Homework 5 - Optimal Control Systems

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1 State the fundamental theorem of the calculus of variations

According to Kirk in [1] the fundamental theorem of calculus of variations can be stated that the variation of a functional must be zero for an extremal.

It means that for the following conditions:

1. Vector function of t in the class Ω;
2. $J(x)$ be a differentiable functional of x;
3. Ω includes functions that are not constrained by any boundaries;
4. x^* is an extremal (maximum or minimum);

Then,

$$\delta J(x^*, \delta x) = 0 \tag{1}$$

for all admissible δx. Where δJ is the variation of the functional J.

2 Find the extremals

a $J(x) = \int_a^b 12xy + (y')^2 dx$

The solution of this problem can be found using the Euler Equation:

$$\frac{\partial g}{\partial y} - \frac{d}{dt}\frac{\partial q}{\partial \dot{y}} = 0 \tag{2}$$

Therefore, for $g(y, \dot{y}, x) = 12xy + y'^2$, we have:

$$\frac{\partial g}{\partial y} = 12x$$
$$\frac{\partial q}{\partial \dot{y}} = 2y'$$
$$\frac{d}{dt}\frac{\partial q}{\partial \dot{y}} = 2y''$$

Therefore:

$$\frac{\partial g}{\partial y} - \frac{d}{dt}\frac{\partial q}{\partial \dot{y}} = 12x - 2y'' = 0$$

The second-order linear ordinary differential equation also can be written as :

$$\frac{d^2 y(x)}{dx^2} = 6x \tag{3}$$

Integrate both side with respect to x:

$$\frac{dy(x)}{dx} = \int_a^b 6x \ dx = 3x^2 + C_1$$

Integrate again both side with respect to x:

$$y(x) = \int_a^b 3x^2 + C_1 \ dx = x^3 + C_1 x + C_2$$

Therefore $y(x) = x^3 + C_1 x + C_2$ with initial and final conditions:

$$y(a) = y_a$$
$$y(b) = y_b$$

The constants $C1$ and $C2$ can be found by applying the boundary conditions.

b $J(x) = \int_a^b y^2 + (y')^2 dx$

The solution of this problem can be found using the Euler Equation:

$$\frac{\partial g}{\partial y} - \frac{d}{dt}\frac{\partial q}{\partial \dot{y}} = 0 \tag{4}$$

Therefore, for $g(y, \dot{y}, x) = y^2 + (y')^2$, we have:

$$\frac{\partial g}{\partial y} = 2y$$
$$\frac{\partial q}{\partial \dot{y}} = 2y'$$
$$\frac{d}{dt}\frac{\partial q}{\partial \dot{y}} = 2y''$$

Therefore:

$$\frac{\partial g}{\partial y} - \frac{d}{dt}\frac{\partial q}{\partial \dot{y}} = 2y - 2y'' = 0$$

The second-order linear ordinary differential equation also can be written as :

$$\frac{d^2y(x)}{dx^2} - y = 0 \tag{5}$$

The solution for this second-order linear ordinary differential equation is:

$$y(x) = C_1 e^x + C_2 e^{-x}$$

with initial and final conditions:

$$y(a) = y_a$$
$$y(b) = y_b$$

The constants C_1 and C_2 can be found by applying the boundary conditions.

3 Verification of solution of Brachistochrone

$$\begin{cases} x = x_f + \dfrac{y_f}{2\cos^2\theta_f}[2(\theta_f - \theta) + \sin(2\theta_f) - \sin(2\theta)] \\ y = \dfrac{y_f \cos^2\theta}{\cos^2\theta_f} \end{cases}$$

The aim if verify if the following equation is true:

$$1 + \dot{y}^2 + 2y\ddot{y} = 0 \tag{6}$$

To find $\dot{y}$ we can use the following expression:

$$\dot{y} = \frac{dy}{d\theta}\frac{d\theta}{dx}$$

where:

$$\frac{d\theta}{dx} = \left(\frac{d\theta}{dx}\right)^{-1}$$

$$\frac{dy}{d\theta} = -2y_f \frac{\cos\theta \sin\theta}{\cos^2\theta_f}$$
$$\frac{dx}{d\theta} = -y_f \frac{1 + \cos(2\theta)}{\cos^2(\theta_f)}$$

