **N=10** | 1.30129156448336 | 1.27E-06 |
| **N=20** | 1.30129029317366 | 8.60E-09 |
| $I_{36} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} x^2y^2z\ dzdydx = 0.0163624617374468$ | | |
| **N=5** | 0.0211118758325485 | 4.74E-03 |
| **N=10** | 0.0153837186361798 | 9.78E-04 |
| **N=20** | 0.0163624617302494 | 7.19E-12 |

**Paraboloid**

Integral of a function $f(x,y,z)$ over a paraboloid $z = a^2 - x^2 - y^2$ above the $z = 0$ plane is given by,

$$I = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{0}^{a^2-x^2-y^2} f(x,y,z)dzdydx$$

(4.20)

This paraboloid is transformed to a zero-one cube using the transformation

$$x = \mathrm{a}\xi cos(2\pi\eta), \quad \mathrm{y} = \mathrm{a}\xi sin(2\pi\eta), \quad \mathrm{z} = \mathrm{a}^2(1-\xi^2)\gamma$$

The Jacobian of the transformation is found to be

$$|J| = \begin{vmatrix} \mathrm{a}cos(2\pi\eta) & -2\pi\mathrm{a}\xi sin(2\pi\eta) & 0 \\ \mathrm{a}sin(2\pi\eta) & 2\pi\mathrm{a}\xi cos(2\pi\eta) & 0 \\ 0 & 0 & a^2(1-\xi^2) \end{vmatrix} = 2\pi a^4\xi(1-\xi^2)$$

Integral in Eq.(4.20) will now be

$$I = \int_0^1 \int_0^1 \int_0^1 f(\mathrm{a}\xi cos(2\pi\eta), \mathrm{a}\xi sin(2\pi\eta), \mathrm{a}^2(1-\xi^2)\gamma)\ |J|\ d\gamma\ d\eta\ d\xi$$

$$\approx \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_1^i w_2^j w_3^k\ \ |J| f(\,\mathrm{a}\xi_i cos(2\pi\eta_j), \mathrm{a}\xi_i sin(2\pi\eta_j), \mathrm{a}^2(1-{\xi_i}^2)\gamma_k)$$

$$\approx \sum_{m=1}^{N^3} c_m\, f(x_m\,, y_m\,, z_m\,)$$

(4.21)

where, $c_m = \ 2\pi a^4 \xi_i(1-{\xi_i}^2) w_1^i w_2^j w_3^k$ ;

$x_m = \mathrm{a}\xi_i cos(2\pi\eta_j)$; $y_m = \mathrm{a}\xi_i sin(2\pi\eta_j)$; $z_m = \mathrm{a}^2(1-{\xi_i}^2)\gamma_k$ (4.21a)

Table 4.9 gives the integration results of some functions over a paraboloid using Eqs. (4.21 and 4.21a)

**Table 4. 9: Integration results over a paraboloid**

| **Integrals with exact values** | **Computed Value using Eqs.(4.21 and 4.21a)** | **Abs. Error** |
|---|---|---|
| $I_{37} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2} dzdydx = 1.57079632679490$ | | |
| **N=5** | 1.57079632679486 | 4.01E-14 |
| $I_{38} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2} e^{x/3}(y^2+z^2)\, dzdydx = 0.526879697108398$ | | |
| **N=5** | 0.523909762178377 | 2.96E-03 |
| **N=10** | 0.526883295301921 | 3.59E-06 |
| **N=20** | 0.526879697125928 | 1.75E-11 |
| $I_{39} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2} Cos(x)(y^2+z^2)\, dzdydx = 0.494819261413857$ | | |
| **N=5** | 0.500292661917175 | 5.47E-03 |
| **N=10** | 0.495734673097876 | 9.15E-04 |
| **N=20** | 0.494819260552759 | 8.61E-10 |
| $I_{40} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2} (x^2+y^2+z^2)\, dzdydx = 0.785398163397448$ | | |
| **N=5** | 0.785775882922285 | 3.77E-04 |
| **N=10** | 0.785398163397457 | 9.10E-15 |
| **N=20** | 0.785398163397443 | 4.99E-15 |

## 4.5 Numerical integration over a sphere

In section 3.4, the derivation of a numerical integration method to integrate functions over a circular disc (2-ball) is portrayed, using a combination of polar and linear transformations. Here, the same approach is extended to a sphere (3-ball) $x^2+y^2+z^2 \leq a^2$.

*i.e.* a numerical integration formula is to be derived to evaluate the integral,

$$I = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,\, z)dz\, dy\, dx \tag{4.22}$$

For this derivation, the domain of integration in Eq. (4.22) is primarily transformed to a cuboid

$$0 < r < a,\ 0 < \varphi_1 < \pi,\ 0 < \varphi_2 < 2\pi$$

in the $r$- $\varphi_1$- $\varphi_2$ space using the transformation,

$$x = r\cos\varphi_1, \quad y = r\sin\varphi_1\cos\varphi_2\,, \quad z = r\sin\varphi_1\sin\varphi_2.$$

The Jacobian of this transformation is

$|\,J\,| = r^2\sin\varphi_1.$

Hence, Eq. (4.22) now becomes

$$I = \int_0^a \int_0^\pi \int_0^{2\pi} r^2\sin\varphi_1\, f(r\cos\varphi_1\,, r\sin\varphi_1\cos\varphi_2\,,\ r\sin\varphi_1\sin\varphi_2)\, d\varphi_2\, d\varphi_1\, dr \tag{4.23}$$

Secondly, the cuboid (integration domain in Eq. (4.23)) is transformed to a zero-one cube in the $\xi-\eta-\gamma$ space using the linear transformation,

$$r = a\xi,\ \ \varphi_1 = \pi\eta,\ \varphi_2 = 2\pi\gamma,$$

which gives the Jacobian as $|J| = 2\pi^2 a$ and then Eq. (4.23) will be

$$I = \int_0^1 \int_0^1 \int_0^1 2\pi^2 a\, (a\xi)^2 \; \sin(\pi\eta)\, f(\bar{x}(\xi,\eta,\gamma))d\gamma d\eta d\xi$$

where, $\bar{x} = (x, y, z)$

Applying the generalized Gaussian quadrature rule to this integral with different node points $N_1$, $N_2$ and $N_3$ along each direction ξ, η and γ respectively, one can approximate the integral *I* in the following way.

$$I \approx \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \sum_{i_3=1}^{N_3} 2\pi^2 a^3 (\xi_i)^2 \sin(\pi\eta_j)\, w_1^i w_2^j w_3^k$$

$$f\left[a\xi_i \cos\left(\pi\eta_j\right),\; a\xi_i \sin\left(\pi\eta_j\right)\; \cos(2\pi\gamma_k),\; a\xi_i \sin(\pi\eta_j) \sin(2\pi\gamma_k)\right]$$

$$\therefore I \approx \sum_{m=1}^{N_1 N_2 N_3} c_m f(x_m\,,\, y_m\,, z_m) \tag{4.24}$$

where $c_m = 2\pi^2 a^3 (\xi_i)^2 \sin(\pi\eta_j)\; w_1^i w_2^j w_3^k;$ (4.24a)

$$x_m = a\xi_i \cos\left(\pi\eta_j\right); \tag{4.24b}$$

$$y_m = \; a\xi_i \sin\left(\pi\eta_j\right)\; \cos(2\pi\gamma_k)\,; \tag{4.24c}$$

$$z_m = a\xi_i \sin(\pi\eta_j) \sin(2\pi\gamma_k) \tag{4.24d}$$

By taking the generalized Gaussian quadrature points in one dimension as $\xi_i$ , $\eta_j$ and $\gamma_k$ and their corresponding weights as $w_1^i$ , $w_2^j$ and $w_3^k$ in Eqs. (4.24a-d), the quadrature points ( $x_m$ , $y_m$ , $z_m$) and the weights $c_m$ at these points, can be evaluated. Using these nodes and weights in Eq. (4.24), one can evaluate the integral of any function over the sphere.

