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Shape Optimization with Shape Derivatives

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Abstract

Shape optimization is widely used in practice. The typical problem is to find the optimal shape which minimizes a certain cost functional and satisfies some given constraints. Usually shape optimization problems are solved numerically, by some iterative method. But also some gradient information is needed. In this thesis the shape derivative approach, for getting gradient information of the functional, is presented.

Starting with an introduction in shape derivatives some necessary rules and examples for them are shown. According to this the process is done for a special Dirichlet problem. At the end this theory is worked out for the practical example, which was given by the ACCM (Austrian Center of Competence in Mechatronics).

Zusammenfassung

Formoptimierung ist ein weit verbreitetes Gebiet der Optimierung. Das Problem besteht darin, eine optimale Form (Geometrie) zu finden, die ein gegebenes Zielfunktional minimiert und vorgegebene Nebenbedingungen erfüllt. Solche Probleme werden meistens numerisch gelöst, mit iterativen Verfahren. Für diese benötigt man aber auch Gradienten - Information. Um diese zu bewerkstelligen werden in dieser Arbeit die sogenannten shape derivatives (Form - Ableitungen) präsentiert.

Zuerst folgt eine Einführung in die Theorie der shape derivatives, einige wichtige Beispiele und nützliche Rechenregeln werden bewiesen. Dann wird die gesamte Prozedur an einem speziellen Dirichlet-Problem gezeigt. Zum Schluss wird sie ausgeführt für das vom ACCM (Austrian Center of Competence in Mechatronics) gegebene praktische Beispiel.

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Contents

Chapter 1

Introduction

The whole thesis is mainly based on the books from Sokolowski and Zolesio [2] and Delfour and Zolesio [3].
In this books one can find the exact mathematical approach for introducing shape derivatives and working with it. This means that in this thesis the concentration is about on how to calculate and work with shape derivatives and not about all the assumptions which have to be satisfied. It is assumed that everything is smooth enough, well defined and exists. So in this thesis it is shown how someone can work with shape derivatives.

In the first chapter some basic knowledge about shape derivatives is introduced, especially the definition of the change of the geometry of some domain $\Omega \subseteq \mathbb{R}^N$. Then the term Eulerian derivative (directional derivative of a functional) is defined and some examples for this follow.

In the next chapter it is shown how to get information about the shape derivative, which is needed for computing the Eulerian derivative of a functional. It starts with a simple Dirichlet boundary value problem and it is presented, how to get a Dirichlet boundary value problem for the shape derivative out of the given one.

In the last chapter the main problem, an optimization problem, which was given by the ACCM, is introduced. It is shown, by techniques of the previous chapter, how to get information on the shape derivative. Then the cost functional for the problem is derived and the derivative of this functional is computed.

Chapter 2

Introduction to Shape Derivatives

In this chapter the basic knowledge in shape derivatives is presented and it is shown, how to compute the derivative of a functional (Eulerian derivative) using shape derivatives.

2.1 The Geometry

We want to study the geometric change of a domain $\Omega \subseteq \mathbb{R}^N, N \in \mathbb{N}$.

Let

$$T_t : \mathbb{R}^N \to \mathbb{R}^N \qquad t \in [0, \varepsilon)$$

be a family of transformations, which describes the change of the domain Ω:

$$X \in \Omega \mapsto x = T_t(X) \equiv x(t, X)$$

The transformed geometry is given by:

$$\Omega_t = T_t(\Omega)$$

Remark 2.1 If Ω is a circle with radius r_0 then Ω_t is another circle with another radius r_t. So $T_0(\Omega)$ corresponds to the unchanged domain Ω.

The point X may be thought of as the Lagrangian coordinate while x is the Eulerian coordinate.

Definition 2.2 The *Eulerian velocity field* $V(t,x)$ at the point $x(t)$ is given by:

$$V(t,x) = \frac{\partial x}{\partial t}(t, T_t^{-1}(x))$$

By this definition one can see, that $x(t,X)$ satisfies the following initial value problem

$$\begin{aligned} \frac{d}{dt}x(t,X) &= V(t, x(t,X)) \\ x(0,X) &= X \end{aligned}$$

and determines the family of transformations $T_t(V)(X)$.

2.2 Shape functionals

In this section the so called Eulerian derivative of some examples is calculated and the term Shape Derivative is defined.

Definition 2.3 Let $\Omega \subseteq \mathbb{R}^N$ and let J be a functional with $\Omega \mapsto J(\Omega)$. Then the *Eulerian derivative* of the functional J at Ω in the direction of a vector field V is given by:

$$dJ(\Omega; V) = \lim_{t \downarrow 0} \frac{1}{t}(J(\Omega_t) - J(\Omega)) \tag{2.1}$$

with $\Omega_t = T_t(V)(\Omega)$.

The Eulerian derivative is a directional derivative for the, in this sense often called, shape functional J.

In the next examples the Eulerian derivative of some important functional examples is calculated. Let $\Gamma = \partial\Omega$.

2.2.1 Example 1

$$J(\Omega) = \int_\Omega 1 \; dx$$

So

$$J(\Omega_t) = \int_{\Omega_t} 1 \;\; dx.$$

Transforming the integral to an integral over Ω (substitution rule) leads to

$$J(\Omega_t) = \int_{\Omega} \det(DT_t) \;\; dx$$

with $DT_t = (\frac{\partial T_{t,i}(V)}{\partial X_j}(X))_{i,j=1,\ldots,N}$.

With (2.1) we have

$$dJ(\Omega; V) = \lim_{t \downarrow 0} \frac{1}{t} \int_{\Omega} (\gamma(t) - \gamma(0)) \;\; dx$$

with $\gamma(t) := \det(DT_t)$ and $\gamma(0) = 1$.

