Final Problem Set **MATH 545** **Erise He**
Differential Geometry

Problem 1 **Show that the helicoid $\sigma(u,t) = (u\cos t, u\sin t, t)$ and the logarithmic cylinder $\sigma(u,t) = (u\cos t, u\sin t, \ln u)$ have the same Gaussian curvature function $K(u,t)$, even though these surfaces are not local isometric.**

Part A *Proof.* First, we show that the Gaussian curvature $K(u,t)$ is the same for helicoid and logarithmic cylinder.

(1) Helicoid:

$$\sigma(u,t) = (u\cos t, u\sin t, t)$$

we compute the first derivatives, and second derivatives:

$$\begin{aligned} \sigma_u &= (\cos t, \sin t, 0) \\ \sigma_t &= (-u\sin t, u\cos t, 1) \end{aligned} \quad \text{and} \quad \begin{aligned} \sigma_{uu} &= (0,0,0) \\ \sigma_{ut} &= (-\sin t, \cos t, 0) \\ \sigma_{tt} &= (-u\cos t, -u\sin t, 0) \end{aligned}$$

The *first fundamental form* is:

$$\begin{aligned} g_{uu} &= \langle \sigma_u, \sigma_u \rangle = \cos^2 t + \sin^2 t = 1 \\ g_{ut} &= \langle \sigma_u, \sigma_t \rangle = -u\cos t \sin t + u \sin t \cos t = 0 \\ g_{tt} &= \langle \sigma_t, \sigma_t \rangle = u^2\cos^2 t + u^2 \sin^2 t + 1 = 1 + u^2 \end{aligned} \quad \Rightarrow \quad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1+u^2 \end{pmatrix}$$

Since we know that:

$$\begin{aligned} \sigma_u \times \sigma_t &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -u\sin t & u\cos t & 1 \end{pmatrix} \\ &= \mathbf{i} \begin{vmatrix} \sin t & 0 \\ u\cos t & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} \cos t & 0 \\ -u\sin t & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} \cos t & \sin t \\ -u\sin t & u\cos t \end{vmatrix} \\ &= \mathbf{i}(\sin t - 0) - \mathbf{j}(\cos t - 0) + \mathbf{k}\left(u\cos^2 t + u\sin^2 t\right) \\ &= (\sin t, -\cos t, u) \\ \|\sigma_u \times \sigma_t\| &= \sqrt{\sin^2 t + (-\cos t)^2 + u^2} = \sqrt{1+u^2} \end{aligned}$$

Then, the normal vector $N(u,t)$ is:

$$N(u,t) = \frac{\sigma_u \times \sigma_t}{\|\sigma_u \times \sigma_t\|} = \frac{(\sin t, -\cos t, u)}{\sqrt{1+u^2}}$$

Hence, we have the *second fundamental form*:

$$\begin{aligned} \mathrm{II}_{uu} &= \langle \sigma_{uu}, N \rangle = 0 \\ \mathrm{II}_{ut} &= \langle \sigma_{ut}, N \rangle = \frac{-\sin^2 t - \cos^2 t}{\sqrt{1+u^2}} = -\frac{1}{\sqrt{1+u^2}} \\ \mathrm{II}_{tt} &= \langle \sigma_{tt}, N \rangle = \frac{-u\cos t \sin t + u \sin t \cos t}{\sqrt{1+u^2}} = 0 \end{aligned} \quad \Rightarrow \quad \mathrm{II}_{ij} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{1+u^2}} \\ -\frac{1}{\sqrt{1+u^2}} & 0 \end{pmatrix}$$

According to the formula, the shape operator W is:

$$\left[W^i{}_j\right] = \left[g^{ik}\right]\left[\mathrm{II}_{kj}\right] = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+u^2} \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{1+u^2}} \\ -\frac{1}{\sqrt{1+u^2}} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\frac{1}{\sqrt{1+u^2}} \\ -\frac{1}{(1+u^2)\sqrt{1+u^2}} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\left(1+u^2\right)^{-1/2} \\ -\left(1+u^2\right)^{-3/2} & 0 \end{pmatrix}$$