Therefore:

$$\dot{y} = 2{\color{blue}y_f} \frac{\cos\theta \sin\theta}{{\color{red}\cos^2\theta_f}} \frac{{\color{red}\cos^2(\theta_f)}}{{\color{blue}y_f}\left[1 + \cos(2\theta)\right]} = 2\frac{\cos\theta \sin\theta}{1 + \cos(2\theta)} \tag{7}$$

According to Figure 1 and some mathematic derivation, we can write $\cos(2\theta)$ equal to $2\cos^2(2\theta) - 1$

Therefore:

$$\frac{d\dot{y}}{d\theta} = \frac{d}{d\theta}\left[2\frac{\cos\theta \sin\theta}{1 + 2\cos^2(2\theta) - 1}\right] = \frac{\sin\theta}{\cos\theta}$$

To find $\ddot{y}$ we can use the following expression:

$$\ddot{y} = \frac{d\dot{y}}{d\theta}\frac{d\theta}{dx}$$

where:

$$\frac{d\dot{y}}{d\theta} = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

Therefore:

$$\begin{aligned}\ddot{y} = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) &= \left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right)\left(-\frac{\cos^2(\theta_f)}{y_f\left[1 + \cos(2\theta)\right]}\right) \\ &= -\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\frac{\cos^2(\theta_f)}{2y_f\cos^2\theta}\end{aligned}$$

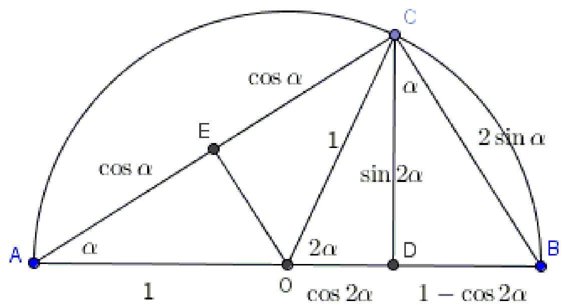

Figure 1: Illustration of trigonometric identities

Replacing the derived equations in:

$$1 + \dot{y}^2 + 2y\ddot{y} = 0$$

$$1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 2\left(\frac{y_f\cos^2\theta}{\cos^2\theta_f}\right)\left(-\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\frac{\cos^2(\theta_f)}{2y_f\cos^2\theta}\right) = 0$$

$$1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 + {\color{magenta}2}\left(\frac{{\color{red}y_f}{\color{blue}\cos^2\theta}}{{\color{green}\cos^2\theta_f}}\right)\left(-\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\frac{{\color{green}\cos^2(\theta_f)}}{{\color{magenta}2y_f}{\color{blue}\cos^2\theta}}\right) = 0$$

$$1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 - \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = 0$$

Knowing that $\cos^2\theta + \sin^2\theta = 1$ and $\cos^2\theta = \sin^2\theta - 1$, we have

$$1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 - \frac{1}{\sin^2\theta - 1} = 0$$

$$\frac{\sin^2\theta\cos^2\theta {\color{red}- \cos^2\theta} + \sin^4\theta - \sin^2\theta {\color{red}+ \cos^2\theta}}{\cos^2\theta\left[\sin^2\theta - 1\right]} = 0$$

$$\frac{\sin^2\theta\left(\cos^2\theta + \sin^2\theta - 1\right)}{\cos^2\theta\left[\sin^2\theta - 1\right]} = 0$$

$$1 - 1 = 0$$

4 Plot the Brachistochrone Solution

The figure 2-4 contains the brachistochrone solution for the desired final conditions: (1, 1), (1, 3), and (3, 1).

The figure 4 is an additional figure with the goal to verify the principle of optimality. This figure contains two solution: the first one is the final condition equal to (1,1) and the second one is a point in the optimal path from the previous solution. We can verify that the optimal path is the same for both.

