- Expressions for w_1, w_2 and k (Eqs. B5-B7)

Math:

off by some prefactor, see Eqs. B2-B4 for the correct one

$$\frac{2}{\pi}\cdot w_1 = \frac{m_1}{M(m_1+m_3)}\left[(\alpha_{12})^{1/2} m_2^2\, \underline{b_{12}^{(1)}} + (m_1+m_3)^2\, \alpha_{12}\, \underline{b_{12}^{(1)}} + 2m_2(m_1+m_3)(\alpha_{12})^{3/4}\, b_{12}^{(2)}\right]$$

$$+ \frac{m_3}{M(m_1+m_3)}\left[(m_1+m_3)^2(\alpha_{23})^2\, \underline{b_{23}^{(1)}} + m_2^2(\alpha_{23})^{5/2}\, \underline{b_{23}^{(1)}} + 2m_2(m_1+m_3)(\alpha_{23})^{9/4}\, b_{23}^{(2)}\right]$$

$$+ \frac{m_1 m_2 m_3}{M(m_1+m_3)}\left[\alpha_{13}\alpha_{23}(\alpha_{12})^{-1/2}\, \underline{b_{13}^{(1)}} + \alpha_{13}(\alpha_{23})^{3/2}\, \underline{b_{13}^{(1)}} - 2(\alpha_{23})^{5/4}(\alpha_{12})^{-1/4}\, b_{13}^{(2)}\right]$$

$$\Big\downarrow \quad b_{ij}^{(1)} = b_{ij}^{(2)} = \frac{2}{\pi(1-\alpha_{ij})^2}$$

$$w_1 = \frac{m_1}{M(m_1+m_3)}(\alpha_{12})^{1/2}\left[m_2 + (m_1+m_3)(\alpha_{12})^{1/4}\right]^2 \frac{1}{(1-\alpha_{12})^2}$$

$$+ \frac{m_3}{M(m_1+m_3)}(\alpha_{23})^{5/2}\left[m_2 + (m_1+m_3)(\alpha_{23})^{-1/4}\right]^2 \frac{1}{(1-\alpha_{23})^2}$$

$$+ \frac{m_1 m_2 m_3}{M(m_1+m_3)}\underbrace{\left[(\alpha_{12})^{1/2}\alpha_{23}^2\, b_{13}^{(1)} + \alpha_{12}(\alpha_{23})^{5/2}\, b_{13}^{(1)} - 2(\alpha_{12})^{-1/4}(\alpha_{23})^{5/4}\, b_{13}^{(2)}\right]}_{0}\cdot\frac{\pi}{2}$$

$$\Big\downarrow \quad (\alpha_{12})^{1/4} \simeq (\alpha_{23})^{-1/4} \approx 1$$

$$w_1 = \frac{M}{m_1+m_3}\left[(\alpha_{12})^{1/2}\frac{m_1}{(1-\alpha_{12})^2} + (\alpha_{23})^{5/2}\frac{m_3}{(1-\alpha_{23})^2}\right]$$

$$\Big\downarrow \quad \text{Eq. B8}$$

$$\boxed{w_1 = \frac{M}{m_1+m_3}\left[\frac{m_1}{\ell_{c,12}^2} + \frac{m_3}{\ell_{c,23}^2}\right]}$$

$$\frac{2}{\pi} w_2 = \frac{m_2 m_3}{m_1+m_3} (\alpha_{12})^{1/2} b_{12}^{(1)} + \frac{m_1 m_2}{m_1+m_3} (\alpha_{23})^{5/2} b_{23}^{(1)}$$

$$+ \frac{1}{m_1+m_3} \Big(\underbrace{m_3^2 \alpha_{13} \alpha_{23} (\alpha_{12})^{-1/2} b_{13}^{(1)} + m_1^2 \alpha_{13} (\alpha_{23})^{3/2} b_{13}^{(1)} + 2 m_1 m_3 \alpha_{13} (\alpha_{23})^{5/4} (\alpha_{12})^{-1/4} b_{13}^{(2)}} \Big)$$

$$= \left[m_1 (\alpha_{12})^{1/2} (\alpha_{23})^{5/4} + m_3 (\alpha_{12})^{1/4} (\alpha_{23}) \right]^2 \frac{1}{(1-\alpha_{13})^2} \cdot \frac{2}{\pi}$$

$$= (\alpha_{12})^{1/2} (\alpha_{23})^{5/2} \left[m_1 (\alpha_{12})^{1/4} + m_3 (\alpha_{23})^{-1/4} \right]^2 \frac{1}{(1-\alpha_{13})^2} \cdot \frac{2}{\pi}$$

$$\boxed{w_2 = \frac{1}{m_1+m_3} \left[\frac{m_2 m_3}{e_{c,12}^2} + \frac{m_1 m_2}{e_{c,23}^2} + \frac{(m_1+m_3)^2}{e_{c,13}^2} \right]}$$

$$\frac{2}{\pi} k = - \frac{1}{m_1+m_3} \sqrt{\frac{m_1 m_2 m_3}{M}} \left((\alpha_{12})^{1/2} m_2 b_{12}^{(1)} + (m_1+m_3)(\alpha_{12})^{3/4} b_{12}^{(2)} \right)$$

$$+ \frac{1}{m_1+m_3} \sqrt{\frac{m_1 m_2 m_3}{M}} \left(m_2 (\alpha_{23})^{5/2} b_{23}^{(1)} + (m_1+m_3)(\alpha_{23})^{9/4} b_{23}^{(2)} \right) \rightarrow (\alpha_{23})^{5/2} \left[m_2 + \cancel{(\alpha_{23})}^{-1/4} (m_1+m_3) \right]$$

$$- \frac{1}{m_1+m_3} \sqrt{\frac{m_1 m_2 m_3}{M}} \Big(m_3 \alpha_{13} \alpha_{23} (\alpha_{12})^{-1/2} b_{13}^{(1)} - m_1 \alpha_{13} (\alpha_{23})^{3/2} b_{13}^{(1)}$$

$$+ (m_1 - m_3) \alpha_{13} (\alpha_{23})^{5/4} (\alpha_{12})^{-1/4} b_{13}^{(2)} \Big)$$

$$\boxed{k = - \frac{\sqrt{M m_1 m_2 m_3}}{m_1+m_3} \left[\frac{1}{e_{c,12}^2} - \frac{1}{e_{c,23}^2} \right]}$$

Code : "Test_approximation_secular.ipynb" in POEL

— compares expressions to celmech

- Expression for ψ (Eq. 26)

$$\omega_1 = \frac{n_3}{2\pi}\frac{m_{tot}}{M_*}\left(\frac{\tilde m_1}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,12}^2}+\frac{\tilde m_3}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,23}^2}\right),$$

$$\omega_2 = \frac{n_3}{2\pi}\frac{m_{tot}}{M_*}\left[\tilde m_2\left(\frac{\tilde m_3}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,12}^2}+\frac{\tilde m_1}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,23}^2}\right)+(\tilde m_1+\tilde m_3)\frac{1}{e_{c,13}^2}\right],$$

$$k = \frac{n_3}{2\pi}\frac{m_{tot}}{M_*}\frac{\sqrt{\tilde m_1\tilde m_2\tilde m_3}}{\tilde m_1+\tilde m_3}\left(\frac{1}{e_{c,23}^2}-\frac{1}{e_{c,12}^2}\right).$$

Define $\mu = \dfrac{\tilde m_3-\tilde m_1}{\tilde m_1+\tilde m_3}$, $\Delta = \dfrac{e_{c,12}}{e_{c,13}} - \dfrac{e_{c,23}}{e_{c,13}}$.

