Strong, Weak and Finite Element Formulations of 1-D Scalar Problems
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1. From Strong to Weak

Strong statement:

$$\frac{d}{dx}\left(-a\frac{du}{dx}\right) = f \qquad u(0) = 0; -au_{,x}(1) = -Q \tag{1.1}$$

To convert into weak form:

1. **Multiply ODE** by a virtual function $u^v(x)$ satisfying $u^v(0) = 0$:

$$\left(-au_{,x}\right)_{,x} u^v = fu^v \tag{1.2}$$

2. **Shift Derivatives:** Recall that

$$\left(-au_{,x}u^v\right)_{,x} \equiv \left(-au_{,x}\right)_{,x} u^v + \left(-au_{,x}\right)u^v_{,x} \tag{1.3}$$

Thus, replacing LHS

$$\left(-au_{,x}u^v\right)_{,x} - \left(-au_{,x}\right)u^v_{,x} = fu^v \tag{1.4}$$

3. **Integrate**:

$$\int_0^1 \left(-au_{,x}u^v\right)_{,x} dx - \int_0^1 \left(-au_{,x}\right)u^v_{,x} dx = \int_0^1 fu^v(x)dx \tag{1.5}$$

i.e.,

$$\left[-au_{,x}u^v\right]_0^1 + \int_0^1 au_{,x}u^v_{,x} dx = \int_0^1 fu^v(x)dx \tag{1.6}$$

4. **Apply boundary conditions**:

$$-Qu^v(1) + \int_0^1 au_{,x}u^v_{,x} dx = \int_0^1 fu^v(x)dx \tag{1.7}$$

i.e.,

$$\int_0^1 au_{,x}u^v_{,x} dx = \int_0^1 fu^v(x)dx + Qu^v(1) \tag{1.8}$$

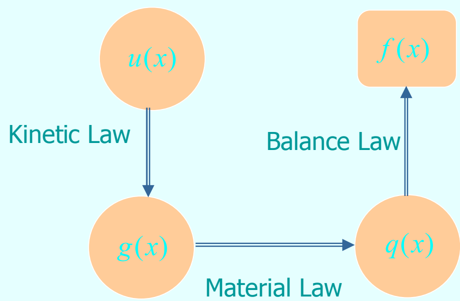
5. **Arrive at weak statement**:

$$\boxed{\begin{gathered}\text{Find } u(x), where, u(0) = 0, s.t. \\ \int_0^1 au_{,x}u^v_{,x} dx = \int_0^1 fu^v(x)dx + Qu^v(1) \\ \forall u^v(x) where, u^v(0) = 0\end{gathered}} \tag{1.9}$$

2. Graphical Interpretation of Strong & Weak Forms

Strong form:

Strong Form



$$\text{Kinetic Law}: \quad g(x) = -\frac{du}{dx}$$

$$\text{Fourier Law}: \quad q(x) = -a\frac{du}{dx}$$

$$\text{Balance Law}: \quad \text{Heat balance in x}$$

$$q(x+dx) - q(x) = f(x)dx$$

$$\frac{dq}{dx} = f$$

$$\frac{d}{dx}\left(-a\frac{du}{dx}\right) = f$$

$$\boxed{\text{Differential Equation}: \quad \frac{d}{dx}\left(-a\frac{du}{dx}\right) = f}$$

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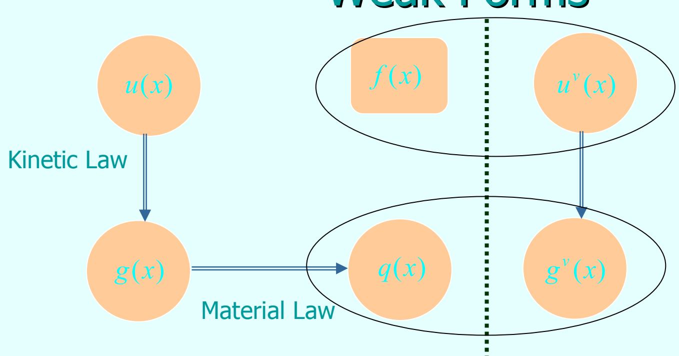
Weak Forms



$$\text{Kinetic Law}: \quad g = -\frac{du}{dx} \;\&\; g^v = -\frac{du^v}{dx}$$

$$\text{Fourier Law}: \quad q = -a\frac{du}{dx}$$

$$\boxed{\text{Virtual Work Law}: \int_0^1 qg^v dx = \int_0^1 fu^v dx}$$

$$\int_0^1 \left(-a\frac{du}{dx}\right)\left(-\frac{du^v}{dx}\right)dx = \int_0^1 fu^v dx$$

$$\int_0^1 \left(a\frac{du}{dx}\right)\left(\frac{du^v}{dx}\right)dx = \int_0^1 fu^v dx$$

