A brief introduction to PDE constrained optimization

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Frontiers in PDE-constrained Optimization
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Outline

Motivation and examples

Linear elliptic optimal control problems

Differentiation in Banach spaces

First-order necessary conditions

Optimal control of semilinear PDEs

An abstract problem

Discretization using finite element methods

Outline

Controlling pollutants in river: Antil, Heinkenschloss, Hoppe 2010

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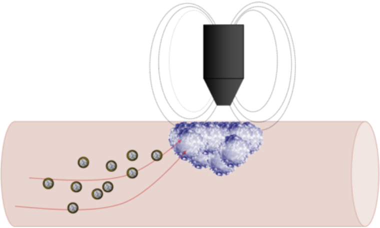
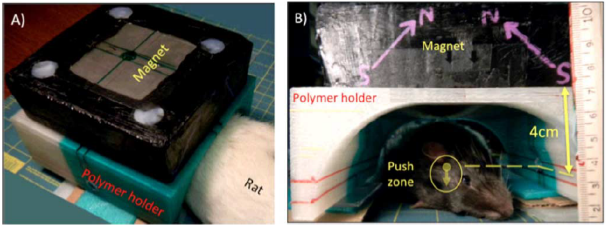
Magnetic drug targeting: Antil, Nochetto, Venegas 2016

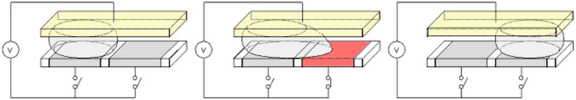
The magnetic field exerts a force on magnetic materials such as magnetic nanoparticles (MNPs)

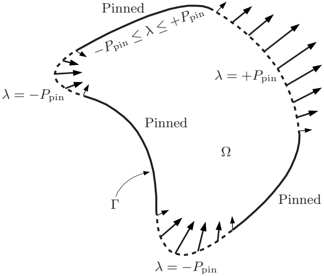




Experimental setup (Shapiro et al., 2013)

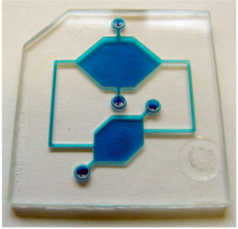
Electrowetting on Dielectric (EWOD)

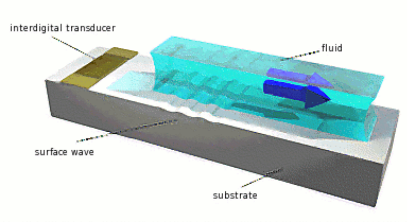




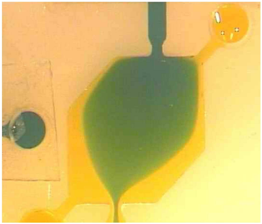
- **Forward problem:** Walker, Nochetto, Shapiro 2009.
- **Optimal control:** Antil, Hintermüller, Nochetto, Surowiec, Wegner 2015.

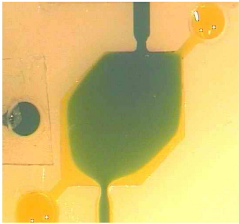
Shape optimization of microfluidic biochip: Antil, Heinkenschloss, Hoppe, Linsenmann, Wixforth 2010

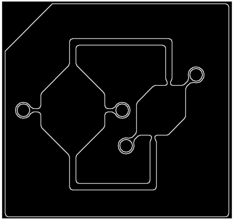




Microfluidic Biochip







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Today:

- Control constraints are pointwise a.e.
- State equation is a boundary value problem or a variational inequality.

Steps to analyze OCP

Analysis

- Existence of solution to PDE. Is it unique?
- Existence of solution to optimization problem.
- First-order necessary conditions
  - Adjoint equations
- If possible: second order necessary/sufficient conditions.

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Solver development (function space)

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Discrete problem / solver

- FD / FV / FE for problem and solver.
- Analysis of discrete solver: mesh independence?
- Efficiency
  - Adaptive finite element methods (AFEM).
  - Model reduction: Proper orthogonal decomposition, reduced basis.
- Software development: Trilinos, Rapid Optimization Library (ROL), Dolfin-Adjoint.

