Ch 1-2.

For $c = 1$, we have

$$\beta = \frac{v}{c} = v \qquad \Rightarrow \qquad \gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2}}$$

S.R. Time Dilation:

$$d\tau = dt\sqrt{1-v^2} \quad \text{where } v^2 = v_x^2 + v_y^2 + v_z^2$$

Lorentz transformation matrix:

$$\Lambda^\mu_\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Leftrightarrow (\Lambda^{-1})^\mu_\nu = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ch 3. Four Vector

3.1 Four Vector Notation

Proper time: τ

Four-position: $x^\mu(\tau) = (t(\tau),\, x(\tau),\, y(\tau),\, z(\tau))$

Four-displacement (arc length s):

$$d\mathbf{s} = \begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix}$$

Four-velocity:

$$\mathbf{u} \equiv \begin{bmatrix} u^t \\ u^x \\ u^y \\ u^z \end{bmatrix} \equiv \begin{bmatrix} \frac{dt}{d\tau} \\ \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{bmatrix} = \begin{bmatrix} \frac{dt}{dt\sqrt{1-v^2}} \\ \frac{dx}{dt\sqrt{1-v^2}} \\ \frac{dy}{dt\sqrt{1-v^2}} \\ \frac{dz}{dt\sqrt{1-v^2}} \end{bmatrix} = \begin{bmatrix} \gamma \\ v_x\gamma \\ v_y\gamma \\ v_z\gamma \end{bmatrix}.$$

3.2 Lorentz Transformation

$$u'^\mu = \begin{bmatrix} u'^t \\ u'^x \\ u'^y \\ u'^z \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u^t \\ u^x \\ u^y \\ u^z \end{bmatrix},$$

3.3 Scalar product, magnitude, the interval

Minkowski metric: $\eta_{\mu\nu} = \mathrm{diag}(-,1,1,1)$

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= \eta_{\mu\nu}A^\mu B^\nu \quad \text{or} \quad A_\mu B^\mu \\ &= \eta_{tt}A^tB^t + \eta_{xx}A^xB^x + \eta_{yy}A^yB^y + \eta_{zz}A^zB^z \\ &= -A^tB^t + A^xB^x + A^yB^y + A^zB^z \end{aligned}$$

$$A^2 = \mathbf{A}\cdot\mathbf{A} = -(A^t)^2 + (A^x)^2 + (A^y)^2 + (A^z)^2.$$

Interval

$$ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2.$$

Normalization of 4-velocity

$$\begin{aligned} ds^2 &= -d\tau^2 \quad \text{(rest frame)} \\ \Rightarrow \quad \mathbf{u}\cdot\mathbf{u} &= \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ &= \frac{1}{d\tau^2}\eta_{\mu\nu}dx^\mu dx^\nu \\ &= \frac{ds^2}{d\tau^2} = \boxed{-1}. \end{aligned}$$

3.4 Relation Between u and v

Velocity relation: $v_i = \dfrac{u^i}{u^t}$ for $i = x, y, z$

At rest:

$$u^\mu = (1, 0, 0, 0)$$

Non-relativistic limit ($v \ll 1$, $\gamma \approx 1$):

$$u^\mu \approx (1,\, v_x,\, v_y,\, v_z)$$

3.5 Four Momentum

4-Momentum for mass m and light

$$\mathbf{p} = m\mathbf{u} \quad \Rightarrow \quad p^\mu = \begin{bmatrix} mu^t \\ mu^x \\ mu^y \\ mu^z \end{bmatrix} = \begin{bmatrix} \gamma m \\ \gamma m v_x \\ \gamma m v_y \\ \gamma m v_z \end{bmatrix} = \underbrace{\begin{bmatrix} E \\ Ev_x \\ Ev_y \\ Ev_z \end{bmatrix}}_{\text{light}}$$

Relativistic energy: $\boxed{E = p^t = \gamma m}$

Invariant mass relation:

$$\begin{aligned} \mathbf{p}\cdot\mathbf{p} &= -(p^t)^2 + \vec{p}^{\,2} = -E^2 + \vec{p}^{\,2} \quad (1) \\ \mathbf{p}\cdot\mathbf{p} &= \eta_{\mu\nu}p^\mu p^\nu = m^2\eta_{\mu\nu}u^\mu u^\nu = -m^2 \quad (2) \end{aligned}$$

where $\vec{p} = (p^x, p^y, p^z)$ for spatial 3-momentum.