Finally the Gaussian curvature function $K(u,t)$ for helicoid is:

$$K_{\text{hel}} = \det(W) = -\left(-\frac{1}{\sqrt{1+u^2}} \cdot \frac{-1}{(1+u^2)\sqrt{1+u^2}}\right) = \boxed{-\frac{1}{\left(1+u^2\right)^2}}$$

(2) Logarithmic Cylinder:

$$\sigma(u,t) = (u\cos t, u\sin t, \ln u)$$

we compute the first derivatives, and second derivatives:

$$\begin{aligned} \sigma_u &= \left(\cos t, \sin t, \frac{1}{u}\right) \\ \sigma_t &= (-u\sin t, u\cos t, 0) \end{aligned} \quad \text{and} \quad \begin{aligned} \sigma_{uu} &= \left(0, 0, -\frac{1}{u^2}\right) \\ \sigma_{ut} &= (-\sin t, \cos t, 0) \\ \sigma_{tt} &= (-u\cos t, -u\sin t, 0) \end{aligned}$$

The *first fundamental form* is:

$$\left.\begin{aligned} g_{uu} &= \langle\sigma_u, \sigma_u\rangle = \cos^2 t + \sin^2 t + \frac{1}{u^2} = 1 + \frac{1}{u^2} = \frac{u^2+1}{u^2} \\ g_{ut} &= \langle\sigma_u, \sigma_t\rangle = -u\sin t\cos t + u\sin t\cos t + 0 = 0 \\ g_{tt} &= \langle\sigma_t, \sigma_t\rangle = u^2\sin^2 t + u^2\cos^2 t + 0 = u^2 \end{aligned}\right. \quad \Rightarrow \quad g_{ij} = \begin{pmatrix} \frac{u^2+1}{u^2} & 0 \\ 0 & u^2 \end{pmatrix}$$

Since we know that:

$$\begin{aligned} \sigma_u \times \sigma_t &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & \frac{1}{u} \\ -u\sin t & u\cos t & 0 \end{pmatrix} \\ &= \mathbf{i}\begin{vmatrix} \sin t & \frac{1}{u} \\ u\cos t & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \cos t & \frac{1}{u} \\ -u\sin t & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \cos t & \sin t \\ -u\sin t & u\cos t \end{vmatrix} \\ &= \mathbf{i}(0-\cos t) - \mathbf{j}(0+\sin t) + \mathbf{k}\left(u\cos^2 t + u\sin^2 t\right) \\ &= (-\cos t, -\sin t, u) \\ \|\sigma_u \times \sigma_t\| &= \sqrt{(-\cos t)^2 + (-\sin t)^2 + u^2} = \sqrt{1+u^2} \end{aligned}$$

Then, the normal vector $N(u,t)$ is:

$$N(u,t) = \frac{\sigma_u \times \sigma_t}{\|\sigma_u \times \sigma_t\|} = \frac{(-\cos t, -\sin t, u)}{\sqrt{1+u^2}}$$

Hence, we have the *second fundamental form*:

$$\begin{aligned}
\mathrm{II}_{uu} &= \langle \sigma_{uu}, N\rangle = \frac{-1/u^2 \cdot u}{\sqrt{1+u^2}} = -\frac{1}{u\sqrt{1+u^2}} \\
\mathrm{II}_{ut} &= \langle \sigma_{ut}, N\rangle = \frac{\sin t\cos t - \cos t \sin t}{\sqrt{1+u^2}} = 0 \\
\mathrm{II}_{tt} &= \langle \sigma_{tt}, N\rangle = \frac{u\cos^2 t + u\sin^2 t}{\sqrt{1+u^2}} = \frac{u}{\sqrt{1+u^2}}
\end{aligned} \quad \Rightarrow \quad \mathrm{II}_{ij} = \begin{pmatrix} -\frac{1}{u\sqrt{1+u^2}} & 0 \\ 0 & \frac{u}{\sqrt{1+u^2}} \end{pmatrix}$$