The distribution of nodal points $(x_m, y_m, z_m)$ in a unit sphere, $x^2+y^2+z^2 \leq 1$, is plotted in Fig. 4.6 and these points along with their weights (for $N_1= N_2= N_3 =5$) is tabulated in Table 4.14.

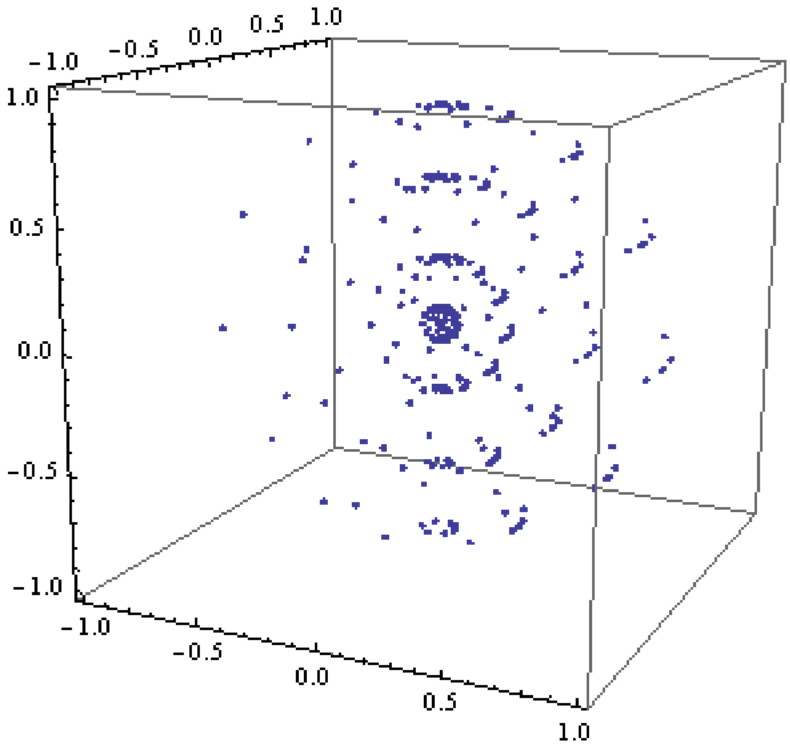


**Fig.4. 6: Distribution of integration points ($x_m$ , $y_m$ , $z_m$) in a unit sphere $x^2+y^2+z^2 \leq 1$**

Table 4.10 depicts the numerical results obtained along with the absolute error and number of function evaluations required to get the best integration results by using the proposed formulae for integrating some functions over the unit sphere, $x^2+y^2+z^2 \leq 1$.

**Table 4. 10: Integration results over the unit sphere $x^2 + y^2 + z^2 \leq 1$**

| Integrand | Exact solution | Abs. Error | Number of function evaluations |
|---|---|---|---|
| $\frac{1}{2^3}$ | 0.523598775598299 | 0.3E-14 | 10×20×20 = 4000 |
| $\sqrt{y^2+z^2}exp(x/3)$ | 2.49032688128226 | 5.9E-14 | 10×20×20 = 4000 |
| $x\sqrt{y^2+z^2}$ | 0 | 1.6E-15 | 10×20×20 = 4000 |
| $\sqrt{x^2+y^2+z^2}$ | 3.14159265358979 | 1.3E-13 | 10×20×20 = 4000 |
| $(y^2+z^2)Cos(x)$ | 1.55911090931194 | 3.1E-13 | 10×20×20 = 4000 |

## 4.6 Numerical evaluation of triple integrals over a closed region bounded by an irregular surface

This section describes an integration method to evaluate the integral

$$I = \iiint_D f(x,y,z)dxdydz \tag{4.25}$$

of a function $f(x,y,z)$ over a three dimensional closed region, D which has the surface in polar coordinates as, $S: r = u(\varphi_1, \varphi_2)$.

Under the transformation,

$$x = r\cos\varphi_1, \quad y = r\sin\varphi_1\cos\varphi_2, \quad z = r\sin\varphi_1\sin\varphi_2 \tag{4.26}$$

and with the Jacobian, $|J_1| = r^2\sin\varphi_1$, the integral in Eq. (4.25) becomes,

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{u(\varphi_1,\varphi_2)} |J_1| f(r\cos\varphi_1, r\sin\varphi_1\cos\varphi_2, r\sin\varphi_1\sin\varphi_2)\, dr\, d\varphi_1\, d\varphi_2 \tag{4.27}$$

Next, the domain of integration in the integral in Eq. (4.27) is transformed to a zero-one cube in the $\xi - \eta - \gamma$ space using the following transformation,

$$\varphi_2 = 2\pi\gamma, \varphi_1 = \pi\eta,\ r = u(\varphi_1, \varphi_2)\xi,$$

which gives a Jacobian $|J_2| = 2\pi^2 u(\varphi_1, \varphi_2)$.

Now the integral in Eq.(4.27) will be

$$I = \int_0^1\int_0^1\int_0^1 |J_1||J_2| f(\overline{\boldsymbol{x}}(\xi,\eta,\gamma))d\xi d\eta d\gamma \tag{4.28}$$

where, $\overline{\boldsymbol{x}} = (x, y, z)$.

After applying the generalized Gaussian quadrature rule for one dimension to each of these three integrals and by using the product formula, the numerical method can be derived as,

$$I \approx \sum_{m=1}^{N_1 N_2 N_3} c_m f(x_m, y_m, z_m) \tag{4.29}$$

where $c_m = 2\pi^2[u(\pi\eta_j, 2\pi\gamma_k)]^3(\xi_i)^2\sin(\pi\eta_j)\ w_1^i w_2^j w_3^k$; (4.29a)

$x_m = u(\pi\eta_j, 2\pi\gamma_k)\xi_i\cos(\pi\eta_j)$ ; (4.29b)

$y_m = u(\pi\eta_j, 2\pi\gamma_k)\xi_i\sin(\pi\eta_j)\cos(2\pi\gamma_k)$ ; (4.29c)

$z_m = u(\pi\eta_j, 2\pi\gamma_k)\xi_i\sin(\pi\eta_j)\sin(2\pi\gamma_k)$ (4.29d)

Hence, if the boundary of the three-dimensional closed region in polar coordinates is known, one can evaluate the integral of any function over this region using the formulae in Eqs. (4.29, 4.29a-d).

