Problem formulation

Let $\mathcal{M} = (\mathbf{V}, \mathbf{F})$ be mesh living in dimension $d \in \{2,3\}$. Let $\mathbf{V} = \{\mathbf{v}_1^T, \cdots, \mathbf{v}_n^T\}^T \in \mathbb{R}^{n\times d}$ be rest-pose vertex positions. Let $\mathcal{H} = \{\mathbf{h}_1, \cdots, \mathbf{h}_m\}$ be a set of control handles. Let $\mathbf{T}_j \in \mathbb{R}^{d\times(d+1)}$ be affine transformation for each handle $\mathbf{h}_j$. Let $\mathbf{T} = \{\mathbf{T}_1^T, \cdots, \mathbf{T}_m^T\}^T \in \mathbb{R}^{(d+1)m\times d}$. Linear blend skinning is a deformation method whereby deformed vertices $\mathbf{V}' = \{\mathbf{v}_1'^T, \cdots, \mathbf{v}_n'^T\}^T \in \mathbb{R}^{n\times d}$ is a weighted linear combination of handles' transformations,

$$v_i' = \sum_{j=1}^{m} w_j(\mathbf{v}_i)\mathbf{T}_j \begin{pmatrix}\mathbf{v}_i \\ 1\end{pmatrix}$$

where $w_j : \mathcal{M} \to \mathbb{R}$ is computed using bounded biharmonic weights. Equivalently

$$\mathbf{V}' = \mathbf{M}\mathbf{T}$$

where, $\mathbf{M} \in \mathbb{R}^{n\times(d+1)m}$ is a matrix combining $\mathbf{V}$ and $\mathbf{W}$. Note, the energy functions below are function of $\mathbf{V}'$ and hence a function of $\mathbf{T}$ by substituting $\mathbf{MT}$ in place of $\mathbf{V}'$, i.e.

$$E_{arap}(\mathbf{T}) = E_{arap}(\mathbf{T}, \mathbf{M}) = E_{arap}(\mathbf{V}')$$

where $\mathbf{M}$ fixed

arap energy

We can define *as-rigid-as-possible* deformation energy that measures local distortion,

$$E_{arap}(\mathbf{V}', \mathbf{R}) = \frac{1}{2}\sum_{f\in\mathbf{F}}\sum_{(i,j)\in f} c_{ij}||(\mathbf{v}_i' - \mathbf{v}_j') - \mathbf{R}(\mathbf{v}_i - \mathbf{v}_j)||^2$$

We want to reformulate ARAP energy base on $\mathbf{V}'$ only.

$$E_{arap}(\mathbf{V}') \qquad \mathbf{R} = \arg\min_{\mathbf{R}} tr(\mathbf{R}\tilde{\mathbf{K}}\mathbf{T})$$

and in matrix form

$$E_{arap}(\mathbf{V}') = tr(\frac{1}{2}\mathbf{V}'^T\mathbf{L}\mathbf{V}' + \mathbf{V}'^T\mathbf{K}\mathbf{R}) = tr(\frac{1}{2}\mathbf{T}^T\tilde{\mathbf{L}}\mathbf{T} + \mathbf{T}^T\tilde{\mathbf{K}}\mathbf{R})$$

where $\tilde{\mathbf{L}} = \mathbf{M}^T\mathbf{L}\mathbf{M} \in \mathbb{R}^{(d+1)m\times(d+1)m}$, $\tilde{\mathbf{K}} = \mathbf{M}^T\mathbf{K} \in \mathbb{R}^{(d+1)m\times dn}$ and $\mathbf{K}$ as defined in the deformation assignment

overhang energy

An overhanging region that can be 3D printed without support is called *self-supported.* We call the angle between region's tangent plane and printing direction be *self-supported angle* α. Let α_{max} be maximum supporting angle. Let $\tau = sin(\alpha_{max})$ be *maximal supporting coefficient.* Let $\partial\mathcal{M} \subset \mathbf{F}$ be boundary of mesh $\mathcal{M}$, i.e. surface faces. A surface face f is *risky* and thus requires support if $\mathbf{n}(f) \cdot \mathbf{d}_p < -\tau$, where $\mathbf{n}(f)$ is unit normal of f and $\mathbf{d}_p$ is the printing direction. We can define an overhang energy that measures support required for 3D printing,

$$E_{overhang}(\mathbf{V}') = \sum_{f \in \partial\mathcal{M}} min(\mathbf{n}(f) \cdot \mathbf{d}_p + \tau, 0)$$

optimization

Given an objective $E(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$,

$$E = E_{arap} + E_{overhang} + E_{overlap}$$

we find minimizer of E using particle swarm optimization. Let $\mathbf{x} \in \mathbb{R}^{2dm}$ consisting a representation of centroid and rotation (as Euler's angle) about the centroid of for each edge handle. For shape in

Finally, we can convert each component to $\mathbf{T}_j$, affine transformations for edge handles, from which can compute deformed shaped $\mathbf{V}' = \mathbf{M}\mathbf{T}$ with LBS.