Ch 15. Kruskal–Szekeres

Schwarzschild metric (vacuum, $r > 2GM$):

$$ds^2 = -\Big(1-\frac{2GM}{r}\Big)dt^2 + \Big(1-\frac{2GM}{r}\Big)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

Kruskal–Szekeres coordinates (u, v):

$$\boxed{u^2 - v^2 = \Big(\frac{r}{2GM} - 1\Big)e^{r/2GM}}, \qquad t = 2GM \ln\Big|\frac{u+v}{u-v}\Big|.$$

Kruskal–Szekeres metric:

$$ds^2 = -\frac{32(GM)^3}{r}e^{-r/2GM}(dv^2 - du^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

with $r = r(u, v)$ implicitly via $u^2 - v^2$. The horizon $r = 2GM$ is $u = \pm v$ (null lines); radial photons obey $du = \pm dv$ (45° lines).

Ch 16. Black Hole Thermodynamics

Energy per unit mass at infinity in Schwarzschild:

$$e \equiv \Big(1 - \frac{2GM}{r}\Big)\frac{dt}{d\tau}.$$

Hawking temperature:

$$\boxed{k_B T = \frac{\hbar}{8\pi GM}}, \qquad T = \frac{\hbar}{8\pi k_B GM}.$$

Black-hole lifetime (order of magnitude):

$$\tau_{\text{life}} \approx 2.1 \times 10^{67}\,\text{yr}\Big(\frac{M}{M_\odot}\Big)^3.$$

Entropy (Bekenstein–Hawking):

$$\boxed{S = \frac{4\pi k_B GM^2}{\hbar} = \frac{k_B}{4G\hbar}A}, \qquad A = 4\pi(2GM)^2.$$

Ch 17–19. Covariant Derivative and Curvature

Christoffel symbols (Levi–Civita connection):

$$\boxed{\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})}.$$

Absolute gradient (covariant derivative):

$$\nabla_\alpha A^\mu = \partial_\alpha A^\mu + \Gamma^\mu_{\alpha\nu}A^\nu, \qquad \nabla_\alpha B_\mu = \partial_\alpha B_\mu - \Gamma^\nu_{\alpha\mu}B_\nu.$$

General tensor (example):

$$\nabla_\alpha T^{\mu\nu}{}_\sigma = \partial_\alpha T^{\mu\nu}{}_\sigma + \Gamma^\mu_{\alpha\beta}T^{\beta\nu}{}_\sigma + \Gamma^\nu_{\alpha\beta}T^{\mu\beta}{}_\sigma - \Gamma^\beta_{\alpha\sigma}T^{\mu\nu}{}_\beta.$$

Metric compatibility and geodesics:

$$\nabla_\alpha g_{\mu\nu} = 0, \qquad \boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0}.$$

Riemann tensor:

$$\boxed{R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\gamma\mu}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\beta\mu}}.$$

Geodesic deviation:

$$\boxed{\Big(\frac{d^2n}{d\tau^2}\Big)^\alpha = -R^\alpha{}_{\mu\nu\sigma}u^\mu u^\nu n^\sigma}.$$

Ricci tensor, scalar, and key symmetries:

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}, \qquad R \equiv g^{\mu\nu}R_{\mu\nu},$$

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta},$$

$$R_{\alpha[\beta\mu\nu]} = 0, \qquad \nabla_\sigma R^\alpha{}_{\beta\mu\nu} + \nabla_\mu R^\alpha{}_{\beta\nu\sigma} + \nabla_\nu R^\alpha{}_{\beta\sigma\mu} = 0.$$

Ch 20. Stress–Energy as Source

Dust:

$$T^{\mu\nu}_{\text{dust}} = \rho_0 u^\mu u^\nu.$$

Perfect fluid (rest-frame density ρ_0, pressure p_0):

$$\boxed{T^{\mu\nu} = (\rho_0 + p_0)u^\mu u^\nu + p_0 g^{\mu\nu}}.$$

Physical meanings:

- T^{tt}: energy density
- T^{it}: i-momentum density and i-flux of energy
- T^{ij}: i-flux of j-momentum (stresses).

