The following supplemental material should help you with understanding why an undamped second-order system has an oscillating output (see page 4).

Damping Conditions

For a mass-spring –damper system, we have the transfer function as below.

$$\frac{x(s)}{r(s)} = \frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{1}$$

In equation (1), ζ is damping constant, and ω_n is natural frequency of the system.

As r(t)=1, whose Laplace transform is r(s)=1/s, we have

$$x(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} r(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}. \tag{2}$$

There are **two poles** of the transfer function (1), which are $s_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}\,\omega_n$ and $s_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}\,\omega_n$

1. As $\zeta > 1$ Over-damped condition

We have two negative real poles $s_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}\,\omega_n$ and $s_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}\,\omega_n$.
Therefore,

$$x(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = \frac{k_1}{s} + \frac{k_2}{s - (-\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n)} + \frac{k_3}{s - (-\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n)} \tag{3}$$

Using PFE, we have

$$k_1 = sx(s)\big|_{s=0} = s \times \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}\bigg|_{s=0} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\bigg|_{s=0} = 1$$

$$k_2 = (s + \zeta\omega_n - \sqrt{\zeta^2 - 1}\,\omega_n)x(s)\big|_{s=-\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n} = (s + \zeta\omega_n - \sqrt{\zeta^2 - 1}\,\omega_n) \times \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}\bigg|_{s=-\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n}$$

$$= \frac{\omega_n^2}{s + \zeta\omega_n + \sqrt{\zeta^2 - 1}\,\omega_n}\frac{1}{s}\bigg|_{s=-\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n} = \frac{1}{2\sqrt{\zeta^2 - 1}(-\zeta + \sqrt{\zeta^2 - 1})}$$

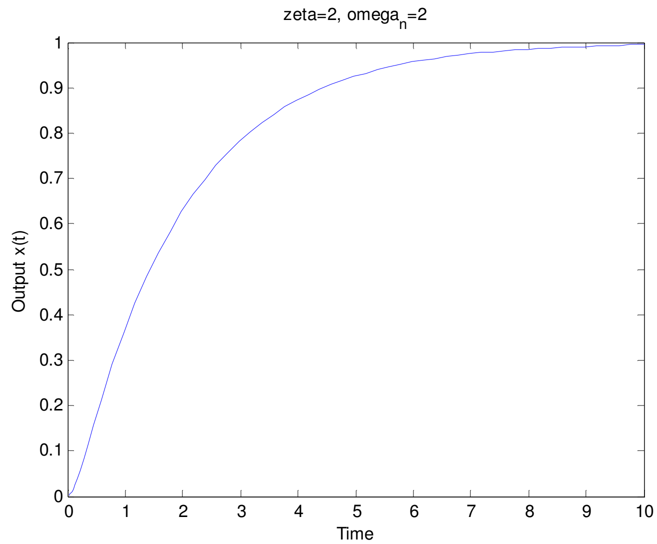
$$k_3 = (s + \zeta\omega_n + \sqrt{\zeta^2 - 1}\,\omega_n)x(s)\big|_{s=-\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n} = (s + \zeta\omega_n + \sqrt{\zeta^2 - 1}\,\omega_n) \times \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}\bigg|_{s=-\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n}$$

$$= \frac{\omega_n^2}{s + \zeta\omega_n - \sqrt{\zeta^2 - 1}\,\omega_n}\frac{1}{s}\bigg|_{s=-\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n} = \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta + \sqrt{\zeta^2 - 1})}$$

Then we do the inverse Laplace transform of (3), we have

x(t)=$1 + k_2 e^{(-\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n)t} + k_3 e^{(-\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n)t}$. As $t=0$, $x(0)=1+k_2+k_3=0$.

As $t \to \infty, x(\infty) = 1$.



From the figure above, we can see that the x(t) goes to steady state 1 after around 10 seconds. It is a slow response.