Optimal heat source

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Linear-quadratic elliptic OCP

$$\min \; J(u,z) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)}$$

subject to

$$\begin{aligned} -\mathrm{div}\,(A\nabla u) &= z \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

and the pointwise control constraints

$$z \in Z_{ad} := \{v \in L^2(\Omega) : z_a(x) \leq v(x) \leq z_b(x)\ ,\ a.e.\ x \in \Omega\}$$

$\lambda > 0$ is a parameter, $z_a, z_b \in L^\infty(\Omega)$ with $z_a < z_b$ are given. A is bounded, symmetric, positive.

Problems involving semilinear elliptic PDEs

- **Heating with radiation boundary condition.** If the heat radiation of the heated body is taken into account, then we obtain:

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\begin{aligned}
-\Delta u &= 0 \quad \text{in } \Omega \\
\partial_{\boldsymbol{\nu}} u &= \alpha(z^4 - u^4) \quad \text{on } \partial\Omega
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- **Simplified supercondutivity.**

$$\begin{aligned} -\Delta u + u + u^3 &= z \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- **Control of stationary flows.**

$$\begin{aligned} -\mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= f \quad \text{in } \Omega \\ \text{div } \boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial\Omega \end{aligned}$$

Optimal control of variational inequalities

Problem:

$$\min \; J(u,z) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)}$$

subject to $u \in \mathcal{K} := \{w \in H^1_0(\Omega) : w(x) \geq 0, \; a.e. \; x \in \Omega\}$

$$\int_\Omega \nabla u \cdot \nabla(u - v)\, dx \leq (z + f, u - v) \quad \forall v \in \mathcal{K}$$

with $z \in Z_{ad}$.

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Regularized problem:

$$\min \; J(u,z) = \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)}$$

subject to (semilinear PDE)

$$u \in H^1_0(\Omega) \;:\; -\Delta u + \gamma\rho_\varepsilon(-u) = z + f \quad \text{in } \Omega$$

and $z \in Z_{ad}$. $\gamma > 0$ and $\varepsilon > 0$ are the penealization and regularization parameters respectively.

Outline

Linear elliptic problems

$\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain.

Sobolev spaces

- $L^p(\Omega)$, $1 \le p \le \infty$ are Lebesgue spaces. $p = 2$: $L^2(\Omega)$ is a Hilbert space with inner product $(u, v) = \int_\Omega u(x)v(x)\, dx$.

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- Note: $L^2(\Omega) \subset H^{-1}(\Omega)$

Model problem:

$$\begin{aligned} -\text{div}\ (A\nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Well-posed

- **Weak solution:** Given $f \in H^{-1}(\Omega)$, a function $u \in H_0^1(\Omega)$ is called a weak solution iff

$$\int_\Omega A\nabla u \cdot \nabla v = \langle f, v\rangle_{-1,1} \quad \forall v \in H_0^1(\Omega)$$

where $\langle \cdot, \cdot \rangle_{-1,1}$ denotes the $H^{-1}(\Omega)$–$H^1(\Omega)$ duality pairing.

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- **Existence and uniqueness:** For each $f \in H^{-1}(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$, such that

$$\|\nabla u\|_{L^2(\Omega)} \le C_{d,\Omega} \|f\|_{H^{-1}(\Omega)}.$$

  - $A = I$, the result is due to the Riesz-representation theorem.
  - $A \neq I$ (maybe non-symmetric), the result is due to Lax-Milgram lemma.
  - For general $W^{1,p}$-spaces use inf-sup conditions combined with Banach-Nečas theorem which is a necessary and sufficient condition.

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Reduced OCP

- **Control to state map:** The solution operator defined as

$$S : H^{-1}(\Omega) \to H_0^1(\Omega), \quad z \mapsto u(z) = Sz.$$

Note: In view of the embedding

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$$

we may also consider S from $L^2(\Omega)$ to $L^2(\Omega)$. We will use the same notation for this operator (for simplicity).

