Homework 2 - Optimal Control Systems

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1. **Suppose that Q is a non-symmetric matrix, S is the symmetric part of Q, and x is an arbitrary vector. Prove the following:**

$$x^T Qx = x^T Sx$$

As the symetric part of Q is : $S = \frac{1}{2}(Q + Q^T)$, So:

$$\begin{aligned}
x^T Sx &= x^T \frac{1}{2}(Q + Q^T)x \\
&= x^T \frac{1}{2}(Q + Q^T)x \\
&= \frac{1}{2}(x^T Q + x^T Q^T)x \\
&= \frac{1}{2}(x^T Qx + x^T Q^T x)
\end{aligned}$$

Knowing that: $x^T Qx = x^T Q^T x$, So $x^T Sx = \frac{1}{2}(x^T Qx + x^T Qx) = x^T Qx$.

It proves that: $x^T Qx = x^T Sx$

3. **Prove the four properties in the left column of Table 1-1 in Kirk's book**

To prove the properties of the linear system State Transition Matrix (STM) it is necessary to use the following representation of STM:

$$\varphi(t) = e^{At}$$

a) $\varphi(0) = I$

$$\begin{aligned}\varphi(t) &= e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3} + ... \\ \varphi(0) &= e^{A0} = I + A0 + \frac{A^20^2}{2} + \frac{A^30^3}{3} + ... \\ &= I\end{aligned}$$

b) $\varphi(t_2 - t_1)\varphi(t_1 - t_0) = \varphi(t_2 - t_0)$

Knowing that

$$\varphi(t_2 - t_1) = e^{A(t_2 - t_1)}$$

And

$$\varphi(t_1 - t_0) = e^{A(t_1 - t_0)}$$

We have:

$$\begin{aligned}\varphi(t_2 - t_1)\varphi(t_1 - t_0) &= e^{A(t_2-t_1)}e^{A(t_1-t_0)} \\ &= e^{A(t_2-t_1+t_1-t_0)} \\ &= e^{A(t_2-t_0)} \\ &= \varphi(t_2 - t_0)\end{aligned}$$

c) $\varphi^{-1}(t_2 - t_1) = \varphi(t_1 - t_2)$

$$\begin{aligned}\varphi^{-1}(t_2 - t_1) &= \left(e^{A(t_2-t_1)}\right)^{-1} \\ &= e^{-A(t_2-t_1)} \\ &= e^{A(t_1-t_2)} \\ &= \varphi(t_1 - t_2)\end{aligned}$$

d) $\frac{d}{dt}\varphi(t) = A\varphi(t)$

$$\begin{aligned}\frac{d}{dt}\varphi(t) &= \frac{d}{dt}e^{At} \\ &= Ae^{At} \\ &= A\varphi(t)\end{aligned}$$

4. **Find the state transition matrices for the systems of Problem 1-12(e), (f), and (h) in Kirk's book**.

e) $\frac{Y(s)}{U(s)} = \frac{5(s+2)}{s(s+1)}$

$$\frac{Y(s)}{X(s)}\frac{X(s)}{U(s)} = 5(s+2)\frac{1}{s(s+1)}$$

For:

$$\frac{Y(s)}{X(s)} \quad = 5s(s+2)$$

Then, $y(t) = 5\ddot{x}(t) + 10x(t)$

For:

$$\frac{X(s)}{U(s)} = \frac{1}{s^2}$$
$$U(s) = s^2 X(s)$$

Then, $u(t) = \ddot{x}(t)$

Assuming the states variables as $x_1 = x(t)$ and $x_2 = \dot{x}(t)$ and $X = [x_1 \quad x_2]^T$ and $\dot{X} = [\dot{x}_1 \quad \dot{x}_2]^T$, it results in:

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} X + u$$
$$Y = \begin{bmatrix} 10 & 5 \end{bmatrix} X$$

For $\varphi(t) = e^{At}$ and knowing that $e^{At} = Qe^{At}Q^{-1}$:

The eigenvalues of A is $\lambda_1 = 0$ and $\lambda_2 = -1$ therefore, the eingevectors can be:

$$Q = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

And Q^{-1} :

$$Q^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Therefore:

$$
\begin{aligned}
\varphi(t) &= Qe^{At}Q^{-1} \\
&= \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}
\end{aligned}
$$

f) $\frac{Y(s)}{U(s)} = \frac{(s+1)(s+2)}{s^2}$

$$
\frac{Y(s)}{X(s)} \frac{X(s)}{U(s)} = (s+1)(s+2)\frac{1}{s^2}
$$

For:

$$
\begin{aligned}
\frac{Y(s)}{X(s)} &= (s+1)(s+2) \\
Y(s) &= (s+1)(s+2)X(s) \\
&= [s^2 + 3s + 2]X(s)
\end{aligned}
$$

Then, $y(t) = \ddot{x}(t) + 3\dot{x}(t) + 2x(t)$

For:

$$
\begin{aligned}
\frac{X(s)}{U(s)} &= \frac{1}{s^2} \\
U(s) &= s^2 X(s)
\end{aligned}
$$

Then, $u(t) = \ddot{x}(t)$

Assuming the states variables as $x_1 = x(t)$ and $x_2 = \dot{x}(t)$ and $X = [x_1 \;\; x_2]^T$ and $\dot{X} = [\dot{x}_1 \;\; \dot{x}_2]^T$, it results in:

$$
\begin{aligned}
\dot{X} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + u \\
Y &= \begin{bmatrix} 2 & 3 \end{bmatrix} X
\end{aligned}
$$

For $\varphi(t) = \mathcal{L}^{-1}\left\{[sI - A]^{-1}\right\}$

$$\begin{aligned}\varphi(t) &= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}^{-1}\right\} \\ &= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s^2} \end{bmatrix}\right\}\end{aligned}$$

Therefore:

$$\varphi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

h) $\frac{Y(s)}{U(s)} = \frac{4}{(s+1)(s+2)}$

$$\frac{Y(s)}{X(s)}\frac{X(s)}{U(s)} = 4\frac{1}{(s+1)(s+2)}$$

For:

$$\frac{Y(s)}{X(s)} = 4$$

Then, $y(t) = 4x(t)$

For:

$$\begin{aligned}\frac{X(s)}{U(s)} &= \frac{1}{(s+1)(s+2)} \\ U(s) &= [s^2 + 3s + 2]X(s)\end{aligned}$$

Then, $u(t) = \ddot{x}(t) + 3\dot{x}(t) + 2x(t)$

Assuming the states variables as $x_1 = x(t)$ and $x_2 = \dot{x}(t)$ and $X = [x_1 \;\; x_2]^T$ and $\dot{X} = [\dot{x}_1 \;\; \dot{x}_2]^T$, it results in:

$$\begin{aligned}\dot{X} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X + u \\ Y &= \begin{bmatrix} 1 & 0 \end{bmatrix} X\end{aligned}$$

For $\varphi(t) = \mathcal{L}^{-1}\left\{[sI - A]^{-1}\right\}$

$$\begin{aligned}
\varphi(t) &= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\right\} \\
&= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}\right\} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-t} - \mathrm{e}^{-2\,t} & \mathrm{e}^{-t} - \mathrm{e}^{-2\,t} \\ 2\,\mathrm{e}^{-2\,t} - 2\,\mathrm{e}^{-t} & 2\,\mathrm{e}^{-2\,t} - \mathrm{e}^{-t} \end{bmatrix}
\end{aligned}$$