Local energy–momentum conservation:

$$\boxed{\nabla_\mu T^{\mu\nu} = 0}.$$

For a perfect fluid this yields (in a LIF, nonrelativistic limit) the continuity and Euler equations:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0,$$

$$\rho\Big(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\Big) = -\boldsymbol{\nabla}p.$$

Ch 21. Einstein Equation

Einstein tensor:

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R, \qquad \nabla_\nu G^{\mu\nu} = 0.$$

Einstein equation (with cosmological constant):

$$\boxed{G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G\, T^{\mu\nu}}.$$

Alternative form:

$$\boxed{R^{\mu\nu} = 8\pi G\Big(T^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}T\Big) + \Lambda g^{\mu\nu}}, \qquad T \equiv T^\alpha{}_\alpha.$$

Vacuum energy:

$$T^{\mu\nu}_{\text{vac}} = -\frac{\Lambda}{8\pi G}g^{\mu\nu}.$$

Newtonian limit (weak field, slow motion, $T^{tt} \approx \rho$):

$$\boxed{R_{tt} \approx \nabla^2\Phi, \qquad \nabla^2\Phi = 4\pi G\rho},$$

fixing the coupling $\kappa = 8\pi G$.

Ch 22. Weak Field and Stationary Sources

Metric perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

Weak-field gauge condition (Lorentz / de Donder):

$$H_\nu \equiv \eta^{\mu\alpha}\Big(\partial_\mu h_{\alpha\nu} - \tfrac{1}{2}\partial_\nu h_{\alpha\mu}\Big) = 0.$$

In this gauge, for stationary sources ($\partial_t = 0$):

$$\nabla^2 h_{\beta\nu} = -16\pi G\Big(T_{\beta\nu} - \tfrac{1}{2}\eta_{\beta\nu}T\Big).$$

For a perfect fluid with rest-frame (ρ_0, p_0):

$$2T_{tt} - \eta_{tt}T \approx \rho_0 + 3p_0 \equiv \rho_g,$$

$$-T_{ti} + \tfrac{1}{2}\eta_{ti}T \approx (\rho_0 + p_0)u_i \equiv \Pi_i,$$

$$2T_{ii} - \eta_{ii}T \approx \rho_0 - p_0 \equiv \rho_c.$$

Spherical, nonrotating, weak field:

$$h_{tt} = h_{xx} = h_{yy} = h_{zz} = \frac{2GM}{r},$$

giving Newtonian acceleration $\ddot{\mathbf{x}} = -\boldsymbol{\nabla}(GM/r)$.

Ch 23. Schwarzschild Solution

Trial metric (spherically symmetric):

$$ds^2 = -A(r,t)\,dt^2 + B(r,t)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

Vacuum Einstein eq: $R_{\mu\nu} = 0$. From $R_{tr} = 0$: $\partial B/\partial t = 0$ (B time-independent).

Key relation (from $R_{tt} = R_{rr} = 0$):

$$\frac{1}{A}\frac{\partial A}{\partial r} = -\frac{1}{B}\frac{dB}{dr}.$$

Solution: $1/B = A = 1 + C/r$. Christoffel symbol:

$$\Gamma^r_{tt} = \frac{1}{2B}\frac{\partial A}{\partial r}.$$

Geodesic for particle at rest ($u^i = 0$, only $u^t \neq 0$):

$$\frac{d^2 r}{d\tau^2} = -\frac{1}{2}\frac{\partial A}{\partial r} = \frac{C}{2r^2}.$$

Matching to Newton ($d^2r/dt^2 = -GM/r^2$) gives $C = -2GM$.
Schwarzschild metric:

$$\boxed{ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2}.$$

Key facts:
- $r = 2GM$: event horizon (Schwarzschild radius).
- **Birkhoff's theorem**: vacuum exterior of *any* spherically symmetric source is Schwarzschild, even if source is time-dependent.
- M = Newtonian mass (from test particle motion at $r \to \infty$).