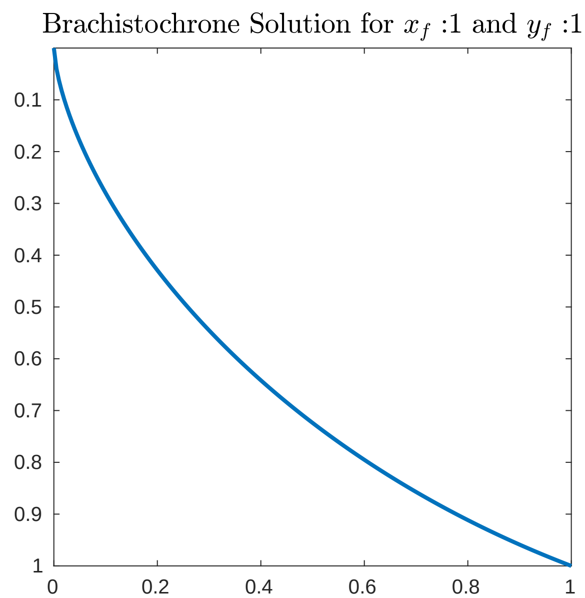

Figure 2: brachistochrone solution for (1,1)

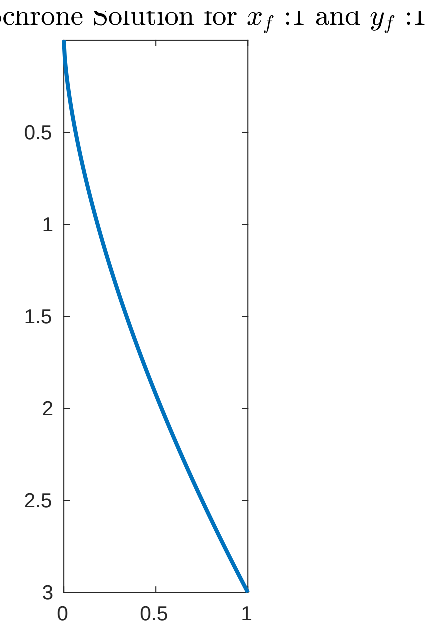

Figure 3: brachistochrone solution for (1,3)

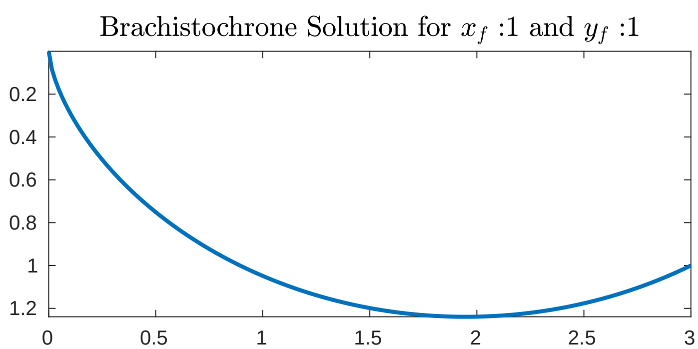

Figure 4: brachistochrone solution for (3,1)

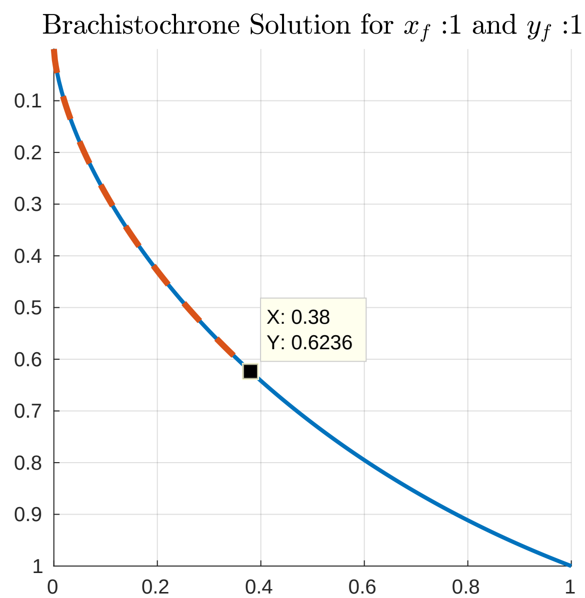

Figure 5: brachistochrone solution to verify the principle of optimality

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% Book: Optimal Control Theory: An introduxtion by Donald E. Kirk
%
% Erivelton Gualter, 02/21/2018

clear all; clc; close all

% parameter
theta0 = pi/2;
X0 = 0;
Y0 = 0;

Xf = [1, 1, 3];
Yf = [1, 3, 1];

