Homework 2 - Question 1

1. For $x(k+1) = A_d x(k) + B_d u(k) + \delta$
$y(k) = Cx(k) + Du(k)$ where
δ : constant disturbance. Obtain the optimization cost function.

State Prediction: " $x_i = x(k+1)$ and $x_0 = x(k)$ "
For $x_1 = A_d x_0 + B_d u_0 + \delta$, So

$x_1 = A_d x_0 + B_d u_0$

$x_2 = A_d(A_d x_0 + B_d u_0) + B_d u_1 = A_d^2 x_0 + A_d B_d u_0 + B_d u_1$

$x_3 = A_d x_2 + B_d u_2 = A_d^3 x_0 + A_d^2 B_d u_0 + A_d B_d u_1 + B_d u_2$

$\vdots$

$x_{ny} = A_d^{ny} X_0 + A^{ny-1} B_d u_0 + \cdots + B_d u_{ny-1}$

Therefore:

$$\underbrace{\hat{x}}_{n\,ny \times 1} = \underbrace{\overset{P_{xx}}{\begin{bmatrix} A_d \\ \vdots \\ A_d^{ny} \end{bmatrix}}}_{n\,ny \times n} \underbrace{X_0}_{n \times 1} + \underbrace{\overset{H_x}{\begin{bmatrix} B & \cdots & 0 \\ \vdots & & \vdots \\ A^{ny-1}B & & B \end{bmatrix}}}_{n\,ny \times ny\,m} \underbrace{\hat{u}}_{ny\,m \times 1}$$

It results in $\hat{x} = P_{xx} X_0 + X_x \hat{u}$

Output Prediction: $y_i = Cx_i + Du_i + \delta$

$y_1 = C(A_d x_0 + B_d u_0) + Du_1 + \delta = CA_d x_0 + CB_d u_0 + Du_1 + \delta$

$y_2 = CA_d^2 x_0 + CA_d B_d u_0 + CB_d u_1 + Du_2 + \delta$

$y_{ny} = CA_d^{ny} X_0 + CA_d^{ny-1} B_d u_0 + \cdots + CB_d u_{ny-1} + Du_{ny} + \delta$

therefore $\hat{y} = PX_0 + H\hat{u} + \Delta$

Error Predictions : $\hat{e} = \hat{r} - \hat{y} = \hat{r} - (Px_0 + H\hat{u} + \Delta)$

Object Function: $J = \sum_{i=1}^{Ny-1} e^T e + \lambda u^T u = \hat{e}^T e + \lambda \hat{u}^T \hat{u}$

$$= (\hat{r} - Px_0 - H\hat{u} - \Delta)^T (\hat{r} - Px_0 - H\hat{u} - \Delta) + \lambda \hat{u}^T \hat{u}$$
$$= (\hat{r}^T - x_0^T P^T - \hat{u}^T H^T - \Delta^T)(\hat{r} - Px_0 - H\hat{u} - \Delta) + \lambda \hat{u}^T \hat{u}$$
$$= \hat{r}^T r - \hat{r}^T P x_0 - \hat{r}^T H\hat{u} - \hat{r}^T \Delta - x_0^T P^T \hat{r} + x_0^T P^T P x_0 +$$
$$+ x_0^T P^T H \hat{u} + x_0^T P^T \Delta - \hat{u}^T H^T \hat{r} + \hat{u}^T H^T P x_0 + \hat{u}^T H^T H \hat{u} +$$
$$\hat{u}^T H^T \Delta + \Delta^T \hat{r} + \Delta^T P x_0 + \Delta^T H \hat{u} + \Delta^T \Delta + \lambda \hat{u}^T \hat{u}$$

Note: $-\hat{r}^T H\hat{u} - \hat{u}^T H^T \hat{r} = -2\hat{r}^T H \hat{u}$

$x_0^T P^T H\hat{u} + \hat{u}^T H^T P x_0 = 2 x_0^T P^T H \hat{u}$

$\hat{u}^T H^T \Delta + \Delta^T H \hat{u} = 2 \Delta^T H \hat{u}$

Therefore we can write as:

$$\boxed{J = J_0 + \hat{u}^T (H^T H + \lambda I)\hat{u} + 2(x_0^T P^T - \hat{r}^T + \Delta^T) H \hat{u}}$$

$$\nabla J = \frac{\partial J}{\partial \hat{u}} = 2(H^T H + \lambda I)\hat{u} + 2(x_0^T P^T - \hat{r}^T + \Delta^T) H = 0$$