Oppenheimer–Volkoff equation (hydrostatic equilibrium):

$$\boxed{\frac{dp}{dr} = -\frac{(\rho + p)(Gm(r) + 4\pi G p r^3)}{r^2(1 - 2Gm(r)/r)}},$$

where $m(r) = \int_0^r 4\pi\rho r'^2 dr'$. Newtonian limit: $p \ll \rho$, $2Gm/r \ll 1$ gives $dP/dr = -GM(r)\rho/r^2$.

Ch 30. Linearized Einstein & Gauge

Perturbation and trace reverse:

$$h_{\mu\nu} = h_{\nu\mu}, \qquad h \equiv \eta^{\mu\nu} h_{\mu\nu},$$

$$H_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h, \qquad H \equiv \eta^{\mu\nu} H_{\mu\nu} = -h.$$

Weak-field Einstein equation (in terms of $H_{\mu\nu}$):

$$\Box^2 H^{\mu\nu} - \partial^\mu \partial_\alpha H^{\alpha\nu} - \partial^\nu \partial_\alpha H^{\alpha\mu} + \eta^{\mu\nu}\partial_\alpha\partial_\beta H^{\alpha\beta} = -16\pi G T^{\mu\nu},$$

where $\Box^2 \equiv \partial^\alpha \partial_\alpha$.
Lorentz gauge:

$$\partial_\mu H^{\mu\nu} = 0 \quad\Longrightarrow\quad \Box^2 H^{\mu\nu} = -16\pi G T^{\mu\nu}.$$

Gauge transformation ($x'^\mu = x^\mu + \xi^\mu$, $|\xi^\mu| \ll 1$):

$$H'_{\mu\nu} = H_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\alpha \xi^\alpha,$$

leaving the weak-field curvature invariant.

Ch 31. Gravitational Waves & TT Gauge

In vacuum and Lorentz gauge:

$$\boxed{\Box^2 H^{\mu\nu} = 0}, \qquad \partial_\mu H^{\mu\nu} = 0.$$

Plane wave:

$$H^{\mu\nu} = A^{\mu\nu}\cos(k_\sigma x^\sigma), \qquad k^\sigma k_\sigma = 0 \Rightarrow \boxed{v_{\rm GW} = 1}.$$

Residual gauge freedom can impose TT gauge:

$$A^\mu{}_\mu = 0, \qquad A^{\mu 0} = 0.$$

For propagation in $+z$ direction, only

$$A^{xx} = A_+, \quad A^{yy} = -A_+, \quad A^{xy} = A^{yx} = A_\times$$

are nonzero ("plus" and "cross" polarizations).
Effect on a ring of test particles in the transverse plane:

$$\Delta s \approx R\Big[1 + \tfrac{1}{2}A_+ \cos\omega t \cos 2\theta\Big] \quad (+\text{ mode}),$$

$$\Delta s \approx R\Big[1 + \tfrac{1}{2}A_\times \cos\omega t \sin 2\theta\Big] \quad (\times\text{ mode}).$$

Ch 32. Gravitational-Wave Energy

In weak field on nearly flat background, expand $G_{\mu\nu}$ to second order in $h_{\mu\nu}$ and define the effective GW stress–energy:

$$T^{GW}_{\mu\nu} \equiv -\frac{\langle G^{(2)}_{\mu\nu}\rangle}{8\pi G}, \qquad \partial_\mu\big(T^{\mu\nu} + T^{\mu\nu}_{GW}\big) = 0.$$

In TT gauge for a plane wave:

$$\boxed{T^{GW}_{tt} = \frac{1}{32\pi G}\Big\langle \dot h^{TT}_{jk}\,\dot h^{jk}_{TT}\Big\rangle}, \qquad j,k = x,y,z,$$

which equals the GW energy flux in the propagation direction. For a single + polarization:

$$T^{GW}_{tt} = \frac{1}{16\pi G}\langle \dot h_+^2\rangle.$$

Ch 33. GW from Sources (Quadrupole)

Small–weak–slow source, Lorentz gauge:

$$\Box^2 H^{\mu\nu} = -16\pi G T^{\mu\nu}.$$

Retarded solution (far zone, distance R):

$$H^{\mu\nu}(t, \mathbf{R}) \approx \frac{4G}{R}\int_{\rm src} T^{\mu\nu}(t - R, \mathbf{r})\, dV.$$