[X(1,:), Y(1,:)] = brachistochrone(X0, Y0, Xf(1), Yf(1), theta0);
[X(2,:), Y(2,:)] = brachistochrone(X0, Y0, Xf(2), Yf(2), theta0);
[X(3,:), Y(3,:)] = brachistochrone(X0, Y0, Xf(3), Yf(3), theta0);

Xf(4) = 0.380000000000000;
Yf(4) = 0.623586752778062;
[X(4,:), Y(4,:)] = brachistochrone(X0, Y0, Xf(4), Yf(4), theta0);

f1 = figure; plot(X(1,:),Y(1,:), 'LineWidth', 2);
set(gca, 'Ydir','reverse');
title_str = strcat('Brachistochrone Solution for $x_f: ...
    $',num2str(Xf(1)),' and $y_f: $',num2str(Yf(1)));
title(title_str, 'Interpreter','Latex', 'FontSize',14);
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axis equal; axis([min(X(1,:)) max(X(1,:)) min(Y(1,:)) max(Y(1,:))]);
saveFigureToPdf('f1',f1);

f2 = figure; plot(X(2,:),Y(2,:), 'LineWidth', 2);
set(gca, 'Ydir','reverse');
title_str = strcat('Brachistochrone Solution for $x_f: ...
    $',num2str(Xf(1)),' and $y_f: $',num2str(Yf(1)));
title(title_str, 'Interpreter','Latex', 'FontSize',14);
axis equal; axis([min(X(2,:)) max(X(2,:)) min(Y(2,:)) max(Y(2,:))]);
saveFigureToPdf('f2',f2);

f3 = figure; plot(X(3,:),Y(3,:), 'LineWidth', 2);
set(gca, 'Ydir','reverse');
title_str = strcat('Brachistochrone Solution for $x_f: ...
    $',num2str(Xf(1)),' and $y_f: $',num2str(Yf(1)));
title(title_str, 'Interpreter','Latex', 'FontSize',14);
axis equal; axis([min(X(3,:)) max(X(3,:)) min(Y(3,:)) max(Y(3,:))]);
saveFigureToPdf('f3',f3);

f4 = figure; hold on;
plot(X(1,:),Y(1,:), 'LineWidth', 2); plot(X(4,:),Y(4,:), '--', ...
    'LineWidth', 3);
set(gca, 'Ydir','reverse');
title_str = strcat('Brachistochrone Solution for $x_f: ...
    $',num2str(Xf(1)),' and $y_f: $',num2str(Yf(1)));
title(title_str, 'Interpreter','Latex', 'FontSize',14);
axis equal; axis([min(X(1,:)) max(X(1,:)) min(Y(1,:)) max(Y(1,:))]);
legend('[0,0]-[1,1]','[0,0]-[0.4,0.6]');
saveFigureToPdf('f4',f4);

function [X,Y] = brachistochrone(X0, Y0, xf, yf, theta0)

    N = 200;

    fun = @(thetaf) f1([X0; Y0], [xf; yf], theta0, thetaf);  % ...
        function of x alone
    tf = fzero(fun,0);

    X = X0:(xf-X0)/N:xf(1);
    for i=1:length(X)

        fun = @(theta) f2(X(i), [xf; yf], theta, tf);  % ...
            function of x alone
        theta_out = fzero(fun,0);

        Y(i) = yf*cos(theta_out)^2/(cos(tf)^2);

    end

    % Functions
    function minxy = f1(P0, Pf, theta, thetaf)

        x = P0(1);
        y = P0(2);
        xf = Pf(1);
        yf = Pf(2);

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        minxy = x - (xf + yf/(2*cos(thetaf)^2) * ...
            (2*(thetaf-theta) + ...
            sin(2*thetaf)) - sin(2*theta));

        minxy = minxy + y - yf/(2*cos(thetaf)^2)*cos(theta)^2;
    end

    function minx = f2(x, Pf, theta, thetaf)

        xf = Pf(1);
        yf = Pf(2);

        minx = x - (xf + yf/(2*cos(thetaf)^2) * ...
            (2*(thetaf-theta) + sin(2*thetaf) ...
            - sin(2*theta)));
    end
end
```

You can access the code at: https://github.com/EriveltonGualter/EEC-744-Optimal-Control-Systems