$$(1) = (2) \quad \Longrightarrow \quad \boxed{E^2 - \vec{p}^{\,2} = m^2}$$

3.6 Energy by observer

In rest frame (IRF): Let $u^\mu_{\text{obs}} = (1, 0, 0, 0)$, for passing object with 4-momentum p^μ:

$$\begin{aligned} E_{(\text{obs})} &= -\,\mathbf{p}\cdot\mathbf{u}_{\text{obs}} \\ &= -\,\eta_{\mu\nu}p^\mu u^\nu_{\text{obs}} = p^t. \end{aligned}$$

Kinetic energy: $E = m + \mathrm{KE}$ for $v \ll 1$

Ch 4. Index Notation

4.1 Useful Identity

- identity matrix: $\delta^\mu{}_\nu = \mathrm{diag}(1,1,1,1)$
- inverse transformation: $(\Lambda^{-1})^\mu{}_\alpha\,\Lambda^\alpha{}_\nu = \delta^\mu{}_\nu$
- selector action: $\delta^\mu{}_\nu A^\nu = A^\mu$
- derivative rule:
$$\frac{d}{d\tau}(A^2) = 2\,\eta_{\mu\nu}A^\mu\frac{dA^\nu}{d\tau}.$$

Ch 5. Arbitrary Coordinates

5.1 Coordinate basis

In any arbitrary curvilinear coordinates ($ds^2 \neq dx^2 + dy^2$), we define a coordinate basis $\mathbf{e}_\mu$ such that

$$ds = dx^\mu\,\mathbf{e}_\mu = \underbrace{du\,\mathbf{e}_u + dw\,\mathbf{e}_w}_{\text{2D case}}$$

Metric tensor:

$$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$$

where $g_{\alpha\beta} \equiv \mathbf{e}_\alpha\cdot\mathbf{e}_\beta$ comprises the metric tensor.

5.2 Transformations Consider u, w and new coordinates $u'(u,w)$ and $w'(u,w)$

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,dx^\nu,$$

For any vector $\mathbf{A}$:

$$\boxed{A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}A^\nu \quad \Longleftrightarrow \quad A^\mu = \frac{\partial x^\mu}{\partial x'^\nu}A'^\nu}$$

5.3 Coordinate Transformations in Flat Spacetime

For *flat spacetime*, the general transformation law = L.T.:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu_\nu \quad \Longleftrightarrow \quad \frac{\partial x^\mu}{\partial x'^\nu} = (\Lambda^{-1})^\mu_\nu$$

5.4 The Metric for a Spherical Surface. In polar coordinates, the metric on a sphere of radius R is:

$$ds^2 = R^2\,d\theta^2 + R^2\sin^2\theta\,d\phi^2$$

that is

$$g_{\mu\nu} = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix}$$

Ch 6. Tensor Equations

6.1 Vectors vs covectors

Vector transformation law:

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$$

Covector transformation law:

A_μ transforms as a covector:

$$\begin{aligned} A'_\mu = g'_{\mu\nu} A'^{\nu} &= \left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}\right)\left(\frac{\partial x'^{\nu}}{\partial x^{\gamma}} A^{\gamma}\right) \\ &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \underbrace{\left(\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x'^{\nu}}{\partial x^{\gamma}}\right)}_{\delta^{\beta}{}_{\gamma}} g_{\alpha\beta} A^{\gamma} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} g_{\alpha\gamma} A^{\gamma} \\ &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}} A_{\alpha} \end{aligned}$$

6.2 Gradient and lowering

Gradient of a scalar is a covector:

$$\partial_\mu \phi = \frac{\partial \phi}{\partial x^{\mu}} \text{ (covector)}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)_\mu$

Lowering an index with the metric produces a covector:

$$A_\mu = g_{\mu\nu} A^{\nu}$$

6.3 Scalar contraction (invariant)

Scalar contraction is invariant:

$$A'^{\mu} B'_{\mu} = \left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha}\right)\left(\frac{\partial x^{\beta}}{\partial x'^{\mu}} B_{\beta}\right) = \underbrace{\left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\mu}}\right)}_{\delta^{\beta}{}_{\alpha}} A^{\alpha} B_{\beta} = A^{\alpha} B_{\alpha}$$