$$\omega_2-\omega_1 = \left[\underset{\downarrow\ 1-(\tilde m_1+\tilde m_3)}{\tilde m_2}\left(\frac{\tilde m_3}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,12}^2}+\frac{\tilde m_1}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,23}^2}\right)+(\tilde m_1+\tilde m_3)\frac{1}{e_{c,13}^2}\right]-\left(\frac{\tilde m_1}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,12}^2}+\frac{\tilde m_3}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,23}^2}\right),$$

$$= \left(\frac{\tilde m_3-\tilde m_1}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,12}^2}-\frac{\tilde m_3-\tilde m_1}{\tilde m_1+\tilde m_3}\frac{1}{e_{c,23}^2}\right)+\left[\tilde m_1\left(\frac{1}{e_{c,13}^2}-\frac{1}{e_{c,23}^2}\right)+\tilde m_3\left(\frac{1}{e_{c,13}^2}-\frac{1}{e_{c,12}^2}\right)\right]$$

$$= \frac{e_{c,13}^2}{e_{c,12}^2\cdot e_{c,23}^2}\left[\mu\underbrace{\left(\frac{e_{c,23}^2}{e_{c,13}^2}-\frac{e_{c,12}^2}{e_{c,13}^2}\right)}+\underbrace{\tilde m_1\left(\frac{e_{c,12}^2}{e_{c,13}^2}\frac{e_{c,23}^2}{e_{c,13}^2}-\frac{e_{c,12}^2}{e_{c,13}^2}\right)}_{①}+\underbrace{\tilde m_3\left(\frac{e_{c,12}^2}{e_{c,13}^2}\frac{e_{c,23}^2}{e_{c,13}^2}-\frac{e_{c,23}^2}{e_{c,13}^2}\right)}_{②}\right]$$

$$\underbrace{\left(\frac{e_{c,23}}{e_{c,13}}+\frac{e_{c,12}}{e_{c,13}}\right)}_{1}\left(\frac{e_{c,23}}{e_{c,13}}-\frac{e_{c,12}}{e_{c,13}}\right) = -\Delta$$

$$\frac{e_{c,12}}{e_{c,13}} = \frac{1}{2}(1+\Delta),\quad \frac{e_{c,23}}{e_{c,13}} = \frac{1}{2}(1-\Delta)$$

① $\tilde m_1\dfrac{e_{c,12}^2}{e_{c,13}^2}\left(\dfrac{e_{c,23}^2}{e_{c,13}^2}-1\right)$

$= \tilde m_1\frac{1}{4}(1+\Delta)^2\left[\frac{1}{4}(1-\Delta)^2-1\right]$

$= \tilde m_1\left[\frac{1}{16}(1-\Delta^2)^2-\frac{1}{4}(1+\Delta)^2\right]$

$= \tilde m_1\left[\frac{1}{16}(1-2\Delta^2+\Delta^4)-\frac{1}{4}(1+2\Delta+\Delta^2)\right]$

② $\tilde m_3\dfrac{e_{c,23}^2}{e_{c,13}^2}\left(\dfrac{e_{c,12}^2}{e_{c,13}^2}-1\right)$

$= \tilde m_3\frac{1}{4}(1-\Delta)^2\left[\frac{1}{4}(1+\Delta)^2-1\right]$

$= \tilde m_3\left[\frac{1}{16}(1-\Delta^2)^2-\frac{1}{4}(1-\Delta)^2\right]$

$= \tilde m_3\left[\frac{1}{16}(1-2\Delta^2+\Delta^4)-\frac{1}{4}(1-2\Delta+\Delta^2)\right]$

$$①+② = (\tilde m_1+\tilde m_3)\left[\frac{1}{16}(-3-6\Delta^2+\Delta^4)+\frac{1}{2}\mu\Delta\right]$$

$$\omega_2-\omega_1=\frac{\ell_{c,13}^2}{\ell_{c,12}^2\cdot\ell_{c,23}^2}\left[-\mu\Delta+(\tilde m_1+\tilde m_3)\left(-\frac{3}{16}-\frac{3}{8}\Delta^2+\frac{1}{2}\mu\Delta+\frac{1}{16}\Delta^4\right)\right]$$

(everything below is under test particle limit)

$\tilde m_1+\tilde m_3=1$

$$\omega_2-\omega_1=-\frac{\ell_{c,13}^2}{\ell_{c,12}^2\cdot\ell_{c,23}^2}\left(\frac{3}{16}+\frac{3}{8}\Delta^2+\frac{1}{2}\mu\Delta-\frac{1}{16}\Delta^4\right)$$

$$k=\sqrt{\tilde m_1\tilde m_2\tilde m_3}\left(\frac{1}{\ell_{c,23}^2}-\frac{1}{\ell_{c,12}^2}\right)=\frac{\ell_{c,13}^2}{\ell_{c,12}^2\cdot\ell_{c,23}^2}\sqrt{\tilde m_1\tilde m_2\tilde m_3}\underbrace{\left(\frac{\ell_{c,12}^2}{\ell_{c,13}^2}-\frac{\ell_{c,23}^2}{\ell_{c,13}^2}\right)}_{\Delta}$$

$$\frac{k}{\omega_2-\omega_1}=-\sqrt{\tilde m_1\tilde m_2\tilde m_3}\,\frac{\Delta}{\frac{3}{16}+\frac{3}{8}\Delta^2+\frac{1}{2}\mu\Delta-\frac{1}{16}\Delta^4}$$

$$\psi=\frac{1}{2}\tan^{-1}\left(\frac{2k}{\omega_2-\omega_1}\right)\approx\frac{k}{\omega_2-\omega_1}=\boxed{-16\sqrt{\tilde m_1\tilde m_2\tilde m_3}\,\frac{\Delta}{3+6\Delta^2+8\mu\Delta-\Delta^4}}$$

Code : "Test_approximation_secular.ipynb" in POEL

— compares expressions to celmech

• Expression for $\vec{e}_-$ (Eq. 29)

$$\vec{S}_1 = \sqrt{M}\sqrt{\frac{\tilde m_2}{\tilde m_1+\tilde m_3}}\left(\tilde m_3 \vec e_{23} - \tilde m_1 \vec e_{12}\right) \qquad \vec S_2 = \sqrt{M}\sqrt{\frac{\tilde m_1 \tilde m_3}{\tilde m_1+\tilde m_3}}\left(\vec e_{23}+\vec e_{12}\right)$$

$$\frac{k}{\delta\omega} = -\frac{16\sqrt{\tilde m_1\tilde m_2\tilde m_3}\,\Delta/(\tilde m_1+\tilde m_3)}{(3+6\Delta^2+8\mu\Delta-\Delta^4)-\tilde m_2(3+6\Delta^2-8\mu\Delta-\Delta^4)}$$

$$(1-\Delta)^3(3+\Delta)+\frac{16\tilde m_3}{\tilde m_1+\tilde m_3}\Delta \equiv a+\frac{16\tilde m_3}{\tilde m_1+\tilde m_3}\Delta$$

$$(1+\Delta)^3(3-\Delta)-\frac{16\tilde m_3}{\tilde m_1+\tilde m_3}\Delta \equiv b-\frac{16\tilde m_3}{\tilde m_1+\tilde m_3}\Delta$$