Assuming everything smooth enough (exchange limit and integral) we get

$$dJ(\Omega; V) = \int_{\Omega} \lim_{t \downarrow 0} \frac{1}{t} (\gamma(t) - \gamma(0)) \;\; dx = \int_{\Omega} \gamma'(0) \;\; dx.$$

It can be shown that (proof: see [2, page 76])

$$\gamma'(0) = \mathrm{div} V(0).$$

Therefore, by Gauß theorem

$$dJ(\Omega; V) = \int_{\Omega} \mathrm{div} V(0) \;\; dx = \int_{\Gamma} V(0) \cdot n \;\; d\Gamma.$$

2.2.2 Example 2

$$J(\Omega) = \int_{\Omega} Y \;\; dx \quad \text{with} \quad Y : \mathbb{R}^N \to \mathbb{R}$$

So

$$J(\Omega_t) = \int_{\Omega_t} Y \ dx = \int_{\Omega} (Y \circ T_t(V))\gamma(t) \ dx.$$

With (2.1) we have

$$dJ(\Omega;V) = \lim_{t \downarrow 0} \frac{1}{t} \int_{\Omega} ((Y \circ T_t(V))\gamma(t) - (Y \circ T_0(V))\gamma(0)) \ dx.$$

Assuming everything smooth enough we get

$$dJ(\Omega;V) = \int_{\Omega} \lim_{t \downarrow 0} \frac{1}{t} ((Y \circ T_t(V))\gamma(t) - (Y \circ T_0(V))\gamma(0)) \ dx.$$

Product rule, chain rule and Gauß theorem leads to

$$dJ(\Omega;V) = \int_{\Omega} (\nabla Y \cdot V(0) + Y \mathrm{div} V(0)) \ dx = \int_{\Omega} \mathrm{div}(YV(0)) \ dx = \int_{\Gamma} YV(0) \cdot n \ d\Gamma.$$

2.2.3 Example 3

$$J(\Omega) = \int_{\Omega} y(\Omega) \ dx \quad \text{with} \quad y(\Omega) : \Omega \to \mathbb{R}$$

So

$$J(\Omega_t) = \int_{\Omega_t} y(\Omega_t) \ dx = \int_{\Omega} (y(\Omega_t) \circ T_t(V))\gamma(t) \ dx.$$

With (2.1) (and assumptions like above) we have

$$dJ(\Omega;V) = \int_{\Omega} \lim_{t \downarrow 0} \frac{1}{t} ((y(\Omega_t) \circ T_t(V))\gamma(t) - (y(\Omega) \circ T_0(V))\gamma(0)) \ dx.$$

Product rule leads to

$$dJ(\Omega;V) = \int_{\Omega} (\dot{y}(\Omega;V) + y(\Omega) \mathrm{div} V(0)) \ dx$$

with the material derivative

$$\dot{y}(\Omega; V) := \lim_{t \downarrow 0} \frac{1}{t}(y(\Omega_t) \circ T_t - y(\Omega)).$$

Now

$$\begin{aligned} dJ(\Omega; V) &= \int_\Omega (\dot{y}(\Omega; V) + y(\Omega)\mathrm{div}V(0)) \ dx \\ &= \int_\Omega (\dot{y}(\Omega; V) - \nabla y(\Omega) \cdot V(0) + \mathrm{div}(y(\Omega)V(0))) \ dx. \end{aligned}$$

Therefore,

$$dJ(\Omega; V) = \int_\Omega y'(\Omega; V) \ dx + \int_\Gamma y(\Omega)V(0) \cdot n \ d\Gamma \tag{2.2}$$

with the so called shape derivative $y'(\Omega; V)$.

Definition 2.4 The *shape derivative* of $y(\Omega)$ in the direction V is given by:

$$y'(\Omega; V) := \dot{y}(\Omega; V) - \nabla y(\Omega) \cdot V(0). \tag{2.3}$$

2.2.4 Example 4

$$J(\Omega) = \int_\Gamma f \ d\Gamma \quad \text{with} \quad f : \mathbb{R}^N \to \mathbb{R}$$

So

$$J(\Omega_t) = \int_{\Gamma_t} f \ d\Gamma.$$

Transforming the boundary integral to an integral over Γ (substitution rule) leads to

$$J(\Omega_t) = \int_\Gamma (f \circ T_t(V))\omega(t) \ d\Gamma$$

with

$$\omega(t) = \gamma(t)\|DT_t^{-T}n\|.$$

We have (everything smooth enough)

$$\begin{aligned}
dJ(\Omega;V) &= \int_\Gamma \lim_{t\downarrow 0}\frac{1}{t}((f\circ T_t(V))\omega(t) - (f\circ T_0(V))\omega(0))\ d\Gamma \\
&= \int_\Gamma (\nabla f\cdot V(0) + f\omega'(0))\ d\Gamma.
\end{aligned}$$

It can be shown that (proof: see [2, page 80])

$$\omega'(0) = \mathrm{div}V(0) - DV(0)n\cdot n.$$

Definition 2.5 Let Ω be a given domain with the boundary Γ and V a vector field. Then the *tangential divergence* is defined as

$$\mathrm{div}_\Gamma V = \mathrm{div}V - DVn\cdot n.$$

Therefore,

$$dJ(\Omega;V) = \int_\Gamma (\nabla f\cdot V(0) + f\mathrm{div}_\Gamma V(0))\ d\Gamma.$$

2.2.5 Example 5

$$J(\Omega) = \int_\Gamma z(\Gamma)\ d\Gamma \quad \text{with} \quad z(\Gamma):\Gamma\to\mathbb{R}$$

So

$$J(\Omega_t) = \int_{\Gamma_t} z(\Gamma_t)\ d\Gamma = \int_\Gamma (z(\Gamma_t)\circ T_t(V))\omega(t)\ d\Gamma.$$

We have (everything smooth enough)

$$\begin{aligned}
dJ(\Omega;V) &= \int_\Gamma \lim_{t\downarrow 0}\frac{1}{t}((z(\Gamma_t)\circ T_t(V))\omega(t) - (z(\Gamma)\circ T_0(V))\omega(0))\ d\Gamma \\
&= \int_\Gamma (\dot{z}(\Gamma;V) + z(\Gamma)\mathrm{div}_\Gamma V(0))\ d\Gamma
\end{aligned}$$

with the material derivative

$$\dot{z}(\Gamma; V) := \lim_{t \downarrow 0} \frac{1}{t}(z(\Gamma_t) \circ T_t - z(\Gamma)).$$

Now

$$\begin{aligned} dJ(\Omega; V) &= \int_\Gamma (\dot{z}(\Gamma; V) - \nabla_\Gamma z(\Gamma) \cdot V(0) + \nabla_\Gamma z(\Gamma) \cdot V(0) + z(\Gamma)\mathrm{div}_\Gamma V(0)) \ d\Gamma \\ &= \int_\Gamma z'(\Gamma; V) \ d\Gamma + \int_\Gamma \mathrm{div}_\Gamma (z(\Gamma)V(0)) \ d\Gamma \end{aligned}$$

with the shape derivative $z'(\Gamma; V)$ and where $\nabla_\Gamma z(\Gamma)$ is the tangential gradient, defined such that

$$\int_\Gamma \nabla_\Gamma z \cdot V \ d\Gamma = -\int_\Gamma z \mathrm{div}_\Gamma V \ d\Gamma$$
$$\forall V \quad \text{with} \quad V \cdot n = 0 \quad \text{on} \quad \Gamma.$$

Definition 2.6 The *shape derivative* of $z(\Gamma)$ in the direction V is given by:

$$z'(\Gamma; V) := \dot{z}(\Gamma; V) - \nabla_\Gamma z(\Gamma) \cdot V(0). \tag{2.4}$$