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$$\mathcal{J}(z) = J(Sz, z).$$

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- **Reduced problem:** $\min_{z \in Z_{ad}} \mathcal{J}(z)$.

Existence of solution: Direct method

There exists a unique solution to the above problem.

- **Minimizing sequence:** $\mathcal{J}$ is bounded below by zero therefore there exists a minimizing sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

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- **Weak compactness:** $\{z_n\} \subset L^2(\Omega)$ is bounded. Since $L^2(\Omega)$ is reflexive therefore there exists a subsequence (not relabeled) such that

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- **Uniqueness:** Exercise.

Outline

Differentiation in Banach spaces

Let $(Z, \| \cdot \|_Z)$, $(V, \| \cdot \|_V)$ B-spaces, $\mathcal{Z} \subset Z$, open and $F : \mathcal{Z} \to V$.

Let $z \in \mathcal{Z}$:

- If $\exists\ \delta F(z,h) := \lim_{t \downarrow 0} \frac{1}{t}(F(z+th) - F(z))$, then $\delta F(z,h)$ is called the **directional derivative** of F at z in direction h.

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- Let $\exists$ the first variation $\delta F(z,\cdot)$. F is said to be **Gâteaux differentiable** at z iff $\exists A \in \mathcal{L}(Z,V)$ such that $\delta F(z,h) = Ah$ $\forall h \in Z$. We write $A = F'_G(z)$.

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- F is said to be **Fréchet differentiable** at z iff $\exists A \in \mathcal{L}(Z,V)$ and a mapping $r(z,\cdot) : Z \to V$ such that: for all $h \in U$ with $z + h \in \mathcal{Z}$, we have

  $$F(z+h) = F(z) + Ah + r(z,h) \quad \text{with} \quad \frac{\|r(z,h)\|_V}{\|h\|_Z} \to 0 \quad \text{as} \quad \|h\|_Z \to 0.$$

  We write $F'(z) =: A$.

Examples

Example 1. $(H, (\cdot,\cdot)_H)$ Hilbert space, $F(z) := \|z\|_H^2 = (z,z)_H$.

$$\forall z, h: \; F(z+h) - F(z) = 2(z,h)_H + \|h\|_H^2$$

Then

$$F'(z)h = (2z, h)_H.$$

Reisz representation theorem (identify H with H^*), we can write

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Example 2. Let $(Z, (\cdot,\cdot)_Z), (H, (\cdot,\cdot)_H)$ Hilbert spaces, $u_d \in H$ fixed. Let $S \in \mathcal{L}(Z, H)$. Consider $E : Z \to \mathbb{R}$,

$$E(z) = \|Sz - u_d\|_H^2.$$

Then $E(z) = G(F(z))$, where $G(v) = \|v\|_H^2$ and $F(z) = Sz - u_d$.

$$\begin{aligned} E'(z)h &= G'(F(z))F'(z)h = (2v, F'(z)h)_H \\ &= 2(Sz - u_d, Sh)_H = 2(S^*(Sz - u_d), h)_Z \end{aligned}$$

we denote the gradient as $\nabla E(z) = 2S^*(Sz - u_d)$.

Outline

First order necessary optimality conditions

Most of the theory is based on these two simple results:

Theorem: Let $(Z, \|\cdot\|_Z)$ be a normed space, $\mathcal{J} : Z \to (\infty, +\infty]$ a mapping with $\mathcal{J} \not\equiv +\infty$. Then $\bar{z} \in Z$ minimizer of $\mathcal{J} \Leftrightarrow 0 \in \partial\mathcal{J}(\bar{z})$.

Proof. $0 \in \partial\mathcal{J}(\bar{z})$ means by definition of $\partial\mathcal{J}(\bar{z})$: $\quad \mathcal{J}(\bar{z}) - \mathcal{J}(z) \leq 0$ $\forall z \in Z$.

