Problem formulation

Let $\mathcal{M} = (\mathbf{V}, \mathbf{F})$ be mesh living in dimension $d \in \{2, 3\}$. Let $\mathbf{V} = \{\mathbf{v}_1^T, \cdots, \mathbf{v}_n^T\}^T \in^{n\times d}$ be rest-pose vertex positions. Let $\mathcal{H} = \{\mathbf{h}_1, \cdots, \mathbf{h}_m\}$ be a set of control handles. Let $\mathbf{T}_j \in^{d\times(d+1)}$ be affine transformation for each handle $\mathbf{h}_j$. Let $\mathbf{T} = \{\mathbf{T}_1^T, \cdots, \mathbf{T}_m^T\}^T \in^{(d+1)m\times d}$. Linear blend skinning is a deformation method whereby deformed vertices $\mathbf{V}' = \{\mathbf{v}_1'^T, \cdots, \mathbf{v}_n'^T\}^T \in^{n\times d}$ is a weighted linear combination of handles' transformations,

$$v_i' = \sum_{j=1}^{m} w_j(\mathbf{v}_i)\mathbf{T}_j \begin{pmatrix} \mathbf{v}_i \\ 1 \end{pmatrix}$$

where $w_j : \mathcal{M} \rightarrow$ is computed using bounded biharmonic weights. Equivalently

$$\mathbf{V}' = \mathbf{M}\mathbf{T}$$

where, $\mathbf{M} \in^{n\times(d+1)m}$ is a matrix combining $\mathbf{V}$ and $\mathbf{W}$.

arap energy

We can define *as-rigid-as-possible* deformation energy that measures local distortion,

$$E_{arap}(\mathbf{V}',) = \frac{1}{2}\sum_{f\in\mathbf{F}}\sum_{(i,j)\in f} c_{ij}||(\mathbf{v}_i' - \mathbf{v}_j') - (\mathbf{v}_i - \mathbf{v}_j)||^2$$

and in matrix form

$$E_{arap}(\mathbf{V}',) = tr(\frac{1}{2}\mathbf{V}'^T\mathbf{L}\mathbf{V}' + \mathbf{V}'^T\mathbf{K}\mathbf{R}) = tr(\frac{1}{2}\mathbf{T}^T\tilde{\mathbf{L}}\mathbf{T} + \mathbf{T}^T\tilde{\mathbf{K}}\mathbf{R})$$

where $\tilde{\mathbf{L}} = \mathbf{M}^T\mathbf{L}\mathbf{M} \in^{(d+1)m\times(d+1)m}$, $\tilde{\mathbf{K}} = \mathbf{M}^T\mathbf{K} \in^{(d+1)m\times dn}$ and $\mathbf{K}$ as defined in the deformation assignment

overhang energy

An overhanging region that can be 3D printed without support is called *self-supported*. We call the angle between region's tangent plane and printing direction be *self-supported angle* α. Let α_{max} be maximum supporting angle. Let $\tau = sin(\alpha_{max})$ be *maximal supporting coefficient*. Let $\partial\mathcal{M} \subset \mathbf{F}$ be boundary of mesh $\mathcal{M}$, i.e. surface faces. A surface face f is *risky* and thus requires support if $\mathbf{n}(f) \cdot \mathbf{d}_p < -\tau$, where $\mathbf{n}(f)$ is unit normal of f and $\mathbf{d}_p$ is the printing direction. We can define an overhang energy that measures support required for 3D printing,

$$E_{support}(\mathbf{V}') = \sum_{f \in \partial \mathcal{M}} min(\mathbf{n}(f) \cdot \mathbf{d}_p + \tau, 0)$$