Using $\int T^{jk} dV = \frac{1}{2} d^2 I^{jk}/dt^2$ with

$$I^{jk} \equiv \int \rho\, x^j x^k\, dV,$$

and the reduced (trace-free) quadrupole

$$\mathcal{I}^{jk} \equiv \int \rho\Big(x^j x^k - \tfrac{1}{3}\eta^{jk} r^2\Big) dV,$$

the TT metric at distance R is

$$\boxed{h^{TT}_{jk}(t, \mathbf{R}) = \frac{2G}{R}\ddot{\mathcal{I}}^{TT}_{jk}(t - R)}.$$

Quadrupole luminosity formula (total GW power):

$$\boxed{L_{GW} = -\frac{dE}{dt} = \frac{G}{5}\Big\langle \dddot{\mathcal{I}}_{jk}\, \dddot{\mathcal{I}}^{jk}\Big\rangle}.$$

Ch 35. Gravitomagnetism

Weak-field, slow-source limit: define potentials

$$\Phi_G \equiv -\tfrac{1}{2} h_{tt} = -\int \frac{G\rho_0}{s}\, dV, \qquad \mathbf{A}_G \equiv -\tfrac{1}{4} h_{ti}\,\hat{\mathbf{e}}_i = -\int \frac{G\mathbf{J}}{s}\, dV,$$

with mass current $\mathbf{J} = \rho_0 \mathbf{v}$.
Gravitoelectric and gravitomagnetic fields:

$$\boxed{\mathbf{E}_G \equiv -\boldsymbol{\nabla}\Phi_G - \frac{\partial \mathbf{A}_G}{\partial t}, \qquad \mathbf{B}_G \equiv \boldsymbol{\nabla}\times\mathbf{A}_G}.$$

Maxwell-like equations:

$$\boldsymbol{\nabla}\cdot\mathbf{E}_G = -4\pi G\rho_0, \qquad \boldsymbol{\nabla}\times\mathbf{B}_G - \frac{\partial \mathbf{E}_G}{\partial t} = -4\pi G\mathbf{J},$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}_G = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E}_G + \frac{\partial \mathbf{B}_G}{\partial t} = 0.$$

Gravitational Lorentz-force law (static fields, $|\mathbf{V}| \ll 1$):

$$\boxed{m\frac{d^2\mathbf{x}}{dt^2} = m\Big(\mathbf{E}_G + \mathbf{V}\times 4\mathbf{B}_G\Big)}.$$

Gyroscope (spin $\mathbf{s}$) precession in $\mathbf{B}_G$:

$$\boldsymbol{\tau} = \mathbf{s}\times 2\mathbf{B}_G, \qquad \boxed{\boldsymbol{\Omega}_{LT} = -2\mathbf{B}_G}.$$

Field of a spinning mass (spin $\mathbf{S}$):

$$\mathbf{B}_G(\mathbf{r}) = \frac{G}{2r^3}\big[3(\mathbf{S}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{S}\big].$$

Ch 36. Kerr Metric

Define (memorize these!):

$$\boxed{\Sigma \equiv r^2 + a^2\cos^2\theta, \qquad \Delta \equiv r^2 - 2GMr + a^2}.$$

Kerr metric in Boyer–Lindquist coordinates:

$$ds^2 = -\Big(1 - \frac{2GMr}{\Sigma}\Big)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 + \Big(r^2 + a^2 + \frac{2GMr\, a^2 \sin^2\theta}{\Sigma}\Big)\sin^2\theta\, d\phi^2 - \frac{4GMra\sin^2\theta}{\Sigma}\, dt\, d\phi.$$

Parameters:
• M: mass, • $a \equiv S/M$: angular momentum per unit mass (S total spin).

Limits:
• $a \to 0 \Rightarrow$ Schwarzschild,
• far from source, first order in GM/r reproduces the weak-field rotating metric.