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3. Equivalence

Equation (1.1) is 'equivalent' to Equation (1.9).
Consider the example:

$$\frac{d}{dx}\left(-\frac{du}{dx}\right) = 1 \qquad u(0) = 0; u_{,x}(1) = -1 \tag{1.10}$$

i.e., $f = 1 \,\&\, Q = -1$ The exact solution is $u(x) = -x^2/2$ as one can easily verify. Let us find the same solution via Equation (1.9). Search for a solution of the form $u(x) = Ax + Bx^2$ (satisfies the essential boundary condition). Similarly, let $u^v(x) = A^v x + B^v x^2$. Substituting in Equation (1.9):

Find $A \,\&\, B, s.t.$

$$\int_0^1 (A + 2Bx)(A^v + 2B^v x)dx = \int_0^1 (A^v x + B^v x^2)dx - (A^v + B^v) \tag{1.11}$$

$\forall A^v \,\&\, B^v$

i.e.,

$$\int_0^1 (A^v + 2B^v x)(A + 2Bx)dx = \int_0^1 (A^v x + B^v x^2)dx - (A^v + B^v) \tag{1.12}$$

i.e.,

$$\int_0^1 \begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{Bmatrix} 1 \\ 2x \end{Bmatrix} \begin{Bmatrix} 1 & 2x \end{Bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} dx = \int_0^1 \begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{Bmatrix} x \\ x^2 \end{Bmatrix} dx - \begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{1.13}$$

i.e.,

$$\begin{Bmatrix} A^v & B^v \end{Bmatrix} \left(\int_0^1 \begin{Bmatrix} 1 \\ 2x \end{Bmatrix} \begin{Bmatrix} 1 & 2x \end{Bmatrix} dx \right) \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} A^v & B^v \end{Bmatrix} \int_0^1 \begin{Bmatrix} x \\ x^2 \end{Bmatrix} dx - \begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{1.14}$$

i.e.,

$$\begin{Bmatrix} A^v & B^v \end{Bmatrix} \left(\int_0^1 \begin{bmatrix} 1 & 2x \\ 2x & 4x^2 \end{bmatrix} dx \right) \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} A^v & B^v \end{Bmatrix} \int_0^1 \begin{Bmatrix} x \\ x^2 \end{Bmatrix} dx - \begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{1.15}$$

i.e.,

$$\begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{Bmatrix} 1/2 \\ 1/3 \end{Bmatrix} - \begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{1.16}$$

Find $A \,\&\, B, s.t.$

$$\begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} A^v & B^v \end{Bmatrix} \begin{Bmatrix} -1/2 \\ -2/3 \end{Bmatrix} \tag{1.17}$$

$\forall A^v \,\&\, B^v$

This is true if and only if:

$$\begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} -1/2 \\ -2/3 \end{Bmatrix} \tag{1.18}$$

i.e., $A = 0, B = -1/2$, i.e., $u(x) = -x^2/2$. We obtain the same solution.

4. Why Weak Formulation?

Consider the example:

$$\frac{d}{dx}\left(-\frac{du}{dx}\right)=\frac{-1}{1+x^2} \qquad u(0)=0; u_{,x}(1)=0 \tag{1.19}$$

It is difficult to integrate Equation (1.19) to obtain the exact solution. So, let us try an approximate solution: $u(x)=Ax+Bx^2$ where A and B are constants (satisfies the essential boundary condition). Substituting in Equation (1.19), we have:

$$-B=\frac{-1}{1+x^2} \qquad A+2B=0 \tag{1.20}$$

The above suggests that B is not a constant (that's about it!!).

Now consider the weak formulation. As before, let $u(x)=Ax+Bx^2$ and $u^v(x)=A^v x+B^v x^2$. Substituting:

$$\text{Find } A \,\&\, B, s.t.$$

$$\begin{Bmatrix} A^v & B^v \end{Bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix}\begin{Bmatrix} A \\ B \end{Bmatrix}=\int_0^1 (A^v x+B^v x^2)\left(\frac{-1}{1+x^2}\right)dx \tag{1.21}$$

$$\forall A^v \,\&\, B^v$$

Evaluate right hand side numerically, we have:

$$\begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix}\begin{Bmatrix} A \\ B \end{Bmatrix}=\begin{Bmatrix} -0.3469 \\ -0.21439 \end{Bmatrix} \tag{1.22}$$

i.e., $u(x)\approx -0.75x+0.4x^2$

Weak formulations naturally promote computing approximate solutions to challenging problems, and are 'equivalent' to strong forms.