First order necessary optimality conditions

Most of the theory is based on these two simple results:

Theorem: Let $(Z, \|\cdot\|_Z)$ be a normed space, $\mathcal{J} : Z \to (\infty, +\infty]$ a mapping with $\mathcal{J} \not\equiv +\infty$. Then $\bar{z} \in Z$ minimizer of $\mathcal{J} \Leftrightarrow 0 \in \partial\mathcal{J}(\bar{z})$.

Proof. $0 \in \partial\mathcal{J}(\bar{z})$ means by definition of $\partial\mathcal{J}(\bar{z})$: $\quad \mathcal{J}(\bar{z}) - \mathcal{J}(z) \le 0$ $\forall z \in Z$.

Theorem: Let $(Z, \|\cdot\|_Z)$ be a normed space; $Z_{ad} \subset Z$ nonempty, convex, closed; $\mathcal{J} : \mathcal{Z} \to \mathbb{R}$ Gâteaux differentiable, where $Z_{ad} \subset \mathcal{Z} \subset Z$, $\mathcal{Z}$ open. If $\bar{z} \in Z_{ad}$ is a solution to

$$\min_{z \in Z_{ad}} \mathcal{J}(z)$$

then $\bar{z}$ solves

$$\mathcal{J}'(\bar{z})(z - \bar{z}) \ge 0 \quad \forall z \in Z_{ad}.$$

Proof. Exercise.

Linear quadratic OCP

- **Reduced funcational**

$$\mathcal{J}(z) = \frac{1}{2}\|Sz - u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)}.$$

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- **Gradient** computation using sensitivity

$$\mathcal{J}'(z)h = (S^*(Sz - u_d) + \lambda z, h)_{L^2(\Omega)} \quad \forall h \in L^2(\Omega),$$

we denote the gradient by

$$\nabla \mathcal{J}(z) = S^*(Sz - u_d) + \lambda z.$$

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- **Gradient** computation using adjoint: $\nabla\mathcal{J}(z) = p + \lambda z$.

Optimality system

- **State:** $\bar{u} \in H_0^1(\Omega)$: $\int_\Omega A\nabla\bar{u} \cdot \nabla v = \int_\Omega \bar{z} v \quad \forall v \in H_0^1(\Omega)$.
- **Adjoint:** $\bar{p} \in H_0^1(\Omega)$: $\int_\Omega A\nabla\bar{p} \cdot \nabla v = \int_\Omega (\bar{u} - u_d) v \quad \forall v \in H_0^1(\Omega)$.
- **Control:** $\bar{z} \in Z_{ad}$: $(\bar{p} + \lambda\bar{z}, z - \bar{z}) \geq 0, \quad \forall z \in Z_{ad}$.

Pointwise interpretation

The variational inequality

$$(\bar{p} + \lambda \bar{z}, z - \bar{z}) \geq 0, \quad \forall z \in Z_{ad}$$

is equivalent to, for a.e. $x \in \Omega$

$$\bar{z}(x) = \begin{cases} z_a(x) & \text{if} \quad \bar{p}(x) + \lambda \bar{z}(x) > 0 \\ \in [z_a(x), z_b(x)] & \text{if} \quad \bar{p}(x) + \lambda \bar{z}(x) = 0 \\ z_b(x) & \text{if} \quad \bar{p}(x) + \lambda \bar{z}(x) < 0. \end{cases}$$

Consequence:

- When $\lambda > 0$, then a.e. in Ω

$$\begin{aligned} \bar{z}(x) &= \mathbb{P}_{[z_a(x), z_b(x)]} \left\{ -\frac{1}{\lambda} \bar{p}(x) \right\} \\ &= \min \left\{ z_b(x), \max \left\{ z_a(x), -\frac{1}{\lambda} \bar{p}(x) \right\} \right\}. \end{aligned}$$

Formal Lagrange method applied to linear OCP

Introduce $L : H_0^1(\Omega) \times Z_{ad} \times H^1(\Omega) \to \mathbb{R}$, defined as

$$L(u,z,p) := J(u,z) - \int_\Omega \big(-\mathsf{div}\,(A\nabla u) - z\big)\, p\; dx.$$

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If $(\bar{u},\bar{z},\bar{p})$ is a stationary point then:

- **State:**

$$D_p L(\bar{u},\bar{z},\bar{p})h = 0 \quad \forall h \in H^1(\Omega).$$

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- **Control:**

$$D_z L(\bar u, \bar z, \bar p)(z - \bar z) \geq 0 \quad \forall z \in Z_{ad}.$$

Outline

Semilinear elliptic OCP

$$\min J(u,z) := \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} -\Delta u + g(x,u) &= z \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

and $z \in Z_{ad}$ with $z_a, z_b \in L^\infty(\Omega)$.