If $\psi \ll 1$, $\cos\psi = 1$, $\sin\psi = -\dfrac{16\sqrt{\tilde m_1\tilde m_2\tilde m_3}\,\Delta/(\tilde m_1+\tilde m_3)}{a+\frac{16\tilde m_3}{\tilde m_1+\tilde m_3}\Delta - \tilde m_2\left(b-\frac{16\tilde m_3}{\tilde m_1+\tilde m_3}\Delta\right)} = -\dfrac{16\sqrt{\tilde m_1\tilde m_2\tilde m_3}\,\Delta}{(\tilde m_1+\tilde m_3)a+16\tilde m_3\Delta-\tilde m_2\left[(\tilde m_1+\tilde m_3)b-16\tilde m_3\Delta\right]}$

$$\vec S_1' = \cos\psi\,\vec S_1 - \sin\psi\,\vec S_2$$

$$\propto (\tilde m_3\vec e_{23}-\tilde m_1\vec e_{12})\left[(\tilde m_1+\tilde m_3)a+16\tilde m_3\Delta-\tilde m_2(\tilde m_1+\tilde m_3)b+16\tilde m_2\tilde m_3\Delta\right]+16\tilde m_1\tilde m_3(\vec e_{23}+\vec e_{12})\Delta$$

$$= \tilde m_3\vec e_{23}\left[(\tilde m_1+\tilde m_3)a+16\tilde m_3\Delta-\tilde m_2(\tilde m_1+\tilde m_3)b+16\tilde m_2\tilde m_3\Delta+16\tilde m_1\Delta\right]$$

$$-\tilde m_1\vec e_{12}\left[(\tilde m_1+\tilde m_3)a+16\tilde m_3\Delta-\tilde m_2(\tilde m_1+\tilde m_3)b+16\tilde m_2\tilde m_3\Delta-16\tilde m_3\Delta\right]$$

$$= \tilde m_3\vec e_{23}\cdot[1] - \tilde m_1\vec e_{12}\cdot[2]$$

$$[1] = (\tilde m_1+\tilde m_3)a+16\tilde m_3\Delta-\tilde m_2(\tilde m_1+\tilde m_3)b+16\tilde m_2\tilde m_3\Delta+16\tilde m_1\Delta$$

$$16(\tilde m_1+\tilde m_3)\Delta = (\tilde m_1+\tilde m_3)(b-a)$$

$$= (\tilde m_1+\tilde m_3)\left[a-\tilde m_2 b+(b-a)\right]+16\tilde m_2\tilde m_3$$

$$= (\tilde m_1+\tilde m_3)^2 b+16\tilde m_2\tilde m_3\Delta$$

$$[2] = (\tilde m_1+\tilde m_3)a+\cancel{16\tilde m_3}\Delta-\tilde m_2(\tilde m_1+\tilde m_3)b+16\tilde m_2\tilde m_3\Delta-\cancel{16\tilde m_3}\Delta$$

$$= 16\tilde m_2(\tilde m_1+\tilde m_3)\Delta-16\tilde m_1\tilde m_2\Delta$$

$$= \tilde m_2(\tilde m_1+\tilde m_3)(b-a)-16\tilde m_1\tilde m_2\Delta$$

$$= (\tilde m_1+\tilde m_3)\left[a-\tilde m_2 b+\tilde m_2(b-a)\right]-16\tilde m_1\tilde m_2\Delta$$

$$= (\tilde m_1+\tilde m_3)^2 a-16\tilde m_1\tilde m_2\Delta$$

$$\psi \ll 1,\quad \vec S_1' \propto (\tilde m_1+\tilde m_3)^2\underbrace{\left(\tilde m_3 b\,\vec e_{23}-\tilde m_1 a\,\vec e_{12}\right)}_{\text{test particle limit}}+16\tilde m_2\Delta\left(\tilde m_3^2\vec e_{23}+\tilde m_1^2\vec e_{12}\right)$$

notations: $\{\vec S_1', a, b\} \longrightarrow \{\vec e_-, \chi_{12}, \chi_{23}\}$ in paper

Code: "error_in_S-.ipynb" & "Nbody.ipynb" in Chaos / Graphs for paper

- Expressions for $\vec{e}_i$ (Eq. 30)

$$\varphi = -\frac{16\sqrt{\tilde{m}_1\tilde{m}_2\tilde{m}_3}\,\eta}{3+6\eta^2+8\mu\eta-\eta^4}, \qquad \vec{S} = \begin{pmatrix} \sqrt{m_2}\,\vec{e}_- \\ \sqrt{M}\cdot\sqrt{\tilde{m}_1\tilde{m}_3}\,\vec{e}_{13} \\ \sqrt{M}\,\vec{e}_{com} \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\varphi\sqrt{\frac{\tilde{m}_1\tilde{m}_2}{\tilde{m}_1+\tilde{m}_3}} + \sin\varphi\sqrt{\frac{\tilde{m}_3}{\tilde{m}_1+\tilde{m}_3}} & -\cos\varphi\sqrt{\tilde{m}_1+\tilde{m}_3} & \cos\varphi\sqrt{\frac{\tilde{m}_2\tilde{m}_3}{\tilde{m}_1+\tilde{m}_3}} - \sin\varphi\sqrt{\frac{\tilde{m}_1}{\tilde{m}_1+\tilde{m}_3}} \\ \sin\varphi\sqrt{\frac{\tilde{m}_1\tilde{m}_2}{\tilde{m}_1+\tilde{m}_3}} - \cos\varphi\sqrt{\frac{\tilde{m}_3}{\tilde{m}_1+\tilde{m}_3}} & -\sin\varphi\sqrt{\tilde{m}_1+\tilde{m}_3} & \sin\varphi\sqrt{\frac{\tilde{m}_2\tilde{m}_3}{\tilde{m}_1+\tilde{m}_3}} + \cos\varphi\sqrt{\frac{\tilde{m}_1}{\tilde{m}_1+\tilde{m}_3}} \\ \sqrt{\tilde{m}_1} & \sqrt{\tilde{m}_2} & \sqrt{\tilde{m}_3} \end{pmatrix}$$

$$\Big\downarrow \quad \vec{G} = R^T\cdot\vec{S}$$

assumption: $\sin\varphi = \varphi$, $\cos\varphi = 1$

$$\sqrt{m_1}\,\vec{e}_1 = \left[\sqrt{\frac{\tilde{m}_1\tilde{m}_2}{\tilde{m}_1+\tilde{m}_3}} + \varphi\sqrt{\frac{\tilde{m}_3}{\tilde{m}_1+\tilde{m}_3}}\right]\sqrt{m_2}\,\vec{e}_- + \left[\varphi\cdot\sqrt{\frac{\tilde{m}_1\tilde{m}_2}{\tilde{m}_1+\tilde{m}_3}} - \sqrt{\frac{\tilde{m}_3}{\tilde{m}_1+\tilde{m}_3}}\right]\sqrt{m_1}\sqrt{\tilde{m}_3}\,\vec{e}_{13} + \sqrt{m_1}\,\vec{e}_{com}$$

$$\sqrt{m_2}\,\vec{e}_2 = -\sqrt{\tilde{m}_1+\tilde{m}_3}\cdot\sqrt{m_2}\,\vec{e}_- - \varphi\sqrt{\tilde{m}_1+\tilde{m}_3}\sqrt{\tilde{m}_1\tilde{m}_3}\sqrt{M}\,\vec{e}_{13} + \sqrt{m_2}\,\vec{e}_{com}$$

$$\sqrt{m_3}\,\vec{e}_3 = \left[\sqrt{\frac{\tilde{m}_2\tilde{m}_3}{\tilde{m}_1+\tilde{m}_3}} - \varphi\sqrt{\frac{\tilde{m}_1}{\tilde{m}_1+\tilde{m}_3}}\right]\sqrt{m_2}\,\vec{e}_- + \left[\varphi\sqrt{\frac{\tilde{m}_2\tilde{m}_3}{\tilde{m}_1+\tilde{m}_3}} + \sqrt{\frac{\tilde{m}_1}{\tilde{m}_1+\tilde{m}_3}}\right]\sqrt{m_3}\sqrt{\tilde{m}_1}\,\vec{e}_{13} + \sqrt{m_3}\,\vec{e}_{com}$$

