Problem Set 4 | **MATH 545** | **Erise He**

General Relativity

Problem 1 **Particle orbits: Book problem P10.2.** P10.2 An object falls radially inward toward a black hole with mass M, starting at rest at infinity. How much time will a clock on the object register between the events of the object passing through the Schwarzschild radial coordinates $r = 10GM$ and $r = 2GM$? (Hint: Argue that an object released from rest at infinity will have $\tilde{E} = 0$, i.e., $e = 1$.)

For radial motion, $d\theta = d\phi = 0$, the Schwarzschild metric is

$$\begin{aligned} ds^2 &= -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &= -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 \end{aligned}$$

We know $ds^2 = -d\tau^2$, hence the 4–velocity normalization

$$\begin{aligned} -d\tau^2 &= -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 \\ -1 &= -\Big(1 - \frac{2M}{r}\Big)\Big(\frac{dt}{d\tau}\Big)^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}\Big(\frac{dr}{d\tau}\Big)^2 \end{aligned}$$

At $r \to \infty$, the metric becomes Minkowski, so

$$e = \lim_{r\to\infty}\Big(1 - \frac{2M}{r}\Big)\frac{dt}{d\tau} = 1$$

for $dt/d\tau \to 1$ released from rest. Then we know

$$\frac{dt}{d\tau} = \frac{e}{1 - \frac{2M}{r}} = \frac{1}{1 - \frac{2M}{r}}$$

Then plugging in back, we have

$$\begin{aligned} -1 &= -\Big(\frac{1}{1 - \frac{2M}{r}}\Big) + \Big(1 - \frac{2M}{r}\Big)^{-1}\Big(\frac{dr}{d\tau}\Big)^2 \\ \Big(\frac{dr}{d\tau}\Big)^2 &= \Big(\frac{1}{1 - \frac{2M}{r}} - 1\Big)\Big(1 - \frac{2M}{r}\Big) \\ \Big(\frac{dr}{d\tau}\Big)^2 &= 1 - 1 + \frac{2M}{r} \\ \frac{dr}{d\tau} &= -\sqrt{\frac{2M}{r}} \quad \text{(inward motion)} \end{aligned}$$

Next, we integrate to get τ:

$$
\begin{aligned}
\Delta\tau = \int_{r_1}^{r_2} d\tau &= \int_{r_1}^{r_2} \left(-\frac{dr}{\sqrt{\frac{2M}{r}}}\right) \\
&= \frac{1}{\sqrt{2M}} \int_{r_2}^{r_1} r^{1/2} dr \\
&= \frac{1}{\sqrt{2M}} \cdot \frac{2}{3} \left[r^{3/2} \right]_{r_2}^{r_1}
\end{aligned}
$$

For $r_1 = 10M,\ r_2 = 2M$,

$$
\begin{aligned}
\Delta\tau &= \frac{2}{3\sqrt{2M}} \left[(10M)^{3/2} - (2M)^{3/2} \right] \\
&= \frac{2}{3\sqrt{2M}} \left(10^{3/2} M^{3/2} - 2^{3/2} M^{3/2} \right) \\
&= \frac{2}{3\sqrt{2M}} \cdot M^{3/2} (10^{3/2} - 2^{3/2}) \\
&= \frac{2M}{3\sqrt{2}} \left(\sqrt{1000} - \sqrt{8} \right) \\
&= \frac{2M}{3} (10\sqrt{5} - 2) \\
&= \frac{1}{3} (20\sqrt{5} - 4) M \approx 13.574\, M
\end{aligned}
$$

Problem 2 **Photon orbits and local orthonormal frame: Book problem P12.7** P12.7 (Important!) Consider an observer who is falling radially into a black hole from rest at infinity.

(a) Use results from chapter 10 to argue that the components of this observer's four-velocity must be $dt/d\tau = (1 - 2GM/r)^{-1}, dr/d\tau = -\sqrt{2GM/r}, d\theta/d\tau = 0$, and $d\phi/d\tau = 0$. These, therefore, are also the Schwarzschild components of the observer's basis vector $\boldsymbol{o}_t$.

