Monte Carlo Simulation on Credit Risk

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Overview

MC is about Sampling and Averaging!

1. To simulate some unknown value μ, we interpret μ probabilistically, i.e. expresses it as the expected value of a function of some random variable X

$$\mu = \mathbb{E}_{\mathbf{X} \sim p(\mathbf{X})} \{ f(\mathbf{X}) \}$$

2. Sample $X_1, \cdots, X_n \overset{i.i.d.}{\sim} p$, and compute the Monte Carlo (MC) estimator

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{X}_i)$$

Gif Demo

link

How to get a better MC estimator?

We evaluate an estimator by computing

$$\mathbb{E}\left\{\hat{\mu}\right\} = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\{f(\mathbf{X}_i)\right\} = \mu \quad \text{(Accuracy)}$$

$$var\left\{\hat{\mu}\right\} = \frac{1}{n^2}\sum_{i=1}^{n} var\left\{f(\mathbf{X}_i)\right\} = \frac{\sigma^2}{n} \quad \text{(Precision)}$$

where $\mathbb{E}\left\{\mathbf{X}_i\right\} = \mu$ and $var\left\{\mathbf{X}_i\right\} = \sigma^2$.

Variance Reduction trick: importance sampling!

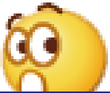
Motivation In the case of estimating rare events, we want to compute $\mu = \mathbb{E}\left\{f(\mathbf{X})\right\}$ where $f(\mathbf{X})$ is close to zero out side of a region $\mathcal{D}$.

1. Consider any *proposal distributions* $q(\mathbf{X})$ to sample from instead of $p(\mathbf{X})$, i.e. $X_1, \cdots, X_n \sim q(\mathbf{X})$.
2. Adjust the estimates such that the MC estimates **remains unbiased**

$$\begin{aligned}\mu = \mathbb{E}_{\mathbf{X}\sim p}\left\{f(\mathbf{X})\right\} &= \int_{\mathcal{D}} f(\mathbf{x})p(\mathbf{x})d\mathbf{x} \\ &= \int_{\mathcal{D}} \frac{f(\mathbf{x})p(\mathbf{x})}{q(\mathbf{x})} q(\mathbf{x})d\mathbf{x} = \mathbb{E}_{\mathbf{X}\sim q}\left\{\frac{f(\mathbf{X})p(\mathbf{X})}{q(\mathbf{X})}\right\}\end{aligned}$$

   where $\frac{p(\mathbf{X})}{q(\mathbf{X})}$ is the *likelihood* $L(\mathbf{X})$, i.e. the adjustment
3. Optimize for $q(\mathbf{X})$ such that **variance is reduced/minimized**

$$var_{\mathbf{X}\sim q}\left\{f(\mathbf{X})L(\mathbf{X})\right\} < var_{\mathbf{X}\sim p}\left\{f(\mathbf{X})\right\}$$



Need to know some basic finance :)

1. **Portfolio** A collection of investments / loans
2. **Credit Risk** Risk of default in a portfolio
3. **Systemic Risk** Overall risk that affects all assets, i.e. government policy, international economic forces, or acts of nature.
4. **Idiosyncratic Risk** Risk endemic to a particular asset, i.e. aspects of a company

A Credit Risk Model

Guassian Copula Factor Model

1. Given probability of default for kth borrower, p_k (i.e. $p_k = 0.01$)
2. Let Y_k be the default indicator for the kth borrower.

$$Y_k = \mathbb{1}_{X_k > x_k} \qquad x_k \text{ picked to match } p_k \text{, i.e. } \mathbb{E}\left\{Y_k\right\} = p_k$$

   where X_k is a weighted sum of the risk factors

$$X_k = \beta_k^T \mathbf{Z} + \sqrt{1 - \beta_k^T \beta_k} \epsilon_k$$

   given systematic risk factors $Z \sim \mathcal{N}\left(0, I_S\right)$ and idiosyncratic risk factors $\epsilon \sim \mathcal{N}\left(0, I_N\right)$

3. L is the total loss from defaults

$$L = c_1 Y_1 + \cdots + c_N Y_N$$

MC and 2 Level IS applied to the Copula Model

We are interested in the *probability that a given portfolio of lenders will result in a loss greater than a specified amount*. And we use *Monte Carlo simulation* to estimate this probability

$$P(L \geq l) = \mathbb{E}_{\mathbf{Z} \sim \mathcal{N}(0, I_S)\ Y_k \sim p_k} \{\mathbb{1}_{L \geq l}\}$$

We can apply *importance sampling* for variance reduction when estimating this probabilistically

The Algorithm with Two level IS!

Find μ, parameter for optimal $\mathbf{Z}$ proposal distribution

Sample $\{Z_i\}_{i=1}^{NZ} \sim \mathcal{N}(\mu, I_S)$

for 1 **to** NZ **do**

Find $p_{k,\theta}$, parameter for optimal Y_k proposal distribution

Sample $\{Y_k^i\}_{i=1,k=1}^{NZ,NE} \sim p_{k,\theta}$

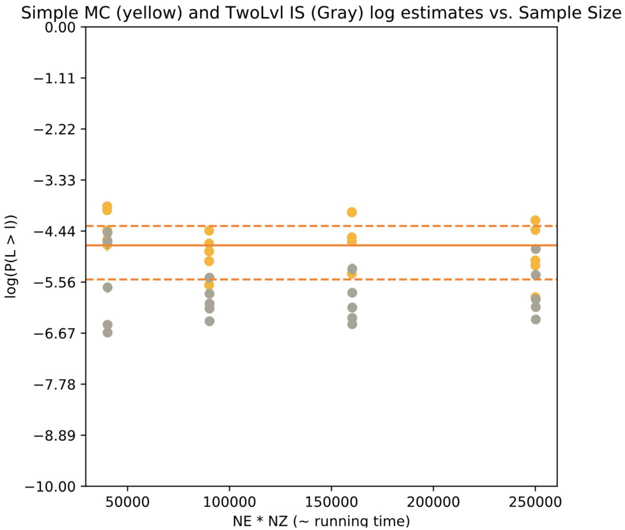
for 1 **to** NE **do**

$\mathcal{L}_{ij} \leftarrow c_1 Y_1^i + \cdots + c_n Y_N^i$

$a_{ij} \leftarrow \mathbb{1}_{\{L_{ij}>l\}} e^{-\theta \mathcal{L}_{ij}+\psi} e^{-\mu^T \mathbf{Z}+\mu^T \mu/2}$

return $mean(a_{ij})$

Problem: Two level IS MC estimator is biased downward



observed from Adam Sturge's master thesis.

Next Steps

1. **Increase Sample Size**
2. **Look into Outer Lever IS: the Selection of μ**

Thanks!