Question 2

For running cost: $l = e^2 + \lambda P + \lambda_u \Delta u^2$

Power. $P = VI = V\left(\frac{V - a\dot{y}/R}{R_m}\right)$ and $V = \frac{R_m}{aR}\left(u + R^2 m_{cw} g\right)$ (

First: $P = \frac{V^2 - Va\dot{y}/R}{R_m}$ (I)

For $V^2 \Rightarrow$ $V^2 = \frac{R_m^2}{(aR)^2}\left(u^2 + 2uR^2 m_{cw} g + R^4 m_{cw}^2 g^2\right)$ (

Replacing II and III to I:

$$P = \frac{R_m}{(aR)^2}\left(u^2 + 2uR^2 m_{cw} g + R^4 m_{cw}^2 g^2\right) - \frac{\dot{y}}{R^2}\left(u + R^2 m_{cw} g\right)$$

$$= \frac{R_m u^2}{a^2 R^2} + \frac{2 R_m u m_{cw} g}{a^2} + \frac{R^2 m_{cw}^2 g^2}{a^2} - \frac{\dot{y} u}{R^2} - \dot{y} m_{cw} g \quad \text{(IV}$$

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Error: $\hat{e} = \hat{y}_f - \hat{y}_{pos} = \hat{y}_f - P_{pos} x_a - H_{pos} \Delta\hat{u}$ (V)

where $x_a = \begin{bmatrix} x \\ u \end{bmatrix}$

Therefore replacing (IV) and (V) to the running cost

$$J = \underbrace{(\hat{y}_f - P_{pos} x_a - H_{pos}\Delta\hat{u})^T (\hat{y}_f - P_{pos} x_a - H_{pos}\Delta\hat{u})}_{\text{Term I}} + \dots$$

$$\underbrace{\lambda\left(\frac{R_m u^2}{a^2 R^2} + \frac{2 R_m u m_{cw} g}{a^2} + \frac{R^2 m_{cw}^2 g^2}{a^2} - \frac{\dot{y} u}{R^2} - \dot{y} m_{cw} g\right)}_{\text{Term II}} + \lambda_u$$

First, solving Term I:

$= \left(\hat{y}_f^T - x_a^T P_{pos}^T - \Delta\hat{u}^T H_{pos}^T\right)\left(\hat{y}_f - P_{pos} x_a - H_{pos}\Delta\hat{u}\right)$

$= \hat{y}_f^T \hat{y}_f - \hat{y}_f^T P_{pos} x_a - \hat{y}_f^T H_{pos}\Delta\hat{u} - x_a^T P_{pos}^T \hat{y}_f + x_a^T P_{pos}^T P_{pos} x_a$

$+ x_a^T P_{pos}^T H_{pos}\Delta\hat{u} - \Delta\hat{u}^T H_{pos}^T \hat{y}_f + \Delta\hat{u}^T H_{pos}^T P_{pos} x_a + \Delta\hat{u}^T H_{pos}^T H_{pos}\Delta\hat{u}$

Note: $-\hat{y}_f^T H_{pos}\Delta\hat{u} - \Delta\hat{u}^T H_{pos}^T \hat{y}_f = -2\hat{y}_f^T H_{pos}\Delta\hat{u}$

$x_a^T P_{pos}^T H_{pos}\Delta\hat{u} + \Delta\hat{u} H_{pos}^T P_{pos} x_a = 2 x_a^T P_{pos}^T H_{pos}\Delta\hat{u}$

Therefore, Term I can be simplied as:

$$\text{Term I} = (\text{Term I})_{\text{constants}} + 2\left(x_a^T P_{pos}^T - \hat{y}_t^T\right) H_{pos}\, \Delta\hat{u} + \Delta\hat{u}^T H_{pos}^T H_{pos} \Delta\hat{u}$$