Let the four-velocity be:

$$u^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}\right)$$

where

$$\frac{d\theta}{d\tau} = 0, \quad \frac{d\phi}{d\tau} = 0 \quad \text{for radial motion}$$

Again, using $e = 1$ for $r \to \infty$, along the geodesic we know that

$$\frac{dt}{d\tau} = \frac{1}{1 - 2GM/r} = (\mathbf{o}_t)^t$$

We then can derive the same way as in Problem 1 to find:

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}} = (\mathbf{o}_t)^r$$

Therefore, we have

$$u^\mu = \left(\frac{1}{1 - 2GM/r}, -\sqrt{\frac{2GM}{r}}, 0, 0\right)$$

By definition of a local orthonormal frame (by the observer), $\mathbf{o}_t$ is tangent to the worldline:

$$\mathbf{o}_t \equiv u^\mu \partial_\mu.$$

Therefore, these four numbers are precisely the Schwarzschild components of $\mathbf{o}_t$.

(b) Let us align the observer's spatial basis vectors so that the $\boldsymbol{o}_z$ vector has no θ or ϕ components, the $\boldsymbol{o}_y$ vector has no r or ϕ components, and the $\boldsymbol{o}_x$ vector has no r or θ components (i.e., the spatial projections of these basis vectors point purely in the r, θ, and ϕ directions, respectively). The observer's metric is $\mathbf{o}_\mu \cdot \mathbf{o}_\nu = \eta_{\mu\nu}$. Prove that satisfying this relation for all μ and ν necessarily implies that

$$(\boldsymbol{o}_z)^\mu = \begin{bmatrix} \frac{-\sqrt{2GM/r}}{1-2GM/r} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and that the $\boldsymbol{o}_y$ and $\boldsymbol{o}_x$ vectors are as given in equation 12.10. (Do not just show that these results work: prove that these basis vector must have these components given our assumptions.)

We are given that the spatial components point purely in r, θ, ϕ directions. So:

$$(\mathbf{o}_z)^\mu = (A, B, 0, 0), \qquad (\mathbf{o}_y)^\mu = (0, 0, C, 0), \qquad (\mathbf{o}_x)^\mu = (0, 0, 0, D)$$

For convenience, let $f = 1 - \frac{2GM}{r}$. First solve for $(\mathbf{o}_z)^\mu$ using $\mathbf{o}_t \cdot \mathbf{o}_z = 0$ and $\mathbf{o}_z \cdot \mathbf{o}_z = 1$. We have

$$\begin{aligned}
0 = \mathbf{o}_t \cdot \mathbf{o}_z &= g_{tt}(\mathbf{o}_t)^t(\mathbf{o}_z)^t + g_{rr}(\mathbf{o}_t)^r(\mathbf{o}_z)^r \\
&= -(f)(f^{-1})A + (f^{-1})\left(-\sqrt{\frac{2GM}{r}}\right)B \\
&= -A - f^{-1}\left(\sqrt{\frac{2GM}{r}}\right)B
\end{aligned}$$

$$\Rightarrow A = f^{-1}\sqrt{\frac{2GM}{r}}\,B$$

Normalize to get:

$$\begin{aligned}
1 = \mathbf{o}_z \cdot \mathbf{o}_z &= g_{tt}A^2 + g_{rr}B^2 \\
&= -fA^2 + f^{-1}B^2 \\
&= -f(f^{-1})^2\frac{2GM}{r}B^2 + f^{-1}B^2 \\
&= -f^{-1}\frac{2GM}{r}B^2 + f^{-1}B^2 \\
&= B^2 f^{-1}\left(1 - \frac{2GM}{r}\right) \\
&= B^2 f^{-1} f \\
&= B^2 \quad \Rightarrow \quad B = \pm 1
\end{aligned}$$

We choose outward-pointing spatial projection (+r), so $B = +1$. Therefore,

$$\boxed{A = -\frac{\sqrt{2GM/r}}{1 - 2GM/r}}$$

(c) Find the critical angle for this observer as a function of r, and draw diagrams like the one shown in problem P12.4 for $r = 4GM, 3GM, 2GM$, and GM.