$$\Big\downarrow$$

assumption: $\tilde{m}_2 \ll 1$, $\tilde{m}_1+\tilde{m}_3 = 1$

$$\vec{e}_1 = \vec{e}_{com} - \tilde{m}_3\,\vec{e}_{13}$$

$$\vec{e}_2 = \vec{e}_{com} + \frac{16\tilde{m}_1\tilde{m}_3\eta}{3+6\eta^2+8\mu\eta-\eta^4}\,\vec{e}_{13} - \vec{e}_-$$

$$\vec{e}_3 = \vec{e}_{com} + \tilde{m}_1\,\vec{e}_{13}$$

$\gamma = \frac{1}{2}(\tilde{x}_{23}-\tilde{x}_{12}) - \frac{1}{2}(\tilde{m}_3-\tilde{m}_1)$

test particle limit

$$\vec{e}_2 = \tilde{m}_1\vec{e}_1 + \tilde{m}_3\vec{e}_3 + \frac{1}{2}(\tilde{x}_{23}-\tilde{x}_{12}-\mu)(\vec{e}_3-\vec{e}_1) - \tilde{x}_{23}(\vec{e}_3-\vec{e}_2) + \tilde{x}_{12}(\vec{e}_2-\vec{e}_1)$$

$$= \Big(\tilde{m}_1 + \cancelto{0}{\frac{\tilde{m}_3-\tilde{m}_1}{2} - \frac{\tilde{x}_{23}-\tilde{x}_{12}}{2}} - \tilde{x}_{12}\Big)\vec{e}_1 + \big(\cancelto{1}{\tilde{x}_{12}+\tilde{x}_{23}}\big)\vec{e}_2 + \Big(\tilde{m}_3 - \cancelto{0}{\frac{\tilde{m}_3-\tilde{m}_1}{2} + \frac{\tilde{x}_{23}-\tilde{x}_{12}}{2}} - \tilde{x}_{23}\Big)\vec{e}_3$$

- Extent of breathing (Eq. 31 in paper, Appendix D in thesis)

$$\gamma = \frac{16\tilde{m}_1\tilde{m}_3\eta}{3+6\eta^2+8\mu\eta-\eta^4}$$

$$\vec{e}_{12} = (\tilde{m}_3+\gamma)\,\vec{e}_{13} - \vec{e}_- \qquad \vec{e}_{23} = (\tilde{m}_1-\gamma)\,\vec{e}_{13} + \vec{e}_-$$

$$\longrightarrow \quad e_{12}^2 = e_-^2 + (\tilde{m}_3+\gamma)^2 e_{13}^2 - \boxed{2(\tilde{m}_3+\gamma)}\, e_- e_{13}\cos\theta(t)$$

$$e_{23}^2 = e_-^2 + (\tilde{m}_1-\gamma)^2 e_{13}^2 + \boxed{2(\tilde{m}_1-\gamma)}\, e_- e_{13}\cos\theta(t)$$

always > 0

$$\theta(t) = (\omega_{13}-\omega_-)t + (\phi_{13}-\phi_-)$$

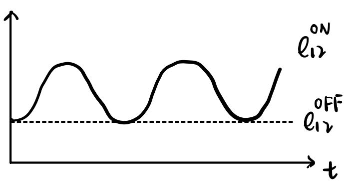
$$\tilde{m}_1-\gamma = \tilde{m}_1\left[1-\frac{16(1-\tilde{m}_1)\eta}{3+6\eta^2+8(1-2\tilde{m}_1)\eta-\eta^4}\right] = \tilde{m}_1\frac{3+6\eta^2-8\eta-\eta^4}{3+6\eta^2+8\mu\eta-\eta^4} = \frac{\chi_{12}}{\chi_{12}+\chi_{23}} = \tilde{\chi}_{12}$$

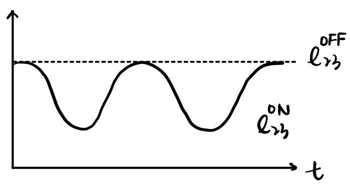
$$\tilde{m}_3+\gamma = \tilde{m}_3\left[1+\frac{16(1-\tilde{m}_3)\eta}{3+6\eta^2+8(2\tilde{m}_3-1)\eta-\eta^4}\right] = \tilde{m}_3\frac{3+6\eta^2+8\eta-\eta^4}{3+6\eta^2+8\mu\eta-\eta^4} = \frac{\chi_{23}}{\chi_{12}+\chi_{23}} = \tilde{\chi}_{23}$$

$$\boxed{\begin{aligned}\vec{e}_{12} &= \tilde{\chi}_{23}\vec{e}_{13} - \vec{e}_- \\ \vec{e}_{23} &= \tilde{\chi}_{12}\vec{e}_{13} + \vec{e}_-\end{aligned}}$$

(checked with sim)

- Scenario 1





if $\boxed{\phi_{13}-\phi_- = 0}$

$$\frac{e_{12,max}^{ON}}{e_{12}^{OFF}} = \frac{\tilde{\chi}_{23}e_{13}^{ON}+e_-^{ON}}{\tilde{\chi}_{23}e_{13}^{OFF}} = (\tilde{\chi}_{12}-\tilde{\chi}_{23}) + \frac{1-(\tilde{\chi}_{12}-\tilde{\chi}_{23})^2}{2\tilde{\chi}_{23}} = 4\tilde{\chi}_{12}-1$$

To turn off breathing, we set $e_-^{OFF} = 0$.

$$e_{12}^{ON}(0) = e_{12}^{OFF} \longrightarrow |e_-^{ON} - \tilde{\chi}_{23}e_{13}^{ON}| = \tilde{\chi}_{23}e_{13}^{OFF}$$

$$e_{23}^{ON}(0) = e_{23}^{OFF} \longrightarrow e_-^{ON} + \tilde{\chi}_{12}e_{13}^{ON} = \tilde{\chi}_{12}e_{13}^{OFF}$$

① $e_-^{ON} > \tilde{\chi}_{23}e_{13}^{ON}$

$$\begin{cases} e_-^{ON} - \tilde{\chi}_{23}e_{13}^{ON} = \tilde{\chi}_{23}e_{13}^{OFF} \\ e_-^{ON} + \tilde{\chi}_{12}e_{13}^{ON} = \tilde{\chi}_{12}e_{13}^{OFF} \end{cases} \longrightarrow \boxed{\begin{aligned} e_-^{ON} &= \tfrac{1}{2}\left[1-(\tilde{\chi}_{12}-\tilde{\chi}_{23})^2\right]e_{13}^{OFF} \\ e_{13}^{ON} &= \underbrace{(\tilde{\chi}_{12}-\tilde{\chi}_{23})}_{>0}e_{13}^{OFF}\end{aligned}}$$

② $e_-^{ON} < \tilde{\chi}_{23}e_{13}^{ON} \longrightarrow$ no non-trivial solution

Note: $e_{13}^{OFF} = e_{12}^{OFF}/\tilde{\chi}_{23} = e_{23}^{OFF}/\tilde{\chi}_{12}$, which will blow up if either $\tilde{\chi}_{12}\to 0$ or $\tilde{\chi}_{23}\to 0$. See next page.