State space

$(\mathbf{A}1)$ $\Omega \subset \mathbb{R}^d$, open, bounded, $\partial\Omega$ is Lipschitz.

$(\mathbf{A}2)$ $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is bounded and measurable with respect to $x \in \Omega$ for every $u \in \mathbb{R}$.

$(\mathbf{A}3)$ g is continuous, monotone increasing and Lipschitz continuous in u for a.e. $x \in \Omega$.

$(\mathbf{A}4)$ $g(x, 0) = 0$ for a.e. $x \in \Omega$ (not really necessary!)

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Theorem: Suppose $(A1) - (A4)$ hold. For every $z \in L^r(\Omega)$ with $r > d/2$, the state equation has a unique weak solution $u \in H_0^1(\Omega) \cap C(\bar{\Omega})$, i.e., we have

$$\int_\Omega \nabla u \cdot \nabla v \; dx + \int_\Omega g(x, u(x))v(x) \; dx = \int_\Omega zv \; dx \quad \forall v \in H_0^1(\Omega).$$

Moreover, there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} + \|u\|_{C(\bar{\Omega})} \le C\|z\|_{L^r(\Omega)}.$$

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Remark:

- The proof of the theorem uses the Browder–Minty theorem on monotone operators.

Control-to-state map and reduced problem

- **Control to state map**

$$S : L^\infty(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega}),$$

is well defined, and is globally Lipschitz continuous.

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$$\min \mathcal{J}(z) := \frac{1}{2}\|S(u) - u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)}$$

First-order necessary conditions

- **Lagrangian functional:**

$$
\begin{aligned}
L(u,z,p) &= J(u,z) - \int_\Omega \big(-\Delta u + g(x,u(x)) - z\big)\, p \; dx \\
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- **Adjoint system:** For all $h \in H_0^1(\Omega)$

$$0 = D_u L(\bar u, \bar z, \bar p)h = \int_\Omega (\bar u - u_d) h \; dx - \int_\Omega \big(\nabla h \cdot \nabla \bar p + g_u(x,u(x))\bar p h\big)$$

which yields

$$-\Delta \bar p + g_u(x,\bar u(x))\bar p = \bar u - u_d \quad \text{in } \Omega$$
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- **Gradient:**

$$\nabla \mathcal{J}(\bar z) = \bar p + \lambda \bar z.$$

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Differentiability of Nemytskii operator

The rigorous proof uses:

Theorem. In addition to (A1)-(A4) if

$(\mathbf{A}5)$ g is continuously differentiable with respect to u for a.e. $x \in \Omega$, and we have:

(i) $|g_u(x,0)| \leq K$ for a.e. $x \in \Omega$.
(ii) g_u is locally Lipschitz with respect to $u \in \mathbb{R}$.

then the Nemytskii operator $G : u \mapsto g(\cdot, u(\cdot))$ is continuously Fréchet differentiable from $L^\infty(\Omega)$ into itself, and

$$(G'(u)h)(x) = g_u(x, u(x))h(x) \quad \text{a.e. in } \Omega, \quad \forall h \in L^\infty(\Omega).$$

Remark: (A5) holds if $g(x,u) = g(u)$ and $g \in C^2(\mathbb{R})$.

Outline

Gradient computation via adjoint: general form

- **Abstract problem:**

$$\min J(u, z)$$

subject to

$$c(u, z) = 0$$

and $z \in Z_{ad}$.