Using eq. 12.13 and eq. 12.12(a)–(b), we directly have

$$
\begin{aligned}
\mathbf{o}_x \cdot \mathbf{p} &= g_{\phi\phi}(\mathbf{o}_x)^\phi \mathbf{p}^\phi = r^2\left(\frac{1}{r}\right)\left(\frac{Eb}{r^2}\right) = \frac{Eb}{r}, \\
\mathbf{o}_t \cdot \mathbf{p} &= g_{tt}(\mathbf{o}_t)^t \mathbf{p}^t + g_{rr}(\mathbf{o}_t)^r \mathbf{p}^r \\
&= -f(f^{-1})^2 E + (f^{-1})\left(-\sqrt{\frac{2GM}{r}}\right)\mathbf{p}^r \\
&= -f^{-1}\left(E - \sqrt{\frac{2GM}{r}}\,\mathbf{p}^r\right)
\end{aligned}
$$

Hence,

$$
\sin\psi = \frac{\mathbf{o}_x \cdot \mathbf{p}}{-\mathbf{o}_t \cdot \mathbf{p}} = \frac{Eb/r}{E - \sqrt{2GM/r}\,\mathbf{p}^r} f(\mathbf{p}^r)^{-1} = \frac{b/r}{1 - \sqrt{\frac{2GM}{r}}\frac{\mathbf{p}^r}{E}} f.
$$

where

$$
\frac{\mathbf{p}^r}{E} = +\sqrt{1 + \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)}
$$

For the critical angle $b = b_c = \sqrt{27}\,GM$, so finally

$$
\sin\psi_c(r) = \frac{(\sqrt{27}\,GM/r)f}{1 + \sqrt{\frac{2GM}{r}}\sqrt{1 - 27(GM/r)^2 f}}
$$

(d) Imagine that such a falling observer receives radial signals from infinity. Will these signals be red-shifted or blue-shifted? What is the fractional change in wavelength of these signals?

The signal should be red-shifted, because $\lambda \propto 1/E$:

$$
\frac{\lambda_{\text{obs}}}{\lambda_\infty} = \left(\frac{E_{\text{obs}}}{E_\infty}\right)^{-1} = \frac{1 - 2GM/r}{1 - \sqrt{2GM/r}}
$$

And the fractional change should be

$$
\begin{aligned}
\frac{\Delta\lambda}{\lambda_\infty} = \frac{\lambda_{\text{obs}} - \lambda_\infty}{\lambda_\infty} &= \frac{1 - 2GM/r}{1 - \sqrt{2GM/r}} - 1 \\
&= \frac{1 - s^2}{1 - s} - 1 \quad (s = \sqrt{2GM/r}) \\
&= \frac{1 - s^2 - 1 + s}{1 - s} \\
&= \frac{s - s^2}{1 - s} \\
&= \frac{s(1 - s)}{1 - s} = s = \sqrt{\frac{2GM}{r}}
\end{aligned}
$$

Problem 3 **More photon orbits and local orthonormal frame: Book problem P12.8. Compare this to what an observer at rest at the same coordinate r observes.** P12.8 Consider an observer who is falling radially from rest at infinity, as in problem P12.7. Imagine that just as the observer falls through $r = 6GM$, an object in a circular orbit at that radius happens to pass through the observer's coordinate system. Use the results of problem P12.7 to determine that object's ordinary velocity components and its speed in the observer's frame.

For a circular geodeisc at fixed r in a equatorial plane ($\theta = \pi/2$), we have

$$\frac{dr}{d\tau} = 0, \quad \frac{d\theta}{d\tau} = 0$$

and the four-velocity is

$$\mathbf{u} = \left(\frac{dt}{d\tau}, 0, 0, \frac{d\phi}{d\tau}\right)$$

where we know from Chapter 10 that $\Omega \equiv \frac{d\phi}{dt} = \sqrt{\frac{GM}{r^3}}$, and

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{3GM}{r}}}, \quad \frac{d\phi}{d\tau} = \frac{\sqrt{GM/r^3}}{\sqrt{1 - \frac{3GM}{r}}}$$