Ch 37. Kerr Equatorial Orbits

Constants of motion (Kerr, equatorial plane):

$$e \equiv -g_{tt}\frac{dt}{d\tau} - g_{t\phi}\frac{d\phi}{d\tau}, \qquad \ell \equiv g_{t\phi}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau},$$

(e: energy per unit mass at infinity, ℓ: z-component of angular momentum per unit mass).

Energy-like radial equation (equatorial, massive particle):

$$\boxed{\tilde{E} \equiv \tfrac{1}{2}(e^2 - 1) = \tfrac{1}{2}\Big(\frac{dr}{d\tau}\Big)^2 - \frac{GM}{r} + \frac{\ell^2 + a^2(1-e^2)}{2r^2} - \frac{GM(\ell - ea)^2}{r^3}}.$$

Radial "force" form:

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} + \frac{\ell^2 + a^2(1-e^2)}{r^3} - \frac{3GM(\ell - ea)^2}{r^4}.$$

Azimuthal motion:

$$\frac{d\phi}{d\tau} = \frac{(2GMa/r)e + (1 - 2GM/r)\ell}{r^2 - 2GMr + a^2}.$$

Zero-angular-momentum trajectories ($\ell = 0$) are dragged:

$$\frac{d\phi}{d\tau} = \frac{2GMae}{r(r^2 - 2GMr + a^2)} \neq 0.$$

Kepler's third law in Kerr (equatorial circular orbits):

$$\boxed{\Omega \equiv \frac{d\phi}{dt} = \frac{\sqrt{GM}}{r^{3/2} \pm a\sqrt{GM}}},$$

$$T \equiv \Big|\frac{2\pi}{\Omega}\Big| = \sqrt{\frac{4\pi^2 r^3}{GM}} \pm 2\pi a,$$

upper sign: co-rotating, lower: counter-rotating.

ISCO condition (equatorial):

$$\boxed{r^2 - 6GMr - 3a^2 \pm 8a\sqrt{GMr} = 0}.$$

Ch 38. Ergoregion and Horizons

Infinite-redshift surfaces ($g_{tt} = 0$):

$$r = GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta}.$$

The outer one

$$r_e(\theta) = GM + \sqrt{(GM)^2 - a^2\cos^2\theta}$$

is the *static limit* bounding the ergoregion.

Kerr horizons from $\Delta = 0$:

$$\boxed{r_\pm = GM \pm \sqrt{(GM)^2 - a^2}}.$$

Outer horizon: r_+. For $a \to 0$, $r_+ \to 2GM$.

Horizon area:

$$A = 8\pi GMr_+ = 4\pi(2GM_{\rm ir})^2,$$

where $M_{\rm ir}$ is the irreducible mass (Ch. 39).

Condition for a horizon (cosmic censorship):

$$\boxed{a \leq GM};$$

if $a > GM$ there is no horizon (naked singularity, excluded by cosmic censorship conjecture).

Ch 39. Penrose Process and Irreducible Mass

Equatorial geodesic "energy" equation (massive particle):

$$\frac{1}{2}(e^2 - 1) = \tfrac{1}{2}\Big(\frac{dr}{d\tau}\Big)^2 - \frac{GM}{r} + \frac{\ell^2 + a^2(1-e^2)}{2r^2} - \frac{GM(\ell - ea)^2}{r^3}.$$

Rewrite as quadratic in e:

$$0 = -Ae^2 + Be + C + \Big(\frac{dr}{d\tau}\Big)^2,$$

$$A = 1 + \frac{a^2}{r^2} + \frac{2GMa^2}{r^3}, \quad B = \frac{4GMa}{r^3}, \quad C = 1 - \frac{2GM}{r} + \frac{\ell^2 + a^2}{r^2} - \frac{2GM\ell^2}{r^3}.$$

Effective potentials for e:

$$V_\pm(r) = \frac{\frac{1}{2}B \pm \sqrt{\frac{1}{4}B^2 + AC}}{A}, \qquad e \geq V_-(r),$$

with turning points at $e = V_-(r)$. Inside the ergoregion, $V_-(r)$ can be negative $\Rightarrow$ negative-energy orbits.