- Scenario 2

$$-\ell_{12}^2 = \tilde{\chi}_{23}^2 \ell_{13}^2 + \ell_-^2 - 2\tilde{\chi}_{23}\ell_{13}\ell_- \cos\phi(t)$$

$$\ell_{23}^2 = \tilde{\chi}_{12}^2 \ell_{13}^2 + \ell_-^2 + 2\tilde{\chi}_{12}\ell_{13}\ell_- \cos\phi(t)$$

$\ell_{13}^{OFF} = 0$

$$(\ell_-^{OFF})^2 = (\tilde{\chi}_{23}\ell_{13}^{ON})^2 + (\ell_-^{ON})^2 - 2\tilde{\chi}_{23}\ell_{13}^{ON}\ell_-^{ON}\cos\phi_0$$

$$(\ell_-^{OFF})^2 = (\tilde{\chi}_{12}\ell_{13}^{ON})^2 + (\ell_-^{ON})^2 + 2\tilde{\chi}_{12}\ell_{13}^{ON}\ell_-^{ON}\cos\phi_0$$

$$\Downarrow$$

$$(\tilde{\chi}_{23}-\tilde{\chi}_{12})(\ell_{13}^{ON})^2 = 2\ell_{13}^{ON}\ell_-^{ON}\cos\phi_0$$

$$\ell_-^{ON} = \frac{\tilde{\chi}_{23}-\tilde{\chi}_{12}}{2\cos\phi_0}\ell_{13}^{ON}$$

$$(\ell_-^{OFF})^2 = (\tilde{\chi}_{23}\ell_{13}^{ON})^2 + \left(\frac{\tilde{\chi}_{23}-\tilde{\chi}_{12}}{2\cos\phi_0}\ell_{13}^{ON}\right)^2 - \tilde{\chi}_{23}(\tilde{\chi}_{23}-\tilde{\chi}_{12})(\ell_{13}^{ON})^2$$

$$= \left[\left(\frac{\tilde{\chi}_{23}-\tilde{\chi}_{12}}{2\cos\phi_0}\right)^2 + \tilde{\chi}_{12}\tilde{\chi}_{23}\right](\ell_{13}^{ON})^2$$

maximize: $(\tilde{\chi}_{23}\ell_{13}^{ON} + \ell_-^{ON})$ wrt. ϕ_0

Solution:

$$\cos(\phi_- - \phi_{13}) = \frac{1-2\tilde{\chi}_{12}}{2\tilde{\chi}_{12}} = \frac{\tilde{\chi}_{23}-\tilde{\chi}_{12}}{2\tilde{\chi}_{12}} \quad (\text{if } \tilde{\chi}_{12} > 1/2)$$

$$\ell_{13}^{ON} = \ell_-^{OFF} / \sqrt{\tilde{\chi}_{12}}$$

$$\ell_-^{ON} = \ell_-^{OFF} \cdot \sqrt{\tilde{\chi}_{12}}$$

$$\frac{\ell_{12,max}^{ON}}{\ell_{12}^{OFF}} = \frac{1}{\sqrt{1-\tilde{\chi}_{23}}} = \frac{1}{\sqrt{\tilde{\chi}_{12}}}$$

In[ ]:=
```
f = (((B - A) / (2 * Cos[x]))^2 + A * B)^(-1/2) * (B + (B - A) / (2 * Cos[x]))
```

Out[ ]=

$$\frac{B + \frac{1}{2}(-A+B)\,\mathrm{Sec}[x]}{\sqrt{A B + \frac{1}{4}(-A+B)^2\,\mathrm{Sec}[x]^2}}$$

In[ ]:=
```
D[f, x] == 0
```

Out[ ]=

$$-\frac{(-A+B)^2\,\mathrm{Sec}[x]^2\left(B+\frac{1}{2}(-A+B)\,\mathrm{Sec}[x]\right)\mathrm{Tan}[x]}{4\left(A B+\frac{1}{4}(-A+B)^2\,\mathrm{Sec}[x]^2\right)^{3/2}} + \frac{(-A+B)\,\mathrm{Sec}[x]\,\mathrm{Tan}[x]}{2\sqrt{A B+\frac{1}{4}(-A+B)^2\,\mathrm{Sec}[x]^2}} == 0$$

In[ ]:=

$$\mathrm{Solve}\left[-\frac{(-A+B)^2\,\mathrm{Sec}[x]^2\left(B+\frac{1}{2}(-A+B)\,\mathrm{Sec}[x]\right)\mathrm{Tan}[x]}{4\left(A B+\frac{1}{4}(-A+B)^2\,\mathrm{Sec}[x]^2\right)^{3/2}} + \frac{(-A+B)\,\mathrm{Sec}[x]\,\mathrm{Tan}[x]}{2\sqrt{A B+\frac{1}{4}(-A+B)^2\,\mathrm{Sec}[x]^2}} == 0,\ \{x\},\ \mathbb{R},\ \mathrm{Assumptions} \to \{A+B == 1,\ 0<A<1,\ 0<B<1\}\right]$$

Out[ ]=

$$\left\{\{x \to \pi c_1 \text{ if } c_1 \in \mathbb{Z}\},\right.$$
$$\left\{x \to -\mathrm{ArcCos}\left[\frac{1-2A}{2A}\right] + 2\pi c_1 \text{ if } \left(c_1 \in \mathbb{Z}\ \&\&\ \frac{1}{4} \le A < \frac{1}{2}\right)\ ||\ \left(c_1 \in \mathbb{Z}\ \&\&\ A > \frac{1}{2}\right)\right\},$$
$$\left.\left\{x \to \mathrm{ArcCos}\left[\frac{1-2A}{2A}\right] + 2\pi c_1 \text{ if } \left(c_1 \in \mathbb{Z}\ \&\&\ \frac{1}{4} \le A < \frac{1}{2}\right)\ ||\ \left(c_1 \in \mathbb{Z}\ \&\&\ A > \frac{1}{2}\right)\right\}\right\}$$

In[ ]:=

$$\mathrm{Simplify}\left[D[f, \{x, 2\}]\ /.\ x \to \mathrm{ArcCos}\left[\frac{1-2A}{2A}\right],\ \mathrm{Assumptions} \to A+B == 1\right]$$

Out[ ]=

$$\frac{B\left(-3+7B-4B^2\right)}{(1-2B)^2\sqrt{1-B}}$$

In[ ]:=

$$\mathrm{Simplify}\left[f\ /.\ x \to -\mathrm{ArcCos}\left[\frac{1-2A}{2A}\right],\ \mathrm{Assumptions} \to A+B == 1\right]$$

Out[ ]=

$$\frac{1}{\sqrt{1-B}}$$

Code: "Stability_boundary_secular.ipynb" in POEL

- Unsimplify and test

I. In paper, we assume that $a_i = \text{const}$ such that

$$\vec{G}_i = \sqrt{m_i}\,\vec{e}_i\,, \quad \vec{S} = R\cdot\vec{G}.$$

II. If we try including the a_i factors, we have

$$\vec{G}_i = a_i^{1/4}\sqrt{m_i}\,\vec{e}_i$$

If we keep all the rotation matrices (R_1, R_2) the same, $\vec{S} = R\cdot\vec{G}$ is equivalent to re-scaling

$$\vec{e}_i \longrightarrow a_i^{1/4}\,\vec{e}_i$$

in all the eigenmodes. For example, $\vec{e}_{com} \rightarrow a_1^{1/4}\tilde{m}_1\vec{e}_1 + a_2^{1/4}\tilde{m}_2\vec{e}_2 + a_3^{1/4}\tilde{m}_3\vec{e}_3$.

How does this change our graphs?