Using the same results as in Problem 2, the measured ordinary velocity components in this LOF are obtained as

$$v_x = \frac{\mathbf{o}_x \cdot \mathbf{u}}{-\mathbf{o}_t \cdot \mathbf{u}}, \quad v_y = \frac{\mathbf{o}_y \cdot \mathbf{u}}{-\mathbf{o}_t \cdot \mathbf{u}}, \quad v_z = \frac{\mathbf{o}_z \cdot \mathbf{u}}{-\mathbf{o}_t \cdot \mathbf{u}}$$

where $\mathbf{u}$ is the object's four-velocity. Therefore, compute:

$$\begin{aligned}
\mathbf{o}_x \cdot u &= g_{\phi\phi} (\mathbf{o}_x)^\phi u^\phi = r^2 \left(\frac{1}{r}\right) \frac{d\phi}{d\tau} = r\frac{d\phi}{d\tau} \\
\mathbf{o}_y \cdot u &= g_{\theta\theta} \left(\mathbf{o}_y\right)^\theta u^\theta = r^2 \left(\frac{1}{r}\right) 0 = 0 \\
\mathbf{o}_z \cdot u &= g_{tt} (\mathbf{o}_z)^t u^t + g_{rr} (\mathbf{o}_z)^r u^r \\
&= -\left(1 - \frac{2GM}{r}\right) \left(-\frac{\sqrt{2GM/r}}{1 - \frac{2GM}{r}}\right) \frac{dt}{d\tau} = \sqrt{\frac{2GM}{r}} \frac{dt}{d\tau} \\
\mathbf{o}_t \cdot u &= g_{tt} (\mathbf{o}_t)^t u^t + g_{rr} (\mathbf{o}_t)^r u^r \\
&= -\left(1 - \frac{2GM}{r}\right) \left(\frac{1}{1 - \frac{2GM}{r}}\right) \frac{dt}{d\tau} = -\frac{dt}{d\tau}
\end{aligned}$$

Therefore the components are

$$
\begin{aligned}
v_x &= \frac{\mathbf{o}_x \cdot u}{-\mathbf{o}_t \cdot u} = \frac{r\frac{d\phi}{d\tau}}{\frac{dt}{d\tau}} = r\frac{d\phi}{dt} = r\Omega = \sqrt{\frac{GM}{r}} \\
v_y &= 0 \\
v_z &= \frac{\mathbf{o}_z \cdot u}{-\mathbf{o}_t \cdot u} = \frac{\sqrt{\frac{2GM}{r}}\frac{dt}{d\tau}}{\frac{dt}{d\tau}} = \sqrt{\frac{2GM}{r}}
\end{aligned}
$$

Then, we evaluate at $r = 6GM$

$$
(v_x, v_y, v_z)\,|_{r=6GM} = \left(\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{3}}\right)
$$

Hence the speed in the observer's frame is

$$
\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\frac{1}{6} + 0 + \frac{1}{3}} = \frac{1}{\sqrt{2}} \approx 0.7071
$$

Problem 4 **Behavior at event horizon for local observer: Book problem P14.3** P14.3 As discussed in the chapter, a falling object appears to a distant observer to "freeze" at the event horizon. This, however, is purely an artifact of the coordinates. Consider an inward-falling object moving in the equatorial plane with arbitrary e and ℓ. Show that an observer at rest at a given R will measure its squared speed to be

$$v_{\text{obs}}^2 = 1 - \frac{1}{e^2}\left(1 - \frac{2GM}{R}\right)$$

Note that this approaches 1 (the speed of light) as R approaches $2GM$, no matter what the particle's energy-per-unit-mass-at-infinity e might be. So compared to any local observer, the particle does not "freeze." [Hints: Use the equations in table 14.1 to evaluate the particle's fourvelocity $\boldsymbol{u} = [dt/d\tau, dr/d\tau, 0, d\phi/d\tau]$ at R for arbitrary e and ℓ. Then use the techniques discussed in chapter 12 to evaluate the components of $\boldsymbol{u}$ in the orthonormal coordinate system at rest at R, and finally use $v_{\text{obs},x} = u^x/u^t$ and $v_{\text{obs},z} = u^z/u^t$ in that frame.]