Penrose process: particle P splits into $Q + R$ in the ergoregion, with $e_Q < 0$ (falls in) and $e_R = e_P - e_Q > e_P$ (escapes), extracting rotational energy from the hole.

At the horizon $r = r_+$, one finds

$$e \geq \frac{B}{2A},$$

leading to the bound (for a particle of mass m)

$$\Delta M \geq \frac{a\,\Delta S}{r_+^2 + a^2} = \frac{a\,\Delta S}{2GMr_+},$$

with $\Delta M = me$, $\Delta S = m\ell$.

Irreducible mass:

$$\boxed{M_{\rm ir} \equiv \frac{\sqrt{2GMr_+}}{2G}}, \qquad \Delta M_{\rm ir} \geq 0,$$

and the mass decomposition

$$\boxed{M^2 = M_{\rm ir}^2 + \Big(\frac{S}{2GM_{\rm ir}}\Big)^2}.$$

For Schwarzschild ($a = 0$), $M_{\rm ir} = M$. Black-hole entropy is proportional to $A \propto M_{\rm ir}^2$ and cannot decrease in any classical process.

Minkowski metrics

Cartesian (t, x, y, z):
Motivation: Special relativity in the absence of gravity.
- Coordinates: (t, x, y, z)
- Line element:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

- Metric:

$$g_{\mu\nu} = \mathrm{diag}(-1,\,1,\,1,\,1).$$

Spherical spatial coords (t, r, θ, ϕ):
Motivation: Same flat spacetime as above.
- Coordinates: (t, r, θ, ϕ)
- Line element:

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

- Metric:

$$g_{\mu\nu} = \mathrm{diag}\big(-1,\,1,\,r^2,\,r^2\sin^2\theta\big).$$

Useful when comparing with Schwarzschild and Kerr.

2-sphere of radius r (angular part):
Motivation: Spatial geometry of a sphere of areal radius r; appears as the angular part of many 4D metrics (Schwarzschild, FRW, etc.).
- Coordinates: (θ, ϕ) (spatial 2D)
- Line element:

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

- Metric:

$$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}, \qquad i, j = \theta, \phi.$$

Curved even though embedded in flat space; Gaussian curvature $K = 1/r^2$.

Schwarzschild

Schwarzschild metric (exterior to spherical mass M):
Motivation: Unique spherically symmetric, vacuum ($T_{\mu\nu} = 0$), static solution of Einstein's equations outside a spherical mass (Birkhoff's theorem).
- Coordinates: (t, r, θ, ϕ)
- Line element:

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

- Metric:

$$g_{\mu\nu} = \begin{pmatrix} -\Big(1 - \dfrac{2GM}{r}\Big) & 0 & 0 & 0 \\ 0 & \Big(1 - \dfrac{2GM}{r}\Big)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

Key properties: Static? yes; Stationary? yes; Spherically symmetric? yes; Asymptotically flat? yes ($g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$); Horizons / singularities: event horizon at $r = 2GM$; curvature singularity at $r = 0$.

Weak-field Schwarzschild (Newtonian limit):
Motivation: Expansion of Schwarzschild for $GM/r \ll 1$; used to connect g_{tt} to Newtonian potential $\Phi_N = -GM/r$ and to derive geodesic equation $\Rightarrow$ Newtonian gravity.

For $GM/r \ll 1$:
- Coordinates: (t, x, y, z)
- Line element:

$$ds^2 \approx -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 + \frac{2GM}{r}\Big)(dx^2 + dy^2 + dz^2)$$

where $r = \sqrt{x^2 + y^2 + z^2}$.
- Metric:

$$g_{\mu\nu} \approx \begin{pmatrix} -\Big(1 - \dfrac{2GM}{r}\Big) & 0 & 0 & 0 \\ 0 & 1 + \dfrac{2GM}{r} & 0 & 0 \\ 0 & 0 & 1 + \dfrac{2GM}{r} & 0 \\ 0 & 0 & 0 & 1 + \dfrac{2GM}{r} \end{pmatrix}.$$

Key properties (within the approximation): Static? yes (to this order); Stationary? yes; Spherically symmetric? yes; Asymptotically flat? yes; Horizons: not captured in this expansion (valid only for $r \gg 2GM$).

