Monte Carlo Simulation on Credit Risk

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Overview

1. Monte Carlo Simulation

2. Problem Setup

MC - Monte Carlo Simulation

1. To simulate some unknown value μ, we interpret μ probabilistically, i.e. expresses it as the expected value of a function of some random variable X

$$\mu = \mathbb{E}_{\mathbf{X} \sim p(\mathbf{X})} \{f(\mathbf{X})\}$$

2. Sample $X_1, \cdots, X_n \overset{i.i.d.}{\sim} p$, and compute the Monte Carlo (MC) estimator

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{X}_i)$$

MC - Variance Reduction

We evaluate an estimator by computing

$$\mathbb{E}\left\{\hat{\mu}\right\} = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\{f(\mathbf{X}_i)\right\} = \mu \quad \text{(unbiased)}$$

$$var\left\{\hat{\mu}\right\} = \frac{1}{n^2}\sum_{i=1}^{n} var\left\{f(\mathbf{X}_i)\right\} = \frac{\sigma^2}{n}$$

where $\mathbb{E}\left\{\mathbf{X}_i\right\} = \mu$ and $var\left\{\mathbf{X}_i\right\} = \sigma^2$. Intuitively, estimator is more precise, i.e. smaller CI, with decreased variance. The strategies and trade-offs for variance reduction varies

1. Increase sample size n comes at increased runtime cost
2. Decrease random variable's variance, the tricks!

MC - Importance Sampling

Motivation In the case of estimating rare events, we want to compute $\mu = \mathbb{E}\left\{f(\mathbf{X})\right\}$ where $f(\mathbf{X})$ is close to zero out side of a region $\mathcal{D}$. We can exploit this to do variance reduction with **importance sampling**

1. Find an alternate *proposal distribution* to sample $X_1, \cdots, X_n \sim q(\mathbf{X})$ such that the **variance is reduced**

$$var_{\mathbf{X}\sim q}\left\{\frac{f(\mathbf{X})p(\mathbf{X})}{q(\mathbf{X})}\right\} < var_{\mathbf{X}\sim p}\left\{f(\mathbf{X})\right\}$$

2. Adjust the estimates to account for sampling from the $q(\mathbf{X})$ such that the estimator remains **unbiased**

$$\begin{aligned}\mu = \mathbb{E}_{\mathbf{X}\sim p}\left\{f(\mathbf{X})\right\} &= \int_{\mathcal{D}} f(\mathbf{x})p(\mathbf{x})d\mathbf{x} \\ &= \int_{\mathcal{D}} \frac{f(\mathbf{x})p(\mathbf{x})}{q(\mathbf{x})}q(\mathbf{x})d\mathbf{x} = \mathbb{E}_{\mathbf{X}\sim q}\left\{\frac{f(\mathbf{X})p(\mathbf{X})}{q(\mathbf{X})}\right\}\end{aligned}$$

Problem Setup - Finance Terminologies

1. **Portfolio** A collection of investments
2. **Credit Risk** Risk of default in a portfolio
3. **Systemic Risk** Overall risk that affects all assets, i.e. government policy, international economic forces, or acts of nature.
4. **Idiosyncratic Risk** Risk endemic to a particular asset, i.e. aspects of a company

Problem Setup - A Credit Risk Model

Guassian Copula Factor Model is a model that measure credit risk by modeling a set of underlying risk factors as Gaussian random variables. Let

1. Y_k be the default indicator for the kth borrower, i.e. 1 if kth borrower defaults and 0 otherwise. Given systematic risk factors $Z_i \sim \mathcal{N}(0, I_S)$ and idiosyncratic risk factors $\epsilon_k \sim \mathcal{N}(0, I_N)$, we have

$$Y_k = \mathbb{1}_{X_k > x_k} \qquad x_k \text{ picked to match } p_k$$

   where l is the tail probability supplied, and X_k is a weighted sum of the risk factors

$$X_k = \beta_k^T \mathbf{Z} + \sqrt{1 - \beta_k^T \beta_k} \epsilon_k$$

2. p_k be probability of default for kth borrower
3. c_k be loss resulting from default of kth borrower
4. L be total loss from defaults

$$L = c_1 Y_1 + \cdots + c_N Y_N$$

Problem Setup - MC+IS applied to Risk Model

We are interested in the probability that a given portfolio of lenders will result in a loss greater than a specified amount. And we use Monte Carlo simulation to estimate this probability

$$P(L \geq l) = \mathbb{E}_{\mathbf{Z} \sim \mathcal{N}(0, I_S)\ Y_k \sim p_k} \{\mathbb{1}_{L \geq l}\}$$