1. **The risk modeling**
   Given
$$y_n = \beta_n^T \mathcal{Z} + \sqrt{1 - \beta_n^T \beta_n} \epsilon_n$$
   where $\mathcal{Z} \sim \mathcal{N}(0, I_S)$ and $\epsilon \sim \mathcal{N}(0, I_N)$, the resulting $y_n \sim \mathcal{N}(0, 1)$ as shown below
$$\mathbb{E}\{y_n\} = \sum_i \beta_{n,i} \mathbb{E}\{\mathcal{Z}_i\} + \sqrt{1 - \beta_n^T \beta_n} \mathbb{E}\{\epsilon_n\} = 0$$
$$\begin{aligned} var\{y_n\} &= \mathbb{E}\left\{(y_n - \mathbb{E}\{y_n\})^2\right\} \\ &= \mathbb{E}\left\{(\beta_n^T \mathcal{Z} + \sqrt{1 - \beta_n^T \beta_n} \epsilon_n)^2\right\} \\ &= \beta_n^T \beta_n \mathbb{E}\left\{(\mathcal{Z} - 0)^2\right\} + (1 - \beta_n^T \beta_n) \mathbb{E}\left\{(\epsilon_n - 0)^2\right\} \\ &= \beta_n^T \beta_n var\{\mathcal{Z}\} + (1 - \beta_n^T \beta_n) var\{\epsilon_n\} \\ &= 1 \end{aligned}$$

2. **Motivation for threshold between different states $H_{c(n)}^c$**
   The motivation is to model discrete probability in the credit state matrix with a continuous distribution such as the gaussian. In this cawe, we want to set $H_{c(n)}^c$ such that
$$p(H_{c(n)}^{c-1} \leq y_n \leq H_{c(n)}^c) = p_{c(n)}^c$$
   therefore we can write
$$p(y_n \leq H_{c(n)}^c) = \sum_{\gamma=1}^{c} p_{c(n)}^{\gamma} \qquad \xrightarrow{y_n \sim \mathcal{N}(0,1)} \qquad \Phi(H_{c(n)}^c) = \sum_{\gamma=1}^{c} p_{c(n)}^{\gamma}$$

3. **Confidence interval for monte carlo estimation**
$$p(L_N(\mathcal{Z}, \epsilon) \geq l) \in p(L_N(\mathcal{Z}, \epsilon) \geq l) \pm CI$$
   Idea is that for the two naive algorithms that are purported to be equivalent, the CI should be approximately the same

1. **Likelihood Function for Two Level IS**
Likelihood for the inner sampling conditioned on Z is given by

$$e^{\theta_x(Z)L+\psi(\theta_x(Z),Z)} \qquad \text{where} \qquad \psi(\theta)=\sum_{k=1}^{m}\log(1+p_k(e^{\theta c_k}-1))$$

The likelihood function for the outer sampling of Z consists of the following change of distribution

$$Z\sim\mathcal{N}(0,I) \qquad \longrightarrow \qquad Z\sim\mathcal{N}(\mu,I)$$

where μ is the twisting parameter for the outer importance sampling such that the resulting shifted normal distribution resembles the zero variance IS distribution, in other words,

$$\mu=\max_z P(L>x|Z=z)e^{\frac{-z^Tz}{2}}$$

Then the likelihood for the outer IS is then

$$\frac{\mathcal{N}(0,I)}{\mathcal{N}(\mu,I)}=\frac{exp(-\frac{1}{2}z^Tz)}{exp(-\frac{1}{2}(z-\mu)^T(z-\mu))}=exp\left(-\frac{1}{2}z^Tz-\frac{1}{2}z^Tz+z^T\mu-\frac{1}{2}\mu^T\mu\right)=e^{-\mu^TZ+\mu^T\mu/2}$$

Therefore, the estimator for probability of tail event is given by

$$\mathbb{1}_{L>x}e^{\theta_x(Z)L+\psi(\theta_x(Z),Z)}e^{-\mu^TZ+\mu^T\mu/2}$$