5. Finite Element Solutions of Weak Formulation

Consider the model problem:

$$\text{Find } u(x), where, u(0)=0, s.t.$$

$$\int_0^1 a u_{,x} u^v_{,x} dx=\int_0^1 f u^v(x) dx+Q u^v(1) \tag{1.23}$$

$$\forall u^v(x) where, u^v(0)=0$$

Break up the domain into finite elements; thus:

$$\sum_e \int_{x_i}^{x_{i+1}} a u_{,x} u^v_{,x} dx=\sum_e \int_{x_i}^{x_{i+1}} f u^v dx+Q u^v(1) \tag{1.24}$$

Define linear shape functions:

$$L_1(\xi)=\left(\frac{1-\xi}{2}\right); L_2(\xi)=\left(\frac{1+\xi}{2}\right), -1\le \xi \le 1 \tag{1.25}$$

Now, let (over a single element):

$$x = L_1(\xi)x_i + L_2(\xi)x_{i+1}$$
$$u = L_1(\xi)u_i + L_2(\xi)u_{i+1} \tag{1.26}$$
$$u^v = L_1(\xi)u_i^v + L_2(\xi)u_{i+1}^v$$

Note that:

$$\frac{dx}{d\xi} = L_{1,\xi}x_i + L_{2,\xi}x_{i+1} = \frac{x_{i+1} - x_i}{2} = \frac{h}{2}$$
$$u_{,x} = \frac{2}{h}\left(L_{1,\xi}u_i + L_{2,\xi}u_{i+1}\right) = \frac{2}{h}\begin{Bmatrix} L_{1,\xi} & L_{2,\xi} \end{Bmatrix}\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \tag{1.27}$$
$$u_{,x}^v = \frac{2}{h}\left(L_{1,\xi}u_i^v + L_{2,\xi}u_{,i+1}^v\right) = \frac{2}{h}\begin{Bmatrix} u_i^v & u_{,i+1}^v \end{Bmatrix}\begin{Bmatrix} L_{1,\xi} \\ L_{2,\xi} \end{Bmatrix}$$

Considering integration over a single element:L

$$\int_{x_i}^{x_{i+1}} a u_{,x} u_{,x}^v dx = \int_{-1}^{1} a \frac{2}{h}\begin{Bmatrix} u_i^v & u_{,i+1}^v \end{Bmatrix}\begin{Bmatrix} L_{1,\xi} \\ L_{2,\xi} \end{Bmatrix}\frac{2}{h}\begin{Bmatrix} L_{1,\xi} & L_{2,\xi} \end{Bmatrix}\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}\frac{h}{2}d\xi$$
$$= \frac{2}{h}\begin{Bmatrix} u_i^v & u_{,i+1}^v \end{Bmatrix}\left(\int_{-1}^{1} a \begin{bmatrix} L_{1,\xi}L_{1,\xi} & L_{1,\xi}L_{2,\xi} \\ L_{2,\xi}L_{1,\xi} & L_{2,\xi}L_{2,\xi} \end{bmatrix} d\xi\right)\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \tag{1.28}$$
$$and$$
$$\int_{x_i}^{x_{i+1}} f u^v dx = \int_{-1}^{1} f \begin{Bmatrix} u_i^v & u_{,i+1}^v \end{Bmatrix}\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix}\frac{h}{2}d\xi$$

Thus, from Equation (1.24) and Equation (1.28), we have:

$$\sum_e \frac{2}{h}\begin{Bmatrix} u_i^v & u_{,i+1}^v \end{Bmatrix}\left(\int_{-1}^{1} a \begin{bmatrix} L_{1,\xi}L_{1,\xi} & L_{1,\xi}L_{2,\xi} \\ L_{2,\xi}L_{1,\xi} & L_{2,\xi}L_{2,\xi} \end{bmatrix} d\xi\right)\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}$$
$$= \sum_e \int_{-1}^{1} f \begin{Bmatrix} u_i^v & u_{,i+1}^v \end{Bmatrix}\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix}\frac{h}{2}d\xi + Qu^v(1) \tag{1.29}$$