Remark 2.7 The tangential gradient of z in a point of Γ can also be defined as:

$$\nabla_\Gamma z(\Gamma) = \nabla z|_\Gamma - \frac{\partial z}{\partial n} n. \tag{2.5}$$

Observe that (see [2, page 115])

$$\int_\Gamma \mathrm{div}_\Gamma (z(\Gamma)V(0)) \ d\Gamma = \int_\Gamma z(\Gamma)\kappa V(0) \cdot n \ d\Gamma$$

with the mean curvature on the manifold Γ

$$\kappa = \mathrm{div}_\Gamma n.$$

So finally we get

$$dJ(\Omega; V) = \int_\Gamma (z'(\Gamma; V) + \kappa z(\Gamma) V(0) \cdot n) \ d\Gamma \tag{2.6}$$

If $z(\Gamma) = y(\Omega)|_\Gamma$ then we have

$$\dot{z}(\Gamma; V) = \dot{y}(\Omega; V)|_\Gamma$$

$$z'(\Gamma; V) = y'(\Omega; V)|_\Gamma + \frac{\partial y}{\partial n}(\Omega) V(0) \cdot n$$

and

$$dJ(\Omega; V) = \int_\Gamma y'(\Omega; V)|_\Gamma \ d\Gamma + \int_\Gamma (\frac{\partial y}{\partial n}(\Omega) + \kappa y(\Omega)) V(0) \cdot n \ d\Gamma \tag{2.7}$$

These examples can be found in [2, page 54-55;77;113;80;115-116]. There one can also find the theory about which assumptions on the data have to be satisfied to guarantee the well definedness of all the steps made.

With the examples above it is shown, how to compute the exact gradient of functionals (Eulerian derivative) which consist of integrals. This information about the gradient can be taken for the optimization part. The only thing left is how to compute the occuring shape derivatives. Before we concentrate on this, some rules for shape derivatives are shown.

2.3 Rules for Shape Derivatives

Theorem 2.8 Let $\Omega \subseteq \mathbb{R}^n$, $u(\Omega), v(\Omega) : \Omega \to \mathbb{R}$ and $y(\Gamma), z(\Gamma) : \Gamma \to \mathbb{R}$ such that the shape derivatives $u'(\Omega; V), v'(\Omega; V)$ and $y'(\Gamma; V), z'(\Gamma; V)$ exist for any vector field V.

Then:

$(uv)'(\Omega; V)$ and $(yz)'(\Gamma; V)$ exist and

$$\begin{aligned} (uv)'(\Omega; V) &= u'(\Omega; V) v(\Omega) + u(\Omega) v'(\Omega; V) \\ (yz)'(\Gamma; V) &= y'(\Gamma; V) z(\Gamma) + y(\Gamma) z'(\Gamma; V) \end{aligned}$$

Proof:
By definition we have

$$(uv)'(\Omega; V) = (\dot{uv})(\Omega; V) - \nabla(u(\Omega)v(\Omega)) \cdot V(0).$$

The ∇ in this definition is the derivative with respect to x, so we can apply the product rule for derivatives to this term. But also the material derivative is only a derivative with respect to the change t, so we can also apply the product rule for derivatives to this term:

$$\begin{aligned} (uv)'(\Omega; V) &= \dot{u}(\Omega; V)v(\Omega) + \dot{v}(\Omega; V)u(\Omega) - \nabla u(\Omega) \cdot V(0)v(\Omega) - \nabla v(\Omega) \cdot V(0)u(\Omega) \\ &= u'(\Omega; V)v(\Omega) + u(\Omega)v'(\Omega; V) \end{aligned}$$

The result for $(yz)'(\Gamma; V)$ follows with the same argument. ■

Theorem 2.9 Let $\Omega \subseteq \mathbb{R}^n$, $u(\Omega) : \Omega \to \mathbb{R}^m$, $z(\Gamma) : \Gamma \to \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^k$, such that the shape derivatives $u'(\Omega; V)$ and $z'(\Gamma; V)$ exist for any vector field V and f is differentiable in $u(\Omega)$ and $z(\Gamma)$.

Then:

$(f \circ u)'(\Omega; V)$ and $(f \circ z)'(\Gamma; V)$ exist and

$$\begin{aligned} (f \circ u)'(\Omega; V) &= (\frac{\partial f}{\partial u} \circ u(\Omega))u'(\Omega; V) \\ (f \circ z)'(\Gamma; V) &= (\frac{\partial f}{\partial z} \circ z(\Gamma))z'(\Gamma; V) \end{aligned}$$

Proof:
By definition we have

$$(f \circ u)'(\Omega; V) = (f \dot{\circ} u)(\Omega; V) - \nabla(f \circ u(\Omega)) \cdot V(0).$$

We can apply the chain rule for derivatives to the ∇ term. But the material derivative is defined by a limit process, like every derivative is introduced and so the proof of the chain rule for derivatives works exactly the same for the material derivative. So the first statement follows.

The result for $(f \circ z)'(\Gamma; V)$ follows with the same argument. ■

Chapter 3

Shape derivative for a Dirichlet BVP

In this chapter it is presented, how to get information about the shape derivative, which is needed for computing the Eulerian derivative of a functional which is of the form as in chapter 2. We start with a simple Dirichlet boundary value problem.

Let $\Omega \subseteq \mathbb{R}^N$, $h(\Omega) \in L^2(\Omega)$ and $z(\Gamma) \in H^{\frac{1}{2}}(\Gamma)$ given.
$y(\Omega) \in H^1(\Omega)$ is a solution of the Dirichlet boundary value problem

$$\begin{aligned} -\Delta y(\Omega) &= h(\Omega) \quad \text{in} \quad \Omega & (3.1)\\ y(\Omega) &= z(\Gamma) \quad \text{on} \quad \Gamma & (3.2) \end{aligned}$$

It is assumed that for any vector field V there exist the shape derivatives $h'(\Omega) \in L^2(\Omega)$, $z'(\Gamma) \in H^{\frac{1}{2}}(\Gamma)$ and $y'(\Omega) \in H^1(\Omega)$.

