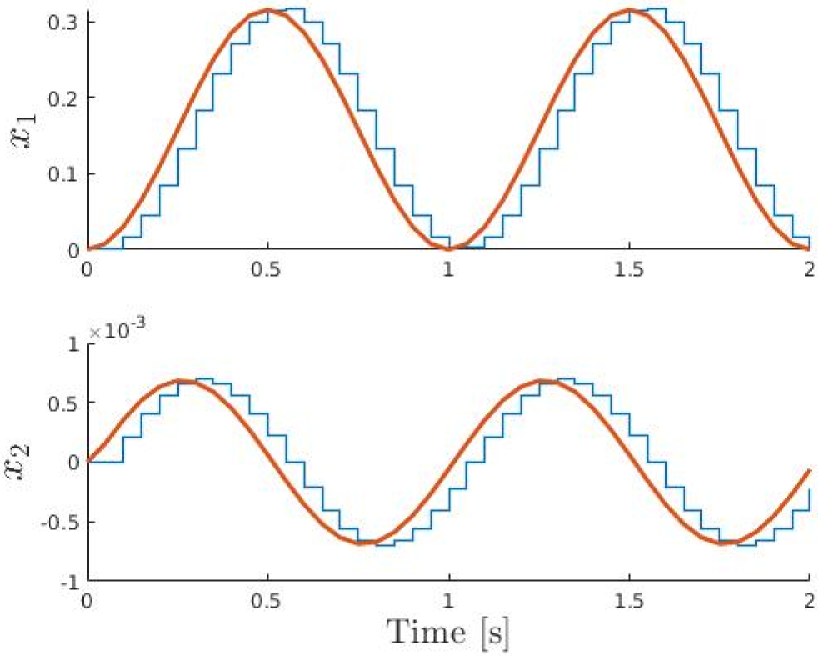
Homework 2 - Model Predictive Control

Erivelton Gualter dos Santos

1 Mini-Project - Elevator

In order to verify that the discrete states have the same physical meaning as their continuous counterparts, a simulation was done in order to compare both states after discretization for a sinusoidal control input.



The code to verify the discretization can be found here: `https://github.com/EriveltonGualter/ESC-794-Model-Predictive-Control/tree/master/HW2`.

1.1 Tuning Control System

In order to help the task to control the system, it was developed a interface, which is despited in the following figure, to accelerate and simplify the process to tune the parameters.

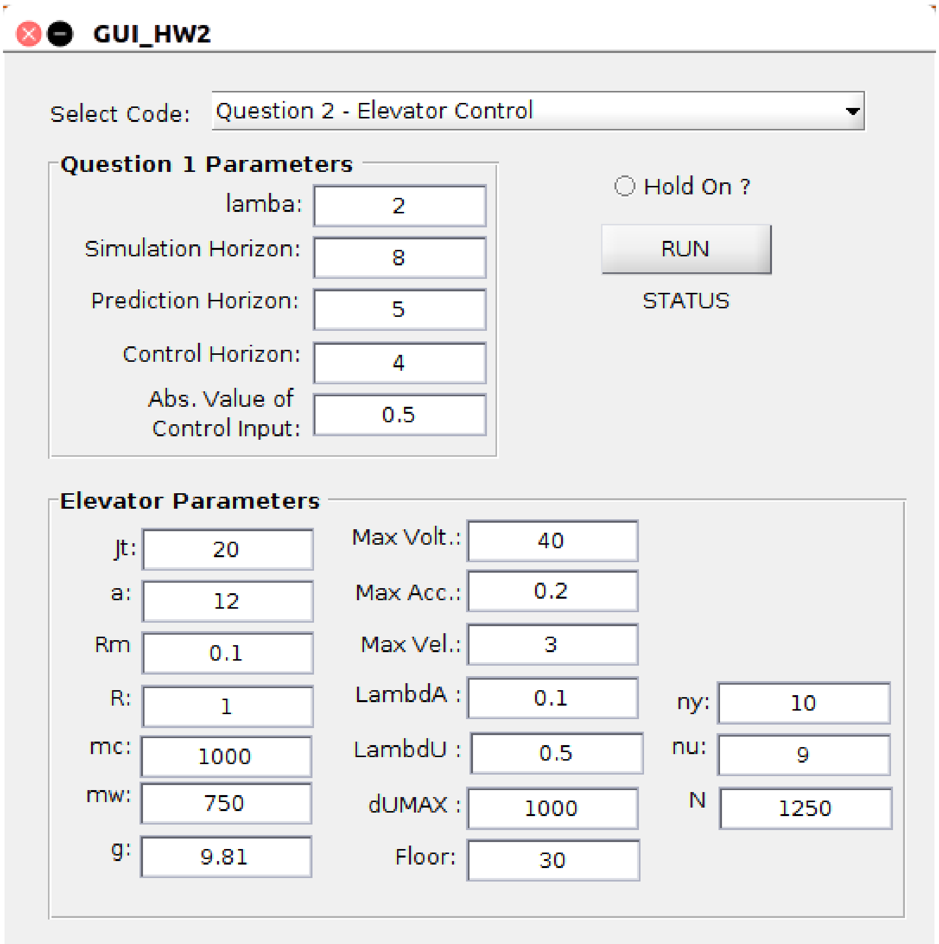


Figure 1: Interface for Homework 2

After some tests varying the parameters of the MPC controller (λ and λ_u), it was clear that for the current specifications of the problem we ca not achieve it. In order to reach all the requirements, it is necessary to alter some of the constraints for example.

Next figure contains the results of the controller.