For "Term II", note we have the differention of y: $\dot{y}$, which correspond to the velocity state. So this can be written as:

$$\hat{y}_{vel}^T = x_a^T P_{vel}^T + \Delta\hat{u}^T H_{vel}^T$$

or

$$\hat{y}_{vel} = P_{vel}\, x_a + H_{vel}\, \Delta\hat{u}$$

Additionaly, we can write $\hat{u}$ as

$$\hat{u} = P_u x_a + H_u \Delta\hat{u}$$

Before simplify the "term II" lets solve some of the sub terms:

$$\begin{aligned}\hat{u}^T u &= (x_a^T P_u^T + \Delta\hat{u}^T H_u^T)(P_u x_a + H_u \Delta\hat{u}) \\ &= x_a^T P_u^T P_u x_a + x_a^T P_u^T H_u \Delta\hat{u} + \Delta\hat{u}^T H_u^T P_u x_a + \Delta\hat{u}^T H_u^T H_u \Delta\hat{u} \\ &= \underbrace{x_a^T P_u^T P_u x_a}_{\text{constants}} + 2 x_a^T P_u^T H_u \Delta\hat{u} + \Delta\hat{u}^T H_u^T H_u \Delta\hat{u}\end{aligned}$$

"Note we have a constant term here and others terms are in function of Δu. Since we dont need the constants values for J, for next derivation it will be not performed"

For $$\frac{2 R_m m_{cw} g}{a^2}\hat{u} = \frac{2 R_m m_{cw} g}{a^2}\left(P_u x_a + H_u \Delta\hat{u}\right)$$

constant

For $$\frac{\hat{y}_{vel}^T \hat{u}}{R^2} = \frac{1}{R^2}\left(x_a^T P_{vel}^T + \Delta\hat{u}^T H_{vel}^T\right)\left(P_u x_a + H_u \Delta\hat{u}\right)$$

$$\overset{\text{constant}}{=} \frac{1}{R^2}\left(x_a^T P_{vel}^T P_u x_a + x_a^T P_{vel}^T H_u \Delta\hat{u} + \Delta\hat{u}^T H_{vel}^T P_u x_a + \Delta\hat{u}^T H_{vel}^T H_u \Delta\hat{u} \right)$$

For $m_{cw} g \hat{y}_{vel} = m_{cw} g \left(\overset{\text{constant}}{P_{vel} x_a} + H_{uel} \Delta\hat{u} \right)$

Therefore, the second term can be written as:

$$\text{Term II} = \underset{\text{constants}}{(\text{Term I})} + \lambda \Bigg[\frac{R_m}{a^2 R^2} \cdot \left(2x_a^T P_u^T H_u \Delta\hat{u} + \Delta\hat{u}^T H_u^T H_u \Delta\hat{u} \right) + \cdots$$

$$+ \frac{2 R_m m_{cw} g H_u \Delta\hat{u}}{a^2} - \frac{x_a^T P_{vel}^T H_u \Delta\hat{u}}{R^2} - \frac{x_a^T P_u^T H_{vel} \Delta\hat{u}}{R^2} - \frac{\Delta\hat{u}^T H_{vel}^T H_u \Delta\hat{u}}{R^2} + \cdots$$

$$- m_{cw} g H_{vel} \Delta\hat{u} \Bigg]$$

Therefore, cost function can be written as:

$$J = J_0 + \text{Term I} + \text{Term II} + \lambda_u \Delta u^T \Delta u$$

$$\boxed{\begin{aligned} J = J_0 &+ \Delta\hat{u}^T \left(H_{pos}^T H_{pos} + \frac{\lambda R_m}{a^2 R^2} H_u^T H_u - \frac{H_{vel}^T H_u}{R^2} + \lambda_u I \right) \Delta\hat{u} + \cdots \\ &+ \Bigg[2\left(x_a^T P_{pos}^T - \hat{y}_t^T \right) H_{pos} + \frac{2\lambda R_m}{a^2 R^2} x_a^T P_u^T H_u \Delta\hat{u} + \frac{2 R_m m_{cw} g}{a^2} H_u \lambda + \cdots \\ &- \frac{\lambda x_a^T P_{vel}^T H_u}{R^2} - \frac{\lambda x_a^T P_u^T H_{vel}}{R^2} - \lambda m_{cw} g H_{vel} \Bigg] \Delta\hat{u} \end{aligned}}$$