Now let $\phi \in \mathcal{D}(\mathbb{R}^N)$, then (3.2) can be written as integral identity

$$\int_\Gamma y(\Omega)\phi \ d\Gamma \quad = \quad \int_\Gamma z(\Gamma)\phi \ d\Gamma$$

Taking the derivative with respect to t at both sides leads to (using (2.6) and (2.7))

$$\int_\Gamma (y\phi)'(\Omega; V)|_\Gamma \ d\Gamma + \int_\Gamma (\frac{\partial}{\partial n}(y(\Omega)\phi) + \kappa y(\Omega)\phi)V(0) \cdot n \ d\Gamma =$$
$$\int_\Gamma ((z\phi)'(\Gamma; V) + \kappa z(\Gamma)\phi V(0) \cdot n) \ d\Gamma.$$

Product rule for shape - and normal derivatives gives

$$\int_\Gamma y'(\Omega;V)|_\Gamma \phi \ d\Gamma + \int_\Gamma y(\Omega)\phi'(\Gamma;V) \ d\Gamma + \int_\Gamma (\frac{\partial y}{\partial n}(\Omega)\phi + \frac{\partial \phi}{\partial n} y(\Omega))V(0)\cdot n \ d\Gamma +$$
$$\int_\Gamma \kappa y(\Omega)\phi V(0)\cdot n \ d\Gamma = \int_\Gamma (z'(\Gamma;V)\phi + z(\Gamma)\phi'(\Gamma;V) + \kappa z(\Gamma)\phi V(0)\cdot n) \ d\Gamma.$$

For $\phi \in \mathcal{D}(\mathbb{R}^N)$ we have

$$\phi'(\Gamma;V) = \frac{\partial \phi}{\partial n} V(0)\cdot n. \tag{3.3}$$

If it is assumed that $\frac{\partial \phi}{\partial n} = 0$ on Γ, then

$$\int_\Gamma y'(\Omega;V)|_\Gamma \phi \ d\Gamma + \int_\Gamma (\frac{\partial y}{\partial n}(\Omega)\phi + \kappa y(\Omega)\phi)V(0)\cdot n \ d\Gamma =$$
$$\int_\Gamma (z'(\Gamma;V)\phi + \kappa z(\Gamma)\phi V(0)\cdot n) \ d\Gamma.$$

With (3.2) we get

$$\int_\Gamma y'(\Omega;V)|_\Gamma \phi \ d\Gamma + \int_\Gamma \frac{\partial y}{\partial n}(\Omega)\phi V(0)\cdot n \ d\Gamma =$$
$$\int_\Gamma z'(\Gamma;V)\phi \ d\Gamma.$$

In strong form

$$y'(\Omega;V)|_\Gamma = z'(\Gamma;V) - \frac{\partial y}{\partial n}(\Omega)V(0)\cdot n \quad \text{on} \quad \Gamma.$$

Now consider the weak form of (3.1)

$$\int_\Omega \nabla y(\Omega)\cdot\nabla\phi \ dx \ = \ \int_\Omega h(\Omega)\phi \ dx \quad \forall \phi \in \mathcal{D}(\Omega).$$

Taking the derivative with respect to t at both sides (and exchange limit and integral) leads to

$$\int_\Omega \lim_{t\downarrow 0} \frac{1}{t}((A(t)\nabla(y(\Omega_t)\circ T_t))\cdot\nabla(\phi\circ T_t) - \nabla y(\Omega)\cdot\nabla\phi) \ dx =$$
$$\int_\Omega \lim_{t\downarrow 0} \frac{1}{t}((h(\Omega_t)\circ T_t)(\phi\circ T_t)\gamma(t) - h(\Omega)\phi) \ dx$$

with

$$A(t) = \gamma(t) DT_t^{-1} DT_t^{-T}.$$

So the following integral identity holds $\forall \psi = \phi \circ T_t \in \mathcal{D}(\Omega)$

$$\int_\Omega \lim_{t \downarrow 0} \frac{1}{t}(A(t)\nabla(y(\Omega_t) \circ T_t) - \nabla y(\Omega)) \cdot \nabla\psi \ dx =$$
$$\int_\Omega \lim_{t \downarrow 0} \frac{1}{t}((h(\Omega_t) \circ T_t)\gamma(t) - h(\Omega))\psi \ dx.$$

Product rule yields

$$\int_\Omega (A'(0)\nabla y(\Omega)) \cdot \nabla\psi \ dx + \int_\Omega (\dot{\nabla y})(\Omega; V) \cdot \nabla\psi \ dx =$$
$$\int_\Omega \dot{h}(\Omega; V)\psi \ dx + \int_\Omega h(\Omega)\psi \mathrm{div} V(0) \ dx.$$

with

$$A'(0) = (\frac{d}{dt}A(t))_{t=0} = \mathrm{div} V(0) I - DV(0) - DV(0)^T$$

Assuming everything smooth enough we get (exchange material derivative (limit) and gradient)

$$\int_\Omega (A'(0)\nabla y(\Omega)) \cdot \nabla\psi \ dx + \int_\Omega \nabla\dot{y}(\Omega; V) \cdot \nabla\psi \ dx =$$
$$\int_\Omega \dot{h}(\Omega; V)\psi \ dx + \int_\Omega h(\Omega)\psi \mathrm{div} V(0) \ dx.$$

With shape derivatives we have

$$\int_\Omega (A'(0)\nabla y(\Omega)) \cdot \nabla\psi \ dx + \int_\Omega \nabla y'(\Omega; V) \cdot \nabla\psi \ dx + \int_\Omega \nabla(\nabla y(\Omega) \cdot V(0)) \cdot \nabla\psi \ dx =$$
$$\int_\Omega h'(\Omega; V)\psi \ dx + \int_\Omega \mathrm{div}(h(\Omega)V(0))\psi \ dx.$$

Because of

$$\int_\Omega (A'(0)\nabla y(\Omega)) \cdot \nabla\psi \;\, dx + \int_\Omega \nabla(\nabla y(\Omega) \cdot V(0)) \cdot \nabla\psi \;\, dx +$$
$$\int_\Omega \operatorname{div}(h(\Omega)V(0))\psi \;\, dx = 0 \tag{3.4}$$

we have

$$\int_\Omega \nabla y'(\Omega; V) \cdot \nabla\psi \;\, dx = \int_\Omega h'(\Omega; V)\psi \;\, dx.$$

In strong form

$$-\Delta y'(\Omega; V) = h'(\Omega; V) \quad \text{in} \quad \Omega.$$

So in summary

$$\begin{aligned} -\Delta y'(\Omega; V) &= h'(\Omega; V) \quad \text{in} \quad \Omega && (3.5)\\ y'(\Omega; V)|_\Gamma &= z'(\Gamma; V) - \frac{\partial y}{\partial n}(\Omega)V(0) \cdot n \quad \text{on} \quad \Gamma. && (3.6) \end{aligned}$$

So by differentiating the given BVP we get another BVP but now for the shape derivative. So everything appearing in the gradient of the functional can now be computed.

This example can be found in [2, page 118-119].

