$$\underline{II} := (\sigma_{ij} \cdot N)\, du^i du^j$$

Examples:

1) Graph $\quad z = f(x,y)$ with $f(0)=0$, $\nabla f(0) = 0$

$$T_0 S = \{x\text{-}y \text{ plane}\}, \quad N = (0,0,1)$$

$$\sigma(u^1,u^2) = (u^1, u^2, f(u^1,u^2))$$

$$\sigma_{ij} = (0,0,f_{ij})$$

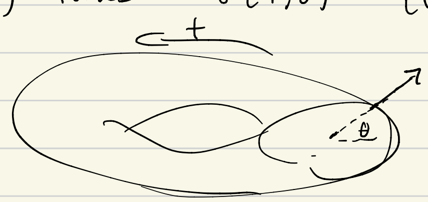
$$\Longrightarrow \underline{II}(0,0) = f_{ij}\, du^i du^j$$

Picture $\underline{II}$ as the quadratic approx. to S at p, relative to N

At a general pt. $\quad N = \dfrac{(-f_1, -f_2, 1)}{\sqrt{1+f_1^2+f_2^2}}$

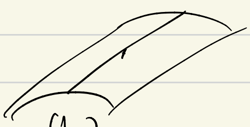
$$\underline{II} = \frac{f_{ij}}{\sqrt{1+f_1^2+f_2^2}}\, du^i du^j$$

2) torus $\quad \sigma(t,\theta) = ((R+\cos\theta)\cos t, (R+\cos\theta)\sin t, \sin\theta)$



$$N = (\cos\theta\cos t, \cos\theta \sin t, \sin\theta)$$

$$\Longrightarrow \underline{II} = -(R+\cos\theta)\cos\theta\, dt^2 - d\theta^2$$

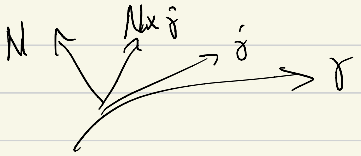
$\theta = 0$	$\theta = \pi/2$	$\theta = \pi$
		
(outside equator)	(top)	(inside equator)

Curves on Surfaces

Suppose S is a regular surface, and $\gamma : I \to S$ is a unit-speed curve

$\Longrightarrow \|\dot\gamma\| = 1$, $\|\ddot\gamma\| = \kappa$

Given a unit normal vector N, we can define an orthonormal frame along the curve



$\dot\gamma,\ N \times \dot\gamma,\ N$

Now consider $\ddot\gamma$ (which is $\perp$ to $\dot\gamma$ since $\dot\gamma \cdot \dot\gamma = 1$)

define: <u>normal curvature</u> $\kappa_n := \ddot\gamma \cdot N$ (at unit speed)

<u>geodesic curvature</u> $\kappa_g := \ddot\gamma \cdot (N \times \dot\gamma)$

This gives $\ddot\gamma = \kappa_n \cdot N + \kappa_g \cdot N \times \dot\gamma$

$\Longrightarrow \kappa = \sqrt{\kappa_n^2 + \kappa_g^2}$

Examples

1) $S =$ plane, $N = e_3$ $\quad + \quad -$

$\ddot\gamma$ lies in the plane $\Longrightarrow \kappa_n = 0,\ \kappa_g = \pm\kappa$

2) $S =$ unit sphere, $N =$ outward

Along the curve γ, $N = \gamma$

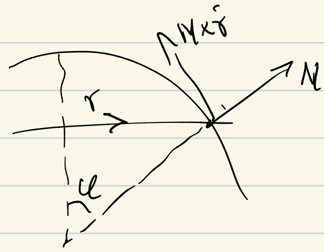
$\Longrightarrow \kappa_n = \ddot\gamma \cdot \gamma$ for a unit speed curve

Consider $\gamma \cdot \dot\gamma = 0$

$\frac{d}{dt}$ gives $\underbrace{\dot\gamma \cdot \dot\gamma}_{1} + \gamma \cdot \ddot\gamma = 0$

$\therefore \kappa_n = -1$ always

Let γ = circle at angle α from pole



γ has radius $\sin\alpha \implies \kappa = \frac{1}{\sin\alpha}$

$$\kappa^2 = \underbrace{\kappa_n^2}_{1} + \kappa_g^2$$

$$\implies \kappa_g^2 = \frac{1}{\sin^2\alpha} - 1 = \cot^2\alpha$$

From the vectors, $\kappa_g > 0$ for $\alpha < \pi/2$, κ_g 0 for $\alpha > \pi/2$

$\therefore$ $\kappa_g = \cot\alpha$

(= 0 at the equator)

Interpretation: κ_n measures curvature caused by the surface
κ_g is intrinsic curvature

Relation to II:

Suppose $\sigma(u^1, u^2)$ is a coord. patch

$$\gamma(t) = \sigma(\gamma^1(t), \gamma^2(t))$$

$$\dot\gamma = \sigma_j \dot\gamma^j$$

unit speed $\implies g_{ij}\dot\gamma^i\dot\gamma^j = 1$

$$\ddot\gamma = \sigma_{ij}\dot\gamma^i\dot\gamma^j + \sigma_j\ddot\gamma^j$$

$\perp$ to N

$$\implies \kappa_n = \ddot\gamma \cdot N$$

$$= (\sigma_{ij}\cdot N)\dot\gamma^i\dot\gamma^j$$

$$= II(\dot\gamma, \dot\gamma)$$

Fact: For a general curve, $\kappa_n = \dfrac{II(\dot\gamma,\dot\gamma)}{g(\dot\gamma,\dot\gamma)}$