The element stiffness matrix and element forcing vector are :

$$K_e = \frac{h}{2}\left(\int_{-1}^{1} a \begin{bmatrix} L_{1,x}L_{1,x} & L_{1,x}L_{2,x} \\ L_{2,x}L_{1,x} & L_{2,x}L_{2,x} \end{bmatrix} d\xi\right)$$
$$f_e = \int_{-1}^{1} f \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix}\frac{h}{2}d\xi \tag{1.30}$$

After assembly, and eliminating the virtual variables, we have:

$$Ku = f \tag{1.31}$$

where

$$K = \sum_e K_e$$
$$f = \sum_e f_e + f_Q \tag{1.32}$$

The essential (Dirichlet) boundary conditions are imposed via the Lagrange multiplier method:

$$\bar{K}\bar{u} = \bar{f} \tag{1.33}$$

where

$$\bar{K} = \begin{bmatrix} K & C^T \\ C & 0 \end{bmatrix}, \bar{u} = \begin{Bmatrix} u \\ \lambda \end{Bmatrix}, \bar{f} = \begin{Bmatrix} f \\ b \end{Bmatrix} \tag{1.34}$$

where C & b capture the boundary condition at $x = 0$. The variables λ are the Lagrange multipliers that contain useful information about the sensitivity of the essential boundary conditions.

6. Non-Linear Problems

Equations (1.31) and (1.33)is valid for both linear and non-linear problems. For linear problems, the equation is solved once since the stiffness matrix and forcing vector are independent of u. On the other hand, for non-linear problems, either the stiffness matrix or the forcing vector or both is dependent on u. So, Equation (1.31) reads:

$$\left[K(u)\right]u = f(u) \tag{1.35}$$

where:

$$\begin{aligned} K(u) &= \sum_e K_e(u) \\ f(u) &= \sum_e f_e(u) + f_Q \end{aligned} \tag{1.36}$$

and

$$\begin{aligned} K_e(u) &= \frac{2}{h}\left(\int_{-1}^{1} a(u)\begin{bmatrix} L_{1,\xi}L_{1,\xi} & L_{1,\xi}L_{2,\xi} \\ L_{2,\xi}L_{1,\xi} & L_{2,\xi}L_{2,\xi} \end{bmatrix} d\xi\right) \\ f_e(u) &= \int_{-1}^{1} f(u)\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix}\frac{h}{2}\,d\xi \end{aligned} \tag{1.37}$$

while Equation (1.33) reads

$$\bar{K}(u)\bar{u} = \bar{f}(u) \tag{1.38}$$

i.e., a non-linear problem.

We will consider two different methods of solving Equation (1.38):

1. Picard iteration
2. Newton Raphson iteration

6.1 Picard Iteration

Theorems (for Picard's Method)

Theorem 1: (Banach's Fixed Point Theorem) If $\varphi(x) : \Re^n \to \Re^n$ is a continuous function such that:

$$d(\varphi(x), \varphi(y)) < d(x, y), \forall x, y \tag{1.39}$$

where $d(.,.)$ is an appropriate metric defined in $\Re^n$, then the iteration:

$$x^{n+1} = \varphi(x^n) \tag{1.40}$$

will converge (to a fixed point of $\varphi(x)$).

Theorem 2: If $\varphi(x)$ is a differentiable function in a range $[a,b]$, then $\varphi(x)$ has a unique fixed point if $\left|\varphi_{,x}(x)\right| < 1$ for all $x \in [a,b]$.
Proof: Mean Value Theorem.

From Equation (1.38), $\varphi(\overline{u}) : \left[\overline{K}(\overline{u})\right]^{-1} \overline{f}(\overline{u})$
Suppose:

$$\left\|\left[\overline{K}(\overline{u})\right]^{-1} \overline{f}(\overline{u}) - \left[\overline{K}(\overline{v})\right]^{-1} \overline{f}(\overline{u})\right\| < \|\overline{u} - \overline{v}\|, \forall u, v \tag{1.41}$$

then the Picard iteration for Equation (1.35) will proceed as follows:

1. Guess u^0 (say 0). Set n = 0.
2. Assemble $\left[\overline{K}(\overline{u}^n)\right]$ & $\overline{f}(\overline{u}^n)$
3. Compute $u^{n+1} = \left[\overline{K}(\overline{u}^n)\right]^{-1} \overline{f}(\overline{u}^n)$
4. Stop when $\left\|u^{n+1} - u^n\right\| < \varepsilon$, else go to step 2.

Of course, establishing Equation (1.41) is not easy … a case-by-case analysis at best! We will later consider stiffening versus softening non-linear problems, and study the convergence of Picard iteration.