In the example some steps where made, which have to be proven:

Showing (3.3):

$$\begin{aligned} \phi'(\Gamma; V) &= \dot\phi(\Gamma; V) - \nabla_\Gamma \phi \cdot V(0)\\ \dot\phi(\Gamma; V) &= \nabla\phi|_\Gamma \cdot V(0) \quad \text{for} \quad \phi \in \mathcal{D}(\mathbb{R}^N) \end{aligned}$$

With (2.5) it follows

$$\begin{aligned} \phi'(\Gamma; V) &= \nabla\phi|_\Gamma \cdot V(0) - \nabla\phi|_\Gamma \cdot V(0) + \frac{\partial\phi}{\partial n} n \cdot V(0)\\ &= \frac{\partial\phi}{\partial n} n \cdot V(0). \quad \blacksquare \end{aligned}$$

Showing (3.4):

$$\int_\Omega (A'(0)\nabla y(\Omega)) \cdot \nabla\psi \ \ dx + \int_\Omega \nabla(\nabla y(\Omega) \cdot V(0)) \cdot \nabla\psi \ \ dx +$$
$$\int_\Omega \mathrm{div}(\Delta y(\Omega) V(0))\psi \ \ dx =$$
$$\int_\Omega (\nabla(\nabla y(\Omega) \cdot V(0)) + (\mathrm{div} V(0) I - DV(0) - DV(0)^T)\nabla y(\Omega)) \cdot \nabla\psi \ \ dx +$$
$$\int_\Omega \mathrm{div}(\Delta y(\Omega) V(0))\psi \ \ dx$$

Since

$$\nabla(\nabla y(\Omega) \cdot V(0)) = \nabla^2 y(\Omega) V(0) + DV(0)^T \nabla y(\Omega)$$

the problem writes

$$\int_\Omega ((\nabla^2 y(\Omega) V(0) + (\mathrm{div} V(0) I - DV(0))\nabla y(\Omega)) \cdot \nabla\psi + \mathrm{div}(\Delta y(\Omega) V(0))\psi) \ \ dx.$$

Using the product rule for $\nabla \times (A \times B)$

$$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$$

we have

$$\nabla \times (\nabla y(\Omega) \times V(0)) = \mathrm{div} V(0) \nabla y(\Omega) - \Delta y(\Omega) V(0) + \nabla^2 y(\Omega) V(0) - DV(0)\nabla y(\Omega).$$

This reduces the problem to

$$\int_\Omega ((\nabla \times (\nabla y(\Omega) \times V(0)) + \Delta y(\Omega) V(0)) \cdot \nabla\psi + \mathrm{div}(\Delta y(\Omega) V(0))\psi) \ \ dx$$

or, equivalently,

$$\int_\Omega (\nabla \times (\nabla y(\Omega) \times V(0)) \cdot \nabla\psi + \mathrm{div}(\Delta y(\Omega)\psi V(0))) \ \ dx$$

or, equivalently,

$$\int_\Omega \nabla \times (\nabla y(\Omega) \times V(0)) \cdot \nabla \psi \;\; dx + \int_\Omega \mathrm{div}(\Delta y(\Omega) \psi V(0)) \;\; dx.$$

Integration by parts leads to

$$-\int_\Omega \mathrm{div}(\nabla \times (\nabla y(\Omega) \times V(0)))\psi \;\; dx + \int_\Gamma (\nabla \times (\nabla y(\Omega) \times V(0))) \cdot n\psi \;\; d\Gamma + \int_\Gamma \Delta y(\Omega) \psi V(0) \cdot n \;\; d\Gamma.$$

Since $\psi \in \mathcal{D}(\Omega)$ and $\mathrm{div}(\nabla \times A) = 0$

$$-\int_\Omega \mathrm{div}(\nabla \times (\nabla y(\Omega) \times V(0)))\psi \;\; dx + \int_\Gamma (\nabla \times (\nabla y(\Omega) \times V(0))) \cdot n\psi \;\; d\Gamma + \int_\Gamma \Delta y(\Omega) \psi V(0) \cdot n \;\; d\Gamma = 0. \quad \blacksquare$$

Chapter 4

The main problem

In this chapter the optimization problem, given by the ACCM, is introduced. Then information about the shape derivative is derived. At last the cost functional and its derivative are computed.

4.1 Introduction

The behaviour of an electric machine can be described by the Maxwell equations:

$$\begin{aligned}
\mathrm{curl} H &= j + \frac{\partial D}{\partial t} \\
\mathrm{curl} E &= -\frac{\partial B}{\partial t} \\
\mathrm{div} B &= 0 \\
\mathrm{div} D &= \rho
\end{aligned}$$

where

H	magnetic field strength
E	electric field strength
B	magnetic flux density
D	electric induction density
j	electric current density
ρ	electric charge density

The given problem is a motor in $2D$, for which a mathematical formulation can be derived by the use of this equations (reduction to $2D$).

So the resulting BVP reads as follows:

Find u:

$$-\mathrm{div}(\nu(|\nabla u|)\nabla u) = J_z - \frac{\partial}{\partial y}H_{0x} + \frac{\partial}{\partial x}H_{0y} \quad \text{in} \quad \Omega \tag{4.1}$$

$$u = 0 \quad \text{on} \quad \Gamma \tag{4.2}$$

where

$$u = A_3$$

$$A = \begin{pmatrix} 0 \\ 0 \\ A_3 \end{pmatrix} \qquad \text{vector potential}$$

$$\nu(|\nabla u|) \qquad \text{relative reluctivity (non-linear function)}$$

and the following identity holds

$$B = \mathrm{curl} A_3.$$

Here, B (contrary to above) denotes the 2D magnetic flux density, and curl denotes the curl operator in 2D.
Or, by variational formulation (under appropriate conditions) we have $\forall \phi \in \mathcal{D}(\Omega)$

$$\int_\Omega \nu(|\nabla u|)\nabla u \cdot \nabla \phi \; dx = \int_\Omega (J_z - \frac{\partial}{\partial y}H_{0x} + \frac{\partial}{\partial x}H_{0y})\phi \; dx. \tag{4.3}$$

Now the task is to maximize the torque of the machine by doing shape optimization (finding the optimal shape respectively the optimal vector of design $d \in \mathbb{R}^M$, where M is the number of design parameters, of the motor). So the functional, which describes the torque, is needed and has to be differentiated then. How this functional is derived will be shown later, first we concentrate on how to get, like in chapter 3, a BVP for the shape derivative. Because, as seen in the examples in chapter 2, for computing the derivative of a functional, the shape derivatives are needed.

4.2 Shape derivative for the derived problem

It is assumed that for any vector field V there exist the shape derivative $u'(\Omega; V)$ and the shape derivative of the right hand side.