$\tilde{m}_2 = 0,\ \Delta = 0$

I.

no α prefac in e_c

$e_{c,12} \equiv 1 - \alpha_{12}$

$\sqrt{m}\,\vec{e}$

dist $(\vec{S}_{-,\text{approx}},\ R_{num}\cdot\vec{G})$

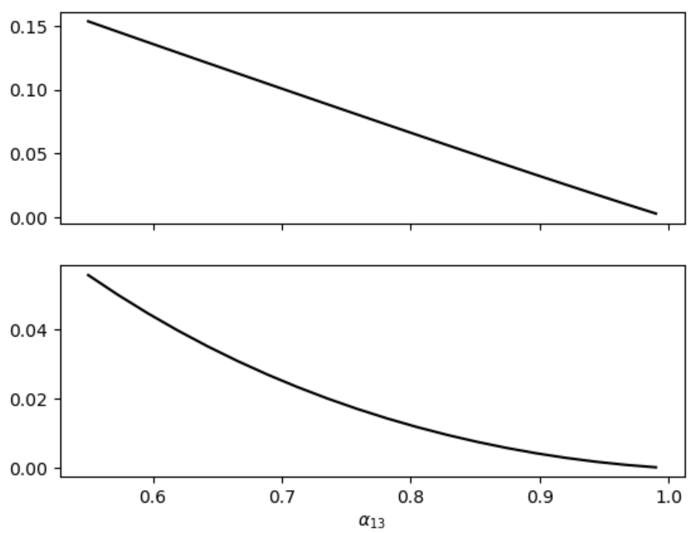
II.

with α prefac in e_c

$e_{c,12} \equiv \alpha_{12}^{-1/4}\,\alpha_{23}^{5/8}\,(1-\alpha_{12})$

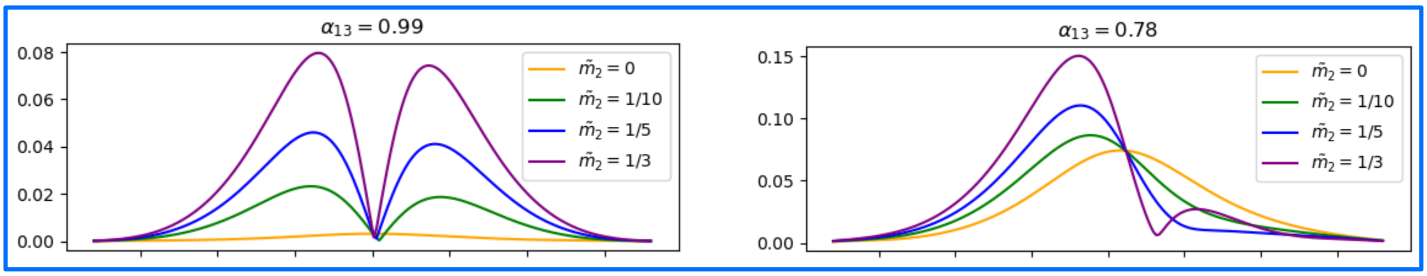
dist $(\vec{S}_{-,\text{approx}},\ R_{num}\cdot\vec{G})$

$a^{1/4}\sqrt{m}\,\vec{e}$

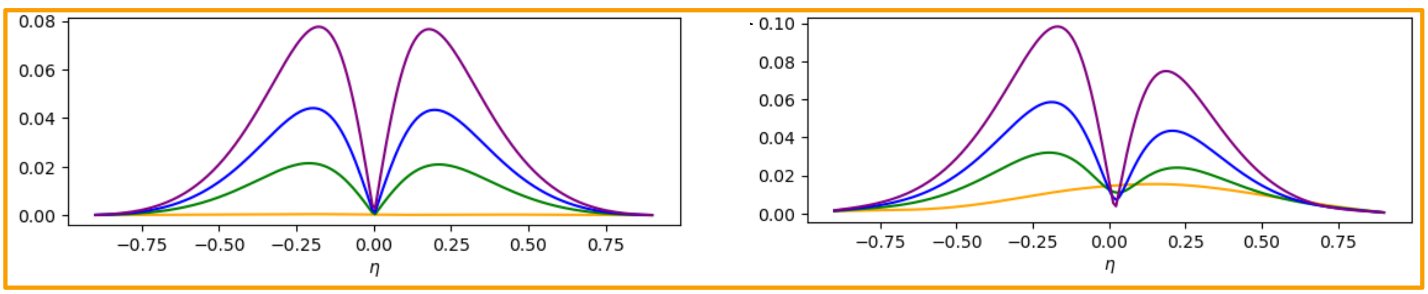


error in mode approximation δ_-

I.

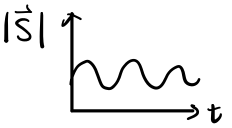


II.



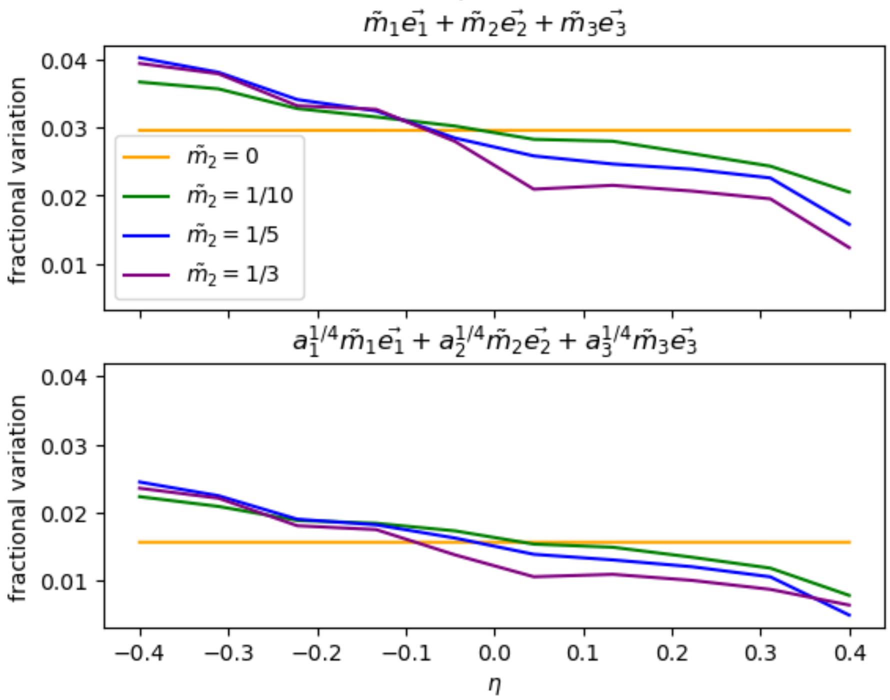
Are the rescaled eigenmodes more conserved than the original ones?

N-body: $\alpha_{13} = 0.78$, ecc = 0.014, 0.012, 0.008, ϖ = 0.72, 1.37, -0.07



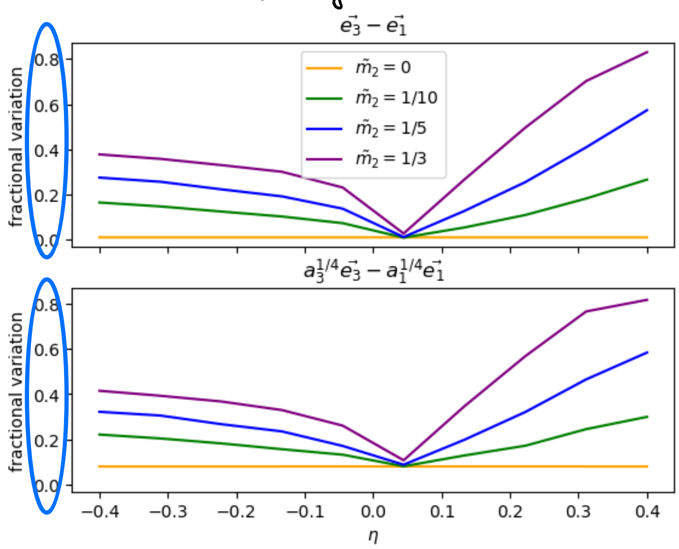
$$\frac{\max - \min}{2 \cdot \text{mean}}$$