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$$z \mapsto S(z) = u(z).$$

Under the assumptions of "**implicit function theorem**"

$$\mathcal{J}'(z)h = (\nabla_u \mathcal{J}(u(z),z), S'(z)h) + (\nabla_z \mathcal{J}(u(z),z), h).$$

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- **Sensitivity of u with respect to z:** Differentiating the state equation:

$$c_u(S(z),z)S'(z)h = -c_z(S(z),z)h,$$

leads to

$$S'(z)h = -c_u(S(z),z)^{-1}\left(c_z(S(z),z)h\right),$$

provided $c_u(S(z),z)^{-1}$ is well-defined.

Adjoint equation and gradient: general form

- **Directional derivative:**

$$\begin{aligned}\mathcal{J}'(z)h = &- \Big(\nabla_u \mathcal{J}(u(z),z), c_u(u(z),z)^{-1} c_z(u(z),z)h\Big)\\ &+ (\nabla_z \mathcal{J}(u(z),z), h).\end{aligned}$$

Adjoint equation and gradient: general form

- **Directional derivative:**

$$\mathcal{J}'(z)h = -\Big(\nabla_u \mathcal{J}(u(z),z), c_u(u(z),z)^{-1} c_z(u(z),z)h\Big) + (\nabla_z \mathcal{J}(u(z),z), h).$$

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$$\nabla \mathcal{J}(z) = -c_z(u(z),z)^* \left(c_u(u(z),z)^{-*} \nabla_u \mathcal{J}(u(z),z)\right) + \nabla_z \mathcal{J}(u(z),z).$$

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- Introducing adjoint variable p solving

$$c_u(u(z),z)^* p = \nabla_u \mathcal{J}(u(z),z)$$

we arrive at following form of gradient

$$\nabla \mathcal{J}(z) = -c_z(u(z),z)^* p + \nabla_z \mathcal{J}(u(z),z),$$

which is tractable.

Hessian computation: general form

- **Lagrangian:** $L(u, z, p) = J(u, z) - \int_\Omega c(u, z)p \; dx$. Then we have

$$D_z \mathcal{J}(z)h = D_z L(u, z, p)h.$$

Hessian computation: general form

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$$D_z\mathcal{J}(z)h = D_z L(u,z,p)h.$$

- **Second order derivative:**

$$\begin{aligned} D_z^2\mathcal{J}(z)[h_1,h_2] = D_u D_z L(u,z,p)[h_1, S'(z)h_2] + D_z^2 L(u,z,p)[h_1,h_2] \\ + D_p D_z L(u,z,p)[h_1, D_z p(z)h_2]. \end{aligned}$$

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- We already know $S'(z)h_2$, moreover

$$D_p D_z L(u,z,p)[h_1, D_z p(z)h_2] = -\left(c_u(u,z)h_1, D_z p(z)h_2\right).$$

It remain to identify $D_z p(z)h_2$.

Outline

Numerical approximation of Poisson equation: Mesh

To avoid technical issues, we will assume that $\partial\Omega$ is polygonal

- **Mesh.** Let $\mathcal{T} = \{K\}$ be a mesh of Ω, where $K \subset \mathbb{R}^d$ is an element that is isoparametrically equivalent to a unit simplex in $\mathbb{R}^d$.

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- **Conforming mesh.** We say that the mesh is conforming if
  - $\bar{\Omega} = \cup_{K \in \mathcal{T}} \bar{K}$.
  - If $\bar{K}_1 \cap \bar{K}_2 = \{x\}$, then x is a node of K_1 and K_2.
  - If $\bar{K}_1 \cap \bar{K}_2 \neq \emptyset$ and $\bar{K}_1 \cap \bar{K}_2 \neq \{x\}$ then $\bar{K}_1 \cap \bar{K}_2$ is an edge/face contained in $\partial K_1 \cap \partial K_2$.

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- **Shape regular.** Let $\mathbb{T}$ be the collection of all conforming refinements of an original mesh $\mathcal{T}^0$. Then $\mathbb{T}$ is called shape regular if $\exists\ \kappa > 0$ such that

$$\frac{h_K}{\rho_K} \leq \kappa \quad \forall K \in \cup_i \mathcal{T}^i$$

  where $h_K = \text{diam } K$, $\rho_K = \max\{\rho > 0 | B_\rho \subset \bar{K}\}$.