It can be noted that, if $u(\varphi_1, \varphi_2) = a$, the integration domain would be the sphere $x^2 + y^2 + z^2 \leq a^2$ and by substituting $u(\varphi_1, \varphi_2) = a$ in Eqs. (4.29, 4.29a-d), the equations derived in section 4.5 (Eqs. (4.24, 4.24a-d)) will be obtained. Hence, integration of a function over a sphere can be considered as a special case of this.

For illustration of the derived formula, we consider two elements $D_1$ and $D_2$ (Fig. 4.7) each with a boundary in polar form,

(i) $r = u(\varphi_1, \varphi_2) = 1 + (0.1)\varphi_1{}^3$

(ii) $r = u(\theta) = 3 - 2sin(\theta)$

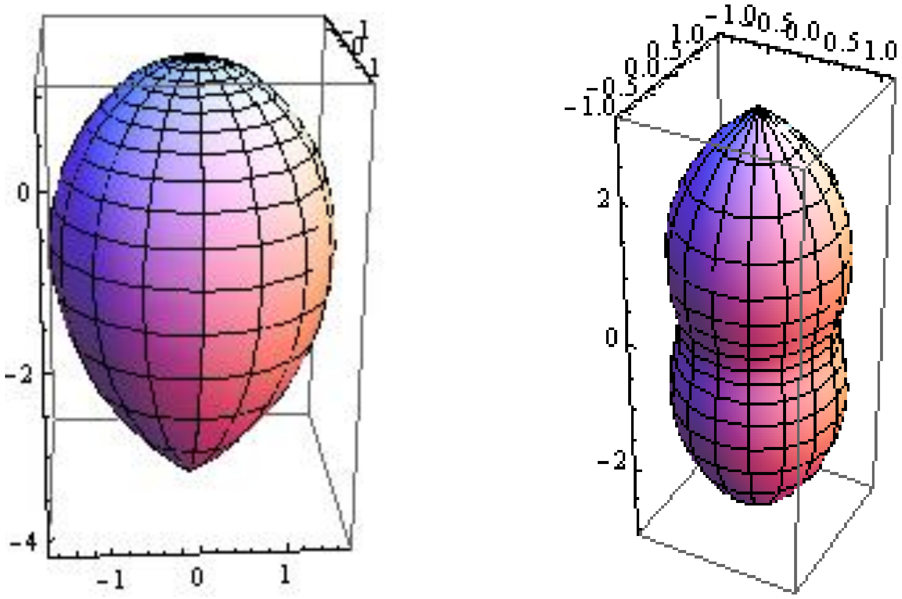


**Fig.4. 7: Domain $D_1$ with boundary u= $1 + (0.1)\varphi_1{}^3$ (left) and domain $D_2$ with boundary $u = 3 - 2sin(\theta)$ (right)**

By using Eqs. (4.29, 4.29a-d), the integration formula over these elements can be obtained as:

(i) $I \approx \sum_{m=1}^{N_1 N_2 N_3} c_m f(x_m\,,\ y_m\,, z_m)$ (4.30)

where $c_m = 2\pi^2[1 + 0.1(\pi\eta_j)^3]^3(\xi_i)^2 \sin(\pi\eta_j)\ w_1^i w_2^j w_3^k$ ; (4.30a)

$x_m = [1 + 0.1(\pi\eta_j)^3]\xi_i \cos(\pi\eta_j)$ ; (4.30b)

$y_m = [1 + 0.1(\pi\eta_j)^3]\xi_i \sin(\pi\eta_j) \cos(2\pi\gamma_k)$ ; (4.30c)

$z_m = [1 + 0.1(\pi\eta_j)^3]\xi_i \sin(\pi\eta_j) \sin(2\pi\gamma_k)$ (4.30d)

(ii) $I \approx \sum_{m=1}^{N_1 N_2 N_3} c_m f(x_m\,,\ y_m\,,\ z_m)$ (4.31)

where $c_m = 2\pi^2[3 - 2\sin(\pi\eta_j)]^3(\xi_i)^2 \sin(\pi\eta_j)\ w_1^i w_2^j w_3^k$ ; (4.31a)

$x_m = [3 - 2\sin(\pi\eta_j)]\xi_i \cos(\pi\eta_j)$ ; (4.31b)

$y_m = [3 - 2\sin(\pi\eta_j)]\xi_i \sin(\pi\eta_j) \cos(2\pi\gamma_k)$ ; (4.31c)

$z_m = [3 - 2\sin(\pi\eta_j)]\xi_i \sin(\pi\eta_j) \sin(2\pi\gamma_k)$ (4.31d)

Table 4.11 presents the volume of both the domains and also integrals of few functions over these domains, obtained using the quadrature rules, Eq. (4.30) and Eq. (4.31), for $N_1 = 10$ , $N_2 = 20$ and $N_3 = 20$.

**Table 4. 11: Integration results over two irregular domains, $D_1$ and $D_2$ in three-dimension**

| Domain | Exact solution | Computed integral value | Abs. error |
|---|---|---|---|
| $\boldsymbol{D_1}$ | | | |
| Volume | 26.65176383712697 | 26.6517638371282 | 1.22E-12 |
| $\iint_{D_1} (x^2+y^2+z^2)d\,D_1$ | 103.7912175650071 | 103.791217575848 | 1.08E-08 |
| $\iint_{D_1} \sqrt{x^2+y^2+z^2}d\,D_1$ | 48.47138691000155 | 48.4713869099198 | 8.17E-11 |
| $\boldsymbol{D_2}$ | | | |
| Volume | 16.2362124223188 | 16.2362124223190 | 2.30E-13 |
| $\iint_{D_2} (x^2+y^2+z^2)d\,D_2$ | 37.7104621546873 | 37.7104621546782 | 9.08E-12 |
| $\iint_{D_2} \sqrt{x^2+y^2+z^2}\,d\,D_2$ | 23.0550874899026 | 23.0550874899030 | 4.36E-13 |

:

**Table 4. 12: 125 cubature points ($x_m$ , $y_m$ , $z_m$) and their corresponding weights $c_m$ over the zero-one cube**

| $x_m$ | $y_m$ | $z_m$ | $c_m$ |
|---|---|---|---|
| 0.56522282050801E-02 | 0.56522282050801E-02 | 0.56522282050801E-02 | 0.93232482322523E-05 |
| 0.56522282050801E-02 | 0.56522282050801E-02 | 0.73430371742652E-01 | 0.57899146685827E-04 |
| 0.56522282050801E-02 | 0.56522282050801E-02 | 0.28495740446256E+00 | 0.12833056628016E-03 |
| 0.56522282050801E-02 | 0.56522282050801E-02 | 0.61948226408478E+00 | 0.15513849272551E-01 |
| 0.56522282050801E-02 | 0.56522282050801E-02 | 0.91575808300470E+00 | 0.92282473241522E-04 |
| 0.56522282050801E-02 | 0.73430371742652E-01 | 0.56522282050801E-02 | 0.57899146685827E-04 |
| 0.56522282050801E-02 | 0.73430371742652E-01 | 0.73430371742652E-01 | 0.35956472502257E-03 |
| 0.56522282050801E-02 | 0.73430371742652E-01 | 0.28495740446256E+00 | 0.79695725097467E-03 |
| 0.56522282050801E-02 | 0.73430371742652E-01 | 0.61948226408478E+00 | 0.96343957847857E-03 |
| 0.56522282050801E-02 | 0.73430371742652E-01 | 0.91575808300470E+00 | 0.57309172958179E-03 |
| 0.56522282050801E-02 | 0.28495740446256E+00 | 0.56522282050801E-02 | 0.12833056628016E-03 |
| 0.56522282050801E-02 | 0.28495740446256E+00 | 0.73430371742652E-01 | 0.79695725097466E-03 |
| 0.56522282050801E-02 | 0.28495740446256E+00 | 0.28495740446256E+00 | 0.17664159348257E-02 |
| 0.56522282050801E-02 | 0.28495740446256E+00 | 0.61948226408478E+00 | 0.21354156971218E-02 |
| 0.56522282050801E-02 | 0.28495740446256E+00 | 0.91575808300470E+00 | 0.12702291898493E-02 |
| 0.56522282050801E-02 | 0.61948226408478E+00 | 0.56522282050801E-02 | 0.15513849272551E-03 |
| 0.56522282050801E-02 | 0.61948226408478E+00 | 0.73430371742652E-01 | 0.96343957847857E-03 |
| 0.56522282050801E-02 | 0.61948226408478E+00 | 0.28495740446256E+00 | 0.21354156971218E-02 |
| 0.56522282050801E-02 | 0.61948226408478E+00 | 0.61948226408478E+00 | 0.25814985641895E-02 |
| 0.56522282050801E-02 | 0.61948226408478E+00 | 0.91575808300470E+00 | 0.15355768126119E-02 |
| 0.56522282050801E-02 | 0.91575808300470E+00 | 0.56522282050801E-02 | 0.92282473241522E-04 |
| 0.56522282050801E-02 | 0.91575808300470E+00 | 0.73430371742652E-01 | 0.57309172958179E-03 |
| 0.56522282050801E-02 | 0.91575808300470E+00 | 0.28495740446256E+00 | 0.12702291898493E-02 |
| 0.56522282050801E-02 | 0.91575808300470E+00 | 0.61948226408478E+00 | 0.15355768126119E-02 |
| 0.56522282050801E-02 | 0.91575808300470E+00 | 0.91575808300470E+00 | 0.91342144448244E-03 |
| 0.73430371742652E-01 | 0.56522282050801E-02 | 0.56522282050801E-02 | 0.57899146685827E-04 |

| | | | |
|---|---|---|---|
| 0.73430371742652E-01 | 0.56522282050801E-02 | 0.73430371742652E-01 | 0.35956472502257E-03 |
| 0.73430371742652E-01 | 0.56522282050801E-02 | 0.28495740446256E+00 | 0.79695725097467E-03 |
| 0.73430371742652E-01 | 0.56522282050801E-02 | 0.61948226408478E+00 | 0.96343957847857E-03 |
| 0.73430371742652E-01 | 0.56522282050801E-02 | 0.91575808300470E+00 | 0.57309172958179E-03 |
| 0.73430371742652E-01 | 0.73430371742652E-01 | 0.56522282050801E-02 | 0.35956472502257E-03 |
| 0.73430371742652E-01 | 0.73430371742652E-01 | 0.73430371742652E-01 | 0.22329654041724E-02 |
| 0.73430371742652E-01 | 0.73430371742652E-01 | 0.28495740446256E+00 | 0.49492562706730E-02 |
| 0.73430371742652E-01 | 0.73430371742652E-01 | 0.61948226408478E+00 | 0.59831432230124E-02 |
| 0.73430371742652E-01 | 0.73430371742652E-01 | 0.91575808300470E+00 | 0.35590087584179E-02 |
| 0.73430371742652E-01 | 0.28495740446256E+00 | 0.56522282050801E-02 | 0.79695725097466E-03 |
| 0.73430371742652E-01 | 0.28495740446256E+00 | 0.73430371742652E-01 | 0.49492562706730E-02 |
| 0.73430371742652E-01 | 0.28495740446256E+00 | 0.28495740446256E+00 | 0.10969779284097E-01 |
| 0.73430371742652E-01 | 0.28495740446256E+00 | 0.61948226408478E+00 | 0.13261338066186E-01 |
| 0.73430371742652E-01 | 0.28495740446256E+00 | 0.91575808300470E+00 | 0.78883651229288E-02 |
| 0.73430371742652E-01 | 0.61948226408478E+00 | 0.56522282050801E-02 | 0.96343957847857E-03 |
| 0.73430371742652E-01 | 0.61948226408478E+00 | 0.73430371742652E-01 | 0.59831432230124E-02 |
| 0.73430371742652E-01 | 0.61948226408478E+00 | 0.28495740446256E+00 | 0.13261338066186E-01 |
| 0.73430371742652E-01 | 0.61948226408478E+00 | 0.61948226408478E+00 | 0.16031597605672E-01 |
| 0.73430371742652E-01 | 0.61948226408478E+00 | 0.91575808300470E+00 | 0.95362243829577E-02 |
| 0.73430371742652E-01 | 0.91575808300470E+00 | 0.56522282050801E-02 | 0.57309172958179E-03 |
| 0.73430371742652E-01 | 0.91575808300470E+00 | 0.73430371742652E-01 | 0.35590087584179E-02 |
| 0.73430371742652E-01 | 0.91575808300470E+00 | 0.28495740446256E+00 | 0.78883651229288E-02 |
| 0.73430371742652E-01 | 0.91575808300470E+00 | 0.61948226408478E+00 | 0.95362243829577E-02 |
| 0.73430371742652E-01 | 0.91575808300470E+00 | 0.91575808300470E+00 | 0.56725210873520E-02 |
| 0.28495740446256E+00 | 0.56522282050801E-02 | 0.56522282050801E-02 | 0.12833056628016E-03 |
| 0.28495740446256E+00 | 0.56522282050801E-02 | 0.73430371742652E-01 | 0.79695725097466E-03 |
| 0.28495740446256E+00 | 0.56522282050801E-02 | 0.28495740446256E+00 | 0.17664159348257E-02 |
| 0.28495740446256E+00 | 0.56522282050801E-02 | 0.61948226408478E+00 | 0.21354156971218E-02 |
| 0.28495740446256E+00 | 0.56522282050801E-02 | 0.91575808300470E+00 | 0.12702291898493E-02 |
| 0.28495740446256E+00 | 0.73430371742652E-01 | 0.56522282050801E-02 | 0.79695725097466E-03 |