6.2 Newton-Raphson Iteration

Theorem (for N-R Method): If $h(x)$ is a second order differentiable in $[a,b]$, and if $h(x^*) = 0$ where $x^* = [a,b]$, and $h_{,x}(x^*) = 0$ then the sequence:

$$x^{n+1} = x^n - h(x^n) / h_{,x}(x^n) \tag{1.42}$$

will converge to x^*
Proof: See popular text-books on numerical methods.

Derivation of the Tangent Matrix
The non-linear equation that we must solve is:

$$R(u) = \left[K(u)\right]u - f = 0 \tag{1.43}$$

Applying Equation (1.42):

$$u^{n+1} = u^n - \left[T(u^n)\right]^{-1} R(u^n) \tag{1.44}$$

where $T(u^n)$ is the tangent or Jacobian matrix given by:

$$T_{ij} = \frac{\partial R_i}{\partial u_j} = \frac{\partial\left(\sum_m K_{im} u_m\right)}{\partial u_j} \tag{1.45}$$

Considering a single element (for linear shape functions $m = 1,2$):

$$T_{ij}^e = \frac{\partial\left(\sum_{m=1}^{2} K_{im}^e u_m\right)}{\partial u_j} = \sum_{m=1}^{2} \frac{\partial\left(K_{im}^e u_m\right)}{\partial u_j} = \sum_{m=1}^{2} \frac{\partial K_{im}^e}{\partial u_j} u_m + \sum_{m=1}^{2} K_{im}^e \frac{\partial u_m}{\partial u_j} \tag{1.46}$$

Considering the definition of element stiffness matrix in Equation (1.30), we have:

$$\frac{\partial K_{im}^{e}}{\partial u_{j}} = \frac{\partial}{\partial u_{j}}\left(\frac{h}{2}\int_{-1}^{1} a(x,u)L_{i,x}L_{m,x}d\xi\right) = \left(\frac{h}{2}\int_{-1}^{1} a_{,u}(x,u)\frac{\partial u}{\partial u_{j}}L_{i,x}L_{m,x}d\xi\right) \tag{1.47}$$

Further, since $u = L_1u_1 + L_2u_2$, we have:

$$\frac{\partial K_{im}^{e}}{\partial u_{j}} = \left(\frac{h}{2}\int_{-1}^{1} a_{,u}(x,u)L_{j}L_{i,x}L_{m,x}d\xi\right) \tag{1.48}$$

The second term in Equation (1.46):

$$\sum_{m=1}^{2} K_{im}^{e}\frac{\partial u_{m}}{\partial u_{j}} = K_{ij}^{e} \tag{1.49}$$

Thus, substituting Equation (1.48) and Equation (1.49) in Equation (1.46), we have:

$$T_{ij}^{e} = \sum_{m=1}^{2}\left(\frac{h}{2}\int_{-1}^{1} a_{,u}(x,u)L_{j}L_{i,x}L_{m,x}d\xi\right)u_{m} + K_{ij}^{e} \tag{1.50}$$

$$T_{ij}^{e} = \left(\frac{h}{2}\int_{-1}^{1} a_{,u}(x,u)L_{j}L_{i,x}d\xi\right)\sum_{m=1}^{2} L_{m,x}u_{m} + K_{ij}^{e} \tag{1.51}$$

Thus, Equation (1.51) is used to assemble the tangent matrix.
Applying zero boundary conditions on Δu^{n}, we have:

$$\begin{bmatrix} T(u^{n}) & C^{T} \\ C & 0 \end{bmatrix}\begin{Bmatrix} \Delta u^{n} \\ \zeta \end{Bmatrix} = \begin{Bmatrix} R(u^{n}) \\ 0 \end{Bmatrix} \tag{1.52}$$

Thus, we must assemble both the stiffness matrix and the tangent matrix $T(u^{n})$ defined via Equation in each step.

Thus, the Newton-Raphson iteration will proceed as follows:

1. Guess u^{0} (say 0). Set n = 0.
2. Assemble $\left[K(u^{n})\right]$ & $f(u^{n})$
3. Assemble/ compute $\left[T(u^{n})\right]$ & $R(u^{n})$
4. Compute $\Delta\overline{u}^{n+1} = \left[\overline{T}(\overline{u}^{n})\right]^{-1}\overline{R}(\overline{u}^{n})$ per Equation (1.52)
5. Compute $u^{n+1} = u^{n} + \Delta u^{n+1}$
6. Stop when $\left\|u^{n+1} - u^{n}\right\| < \varepsilon$, else go to step 2.