1. **The risk modeling**
   Given
$$y_n = \beta_n^T \mathcal{Z} + \sqrt{1-\beta_n^T\beta_n}\epsilon_n$$
   where $\mathcal{Z} \sim \mathcal{N}(0, I_S)$ and $\epsilon \sim \mathcal{N}(0, I_N)$, the resulting $y_n \sim \mathcal{N}(0,1)$ as shown below
$$\mathbb{E}\{y_n\} = \sum_i \beta_{n,i}\mathbb{E}\{\mathcal{Z}_i\} + \sqrt{1-\beta_n^T\beta_n}\mathbb{E}\{\epsilon_n\} = 0$$
$$\begin{aligned} var\{y_n\} &= \mathbb{E}\left\{(y_n - \mathbb{E}\{y_n\})^2\right\} \\ &= \mathbb{E}\left\{(\beta_n^T\mathcal{Z} + \sqrt{1-\beta_n^T\beta_n}\epsilon_n)^2\right\} \\ &= \beta_n^T\beta_n\mathbb{E}\left\{(\mathcal{Z}-0)^2\right\} + (1-\beta_n^T\beta_n)\mathbb{E}\left\{(\epsilon_n-0)^2\right\} \\ &= \beta_n^T\beta_n var\{\mathcal{Z}\} + (1-\beta_n^T\beta_n)var\{\epsilon_n\} \\ &= 1 \end{aligned}$$

2. **Motivation for threshold between different states $H^c_{c(n)}$**
   The motivation is to model discrete probability in the credit state matrix with a continuous distribution such as the gaussian. In this cawe, we want to set $H^c_{c(n)}$ such that
$$p(H^{c-1}_{c(n)} \leq y_n \leq H^c_{c(n)}) = p^c_{c(n)}$$
   therefore we can write
$$p(y_n \leq H^c_{c(n)}) = \sum_{\gamma=1}^{c} p^\gamma_{c(n)} \qquad \xrightarrow{y_n\sim\mathcal{N}(0,1)} \qquad \Phi(H^c_{c(n)}) = \sum_{\gamma=1}^{c} p^\gamma_{c(n)}$$

3. **Confidence interval for monte carlo estimation**
$$p(L_N(\mathcal{Z},\epsilon) \geq l) \in p(L_N(\mathcal{Z},\epsilon) \geq l) \pm CI$$
   Idea is that for the two naive algorithms that are purported to be equivalent, the CI should be approximately the same

1. **Likelihood Function for Two Level IS**
   Likelihood for the inner sampling conditioned on Z is given by

$$e^{\theta_x(Z)L+\psi(\theta_x(Z),Z)} \qquad \text{where} \qquad \psi(\theta) = \sum_{k=1}^{m} \log(1 + p_k(e^{\theta c_k} - 1))$$

   The likelihood function for the outer sampling of Z consists of the following change of distribution

$$Z \sim \mathcal{N}(0, I) \qquad \longrightarrow \qquad Z \sim \mathcal{N}(\mu, I)$$

   where μ is the twisting parameter for the outer importance sampling such that the resulting shifted normal distribution resembles the zero variance IS distribution, in other words,

$$\mu = \max_z P(L > x | Z = z) e^{\frac{-z^T z}{2}}$$

   Then the likelihood for the outer IS is then

$$\frac{\mathcal{N}(0, I)}{\mathcal{N}(\mu, I)} = \frac{exp(-\frac{1}{2}z^T z)}{exp(-\frac{1}{2}(z-\mu)^T(z-\mu))} = exp\left(-\frac{1}{2}z^T z - \frac{1}{2}z^T z + z^T \mu - \frac{1}{2}\mu^T \mu\right) = e^{-\mu^T Z + \mu^T \mu/2}$$

   Therefore, the estimator for probability of tail event is given by

$$\mathbb{1}_{L>x} e^{\theta_x(Z)L+\psi(\theta_x(Z),Z)} e^{-\mu^T Z + \mu^T \mu/2}$$

Glasserman&Li Algorithm

1. Outer Level IS for sysmatic risk Z
   (a) Find shifted parameter μ for outer IS for Z
   (b) Sample $Z \sim \mathcal{N}(\mu, I)$
2. Inner Level IS for each default indicators Y_k
   (a) Calculate conditional default probabilities $p_k(Z)$ for $k = 1, \cdots, m$
   $$p_k(Z) = P(Y_k = 1|Z) = p(X_k > x_k|Z) = P(a_k Z + b_k \epsilon_k > \Phi^{-1}(1 - p_k)|Z)$$
   (b) Compute the twisted parameters $\theta_x(Z)$
   $$\theta_x(Z) = \begin{cases} \text{solution to } \dfrac{\partial}{\partial \theta}\psi_m(\theta, Z) = x & \psi'(0) = \mathbb{E}_p\left\{L|Z\right\} = \displaystyle\sum_{k=1}^{m} p_k(Z)c_k < x \\ 0 & otherwise \end{cases}$$
   (c) Compute default indicators (bernoulli) from twisted conditional default probabilities
   $$p_{k,\theta_x(Z)}(Z) = \frac{p_k(Z)e^{\theta_x(Z)c_k}}{1 + p_k(Z)(e^{\theta_x(Z)c_k} - 1)} \qquad k = 1, \cdots, m$$
   (d) Compute Loss $L = c_1 Y_1 + \cdots + c_m Y_m$ under twisted distribution
3. Return the estimator of tail proabilities
   $$\mathbb{1}_{L>x} e^{\theta_x(Z)L + \psi(\theta_x(Z), Z)} e^{-\mu^T Z + \mu^T \mu/2}$$
   Therefore,
   $$P(L > x) = \mathbb{E}_{Z \sim \mathcal{N}(\mu, I)\, Y_k \sim p_{k,\theta_x(Z)}} \left\{ \mathbb{1}_{L>x} e^{\theta_x(Z)L + \psi(\theta_x(Z), Z)} e^{-\mu^T Z + \mu^T \mu/2} \right\}$$