| | | | |
|---|---|---|---|
| 0.28495740446256E+00 | 0.73430371742652E-01 | 0.73430371742652E-01 | 0.49492562706730E-02 |
| 0.28495740446256E+00 | 0.73430371742652E-01 | 0.28495740446256E+00 | 0.10969779284097E-01 |
| 0.28495740446256E+00 | 0.73430371742652E-01 | 0.61948226408478E+00 | 0.13261338066186E-01 |
| 0.28495740446256E+00 | 0.73430371742652E-01 | 0.91575808300470E+00 | 0.78883651229288E-01 |
| 0.28495740446256E+00 | 0.28495740446256E+00 | 0.56522282050801E-02 | 0.17664159348257E-02 |
| 0.28495740446256E+00 | 0.28495740446256E+00 | 0.73430371742652E-01 | 0.10969779284097E-01 |
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| 0.28495740446256E+00 | 0.28495740446256E+00 | 0.91575808300470E+00 | 0.17484167232086E-01 |
| 0.28495740446256E+00 | 0.61948226408478E+00 | 0.56522282050801E-02 | 0.21354156971218E-02 |
| 0.28495740446256E+00 | 0.61948226408478E+00 | 0.73430371742652E-01 | 0.13261338066186E-01 |
| 0.28495740446256E+00 | 0.61948226408478E+00 | 0.28495740446256E+00 | 0.29393093354222E-01 |
| 0.28495740446256E+00 | 0.61948226408478E+00 | 0.61948226408478E+00 | 0.35533235235315E-01 |
| 0.28495740446256E+00 | 0.61948226408478E+00 | 0.91575808300470E+00 | 0.21136564963214E-01 |
| 0.28495740446256E+00 | 0.91575808300470E+00 | 0.56522282050801E-02 | 0.12702291898493E-02 |
| 0.28495740446256E+00 | 0.91575808300470E+00 | 0.73430371742652E-01 | 0.78883651229288E-02 |
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| 0.28495740446256E+00 | 0.91575808300470E+00 | 0.61948226408478E+00 | 0.21136564963214E-01 |
| 0.28495740446256E+00 | 0.91575808300470E+00 | 0.91575808300470E+00 | 0.12572859619609E-01 |
| 0.61948226408478E+00 | 0.56522282050801E-02 | 0.56522282050801E-02 | 0.15513849272551E-03 |
| 0.61948226408478E+00 | 0.56522282050801E-02 | 0.73430371742652E-01 | 0.96343957847857E-03 |
| 0.61948226408478E+00 | 0.56522282050801E-02 | 0.28495740446256E+00 | 0.21354156971218E-02 |
| 0.61948226408478E+00 | 0.56522282050801E-02 | 0.61948226408478E+00 | 0.25814985641895E-02 |
| 0.61948226408478E+00 | 0.56522282050801E-02 | 0.91575808300470E+00 | 0.15355768126119E-02 |
| 0.61948226408478E+00 | 0.73430371742652E-01 | 0.56522282050801E-02 | 0.96343957847857E-03 |
| 0.61948226408478E+00 | 0.73430371742652E-01 | 0.73430371742652E-01 | 0.59831432230124E-02 |
| 0.61948226408478E+00 | 0.73430371742652E-01 | 0.28495740446256E+00 | 0.13261338066186E-01 |
| 0.61948226408478E+00 | 0.73430371742652E-01 | 0.61948226408478E+00 | 0.16031597605672E-01 |
| 0.61948226408478E+00 | 0.73430371742652E-01 | 0.91575808300470E+00 | 0.95362243829577E-02 |
| 0.61948226408478E+00 | 0.28495740446256E+00 | 0.56522282050801E-02 | 0.21354156971218E-02 |

| | | | |
|---|---|---|---|
| 0.61948226408478E+00 | 0.28495740446256E+00 | 0.73430371742652E-01 | 0.13261338066186E-01 |
| 0.61948226408478E+00 | 0.28495740446256E+00 | 0.28495740446256E+00 | 0.29393093354222E-01 |
| 0.61948226408478E+00 | 0.28495740446256E+00 | 0.61948226408478E+00 | 0.35533235235315E-01 |
| 0.61948226408478E+00 | 0.28495740446256E+00 | 0.91575808300470E+00 | 0.21136564963214E-01 |
| 0.61948226408478E+00 | 0.61948226408478E+00 | 0.56522282050801E-02 | 0.25814985641895E-01 |
| 0.61948226408478E+00 | 0.61948226408478E+00 | 0.73430371742652E-01 | 0.16031597605672E-01 |
| 0.61948226408478E+00 | 0.61948226408478E+00 | 0.28495740446256E+00 | 0.35533235235315E-01 |
| 0.61948226408478E+00 | 0.61948226408478E+00 | 0.61948226408478E+00 | 0.42956037021088E-01 |
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| 0.61948226408478E+00 | 0.91575808300470E+00 | 0.28495740446256E+00 | 0.21136564963214E-01 |
| 0.61948226408478E+00 | 0.91575808300470E+00 | 0.61948226408478E+00 | 0.25551939221004E-01 |
| 0.61948226408478E+00 | 0.91575808300470E+00 | 0.91575808300470E+00 | 0.15199297775849E-01 |
| 0.91575808300470E+00 | 0.56522282050801E-02 | 0.56522282050801E-02 | 0.92282473241522E-04 |
| 0.91575808300470E+00 | 0.56522282050801E-02 | 0.73430371742652E-01 | 0.57309172958179E-03 |
| 0.91575808300470E+00 | 0.56522282050801E-02 | 0.28495740446256E+00 | 0.12702291898493E-02 |
| 0.91575808300470E+00 | 0.56522282050801E-02 | 0.61948226408478E+00 | 0.15355768126119E-02 |
| 0.91575808300470E+00 | 0.56522282050801E-02 | 0.91575808300470E+00 | 0.91342144448244E-03 |
| 0.91575808300470E+00 | 0.73430371742652E-01 | 0.56522282050801E-02 | 0.57309172958179E-03 |
| 0.91575808300470E+00 | 0.73430371742652E-01 | 0.73430371742652E-01 | 0.35590087584179E-02 |
| 0.91575808300470E+00 | 0.73430371742652E-01 | 0.28495740446256E+00 | 0.78883651229288E-02 |
| 0.91575808300470E+00 | 0.73430371742652E-01 | 0.61948226408478E+00 | 0.95362243829577E-02 |
| 0.91575808300470E+00 | 0.73430371742652E-01 | 0.91575808300470E+00 | 0.56725210873520E-02 |
| 0.91575808300470E+00 | 0.28495740446256E+00 | 0.56522282050801E-02 | 0.12702291898493E-02 |
| 0.91575808300470E+00 | 0.28495740446256E+00 | 0.73430371742652E-01 | 0.78883651229288E-02 |
| 0.91575808300470E+00 | 0.28495740446256E+00 | 0.28495740446256E+00 | 0.17484167232086E-01 |
| 0.91575808300470E+00 | 0.28495740446256E+00 | 0.61948226408478E+00 | 0.21136564963214E-01 |
| 0.91575808300470E+00 | 0.28495740446256E+00 | 0.91575808300470E+00 | 0.12572859619609E-01 |
| 0.91575808300470E+00 | 0.61948226408478E+00 | 0.56522282050801E-02 | 0.15355768126119E-02 |