2. As ζ=1, critically damping condition

$$x(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}\frac{1}{s} = \frac{\omega_n^2}{(s+\omega_n)^2}\frac{1}{s} = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2} \qquad (4)$$

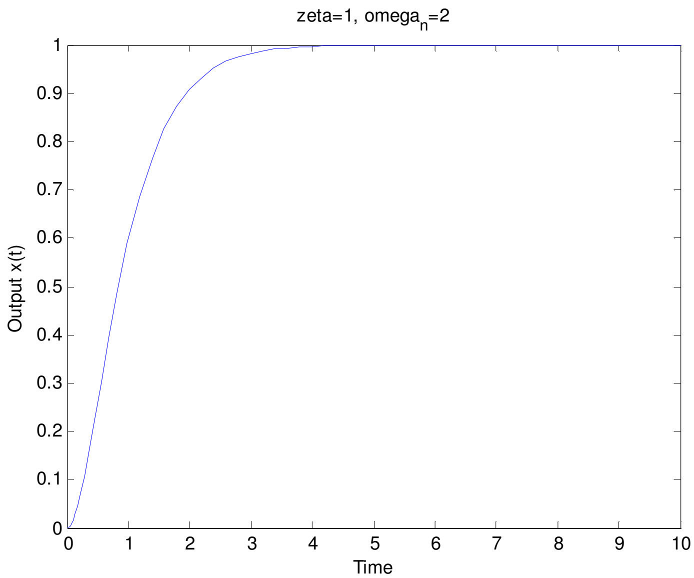
Now we have <u>have two repeated negative real poles</u>, which are $s_1=s_2=-\varsigma\omega_n$.

We do the inverse Laplace transform of (4), we have

$$x(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}.$$

As t=0, x(0)= 0. As $t \to \infty, x(\infty) = 1$. (Actually we can use <u>final value theorem</u> to verify this:

$x(\infty) = \lim_{s\to 0} sx(s) = \lim_{s\to 0} s\frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}\frac{1}{s} = 1$)



From the figure above, we can see the x(t) goes to 1 after around 4 seconds. So the response is faster than the one with ζ=2. As the damping constant decreases, the response speed increases.

3. As $0 < \zeta < 1$, underdamped condition

Now we have two complex poles, which are $s_1 = -\zeta\omega_n + \sqrt{1-\zeta^2}\,\omega_n j$, and $s_2 = -\zeta\omega_n - \sqrt{1-\zeta^2}\,\omega_n j$.

We define the damped frequency $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$.

$$x(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}}(\sqrt{1-\zeta^2}\,\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2} \quad (5)$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

We do the inverse Lapalce transform of (5), then we have

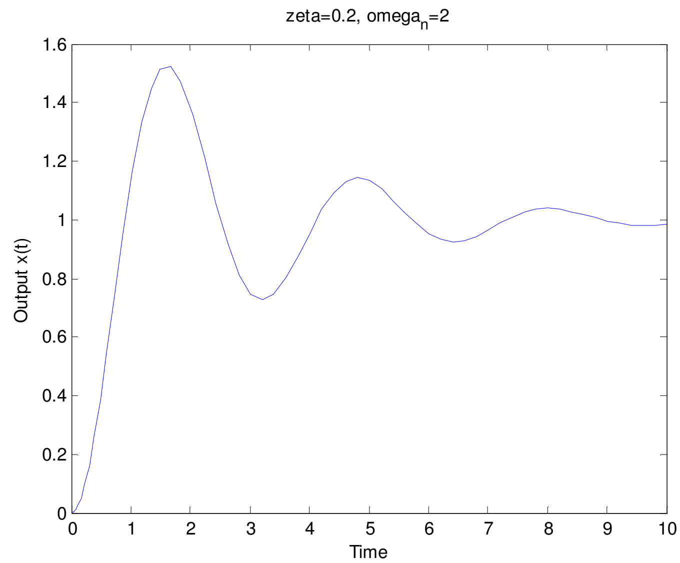
$$x(t) = 1 - e^{-\zeta\omega_n t}[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)]. \quad (6)$$

As t=0, x(0)=0.

As $t \to \infty, x(\infty) = 1$.

As 0<t<∞, we rewrite (6) as $x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_d t + \beta)$, where $\beta = tg^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})$.

Therefore x(t) can be taken as a damped oscillation (the amplitude is decreasing with time).



4. As $\zeta = 0$, undamped condition

Now we have <u>two complex poles on the imaginary axis</u>. They are $s_1 = \omega_n j$, and $s_2 = -\omega_n j$.

$$x(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = \frac{\omega_n^2}{s^2 + \omega_n^2}\frac{1}{s} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2} \qquad (7)$$

We do the inverse of Lapalce transform of (7). Then we have

$x(t) = 1 - \cos(\omega_n t)$. So this x(t) is <u>a same-amplitude oscillation</u>.

As t=0, x(0)=0.