6.4 Inverse metric

For any metric tensor $g_{\mu\nu}$:

$$\boxed{g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \quad\Longleftrightarrow\quad g_{\alpha\beta} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} g'_{\mu\nu}}$$

and also the inverse:

$$\boxed{g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta} \quad\Longleftrightarrow\quad g^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} g'^{\mu\nu}}$$

6.5 Tensor (master law)

$$T'^{\mu\nu\dots}{}_{\alpha\beta\dots} = \frac{\partial x'^{\mu}}{\partial x^{r}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} \cdots \frac{\partial x^{\gamma}}{\partial x'^{\alpha}}\frac{\partial x^{\delta}}{\partial x'^{\beta}} \cdots T^{r\sigma\dots}{}_{\gamma\delta\dots}.$$

Ch 7. Maxwell

7.1 Charge density and current density

Length contracts:

$$V = V'/\gamma \implies \rho = \gamma\rho'$$

Four-current density:

$$\boxed{J^{\mu} = (\rho, \rho v_x, \rho v_y, \rho v_z) \quad \text{with } \vec{J} = \rho\vec{v}}$$

7.2 Gauss's law

(differential form):

$$\nabla\cdot\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi k\rho$$

where E is the electric field:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}_e = q\mathbf{E} \implies \boxed{\frac{dp^{\mu}}{d\tau} = qF^{\mu}{}_{\nu} u^{\nu}}$$

7.3 Maxwell equations

$$\boxed{\partial_\nu F^{\mu\nu} = 4\pi k\, J^{\mu}}, \qquad \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0.$$

where F is the field tensor:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

Charge conservation: Take ∂_μ of both sides and commute derivatives: LHS = 0 because F is antisymmetric,

$$\partial_\mu\partial_\nu F^{\mu\nu} \equiv 0 \implies \boxed{\partial_\mu J^{\mu} = 0}$$

7.4 Potentials

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Ch 8. Geodesics

8.0 Lagrange equation: Lagrangian for our worldline is:

$$L(x,\dot{x}) \equiv \sqrt{-g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}}, \quad \dot{x}^{\mu} \equiv \frac{dx^{\mu}}{d\sigma}$$

Euler-Lagrange equation for our worldline is:

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) - \frac{\partial L}{\partial x^{\mu}} = 0$$

8.1 Extremal proper time Between two a timelike geodesic two timelike-separated events $A \to B$, the *proper time* is:

$$\tau = \int \sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma}}\, d\sigma.$$

8.2 Geodesic equations

$$\boxed{\begin{aligned} 0 &= \frac{d}{d\tau}\Big(g_{\alpha\nu}\frac{dx^{\nu}}{d\tau}\Big) - \frac{1}{2}\,\partial_\alpha g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} \\ &= \frac{d^2x^{\rho}}{d\tau^2} + \underbrace{\frac{1}{2}g^{\rho\alpha}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)}_{\Gamma^{\rho}{}_{\mu\nu}}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} \end{aligned}}$$

8.3 Normalization

$$u\cdot u = g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = -1 \text{ (timelike)}, \quad = 0 \text{ (null)}.$$

Ch 9. Schwarzschild Metric

9.0 Spherical Coordinates for Flat Spacetime:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

9.1 Schwarzschild Metric

$$\boxed{ds^2 = \underbrace{-\left(1 - \frac{2GM}{r}\right)}_{g_{tt}\text{ time part}} dt^2 + \underbrace{\left(1 - \frac{2GM}{r}\right)^{-1}}_{g_{rr}\text{ radial part}} dr^2 + \underbrace{r^2}_{g_{\theta\theta}} d\theta^2 + \underbrace{r^2\sin^2\theta\, d\phi^2}_{g_{\phi\phi}}}.$$

Features: spherically symmetric, static, vacuum, and becomes the flat space metric in the limit as $r \to \infty$.

Units: G has units of m/kg, and GM in units of m.