| | | | |
|---|---|---|---|
| 0.91575808300470E+00 | 0.61948226408478E+00 | 0.73430371742652E-01 | 0.95362243829577E-02 |
| 0.91575808300470E+00 | 0.61948226408478E+00 | 0.28495740446256E+00 | 0.21136564963214E-01 |
| 0.91575808300470E+00 | 0.61948226408478E+00 | 0.61948226408478E+00 | 0.25551939221004E-01 |
| 0.91575808300470E+00 | 0.61948226408478E+00 | 0.91575808300470E+00 | 0.15199297775849E-01 |
| 0.91575808300470E+00 | 0.91575808300470E+00 | 0.56522282050801E-02 | 0.91342144448243E-03 |
| 0.91575808300470E+00 | 0.91575808300470E+00 | 0.73430371742652E-01 | 0.56725210873520E-02 |
| 0.91575808300470E+00 | 0.91575808300470E+00 | 0.28495740446256E+00 | 0.12572859619609E-01 |
| 0.91575808300470E+00 | 0.91575808300470E+00 | 0.61948226408478E+00 | 0.15199297775849E-01 |
| 0.91575808300470E+00 | 0.91575808300470E+00 | 0.91575808300470E+00 | 0.90411397303662E-02 |

**Table 4. 13: 125 cubature points ($x_m$ , $y_m$ , $z_m$) and their corresponding weights $c_m$ for the tetrahedron with corners (0,0,0), (1,0,0), (0,1,0) and (0,0,1)**

| $x_m$ | $y_m$ | $z_m$ | $c_m$ |
|---|---|---|---|
| 0.56522282050801E-02 | 0.56202805213978E-02 | 0.55885134133142E-02 | 0.91660487373935E-05 |
| 0.56522282050801E-02 | 0.56202805213978E-02 | 0.72602627236394E-01 | 0.56922907891682E-04 |
| 0.56522282050801E-02 | 0.56202805213978E-02 | 0.28174521963407E+00 | 0.12616678176089E-03 |
| 0.56522282050801E-02 | 0.56202805213978E-02 | 0.61249914485698E+00 | 0.15252269916492E-03 |
| 0.56522282050801E-02 | 0.56202805213978E-02 | 0.90543519202268E+00 | 0.90726496417081E-04 |
| 0.56522282050801E-02 | 0.73015326524379E-01 | 0.52075812334135E-02 | 0.53042847885419E-04 |
| 0.56522282050801E-02 | 0.73015326524379E-01 | 0.67653783954782E-01 | 0.32940618482384E-03 |
| 0.56522282050801E-02 | 0.73015326524379E-01 | 0.26254050225143E+00 | 0.73011235319256E-03 |
| 0.56522282050801E-02 | 0.73015326524379E-01 | 0.57074910917096E+00 | 0.88263095284167E-03 |
| 0.56522282050801E-02 | 0.73015326524379E-01 | 0.84371763389098E+00 | 0.52502358284391E-03 |
| 0.56522282050801E-02 | 0.28334676018380E+00 | 0.40187399716688E-02 | 0.90727435269522E-04 |
| 0.56522282050801E-02 | 0.28334676018380E+00 | 0.52209068592005E-01 | 0.56343464769359E-03 |
| 0.56522282050801E-02 | 0.28334676018380E+00 | 0.20260500283895E+00 | 0.12488247502632E-02 |
| 0.56522282050801E-02 | 0.28334676018380E+00 | 0.44045251643941E+00 | 0.15097010404456E-02 |
| 0.56522282050801E-02 | 0.28334676018380E+00 | 0.65110492340739E+00 | 0.89802951814240E-03 |
| 0.56522282050801E-02 | 0.61598080895917E+00 | 0.21386164192107E-02 | 0.58367498583671E-04 |
| 0.56522282050801E-02 | 0.61598080895917E+00 | 0.27783626736167E-01 | 0.36247327948323E-03 |

| | | | |
|---|---|---|---|
| 0.56522282050801E-02 | 0.61598080895917E+00 | 0.10781846766405E+00 | 0.80340391664004E-03 |
| 0.56522282050801E-02 | 0.61598080895917E+00 | 0.23439162279237E+00 | 0.97123293608169E-03 |
| 0.56522282050801E-02 | 0.61598080895917E+00 | 0.34649260455877E+00 | 0.57772752500455E-03 |
| 0.56522282050801E-02 | 0.91058200933890E+00 | 0.47346320517390E-03 | 0.76864193768402E-05 |
| 0.56522282050801E-02 | 0.91058200933890E+00 | 0.61509510764516E-02 | 0.47734127838506E-04 |
| 0.56522282050801E-02 | 0.91058200933890E+00 | 0.23869674252292E-01 | 0.10580030979807E-03 |
| 0.56522282050801E-02 | 0.91058200933890E+00 | 0.51891404179037E-01 | 0.12790172339871E-03 |
| 0.56522282050801E-02 | 0.91058200933890E+00 | 0.76709174048143E-01 | 0.76080972295963E-04 |
| 0.73430371742652E-01 | 0.52371829868067E-02 | 0.52075812334135E-02 | 0.49427266034081E-04 |
| 0.73430371742652E-01 | 0.52371829868067E-02 | 0.67653783954782E-01 | 0.30695273311363E-03 |
| 0.73430371742652E-01 | 0.52371829868067E-02 | 0.26254050225143E+00 | 0.68034539913793E-03 |
| 0.73430371742652E-01 | 0.52371829868067E-02 | 0.57074910917096E+00 | 0.82246780961420E-03 |
| 0.73430371742652E-01 | 0.52371829868067E-02 | 0.84371763389098E+00 | 0.48923618052051E-03 |
| 0.73430371742652E-01 | 0.68038352248388E-01 | 0.48526146932012E-02 | 0.28602978543440E-03 |
| 0.73430371742652E-01 | 0.68038352248388E-01 | 0.63042270750031E-01 | 0.17762994281427E-02 |
| 0.73430371742652E-01 | 0.68038352248388E-01 | 0.24464484406144E+00 | 0.39370789475292E-02 |
| 0.73430371742652E-01 | 0.68038352248388E-01 | 0.53184489864962E+00 | 0.47595246507958E-02 |
| 0.73430371742652E-01 | 0.68038352248388E-01 | 0.78620695551754E+00 | 0.28311523369417E-02 |
| 0.73430371742652E-01 | 0.26403287632205E+00 | 0.37448089161908E-02 | 0.48924124321554E-03 |
| 0.73430371742652E-01 | 0.26403287632205E+00 | 0.48650319987778E-01 | 0.30382812727973E-02 |
| 0.73430371742652E-01 | 0.26403287632205E+00 | 0.18879475319253E+00 | 0.67341986639662E-02 |
| 0.73430371742652E-01 | 0.26403287632205E+00 | 0.41042976712825E+00 | 0.81409555083009E-02 |
| 0.73430371742652E-01 | 0.26403287632205E+00 | 0.60672338587242E+00 | 0.48425603192139E-02 |
| 0.73430371742652E-01 | 0.57399345114505E+00 | 0.19928410127134E-02 | 0.31474258569777E-03 |
| 0.73430371742652E-01 | 0.57399345114505E+00 | 0.25889799752959E-01 | 0.19546113847479E-02 |
| 0.73430371742652E-01 | 0.57399345114505E+00 | 0.10046919230525E+00 | 0.43322984917799E-02 |
| 0.73430371742652E-01 | 0.57399345114505E+00 | 0.21841468845988E+00 | 0.52373045450796E-02 |
| 0.73430371742652E-01 | 0.57399345114505E+00 | 0.32287448406548E+00 | 0.31153545973542E-02 |
| 0.73430371742652E-01 | 0.84851362654332E+00 | 0.44119033446378E-03 | 0.41448469921254E-04 |
| 0.73430371742652E-01 | 0.84851362654332E+00 | 0.57316812226058E-02 | 0.25740289007558E-03 |