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$$\frac{h_K}{\rho_K} \leq \kappa \quad \forall K \in \cup_i \mathcal{T}^i$$

  where $h_K = \text{diam } K$, $\rho_K = \max\{\rho > 0 | B_\rho \subset \bar{K}\}$.
- **Quasi-uniform.** A family of shape regular meshes is called-quasi uniform, if there exist a constant $\sigma > 0$ such that

$$\frac{\max h_K}{\min h_K} \leq \sigma,$$

  uniformly for all meshes in the family.

FE space and Galerkin approximation

- **Finite element space:** For $\mathcal{T} \in \mathbb{T}$ we define the finite element space as

$$\mathbb{V}(\mathcal{T}) = \{W \in C^0(\bar{\Omega}) : W_K \in \mathbb{P}_1, \forall K \in \mathcal{T}, W|_{\partial\Omega} = 0\},$$

where $\mathbb{P}_1$ is the set of polynomials of degree at most 1.

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- **Galerkin orthogonality.**

$$\big(\nabla(u - U_{\mathcal{T}}), \nabla W\big)_{L^2(\Omega)} = 0 \quad \forall W \in \mathbb{V}(\mathcal{T}).$$

Best approximation

- **Céa's lemma:**

$$\|\nabla(u - U_{\mathcal{T}})\|_{L^2(\Omega)} \leq C \inf_{W \in \mathbb{V}(\mathcal{T})} \|\nabla(u - W)\|_{L^2(\Omega)}.$$

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- **Nodal interpolation operator:** $\Pi_{\mathcal{T}} : C(\Omega) \to \mathbb{V}(\mathcal{T})$ satisfies

$$\Pi_{\mathcal{T}} u(x_i) = u(x_i) \quad \forall x_i \in \mathring{\mathcal{N}}(\mathcal{T}),$$

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- **Error estimates:** In Céa's lemma replace W by $\Pi_{\mathcal{T}} u$ and then compute the interpolation error.

Algebraic system: state equation

- Recall, $U_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ solves

$$a(U_{\mathcal{T}}, V) = \langle f, V \rangle \qquad \forall V \in \mathbb{V}(\mathcal{T}).$$

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- Now setting $U_{\mathcal{T}} = \sum_{j=1}^{N} U_j \psi_j$, $V = \psi_i$, we get

$$\sum_{j=1}^{N} U_j a(\phi_j, \phi_i) = \langle f, \psi_i \rangle, \qquad i = 1, \ldots, N, ,$$

which is the matrix-vector system:

$$\boldsymbol{A}\boldsymbol{U} = \boldsymbol{F}\,.$$

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- **Stiffness matrix:** $\boldsymbol{A}$ is SPD.

Discrete linear elliptic OCP

$$\min J_{\mathcal{T}}(U_{\mathcal{T}}, Z_{\mathcal{T}}) = \frac{1}{2}\|U_{\mathcal{T}} - u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|Z_{\mathcal{T}}\|^2_{L^2(\Omega)}$$

subject to

$$\text{Find } U_{\mathcal{T}} \in \mathbb{V}(\mathcal{T}) : \int_\Omega \nabla U_{\mathcal{T}} \cdot \nabla V = (Z_{\mathcal{T}}, V)_{L^2(\Omega)} \quad \forall V \in \mathbb{V}(\mathcal{T}),$$

and where

$$Z_{\mathcal{T}} \in \mathbb{Z}_{ad}(\mathcal{T}) := Z_{ad} \cap \mathbb{Z}(\mathcal{T}),$$

is either one of these

$$\mathbb{Z}(\mathcal{T}) = \begin{cases} \mathbb{V}(\mathcal{T}) \\ L^2(\Omega) & \text{variational discretization (Hinze)} \\ \{W_{\mathcal{T}} \in L^\infty(\Omega) : Z_{\mathcal{T}}|_K \in \mathbb{P}_0(K), K \in \mathcal{T}\}. \end{cases}$$

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