Static star interior (Oppenheimer–Volkoff metric):
Motivation: Einstein + perfect-fluid equations $\Rightarrow$ Oppenheimer–Volkoff equation.

Coordinates (t, r, θ, ϕ), functions $A(r)$, $B(r)$:
- Line element:

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

- Metric:

$$g_{\mu\nu} = \begin{pmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

Key properties: Static? yes (fluid at rest in these coordinates); Stationary? yes; Spherically symmetric? yes; Asymptotically flat? matched to exterior Schwarzschild at $r = R$ (stellar surface); Horizons: interior region typically has no horizon; the exterior Schwarzschild horizon location depends on total mass M.

Schwarzschild in ingoing Eddington–Finkelstein coords
Motivation: Coordinate transformation designed so that radially ingoing null geodesics are straight lines and the metric is regular at $r = 2GM$ (HW 7, problem 3).

Coordinate transform:

$$\bar{t} = t + 2GM\ln(r - 2GM).$$

- Coordinates: $(\bar{t}, r, \theta, \phi)$
- Line element:

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)d\bar{t}^2 + \frac{4GM}{r}\,d\bar{t}\,dr + \Big(1 + \frac{2GM}{r}\Big)dr^2 + r^2 d\Omega^2.$$

- Metric (ordering $(\bar{t}, r, \theta, \phi)$):

$$g_{\mu\nu} = \begin{pmatrix} -\Big(1 - \dfrac{2GM}{r}\Big) & \dfrac{2GM}{r} & 0 & 0 \\ \dfrac{2GM}{r} & 1 + \dfrac{2GM}{r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

Key properties: Static? no ($g_{\bar{t}r} \neq 0$); Stationary? yes (metric components independent of $\bar{t}$); Spherically symmetric? yes; Asymptotically flat? yes; Horizons: event horizon at $r = 2GM$; metric is regular there in these coordinates. Ingoing radial null geodesics are straight 45° lines in $(\bar{t}, r)$.

Kruskal–Szekeres (extended Schwarzschild, (v, u, θ, ϕ)):
Motivation: Maximal analytic extension of Schwarzschild that removes the coordinate singularity at $r = 2GM$ and displays full causal structure (two exteriors, black hole, white hole).
- Coordinates: (v, u, θ, ϕ)
- Line element:

$$ds^2 = -\frac{32(GM)^3}{r}e^{-r/2GM}(dv^2 - du^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

with $r = r(u, v)$ determined implicitly by

$$u^2 - v^2 = \Big(\frac{r}{2GM} - 1\Big)e^{r/2GM}.$$

- Metric (ordering (v,u,θ,ϕ)):

$$g_{\mu\nu} = \begin{pmatrix} -\dfrac{32(GM)^3}{r} e^{-r/2GM} & 0 & 0 & 0 \\ 0 & \dfrac{32(GM)^3}{r} e^{-r/2GM} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}.$$

Key properties: Static in (v,u)? not globally (timelike Killing vector changes character across regions); Stationary? only in individual exterior regions; Spherically symmetric? yes; Asymptotically flat? yes in each exterior region; Horizons: event horizons are the null lines $u = \pm v$; $r = 2GM$ is nonsingular in these coordinates.

Rotating metrics

Weak-field rotating sphere (slow rotation, (t,R,θ,ϕ)):
Motivation: Linearized solution for a slowly rotating mass with angular momentum $J = Ma$; used to describe frame dragging / gravitomagnetism (Ch. 36).
- Coordinates: (t,R,θ,ϕ)
- Define:(keep only first order)

$$F_- \equiv 1 - \frac{2GM}{R}, \quad F_+ \equiv 1 + \frac{2GM}{R}$$

- Line element (first order in GM/R and spin $a = S/M$):

$$ds^2 = -F_- \, dt^2 + F_+\big(dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2\big) - \frac{4GMa}{R} \sin^2\theta \, dt \, d\phi.$$

- Metric:

$$g_{\mu\nu} = \begin{pmatrix} -F_- & 0 & 0 & -2GMa \dfrac{\sin^2\theta}{R} \\ 0 & F_+ & 0 & 0 \\ 0 & 0 & F_+ R^2 & 0 \\ -2GMa \dfrac{\sin^2\theta}{R} & 0 & 0 & F_+ R^2 \sin^2\theta \end{pmatrix}.$$

Key properties (to linear order): Static? no ($g_{t\phi} \neq 0$); Stationary? yes; Spherically symmetric? no (only axisymmetric about rotation axis); Asymptotically flat? yes; Horizons: none in this weak-field approximation; describes the far field of a rotating body.