Now let $\phi \in \mathcal{D}(\mathbb{R}^N)$, then (4.2) can be written as integral identity

$$\int_\Gamma u(\Omega)\phi \; d\Gamma \;\; = \;\; 0.$$

Taking the derivative with respect to t leads to ((3.6) with $z = 0$)

$$u'(\Omega; V)|_\Gamma = -\frac{\partial u}{\partial n}(\Omega)V(0) \cdot n \quad \text{on} \quad \Gamma.$$

Now, taking the derivative with respect to t at both sides in (4.3) (and exchange limit and integral) leads to

$$\int_\Omega \lim_{t\downarrow 0} \frac{1}{t}((A(t)\nu(|\nabla(u(\Omega_t) \circ T_t)|)\nabla(u(\Omega_t) \circ T_t)) \cdot \nabla(\phi \circ T_t) -$$
$$(\nu(|\nabla u(\Omega)|)\nabla u(\Omega)) \cdot \nabla\phi) \; dx =$$
$$\int_\Omega \lim_{t\downarrow 0} \frac{1}{t}((J_z(\Omega_t) \circ T_t - \frac{\partial}{\partial y}(H_{0x}(\Omega_t) \circ T_t) + \frac{\partial}{\partial x}(H_{0y}(\Omega_t) \circ T_t))(\phi \circ T_t)\gamma(t) -$$
$$(J_z(\Omega) - \frac{\partial}{\partial y}H_{0x}(\Omega) + \frac{\partial}{\partial x}H_{0y}(\Omega))\phi) \; dx$$

with $A(t)$ like in the previous chapter.

With $\psi = \phi \circ T_t \in \mathcal{D}(\Omega)$ the following identity holds

$$\int_\Omega \lim_{t\downarrow 0} \frac{1}{t}(A(t)\nu(|\nabla(u(\Omega_t) \circ T_t)|)\nabla(u(\Omega_t) \circ T_t) - \nu(|\nabla u(\Omega)|)\nabla u(\Omega))\nabla\psi \; dx =$$
$$\int_\Omega \lim_{t\downarrow 0} \frac{1}{t}((J_z(\Omega_t) \circ T_t - \frac{\partial}{\partial y}(H_{0x}(\Omega_t) \circ T_t) + \frac{\partial}{\partial x}(H_{0y}(\Omega_t) \circ T_t))\gamma(t) -$$
$$(J_z(\Omega) - \frac{\partial}{\partial y}H_{0x}(\Omega) + \frac{\partial}{\partial x}H_{0y}(\Omega)))\psi \; dx.$$

Product rule yields

$$\int_\Omega (A'(0)\nu(|\nabla(u(\Omega)|)\nabla u(\Omega)) \cdot \nabla\psi \; dx + \int_\Omega (\nu(|\dot{\nabla u}|))(\Omega; V)\nabla u(\Omega) \cdot \nabla\psi \; dx +$$
$$\int_\Omega \nu(|\nabla u(\Omega)|)(\dot{\nabla u})(\Omega; V) \cdot \nabla\psi \; dx =$$
$$\int_\Omega (J_z(\Omega) - \frac{\partial}{\partial y}H_{0x}(\Omega) + \frac{\partial}{\partial x}H_{0y}(\Omega))\text{div}V(0)\psi \; dx +$$
$$\int_\Omega ((\dot{J_z})(\Omega; V) - (\dot{\overline{\frac{\partial}{\partial y}H_{0x}}})(\Omega; V) + (\dot{\overline{\frac{\partial}{\partial x}H_{0y}}})(\Omega; V))\psi \; dx.$$

Assuming everything smooth enough we get

$$
\begin{aligned}
&\int_\Omega (A'(0)\nu(|\nabla(u(\Omega)|)\nabla u(\Omega)) \cdot \nabla\psi \;\; dx + \int_\Omega (\nu(|\dot{\overline{\nabla u}}|))(\Omega;V)\nabla u(\Omega) \cdot \nabla\psi \;\; dx + \\
&\int_\Omega \nu(|\nabla u(\Omega)|)\nabla \dot{u}(\Omega;V) \cdot \nabla\psi \;\; dx = \int_\Omega (J_z(\Omega) - \frac{\partial}{\partial y}H_{0x}(\Omega) + \frac{\partial}{\partial x}H_{0y}(\Omega))\mathrm{div}V(0)\psi \;\; dx + \\
&\int_\Omega ((\dot{J_z})(\Omega;V) - (\dot{\overline{\frac{\partial}{\partial y}H_{0x}}})(\Omega;V) + (\dot{\overline{\frac{\partial}{\partial x}H_{0y}}})(\Omega;V))\psi \;\; dx.
\end{aligned}
$$

Written with shape derivatives

$$
\begin{aligned}
&\int_\Omega (A'(0)\nu(|\nabla(u(\Omega)|)\nabla u(\Omega)) \cdot \nabla\psi \;\; dx + \int_\Omega (\nu(|\nabla u|))'(\Omega;V)\nabla u(\Omega) \cdot \nabla\psi \;\; dx + \\
&\int_\Omega \nu(|\nabla u(\Omega)|)\nabla u'(\Omega;V) \cdot \nabla\psi \;\; dx + \int_\Omega (\nabla\nu(|\nabla u(\Omega)|) \cdot V(0))\nabla u(\Omega) \cdot \nabla\psi \;\; dx + \\
&\int_\Omega \nu(|\nabla u(\Omega)|)\nabla(\nabla u(\Omega) \cdot V(0)) \cdot \nabla\psi \;\; dx = \\
&\int_\Omega (J_z'(\Omega;V) - (\frac{\partial}{\partial y}H_{0x})'(\Omega;V) + (\frac{\partial}{\partial x}H_{0y})'(\Omega;V))\psi \;\; dx + \\
&\int_\Omega \mathrm{div}((J_z(\Omega) - (\frac{\partial}{\partial y}H_{0x})(\Omega) + (\frac{\partial}{\partial x}H_{0y})(\Omega))V(0))\psi \;\; dx.
\end{aligned}
$$

It can be shown that (the very same as in (3.4))

$$
\begin{aligned}
&\int_\Omega (A'(0)\nu(|\nabla(u(\Omega)|)\nabla u(\Omega)) \cdot \nabla\psi \;\; dx + \int_\Omega (\nabla\nu(|\nabla u(\Omega)|) \cdot V(0))\nabla u(\Omega) \cdot \nabla\psi \;\; dx + \\
&\int_\Omega \nu(|\nabla u(\Omega)|)\nabla(\nabla u(\Omega) \cdot V(0)) \cdot \nabla\psi \;\; dx - \\
&\int_\Omega \mathrm{div}((J_z(\Omega) - (\frac{\partial}{\partial y}H_{0x})(\Omega) + (\frac{\partial}{\partial x}H_{0y})(\Omega))V(0))\psi \;\; dx = 0
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
&\int_\Omega (\nu(|\nabla u|))'(\Omega;V)\nabla u(\Omega) \cdot \nabla\psi \;\; dx + \int_\Omega \nu(|\nabla u(\Omega)|)\nabla u'(\Omega;V) \cdot \nabla\psi \;\; dx = \\
&\int_\Omega (J_z'(\Omega;V) - (\frac{\partial}{\partial y}H_{0x})'(\Omega;V) + (\frac{\partial}{\partial x}H_{0y})'(\Omega;V))\psi \;\; dx.
\end{aligned}
$$