Nbody $\vec{S}_{com}$

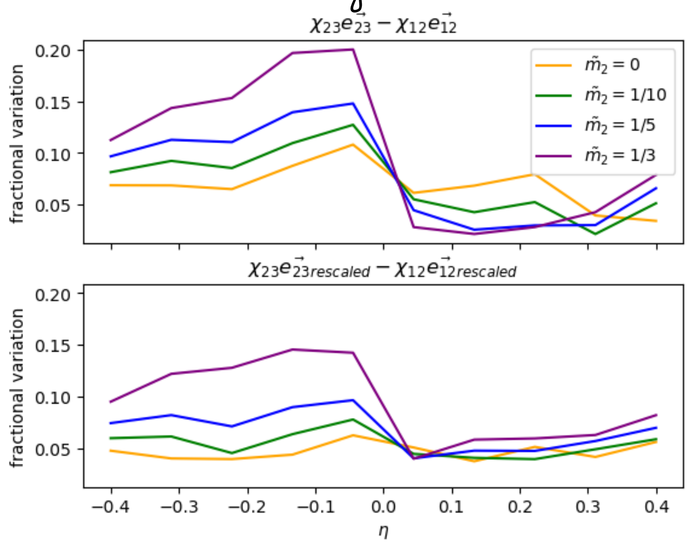


$\vec{e}_{com, rescaled}$ seems more conserved than $\vec{e}_{com}$ in less compact limit

Nbody $\vec{S}_+$



Nbody $\vec{S}_-$



see next page for analysis on the huge error

Consider the first-order correction on $\vec{S}_+$:

$a = (1-\eta)^3(3+\eta),\ b = (1+\eta)^3(3-\eta)$

$$\vec{S}_+^{(1)} = \sin\psi\,\vec{S}_1' + \cos\psi\,\vec{S}_2'$$

$$= \vec{e}_{13} - \underbrace{\frac{16\,\tilde{m}_2\,\eta}{(\tilde{m}_3 - \tilde{m}_1\tilde{m}_2)\,b + (\tilde{m}_1 - \tilde{m}_2\tilde{m}_3)\,a}}_{K}\,(\tilde{m}_3\vec{e}_{23} - \tilde{m}_1\vec{e}_{12})$$

$$= -(1+\tilde{m}_1 K)\,\vec{e}_1 + (0 + (\tilde{m}_1+\tilde{m}_3)K)\,\vec{e}_2 + (1+\tilde{m}_3 K)\,\vec{e}_3$$

$\Uparrow$

HUGE fractional change in the coefficient of $\vec{e}_2$

Because the original mode $\vec{e}_{13}$ is free of $\vec{e}_2$, the fractional error in the coefficient of $\vec{e}_2$ due to the term $\sin\psi\,\vec{S}_1'$ is huge compared to those of $\vec{e}_1$ and $\vec{e}_3$.

For instance, in a system where $\tilde{m}_1 K = \tilde{m}_3 K = 10\%$,

Is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ a good metric? What about sth. like $\{\vec{e}_{12}, \vec{e}_{23}, \vec{e}_2\}$?

$$\vec{S}_{+,\text{numerical}} = 0.62\,\vec{e}_1 + \boxed{0.15}\,\vec{e}_2 - 0.76\,\vec{e}_3$$

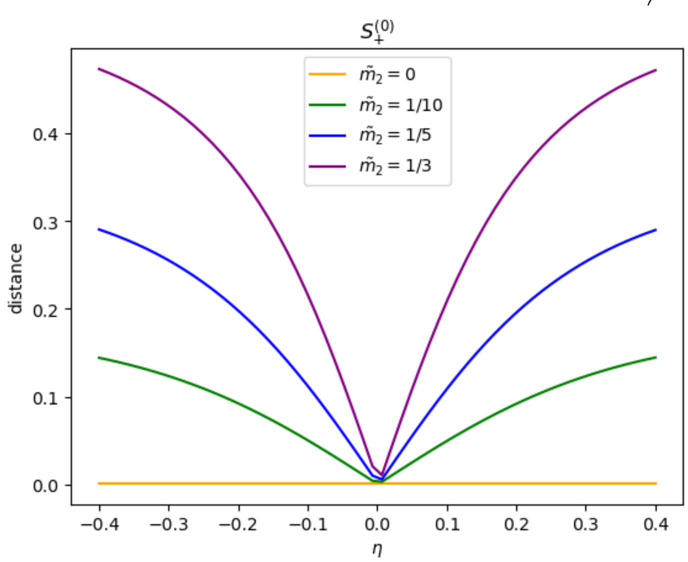
distance = 0.17

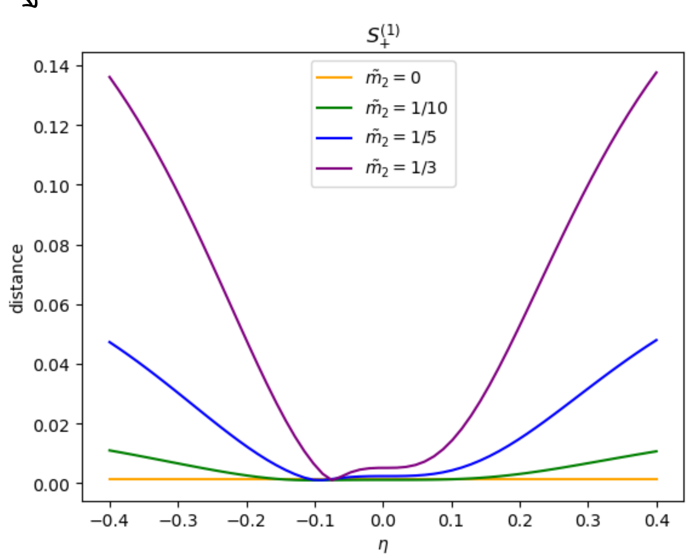
$$\vec{S}_+^{(0)},\ \text{approx} = 0.68\,\vec{e}_1 + \boxed{0.}\,\vec{e}_2 - 0.72\,\vec{e}_3$$

distance = 0.04

$$\vec{S}_+^{(1)},\ \text{approx} = 0.60\,\vec{e}_1 + \boxed{0.14}\,\vec{e}_2 - 0.79\,\vec{e}_3$$

first-order correction





• Rescaling masses Code: "testing2_rescaleMass.ipynb" in Chaos/Graphs for paper

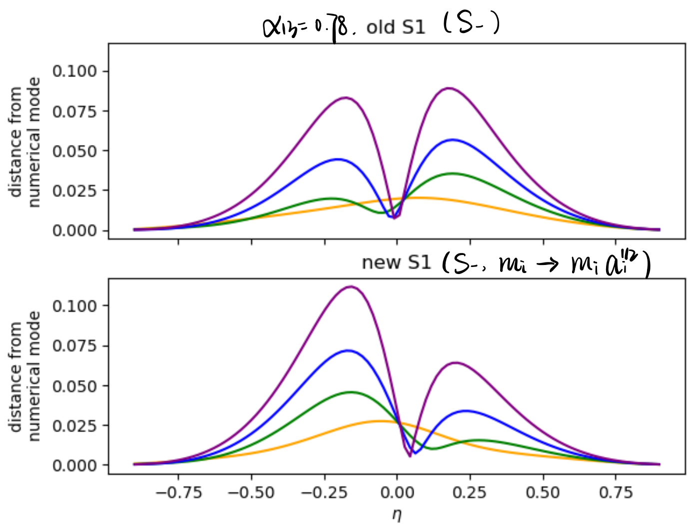
We define $m_i' = m_i a_i^{1/2}$, and $\tilde{m}_i' = m_i' / \sum_j m_j'$. Then