Using Table 14.1 to find the four-velocity of the infalling particle (massive particle $m > 0$) at $r = R$, we have

$$\begin{aligned}
\frac{dt}{d\tau} &= \frac{e}{1 - \frac{2GM}{r}} \\
\frac{d\phi}{d\tau} &= \frac{\ell}{r^2} \\
\frac{dr}{d\tau} &= \pm\sqrt{e^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)}
\end{aligned}$$

we evaluate at the event $r = R$:

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)_{r=R} = \left(\frac{e}{1 - \frac{2GM}{R}}, \frac{dr}{d\tau}, 0, \frac{\ell}{R^2}\right)$$

The orthonormal basis vectors in global components are

$$\begin{aligned}
(\mathbf{o}_t)^\mu &= \left(\frac{1}{\sqrt{1 - \frac{2GM}{R}}}, 0, 0, 0\right) \\
(\mathbf{o}_z)^\mu &= \left(0, \sqrt{1 - \frac{2GM}{R}}, 0, 0\right) \\
(\mathbf{o}_y)^\mu &= (0, 0, 1/R, 0) \\
(\mathbf{o}_x)^\mu &= (0, 0, 0, 1/R)
\end{aligned}$$

where the last one uses $\theta = \pi/2 \Rightarrow \sin\theta = 1$). As in eq. (12.13), measured ordinary components are ratios of dot products:

$$v_{\text{obs},i} = \frac{\mathbf{o}_i \cdot u}{-\mathbf{o}_t \cdot u} \quad (i = x, y, z).$$

then use the Schwarzschild metric to compute the needed dot products at $r = R$:

$$\begin{aligned}
-\mathbf{o}_t \cdot u &= \sqrt{1 - \frac{2GM}{R}} \frac{dt}{d\tau} = \sqrt{1 - \frac{2GM}{R}} \frac{e}{1 - \frac{2GM}{R}} = \frac{e}{\sqrt{1 - \frac{2GM}{R}}} \\
\mathbf{o}_z \cdot u &= \left(1 - \frac{2GM}{R}\right)^{-1} \sqrt{1 - \frac{2GM}{R}} \frac{dr}{d\tau} = \frac{1}{\sqrt{1 - \frac{2GM}{R}}} \frac{dr}{d\tau} \\
\mathbf{o}_x \cdot u &= R^2 \left(\frac{1}{R}\right) \frac{d\phi}{d\tau} = R \frac{d\phi}{d\tau} = \frac{\ell}{R} \\
\mathbf{o}_y \cdot u &= 0
\end{aligned}$$

Therefore

$$v_{\text{obs},z} = \frac{\mathbf{o}_z \cdot u}{-\mathbf{o}_t \cdot u} = \frac{\frac{dr}{d\tau}}{e}, \quad v_{\text{obs},x} = \frac{\mathbf{o}_x \cdot u}{-\mathbf{o}_t \cdot u} = \frac{\sqrt{1 - \frac{2GM}{R}}\ell}{eR}, \quad v_{\text{obs},y} = 0$$

Using the radial equation at $r = R$

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - \left(1 - \frac{2GM}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right)$$

we have

$$\begin{aligned}
v_{\text{obs}}^2 &= v_{\text{obs},x}^2 + v_{\text{obs},y}^2 + v_{\text{obs},z}^2 \\
&= \left(\frac{\sqrt{1 - \frac{2GM}{R}}\ell}{eR}\right)^2 + \left(\frac{dr/d\tau}{e}\right)^2 \\
&= \frac{1}{e^2}\left(1 - \frac{2GM}{R}\right)\frac{\ell^2}{R^2} + \frac{1}{e^2}\left[e^2 - \left(1 - \frac{2GM}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right)\right] \\
&= \frac{1}{e^2}\left(1 - \frac{2GM}{R}\right)\frac{\ell^2}{R^2} + 1 - \frac{1}{e^2}\left(1 - \frac{2GM}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right) \\
&= 1 - \frac{1}{e^2}\left(1 - \frac{2GM}{R}\right)
\end{aligned}$$

which is

$$v_{\text{obs}}^2 = 1 - \frac{1}{e^2}\left(1 - \frac{2GM}{R}\right)$$

Problem 5 Event horizon observations: The Event Horizon Telescope (EHT) released the first ever image of light coming from matter orbiting near a black hole event horizon a few years ago. The press release and links to science papers can be found here: Link to EHT press release.

