example: helicoid to catenoid isometry

S = helicoid $\sigma(\rho, t) = (\rho \cos t, \rho \sin t, t)$

$\tilde{S}$ = catenoid $\eta(z, \theta) = (\cosh z \cos\theta, \cosh z \sin\theta, z)$

define Ψ by $\begin{array}{l} z = \operatorname{arcsinh} \rho \\ \theta = t \end{array}$ $\qquad (\rho = \sinh z)$

check $\Psi^*(d\tilde{s}^2) = ds^2$

Area

Recall from calculus, for a surface patch $\sigma(u^1, u^2)$ area is computed by integrating

$$dA := \|\sigma_1 \times \sigma_2\| \, du^1 du^2$$

In terms of the 1st fund. form:

$$\begin{aligned} \|\sigma_1 \times \sigma_2\|^2 &= \|\sigma_1\|^2 \|\sigma_2\|^2 \sin^2\theta \\ &= \|\sigma_1\|^2 \|\sigma_2\|^2 - (\sigma_1 \cdot \sigma_2)^2 \\ &= g_{11} g_{22} - g_{12}^2 \\ &= \det [g_{ij}] \end{aligned}$$

We write this as

$$dA = \sqrt{\det g} \, du^1 du^2$$

examples

1) cylinder $\sigma(z, \theta) = (\cos\theta, \sin\theta, z)$

$ds^2 = dz^2 + d\theta^2$, $\det g = 1 \Rightarrow dA = dz \, d\theta$

2) sphere in cylindrical coordinates $\{r^2+z^2=1\}$

$$\sigma(z,\theta) = \left(\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z\right)$$

Check: $\det g = 1$

$dA = dz\, d\theta$ (same as for cylinder!)

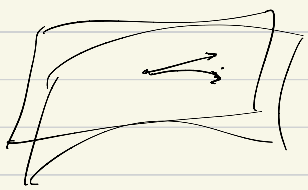
(Archimedes argued that this map preserves area)

Curvature of Surfaces

We start by measuring deviation from the tangent plane.

In a coord. patch, the tangent plane represents a linear approximation

$$\sigma(u+\Delta u) \longrightarrow \sigma(u) + \underbrace{\sigma_i \Delta u^i}_{V} \in \mathbb{R}^3$$



← let d = distance from $\sigma(u+\Delta u)$ to tangent plane

let N = unit normal to T_pS $\quad p = \sigma(u)$

Then $d = [\sigma(u+\Delta u) - \sigma(u)]\cdot N$

Since $\sigma_i \perp N$, the leading term is quadratic in Δu

By Taylor expansion, $d = (\sigma_{ij}\cdot N)\Delta u^i \Delta u^j + O(\Delta u^3)$

define: the <u>second fundamental form</u> is the family of quadratic forms represented in local coord.'s

$$\mathrm{II} := (\sigma_{ij}\cdot N)\, du^i du^j$$