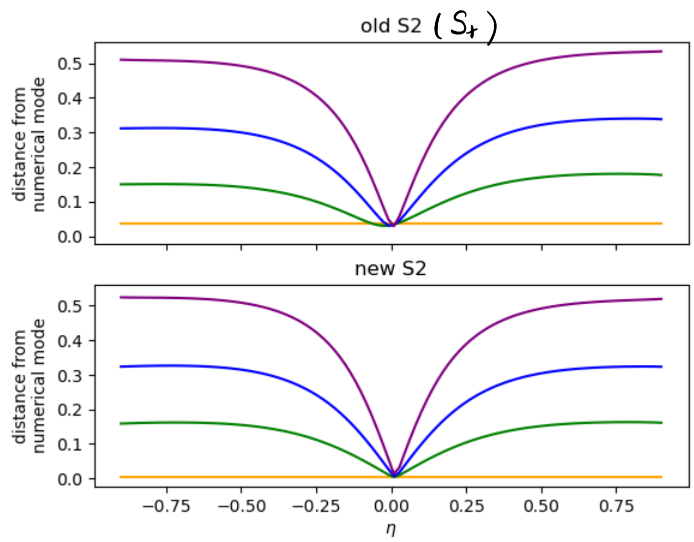
$$R_1' = \begin{pmatrix} \sqrt{\dfrac{m_1' m_2'}{(m_1'+m_3')\, m_{tot}'}} & -\sqrt{\dfrac{(m_1'+m_3')^2}{(m_1'+m_3')\, m_{tot}'}} & \sqrt{\dfrac{m_3' m_2'}{(m_1'+m_3')\, m_{tot}'}} \\ -\sqrt{\dfrac{m_3'}{m_1'+m_3'}} & 0 & \sqrt{\dfrac{m_1'}{m_1'+m_3'}} \\ \sqrt{\dfrac{m_1'}{m_{tot}'}} & \sqrt{\dfrac{m_2'}{m_{tot}'}} & \sqrt{\dfrac{m_3'}{m_{tot}'}} \end{pmatrix}$$

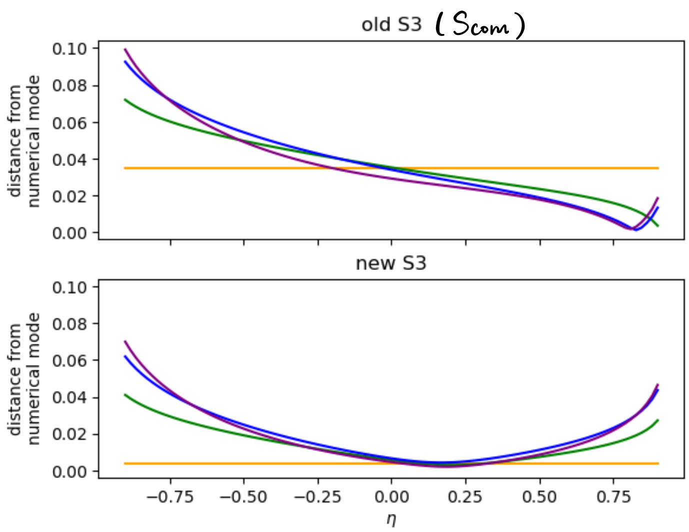
$$= \begin{pmatrix} \sqrt{\dfrac{\tilde{m}_1' \tilde{m}_2'}{\tilde{m}_1'+\tilde{m}_3'}} & -\sqrt{\tilde{m}_1'+\tilde{m}_3'} & \sqrt{\dfrac{\tilde{m}_3' \tilde{m}_2'}{\tilde{m}_1'+\tilde{m}_3'}} \\ -\sqrt{\dfrac{\tilde{m}_3'}{\tilde{m}_1'+\tilde{m}_3'}} & 0 & \sqrt{\dfrac{\tilde{m}_1'}{\tilde{m}_1'+\tilde{m}_3'}} \\ \sqrt{\tilde{m}_1'} & \sqrt{\tilde{m}_2'} & \sqrt{\tilde{m}_3'} \end{pmatrix}$$

$$\vec{G}^{\dagger} = \left(\sqrt{m_1'}\,\vec{e}_1 \quad \sqrt{m_2'}\,\vec{e}_2 \quad \sqrt{m_3'}\,\vec{e}_3 \right)$$

By rescaling the mass, we don't need to approximate constant a_i







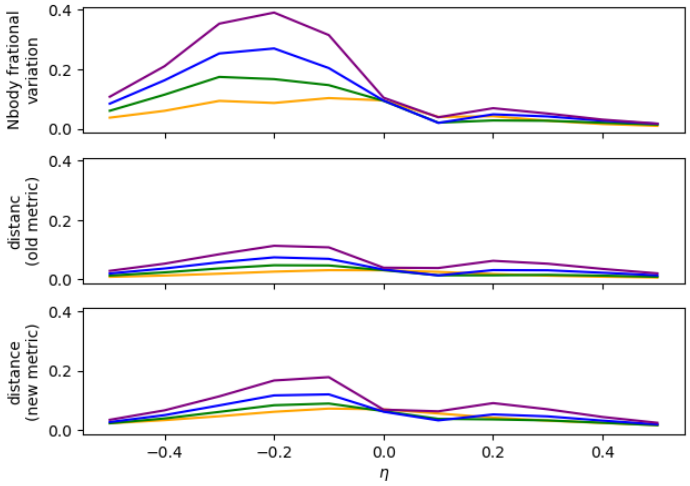
more conserved S_2, S_3 in the test-particle limit

- Change metric

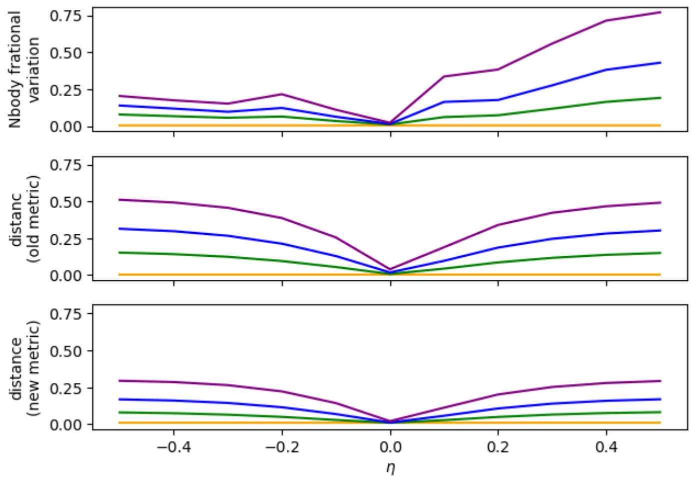
Notes: do not set $m_2 < 1e-20$ in Nbody, this causes numerical errors

always check the oscillations in numerical modes. Fig. 6 in paper is a bad example

error in $\vec{S}_-$



error in $\vec{S}_+$ ($\vec{\ell}_{13}$)

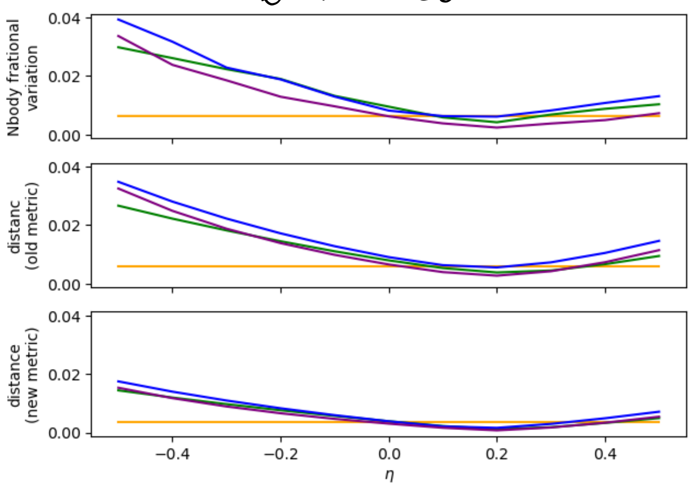


error in $\vec{\varepsilon}_{cm}$

oscillation in Nbody

distance in $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

distance in $\{\vec{e}_{12}, \vec{e}_{23}, \vec{e}_2\}$



Code: "testing1_changeMetric.ipynb" in Chaos / Graphs for paper