| | | | |
|---|---|---|---|
| 0.73430371742652E-01 | 0.84851362654332E+00 | 0.22242635651153E-01 | 0.57052064730402E-03 |
| 0.73430371742652E-01 | 0.84851362654332E+00 | 0.48354308667208E-01 | 0.68970094855110E-03 |
| 0.73430371742652E-01 | 0.84851362654332E+00 | 0.71480414496645E-01 | 0.41026123311597E-03 |
| 0.28495740446255E+00 | 0.40415839263304E-02 | 0.40187399716688E-02 | 0.65242747700303E-04 |
| 0.28495740446255E+00 | 0.40415839263304E-02 | 0.52209068592005E-01 | 0.40516988555754E-03 |
| 0.28495740446255E+00 | 0.40415839263304E-02 | 0.20260500283895E+00 | 0.89803881109693E-03 |
| 0.28495740446255E+00 | 0.40415839263304E-02 | 0.44045251643941E+00 | 0.10856368174861E-02 |
| 0.28495740446255E+00 | 0.40415839263304E-02 | 0.65110492340739E+00 | 0.64577945034528E-03 |
| 0.28495740446255E+00 | 0.52505843602145E-01 | 0.37448089161908E-02 | 0.37755212098927E-03 |
| 0.28495740446255E+00 | 0.52505843602145E-01 | 0.48650319987778E-01 | 0.23446705579587E-02 |
| 0.28495740446255E+00 | 0.52505843602145E-01 | 0.18879475319253E+00 | 0.51968451638151E-02 |
| 0.28495740446255E+00 | 0.52505843602145E-01 | 0.41042976712825E+00 | 0.62824528014786E-02 |
| 0.28495740446255E+00 | 0.52505843602145E-01 | 0.60672338587242E+00 | 0.37370498601489E-02 |
| 0.28495740446255E+00 | 0.20375668210452E+00 | 0.28899046607657E-02 | 0.64578613297536E-03 |
| 0.28495740446255E+00 | 0.20375668210452E+00 | 0.37543914690161E-01 | 0.40104548446395E-02 |
| 0.28495740446255E+00 | 0.20375668210452E+00 | 0.14569470683011E+00 | 0.88889728210724E-02 |
| 0.28495740446255E+00 | 0.20375668210452E+00 | 0.31673255524808E+00 | 0.10745856465159E-01 |
| 0.28495740446255E+00 | 0.20375668210452E+00 | 0.46821420795263E+00 | 0.63920577948237E-02 |
| 0.28495740446255E+00 | 0.44295620600059E+00 | 0.15378943651586E-02 | 0.41545229499566E-03 |
| 0.28495740446255E+00 | 0.44295620600059E+00 | 0.19979404729807E-01 | 0.25800378548010E-02 |
| 0.28495740446255E+00 | 0.44295620600059E+00 | 0.77533031352009E-01 | 0.57185250195044E-02 |
| 0.28495740446255E+00 | 0.44295620600059E+00 | 0.16855269261694E+00 | 0.69131102422025E-02 |
| 0.28495740446255E+00 | 0.44295620600059E+00 | 0.24916531049393E+00 | 0.41121896940852E-02 |
| 0.28495740446255E+00 | 0.65480603655607E+00 | 0.34047077765147E-03 | 0.54710937557648E-04 |
| 0.28495740446255E+00 | 0.65480603655607E+00 | 0.44231929185002E-02 | 0.33976533929572E-03 |
| 0.28495740446255E+00 | 0.65480603655607E+00 | 0.17164853501087E-01 | 0.75307290158843E-03 |
| 0.28495740446255E+00 | 0.65480603655607E+00 | 0.37315479938455E-01 | 0.91038790096040E-03 |
| 0.28495740446255E+00 | 0.65480603655607E+00 | 0.55162115779578E-01 | 0.54153450658072E-03 |
| 0.61948226408477E+00 | 0.21507730794732E-02 | 0.21386164192107E-02 | 0.22336117243971E-04 |
| 0.61948226408477E+00 | 0.21507730794732E-02 | 0.27783626736167E-01 | 0.13871154092270E-03 |