Kerr metric (rotating BH, Boyer–Lindquist (t,r,θ,ϕ)):
Motivation: Exact, stationary, axisymmetric vacuum solution representing a rotating black hole of mass M and specific angular momentum a.
- Define

$$\Sigma = r^2 + a^2 \cos^2\theta, \qquad \Delta = r^2 - 2GMr + a^2.$$

- Line element:

$$ds^2 = -\Big(1 - \frac{2GMr}{\Sigma}\Big) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma \, d\theta^2 + \Big(r^2 + a^2 + \frac{2GMr\, a^2 \sin^2\theta}{\Sigma}\Big) \sin^2\theta \, d\phi^2 - \frac{4GMra\sin^2\theta}{\Sigma} \, dt \, d\phi.$$

- Metric:

$$g_{\mu\nu} = \begin{pmatrix} -\Big(1 - \dfrac{2GMr}{\Sigma}\Big) & 0 & 0 & -\dfrac{2GMra\sin^2\theta}{\Sigma} \\ 0 & \dfrac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\dfrac{2GMra\sin^2\theta}{\Sigma} & 0 & 0 & \Big(r^2 + a^2 + \dfrac{2GMr\, a^2 \sin^2\theta}{\Sigma}\Big) \sin^2\theta \end{pmatrix}.$$

Key properties: Static? no ($g_{t\phi} \neq 0$; no hypersurface-orthogonal timelike Killing vector); Stationary? yes; Spherically symmetric? no (axisymmetric); Asymptotically flat? yes; Horizons: $\Delta = 0$ gives $r_\pm = GM \pm \sqrt{(GM)^2 - a^2}$ (outer horizon r_+, inner horizon r_-); ergosphere where $g_{tt} = 0$.

Cosmological and wormhole metrics

FRW metric (t,r,θ,ϕ) (homogeneous, isotropic):
Motivation: Assumes spatial slices are homogeneous and isotropic with scale factor $a(t)$ and constant spatial curvature $k = +1, 0, -1$ (cosmological principle; HW 3).
- Coordinates: (t,r,θ,ϕ)
- Line element:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right].$$

- Metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{a(t)^2}{1 - kr^2} & 0 & 0 \\ 0 & 0 & a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & a(t)^2 r^2 \sin^2\theta \end{pmatrix}.$$

Key properties: Static? no (explicit t dependence in $a(t)$); Stationary? no (no timelike Killing vector in general); Spherically symmetric about any comoving point? yes (spatial homogeneity and isotropy); Asymptotically flat? generally no (spatial curvature and expansion dominate); Horizons: cosmological particle/event horizons may exist depending on $a(t)$.

Morris–Thorne wormhole (HW 6):
Motivation: Spherically symmetric, static wormhole ansatz with throat radius b (Morris–Thorne / Ellis wormhole); requires exotic matter violating energy conditions.
- Coordinates: (t,r,θ,ϕ), constant b
- Line element:

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)\left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

- Metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b^2 + r^2 & 0 \\ 0 & 0 & 0 & (b^2 + r^2)\sin^2\theta \end{pmatrix}.$$

Key properties: Static? yes; Stationary? yes; Spherically symmetric? yes; Asymptotically flat? yes on both ends ($b^2 + r^2 \sim r^2$ as $r \to \pm\infty$); Horizons: none (traversable wormhole; no g_{tt} zero crossing); Matter content: violates weak/dominant energy conditions.

At $r = 0$ the area radius is b (wormhole throat); asymptotically $b^2 + r^2 \simeq r^2$ gives two flat regions joined at the throat.