According to the formula, the shape operator W is:

$$\begin{aligned}
\left[W^{ii}_{\ j}\right] = \left[g^{ik}\right]\left[\mathrm{II}_{kj}\right] &= \begin{pmatrix} \frac{u^2}{1+u^2} & 0 \\ 0 & \frac{1}{u^2} \end{pmatrix} \begin{pmatrix} -\frac{1}{u\sqrt{1+u^2}} & 0 \\ 0 & \frac{u}{\sqrt{1+u^2}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{u}{(1+u^2)\sqrt{1+u^2}} & 0 \\ 0 & \frac{1}{u\sqrt{1+u^2}} \end{pmatrix} \\
&= \begin{pmatrix} -u(1+u^2)^{-3/2} & 0 \\ 0 & \frac{1}{u}(1+u^2)^{-1/2} \end{pmatrix}
\end{aligned}$$

Finally, the Gaussian curvature function $K(u,t)$ for the logarithmic cylinder is:

$$K_{\text{cyl}} = \left(-u(1+u^2)^{-3/2}\right) \cdot \left(\frac{1}{u}(1+u^2)^{-1/2}\right) = \boxed{-\frac{1}{(1+u^2)^2}} = K_{\text{hel}}$$

Therefore, helicoid and logarithmic cylinder have the same Gaussian curvature function $K(u,t)$. □

Part B *Proof.* Now, we show that these surfaces are not local isometric.

Assume for contradiction there exists a local isometry $\psi : S \to \tilde{S}$ between helicoid and logarithmic cylinder, that is, $\psi(u,t) = (\tilde{u}(u,t), \tilde{t}(u,t))$. Then, by the definition of local isometry, we should have:

$$\psi^* \tilde{g} = g$$

From part A, since

$$K_{\text{hel}}(u,t) = -\frac{1}{\left(1+u^2\right)^2} = K_{\text{cyl}}(\tilde{u},\tilde{t}) = K_{\text{cyl}}(\psi(u,t)) = -\frac{1}{\left(1+\tilde{u}^2\right)^2}$$

we know $u^2 = \tilde{u}^2$. Since $u, \tilde{u} > 0$, so $u = \tilde{u}$, thus $du = d\tilde{u}$. From the metric of the logarithmic cylinder,

$$d\tilde{s}^2 = \left(1 + \frac{1}{\tilde{u}^2}\right) d\tilde{u}^2 + \tilde{u}^2 d\tilde{t}^2$$

we can write $d\tilde{t} = \tilde{t}_u du + \tilde{t}_t dt$, and therefore substitute $\tilde{u}$ and $d\tilde{u}$ to get the pullback:

$$\psi^*\left(d\tilde{s}^2\right) = \left(1 + \frac{1}{u^2}\right) du^2 + u^2 \left(\tilde{t}_u du + \tilde{t}_t dt\right)^2$$

Here, the coefficients of du^2 is:

$$(\psi^*\tilde{g})_{uu} = \left(1 + \frac{1}{u^2}\right) + u^2 \left(\tilde{t}_u\right)^2 \geq 1 + \frac{1}{u^2} > 1$$

However, we have $g_{uu} = 1$ for helicoid, so it is impossible to have $\psi^*\tilde{g} = g$. Hence, the surfaces of helicoid and logarithmic cylinder are not locally isometric.