In strong form

$$
\begin{aligned}
&-\mathrm{div}((\nu(|\nabla u|))'(\Omega;V)\nabla u(\Omega)) - \mathrm{div}(\nu(|\nabla u(\Omega)|)\nabla u'(\Omega;V)) = \\
&J_z'(\Omega;V) - (\frac{\partial}{\partial y}H_{0x})'(\Omega;V) + (\frac{\partial}{\partial x}H_{0y})'(\Omega;V) \quad \text{in} \quad \Omega.
\end{aligned}
$$

Using chain rule for shape derivatives:

$$
\begin{aligned}
(\nu(|\nabla u|))'(\Omega;V) &= (\nu' \circ |\nabla u|)\frac{\partial |\nabla u|}{\partial \nabla u}(\nabla u)'(\Omega;V) \\
&= (\nu' \circ |\nabla u|)\frac{1}{|\nabla u|}\nabla u \cdot (\nabla u)'(\Omega;V).
\end{aligned}
$$

Now

$$
\begin{aligned}
(\nabla u)'(\Omega;V) &= \lim_{t\downarrow 0}(\gamma(t)DT_t^{-T}\nabla(u(\Omega_t)\circ T_t) - \nabla u(\Omega)) - \nabla^2 u(\Omega)V(0) \\
&= (\gamma DT_t^{-T})'(0)\nabla u(\Omega) + (\dot{\nabla u})(\Omega;V) - \nabla^2 u(\Omega)V(0)
\end{aligned}
$$

and assuming everything smooth enough

$$
\begin{aligned}
(\nabla u)'(\Omega;V) &= (\gamma DT_t^{-T})'(0)\nabla u(\Omega) + \nabla \dot{u}(\Omega;V) - \nabla^2 u(\Omega)V(0) \\
&= (\mathrm{div}V(0)I - DV(0)^T)\nabla u(\Omega) + \nabla \dot{u}(\Omega;V) - \nabla^2 u(\Omega)V(0) \\
&= \mathrm{div}V(0)\nabla u(\Omega) + \nabla \dot{u}(\Omega;V) - \nabla(\nabla u(\Omega)\cdot V(0)) \\
&= \mathrm{div}V(0)\nabla u(\Omega) + \nabla u'(\Omega;V).
\end{aligned}
$$

So in summary

$$
\begin{aligned}
&-\mathrm{div}(((\nu' \circ |\nabla u|)\frac{1}{|\nabla u|}\nabla u \cdot (\mathrm{div}V(0)\nabla u(\Omega) + \nabla u'(\Omega;V)))\nabla u(\Omega)) - \\
&\mathrm{div}(\nu(|\nabla u(\Omega)|)\nabla u'(\Omega;V)) = \\
&J_z'(\Omega;V) - (\frac{\partial}{\partial y}H_{0x})'(\Omega;V) + (\frac{\partial}{\partial x}H_{0y})'(\Omega;V) \quad \text{in} \quad \Omega \qquad (4.4) \\
&u'(\Omega;V)|_\Gamma = -\frac{\partial u}{\partial n}(\Omega)V(0)\cdot n \quad \text{on} \quad \Gamma. \qquad (4.5)
\end{aligned}
$$

This is now a BVP for the shape derivative $u'(\Omega;V)$. The only thing left is the functional and its derivative.

4.3 The cost functional

The derivation of the functional is based on the paper [1].

For a plane problem the normal stress σ_n and the stress in rotor peripheral direction σ_t can be calculated as follows ([1, page 1-2])

$$\begin{pmatrix} \sigma_n \\ \sigma_t \end{pmatrix} = \frac{1}{2\mu_0} \begin{pmatrix} B_n^2 - B_t^2 \\ 2B_nB_t \end{pmatrix}$$

where B_n denotes the normal component, B_t the tangential component of the magnetic flux density and μ_0 is the absolute permeability of vacuum.

The force acting on the surface of a body inside a closed area Ω can be calculated via the integral

$$F = \int_{\partial\Omega} \sigma \ dS.$$

with $\sigma = (\sigma_n, \sigma_t)^T$.

For the area Ω a cylinder with radius r is chosen and for the torque only the force in peripheral direction is of importance. So now it follows for the torque M (with $\Gamma = \partial\Omega$):

$$\begin{aligned} M &= F_t r \\ &= \int_\Gamma r\sigma_t \ d\Gamma = \frac{1}{\mu_0} \int_\Gamma rB_n(\Gamma)B_t(\Gamma) \ d\Gamma \end{aligned}$$

where $r = \sqrt{x_1^2 + x_2^2}$. We only consider a constant change of the geometry, which means that it always stays a circle. Because of

$$B = \text{curl} A_3 = \text{curl} u = \begin{pmatrix} \frac{\partial u}{\partial x_2} \\ -\frac{\partial u}{\partial x_1} \end{pmatrix}$$

B_n, B_t can be calculated (with $n = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2}} \end{pmatrix}$)

$$\begin{aligned} B_n &= B \cdot n = \frac{1}{\sqrt{x_1^2 + x_2^2}} (\frac{\partial u}{\partial x_2} x_1 - \frac{\partial u}{\partial x_1} x_2) \\ B_t &= B \cdot n^\perp = \frac{1}{\sqrt{x_1^2 + x_2^2}} (\frac{\partial u}{\partial x_2} x_2 + \frac{\partial u}{\partial x_1} x_1). \end{aligned}$$

So the functional (torque) becomes

$$M = J(\Omega) = \frac{1}{\mu_0} \int_\Gamma \nabla u(\Omega)|_\Gamma^T Q \nabla u(\Omega)|_\Gamma \; d\Gamma \tag{4.6}$$

with

$$Q = \frac{1}{\sqrt{x_1^2 + x_2^2}} \begin{pmatrix} -x_1 x_2 & \frac{x_1^2 - x_2^2}{2} \\ \frac{x_1^2 - x_2^2}{2} & x_1 x_2 \end{pmatrix}.$$

Now the derivative with respect to t (Eulerian derivative) of the functional has to be calculated. With (2.1) (and exchanging limit and integral) we have

$$\begin{aligned} dJ(\Omega; V) = \quad & \frac{1}{\mu_0} \int_\Gamma \lim_{t \downarrow 0} \frac{1}{t} ((\nabla(u(\Omega_t)|_{\Gamma_t} \circ T_t)^T DT_t^{-1})(Q \circ T_t) \\ & (DT_t^{-T} \nabla(u(\Omega_t)|_{\Gamma_t} \circ T_t))\omega(t) - \nabla u(\Omega)|_\Gamma^T Q \nabla u(\Omega)|_\Gamma) \; d\Gamma \end{aligned}$$

with $\omega(t)$ like in chapter 2.