9.2 Meaning of r

$$r = \frac{C}{2\pi} \text{ (circumference)}, \qquad ds = \frac{dr}{\sqrt{1 - 2GM/r}} \ (t = \text{const}).$$

9.3 Newtonian limit and r_s

$$\left.\frac{d^2r}{d\tau^2}\right|_{\text{rest}} = -\frac{1}{2}\frac{r_s}{r^2} \ \Rightarrow\ r_s = 2GM.$$

9.4 Gravitational time dilation & redshift

Time dilation in Schwarzschild metric:

$$\boxed{\Delta\tau_r = \sqrt{1 - \tfrac{2GM}{r}}\,\Delta t}$$

Redshift in Schwarzschild metric:

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1-\frac{2GM}{r_R}}{1-\frac{2GM}{r_E}}} \approx 1 + g\,h \quad (\text{if } \frac{2GM}{r} \ll 1 \text{ and } h \equiv r_R - r_E \ll r_E)$$

where $g = GM/r_E^2$.

Ch 10. Particle Orbits

10.1 Conserved quantities (equatorial)

$$e = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}, \qquad l = r^2\frac{d\phi}{d\tau}.$$

10.2 Radial energy form

$$\tfrac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = E - \left[\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}\right], \quad E = \tfrac{1}{2}(e^2-1).$$

10.3 Circular orbits & Kepler

$$\frac{dV}{dr} = 0 \;\Rightarrow\; r_c = \frac{l^2}{2GM}\left(1 \pm \sqrt{1-12(GM/l)^2}\right), \quad \Omega^2 = \frac{GM}{r_c^3}.$$

10.4 Acceleration and ISCO

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} + \frac{l^2}{r^3} - \frac{3GMl^2}{r^4}, \quad r_{\rm ISCO} = 6GM,\; e_{\rm ISCO} = \sqrt{8/9}.$$

Ch 12. Photon Orbits

12.1 Impact parameter

$$\boxed{b = \frac{l}{e} = r^2\left(1 - \frac{2GM}{r}\right)^{-1}\frac{d\phi}{dt}}.$$

12.2 Equation of Radial Motion for a Photon

For null (lightlike) curves, the equation of motion is:

$$1 = \frac{1}{(1-2GM/r)^2}\left(\frac{dr}{dt}\right)^2 + \left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}$$

$$\iff \frac{1}{b^2} = \frac{1}{b^2(1-2GM/r)^2}\left(\frac{dr}{dt}\right)^2 + \frac{1-2GM/r}{r^2}$$

has a peak at $r = 3GM \Rightarrow b_c^2 = 27(GM)^2$.

12.3 Equatorial plane

$$\theta = \frac{\pi}{2}, \quad d\theta = 0, \quad \sin\theta = 1$$

12.4 Photon 4-momentum (equatorial)

For photons in the equatorial plane:

$$p^\mu = \frac{E}{1-2GM/r}\frac{dx^\mu}{dt}$$

Components:

$$p^t = \frac{E}{(1-2GM/r)}$$

$$p^\phi = \frac{E}{(1-2GM/r)}\frac{d\phi}{dt} = E\frac{b}{r^2}$$

$$p^\theta = 0$$

$$p^r = \frac{E}{(1-2GM/r)}\frac{dr}{dt} = \pm E\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)}$$

12.5 Observables for static observers

$$\sin\psi_c = \frac{3\sqrt{3}\,GM}{r}\sqrt{1 - \frac{2GM}{r}}, \qquad E_{\rm obs} = \frac{E}{\sqrt{1-\frac{2GM}{r}}}.$$

Ch 14. Event Horizon

14.1 Horizon signals

$$ds^2\Big|_{\rm static} = -\left(1 - \frac{2GM}{r}\right)dt^2 \Rightarrow ds^2 = 0\;(r = 2GM), \qquad \frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1-\frac{2GM}{r_R}}{1-\frac{2GM}{r_E}}} \to \infty\;(r_E \to 2GM^+).$$

14.2 Coordinate freeze at the horizon

$$\frac{dr}{dt},\; \frac{d\phi}{dt} \;\propto\; (1-2GM/r) \;\to\; 0 \quad (r \to 2GM^+).$$

14.3 Finite distances/times

$$\mathcal{D}(R \to 2GM) = \int_{2GM}^{R}\frac{dr}{\sqrt{1-2GM/r}}$$

$$= \sqrt{R(R-2GM)} + GM\ln\left|\frac{\sqrt{R}+\sqrt{R-2GM}}{\sqrt{2GM}}\right|.$$

$$\Delta\tau\Big|_{\text{rest at } R\to 0} = \frac{\pi R^{3/2}}{\sqrt{8GM}}, \qquad \Delta\tau_{\max}(2GM \to 0) = \pi GM.$$