$\square$

Problem 2 For a vector field Y on a Riemannian manifold, the covariant derivative of Y is a $\binom{1}{1}$ tensor defined by

$$\nabla Y : Z \mapsto \nabla_Z Y.$$

The divergence of X is the trace of this tensor:

$$\operatorname{div} Y := \operatorname{tr}(\nabla Y).$$

If Y is written as $Y^j \partial_j$ in local coordinates, show that

$$\operatorname{div} Y = \frac{1}{\sqrt{g}} \partial_j \left(\sqrt{g} Y^j\right)$$

[You will need Jacobi's matrix derivative formula:

$$\partial_j \det(A) = \det(A) \operatorname{tr}\left(A^{-1} \partial_j A\right)$$

assuming A is smooth and invertible.]

Solution *Proof.* By the formula for divergence, we have

$$\begin{aligned}
\operatorname{div} Y = \operatorname{tr}(\nabla Y) &= \nabla_i Y^i \\
&= (\nabla_i (Y^j \partial_j))^i \\
&= (\underbrace{(\partial_i Y^j)\partial_j}_{(j \to k)} + Y^j \underbrace{(\nabla_i \partial_j)}_{=\Gamma^k_{ij}\partial_k})^i \\
&= \left(\left(\partial_i Y^k + \Gamma^k_{ij} Y^j\right) \partial_k\right)^i \\
&= \partial_i Y^i + \Gamma^i_{ij} Y^j
\end{aligned}$$

Now we calculate the christoffel symbols:

$$\begin{aligned}
\Gamma^i_{ij} &= \frac{1}{2} g^{il} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \\
&= \frac{1}{2} (\underbrace{g^{il} \partial_i g_{jl}}_{\cancel{g^{li} \partial_l g_{ji}}} + g^{il} \partial_j g_{il} - \cancel{g^{il} \partial_l g_{ij}}) = \frac{1}{2} g^{il} \partial_j g_{il}
\end{aligned}$$

using symmetry of the metric. We then can rewrite

$$g^{il} \partial_j g_{il} = \operatorname{tr}\left(g^{-1} \partial_j g\right) = \frac{1}{g} \partial_j g$$

Let $g = \det(g_{ij})$, then by the given Jacobi's matrix derivative formula:

$$\partial_j g = g \operatorname{tr}\left(g^{-1} \partial_j g\right) = g g^{i\ell} \partial_j g_{\ell i} = g g^{i\ell} \partial_j g_{i\ell}$$

Therefore, we have

$$\Gamma^i_{ij} = \frac{1}{2} (g^{il} \partial_j g_{il}) = \frac{1}{2} (\frac{1}{g} \partial_j g) = \frac{1}{\sqrt{g}} (\frac{1}{2\sqrt{g}} \partial_j g) = \frac{1}{\sqrt{g}} \partial_j (\sqrt{g})$$

Finally, we substitute back to get the formula for divergence:

$$\begin{aligned}
\operatorname{div} Y &= \partial_i Y^i + \frac{1}{\sqrt{g}}\partial_j(\sqrt{g})Y^j \\
&= \frac{1}{\sqrt{g}}\left(\sqrt{g}\partial_j Y^j + (\partial_j\sqrt{g})\, Y^j\right) \\
&= \boxed{\frac{1}{\sqrt{g}}\partial_j\left(\sqrt{g}Y^j\right)}
\end{aligned}$$

$\square$

Problem 3 Let (M, g) be a two-dimensional Riemannian manifold, and define a new metric by setting

$$\tilde{g} = e^{2\phi} g$$

for some function $\phi \in C^\infty(M)$. (The metric $\tilde{g}$ is said to be *conformal* to g.) Show that the respective Gaussian curvatures K and $\tilde{K}$ are related by

$$\tilde{K} = e^{-2\phi}(K - \Delta\phi),$$

where Δ is the Laplacian associated to g

$$\Delta\phi := \operatorname{div}(\nabla\phi)$$

Solution *Proof.* We start with the formula for

$$K = \frac{1}{2\sqrt{g}}\partial_i \left(\sqrt{g}\Gamma^j_{ij}\right)$$

$\square$