Using the science papers as references:

a) Using the mass of the black hole in M87 as determined by the EHT, calculate the radius of its event horizon, the photon orbit, and the ISCO in physical units (km or AU). Assume that the BH is a non-rotating (Schwarzschild) black hole.

Since we work in Schwarzschild with G, c kept explicit, we have:

(i) Event horizon

The horizon is the null surface where $g_{tt} = 0 \Rightarrow 1 - 2GM/\left(c^2 r\right) = 0$:

$$r_{\mathrm{H}} = \frac{2GM}{c^2}$$

(ii) Photon sphere r_{ph}

For equatorial null geodesics ($ds^2 = 0$), the effective potential for angular momentum L is

$$V_{\mathrm{eff}}(r) = \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

then a circular null orbit satisfies $dV_{\mathrm{eff}}/dr = 0$

$$\frac{d}{dr}\left[\frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)\right] = L^2\left(-\frac{2}{r^3} + \frac{6GM}{c^2 r^4}\right) = 0$$
$$\Longrightarrow \quad r_{\mathrm{ph}} = \frac{3GM}{c^2}$$

(iii) ISCO r_{ISCO}

For timelike circular orbits, we have

$$E^2 = \left(1 - \frac{2GM}{c^2 r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

and impose $dE^2/dr = 0$ (circular) and $d^2E^2/dr^2 = 0$ (marginal stability). Solving gives

$$r_{\mathrm{ISCO}} = \frac{6GM}{c^2}$$

or we equivalently have $r_{\mathrm{H}} = 2M, r_{\mathrm{ph}} = 3M, r_{\mathrm{ISCO}} = 6M$. Using $GM_{\odot}/c^2 =$

1.4766 km so $r_{S,\odot} = 2GM_\odot/c^2 = 2.9533$ km, then

$$r_H = \frac{2GM}{c^2} = (2.9533 \text{ km}) \times \left(6.5 \times 10^9\right) = 1.92 \times 10^{10} \text{ km}$$
$$r_{ph} = \frac{3GM}{c^2} = \frac{3}{2} r_H = 2.88 \times 10^{10} \text{ km}$$
$$r_{ISCO} = \frac{6GM}{c^2} = 3r_H = 5.76 \times 10^{10} \text{ km}$$

where we use $M = 6.5 \times 10^9 M_\odot$ from EHT results.

b) What is the angular distance from the BH of the innermost light-emitting region measured by the EHT, and how close is this to the photon orbit?

We take $D = 55 \times 10^6$ ly, then for small angles, $\theta \approx r/D$.

(i) Angular radius of the photon orbit

$$\theta_{ph} = \frac{r_{ph}}{D} = \frac{3GM/c^2}{D} \Rightarrow \theta_{ph} \approx 11.4\mu\text{as}.$$

where i convert radians to microarcseconds via $1\text{rad} = 206265 \times 10^6 \mu\text{as}$.

(b) how close

Due to strong lensing, photons orbiting near r_{ph} reach the observer with impact parameter

$$b_c = \sqrt{27}\frac{GM}{c^2}$$

Thus the angular radius is

$$\theta_{sh} = \frac{b_c}{D} = \sqrt{27}\frac{GM}{c^2 D} = \sqrt{3}\theta_{ph}$$

Using $M = 6.5 \times 10^9 M_\odot, D = 55 \times 10^6$, we have

$$\theta_{sh} \approx 19.8\mu\text{as} \quad \Rightarrow \quad \text{predicted diameter } \approx 39.5\mu\text{as}$$

EHT measured a ring diameter $42 \pm 3\mu$as (radius $21 \pm 1.5\mu$as), which is in amazing agreement with the GR prediction above.