Product- and chain rule yields

$$\begin{aligned} dJ(\Omega; V) = \quad & \frac{1}{\mu_0} (\int_\Gamma (\dot{\nabla u})(\Omega; V)|_\Gamma^T Q \nabla u(\Omega)|_\Gamma \; d\Gamma + \\ & \int_\Gamma (\nabla u(\Omega)|_\Gamma^T (DT_t^{-1})'(0)) Q \nabla u(\Omega)|_\Gamma \; d\Gamma + \\ & \int_\Gamma \nabla u(\Omega)|_\Gamma^T \begin{pmatrix} \nabla Q_{11} \cdot V(0) & \nabla Q_{12} \cdot V(0) \\ \nabla Q_{21} \cdot V(0) & \nabla Q_{22} \cdot V(0) \end{pmatrix} \nabla u(\Omega)|_\Gamma \; d\Gamma + \\ & \int_\Gamma \nabla u(\Omega)|_\Gamma^T Q ((DT_t^{-T} \omega)'(0) \nabla u(\Omega)|_\Gamma) \; d\Gamma + \\ & \int_\Gamma \nabla u(\Omega)|_\Gamma^T Q (\dot{\nabla u})(\Omega; V)|_\Gamma \; d\Gamma). \end{aligned}$$

Assuming everything smooth enough we get

$$\begin{aligned} dJ(\Omega; V) = \quad & \frac{1}{\mu_0} (\int_\Gamma \nabla \dot{u}(\Omega; V)|_\Gamma^T Q \nabla u(\Omega)|_\Gamma \; d\Gamma + \\ & \int_\Gamma (\nabla u(\Omega)|_\Gamma^T (DT_t^{-1})'(0)) Q \nabla u(\Omega)|_\Gamma \; d\Gamma + \\ & \int_\Gamma \nabla u(\Omega)|_\Gamma^T \begin{pmatrix} \nabla Q_{11} \cdot V(0) & \nabla Q_{12} \cdot V(0) \\ \nabla Q_{21} \cdot V(0) & \nabla Q_{22} \cdot V(0) \end{pmatrix} \nabla u(\Omega)|_\Gamma \; d\Gamma + \end{aligned}$$

$$\int_\Gamma \nabla u(\Omega)|_\Gamma^T Q((DT_t^{-T}\omega)'(0)\nabla u(\Omega)|_\Gamma)\ d\Gamma +$$
$$\int_\Gamma \nabla u(\Omega)|_\Gamma^T Q\nabla \dot{u}(\Omega;V)|_\Gamma\ d\Gamma).$$

with

$$\begin{aligned}(DT_t^{-1})'(0) &= -DV(0)\\ (DT_t^{-T}\omega)'(0) &= \mathrm{div}_\Gamma V(0)I - DV(0)^T.\end{aligned}$$

For the sake of simplicity let

$$I := \quad \int_\Gamma \nabla u(\Omega)|_\Gamma^T \begin{pmatrix} \nabla Q_{11}\cdot V(0) & \nabla Q_{12}\cdot V(0)\\ \nabla Q_{21}\cdot V(0) & \nabla Q_{22}\cdot V(0)\end{pmatrix} \nabla u(\Omega)|_\Gamma\ d\Gamma$$

Written with shape derivatives the Eulerian derivative becomes

$$\begin{aligned}dJ(\Omega;V) = \quad & \frac{1}{\mu_0}(I - \int_\Gamma (\nabla u(\Omega)|_\Gamma^T DV(0))Q\nabla u(\Omega)|_\Gamma\ d\Gamma +\\ & \int_\Gamma \nabla u(\Omega)|_\Gamma^T Q((\mathrm{div}_\Gamma V(0)I - DV(0)^T)\nabla u(\Omega)|_\Gamma)\ d\Gamma +\\ & \int_\Gamma \nabla u'(\Omega;V)|_\Gamma^T Q\nabla u(\Omega)|_\Gamma\ d\Gamma +\\ & \int_\Gamma \nabla u(\Omega)|_\Gamma^T Q\nabla u'(\Omega;V)|_\Gamma\ d\Gamma +\\ & \int_\Gamma \nabla(\nabla u(\Omega)|_\Gamma \cdot V(0))^T Q\nabla u(\Omega)|_\Gamma\ d\Gamma +\\ & \int_\Gamma \nabla u(\Omega)|_\Gamma^T Q\nabla(\nabla u(\Omega)|_\Gamma \cdot V(0))\ d\Gamma).\end{aligned}$$

Because of

$$\nabla(\nabla u(\Omega)|_\Gamma \cdot V(0)) = \nabla^2 u(\Omega)|_\Gamma V(0) + DV(0)^T\nabla u(\Omega)|_\Gamma$$

we have

$$\begin{aligned}dJ(\Omega;V) = \quad & \frac{1}{\mu_0}(I + \int_\Gamma \nabla u(\Omega)|_\Gamma^T Q(\mathrm{div}_\Gamma V(0)\nabla u(\Omega)|_\Gamma)\ d\Gamma +\\ & \int_\Gamma \nabla u'(\Omega;V)|_\Gamma^T Q\nabla u(\Omega)|_\Gamma\ d\Gamma +\end{aligned}$$

$$\int_\Gamma \nabla u(\Omega)|_\Gamma^T Q \nabla u'(\Omega; V)|_\Gamma \ d\Gamma +$$
$$\int_\Gamma (\nabla^2 u(\Omega)|_\Gamma V(0))^T Q \nabla u(\Omega)|_\Gamma \ d\Gamma +$$
$$\int_\Gamma \nabla u(\Omega)|_\Gamma^T Q (\nabla^2 u(\Omega)|_\Gamma V(0)) \ d\Gamma).$$

This is the gradient of the functional and everything appearing can be computed: $u(\Omega)$ is a solution of the given BVP and a BVP for the shape derivative $u'(\Omega; V)$ was also derived. So this gradient can now be used in some iterative method for solving the optimization problem.

Chapter 5

Conclusions

With the chapters above it is shown, how to compute the exact gradient of a functional consisting of integrals and functions which depend on the integration area. With the shape derivative approach, everything appearing in this gradient can be computed and so this gradient is well defined.

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