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|---|---|---|---|
| 0.61948226408477E+00 | 0.21507730794732E-02 | 0.10781846766405E+00 | 0.30744720112707E-03 |
| 0.61948226408477E+00 | 0.21507730794732E-02 | 0.23439162279237E+00 | 0.37167213360068E-03 |
| 0.61948226408477E+00 | 0.21507730794732E-02 | 0.34649260455877E+00 | 0.22108519375852E-03 |
| 0.61948226408477E+00 | 0.27941558802927E-01 | 0.19928410127134E-02 | 0.12925648807534E-03 |
| 0.61948226408477E+00 | 0.27941558802927E-01 | 0.25889799752959E-01 | 0.80270740161994E-03 |
| 0.61948226408477E+00 | 0.27941558802927E-01 | 0.10046919230525E+00 | 0.17791608564825E-02 |
| 0.61948226408477E+00 | 0.27941558802927E-01 | 0.21841468845988E+00 | 0.21508230002534E-02 |
| 0.61948226408477E+00 | 0.27941558802927E-01 | 0.32287448406548E+00 | 0.12793940593410E-02 |
| 0.61948226408477E+00 | 0.10843134637837E+00 | 0.15378943651586E-02 | 0.22108748158382E-03 |
| 0.61948226408477E+00 | 0.10843134637837E+00 | 0.19979404729807E-01 | 0.13729953560970E-02 |
| 0.61948226408477E+00 | 0.10843134637837E+00 | 0.77533031352009E-01 | 0.30431756188746E-02 |
| 0.61948226408477E+00 | 0.10843134637837E+00 | 0.16855269261694E+00 | 0.36788872074370E-02 |
| 0.61948226408477E+00 | 0.10843134637837E+00 | 0.24916531049393E+00 | 0.21883467108293E-02 |
| 0.61948226408477E+00 | 0.23572398856917E+00 | 0.81840730266856E-03 | 0.14223176517529E-03 |
| 0.61948226408477E+00 | 0.23572398856917E+00 | 0.10632258693631E-01 | 0.88328634292718E-03 |
| 0.61948226408477E+00 | 0.23572398856917E+00 | 0.41260050426136E-01 | 0.19577600545730E-02 |
| 0.61948226408477E+00 | 0.23572398856917E+00 | 0.89697158431248E-01 | 0.23667311131597E-02 |
| 0.61948226408477E+00 | 0.23572398856917E+00 | 0.13259604450068E+00 | 0.14078246912356E-02 |
| 0.61948226408477E+00 | 0.34846219239101E+00 | 0.18118524723671E-03 | 0.18730509656471E-04 |
| 0.61948226408477E+00 | 0.34846219239101E+00 | 0.23538504773955E-02 | 0.11632003129003E-03 |
| 0.61948226408477E+00 | 0.34846219239101E+00 | 0.91344644812954E-02 | 0.25781753859301E-03 |
| 0.61948226408477E+00 | 0.34846219239101E+00 | 0.19857840678845E-01 | 0.31167496174061E-03 |
| 0.61948226408477E+00 | 0.34846219239101E+00 | 0.29355123087404E-01 | 0.18539651772801E-03 |
| 0.91575808300469E+00 | 0.47615453929086E-03 | 0.47346320517390E-03 | 0.65119943092546E-06 |
| 0.91575808300469E+00 | 0.47615453929086E-03 | 0.61509510764510E-02 | 0.40440724556115E-05 |
| 0.91575808300469E+00 | 0.47615453929086E-03 | 0.23869674252292E-01 | 0.89634845764256E-05 |
| 0.91575808300469E+00 | 0.47615453929086E-03 | 0.51891404179037E-01 | 0.10835933535267E-04 |
| 0.91575808300469E+00 | 0.47615453929086E-03 | 0.76709174048143E-01 | 0.64456391766323E-05 |
| 0.91575808300469E+00 | 0.61859152812786E-02 | 0.44119033446378E-03 | 0.37684146514229E-05 |
| 0.91575808300469E+00 | 0.61859152812786E-02 | 0.57316812226058E-02 | 0.23402572498388E-04 |

| | | | |
|---|---|---|---|
| 0.91575808300469E+00 | 0.61859152812786E-02 | 0.22242635651153E-01 | 0.51870632868340E-04 |
| 0.91575808300469E+00 | 0.61859152812786E-02 | 0.48354308667208E-01 | 0.62706275154624E-04 |
| 0.91575808300469E+00 | 0.61859152812786E-02 | 0.71480414496645E-01 | 0.37300157152298E-04 |
| 0.91575808300469E+00 | 0.24005358013931E-01 | 0.34047077765147E-03 | 0.64457058771476E-05 |
| 0.91575808300469E+00 | 0.24005358013931E-01 | 0.44231929185002E-02 | 0.40029060771292E-04 |
| 0.91575808300469E+00 | 0.24005358013931E-01 | 0.17164853501087E-01 | 0.88722413549840E-04 |
| 0.91575808300469E+00 | 0.24005358013931E-01 | 0.37315479938455E-01 | 0.10725629838679E-03 |
| 0.91575808300469E+00 | 0.24005358013931E-01 | 0.55162115779578E-01 | 0.63800262023797E-04 |
| 0.91575808300469E+00 | 0.52186373471091E-01 | 0.18118524723671E-03 | 0.41467030070623E-05 |
| 0.91575808300469E+00 | 0.52186373471091E-01 | 0.23538504773955E-02 | 0.25751815213704E-04 |
| 0.91575808300469E+00 | 0.52186373471091E-01 | 0.91344644812954E-02 | 0.57077611990535E-04 |
| 0.91575808300469E+00 | 0.52186373471091E-01 | 0.19857840678845E-01 | 0.69000978872419E-04 |
| 0.91575808300469E+00 | 0.52186373471091E-01 | 0.29355123087404E-01 | 0.41044494338999E-04 |
| 0.91575808300469E+00 | 0.77145216416258E-01 | 0.40112171175875E-04 | 0.54607956682934E-06 |
| 0.91575808300469E+00 | 0.77145216416258E-01 | 0.52111336166542E-03 | 0.33912580845600E-05 |
| 0.91575808300469E+00 | 0.77145216416258E-01 | 0.20222573772520E-02 | 0.75165541342992E-05 |
| 0.91575808300469E+00 | 0.77145216416258E-01 | 0.43962801422373E-02 | 0.90867430315793E-05 |
| 0.91575808300469E+00 | 0.77145216416258E-01 | 0.64988609179227E-02 | 0.54051519125429E-05 |

**Table 4. 14: The cubature points ($x_m$ , $y_m$ , $z_m$) and weights $c_m$ for integration over a unit sphere $x^2 + y^2 + z^2 \leq 1$, using the derived method in section 4.5**

| $x_m$ | $y_m$ | $z_m$ | $c_m$ |
|---|---|---|---|
| 5.6513371236321E-03 | 1.0029805057610E-04 | 3.5634829549063E-06 | 1.0439581803135E-10 |
| 5.6513371236321E-03 | 8.9867602597929E-05 | 4.4678981226490E-05 | 6.4831790712964E-10 |
| 5.6513371236321E-03 | -2.1866932534506E-05 | 9.7950163969580E-05 | 1.4369642544642E-09 |
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## 4.7 Conclusions

This chapter explicates a general integration formula (derived in section 4.2) to evaluate triple integrals of any function over any bounded three dimensional region. The solver needs to know only the integration limits and have the generalized Gaussian quadrature nodes and weights for one dimension, to evaluate the integral, using this method. The generality and the simplicity of the formula make it unique. Any programming language or any mathematical software can be used to obtain the nodes, weights and then the integral value of a function over a three dimensional bounded region.

As it is found that integration over regions bounded by circular edges should be viewed separately to get a good accuracy, special non-linear transformations are suggested for such regions like cylinder, cone and paraboloid. With the help of these transformations a new set of quadrature rules are derived over these specific regions. Comparison of results over a cylinder using both the derived methods tabulated in the chapter shows that the new quadrature rule is more appropriate for integrating functions over regions with one or more circular edges. An extension of the integration formula given in chapter 3 for a circular disc (2-ball) is provided for a sphere (3-ball), using a combination of few transformations. An integration formula for integration of a function over an irregular domain is also postulated. As a special case of the results obtained over irregular domains, one can obtain effective numerical integration rules for integration over a sphere, an ellipsoid etc.

Tabulated values in this chapter show that the different integration rules proposed here gives a very good accuracy for almost all functions including oscillating functions and functions with end-point singularities. Depending on the integration domain and the integrand function, the integration rule and the value of N can be decided by the solver.