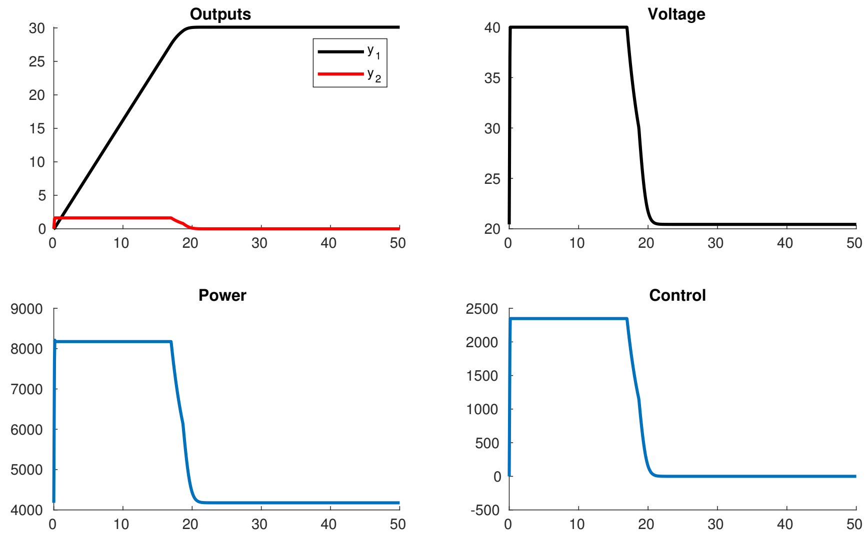


Figure 2: Results

References

1. https://www.thyssenkruppelevator.com/Tools/energy-calculator

Homework 2 - Question 1

1. For $x(k+1) = A_d x(k) + B_d u(k) + \delta$
$y(k) = C x(k) + D u(k)$ where

δ : constant disturbance. Obtain the optimization cost function.

<u>State Prediction:</u> " $x_i = x(k+1)$ and $x_0 = x(k)$ "

For $x_1 = A_d x_0 + B_d u_0 + \delta$, So

$$x_1 = A_d x_0 + B_d u_0$$

$$x_2 = A_d (A_d x_0 + B_d u_0) + B_d u_1 = A_d^2 x_0 + A_d B_d u_0 + B_d u_1$$

$$x_3 = A_d x_2 + B_d u_2 = A_d^3 x_0 + A_d^2 B_d u_0 + A_d B_d u_1 + B_d u_2$$

$$\vdots$$

$$x_{ny} = A_d^{ny} X_0 + A^{ny-1} B_d u_0 + \cdots + B_d u_{ny-1}$$

Therefore:

$$\underset{nny \times 1}{\hat{x}} = \underset{nny \times n}{\overset{P_{xx}}{\begin{bmatrix} A_d \\ \vdots \\ A_d^{ny} \end{bmatrix}}} \underset{n \times 1}{X_0} + \underset{nny \times nym}{\overset{H_x}{\begin{bmatrix} B & \cdots & 0 \\ \vdots & & \vdots \\ A^{ny-1} B & & B \end{bmatrix}}} \underset{nym \times 1}{\hat{u}}$$

It results in $\hat{x} = P_{xx} X_0 + X_x \hat{u}$

<u>Output Prediction:</u> $y_i = C x_i + D u_i + \delta$

$$y_1 = C(A_d x_0 + B_d u_0) + D u_1 + \delta = C A_d x_0 + C B_d u_0 + D u_1 + \delta$$

$$y_2 = C A_d^2 x_0 + C A_d B_d u_0 + C B_d u_1 + D u_2 + \delta$$

$$y_{ny} = C A_d^{ny} X_0 + C A_d^{ny-1} B_d u_0 + \cdots + C B_d u_{ny-1} + D u_{ny} + \delta$$

therefore $\hat{y} = P X_0 + H \hat{u} + \Delta$

Error Predictions : $\hat{e} = \hat{r} - \hat{y} = \hat{r} - \left(P x_0 + H\hat{u} + \Delta \right)$

Object Function: $J = \sum_{i=1}^{N_p-1} e^T e + \lambda u^T u = \hat{e}^T e + \lambda \hat{u}^T \hat{u}$

$$= (\hat{r} - Px_0 - H\hat{u} - \Delta)^T (\hat{r} - Px_0 - H\hat{u} - \Delta) + \lambda \hat{u}^T \hat{u}$$

$$= (\hat{r}^T - x_0^T P^T - \hat{u}^T H^T - \Delta^T)(\hat{r} - Px_0 - H\hat{u} - \Delta) + \lambda \hat{u}^T \hat{u}$$

$$= \hat{r}^T r - \hat{r}^T P x_0 - \hat{r}^T H\hat{u} - \hat{r}^T \Delta - x_0^T P^T \hat{r} + x_0^T P^T P x_0 +$$
$$+ x_0^T P^T H \hat{u} + x_0^T P^T \Delta - \hat{u}^T H^T \hat{r} + \hat{u}^T H^T P x_0 + \hat{u}^T H^T H \hat{u} +$$
$$\hat{u}^T H^T \Delta + \Delta^T \hat{r} + \Delta^T P x_0 + \Delta^T H \hat{u} + \Delta^T \Delta + \lambda \hat{u}^T \hat{u}$$

Note: $-\hat{r}^T H\hat{u} - \hat{u}^T H^T \hat{r} = -2\hat{r}^T H \hat{u}$

$x_0^T P^T H\hat{u} + \hat{u}^T H^T P x_0 = 2 x_0^T P^T H \hat{u}$

$\hat{u}^T H^T \Delta + \Delta^T H\hat{u} = 2\Delta^T H \hat{u}$

Therefore we can write as:

$$\boxed{J = J_0 + \hat{u}^T (H^T H + \lambda I)\hat{u} + 2(x_0^T P^T - \hat{r}^T + \Delta^T) H \hat{u}}$$

$$\nabla J = \frac{\partial J}{\partial \hat{u}} = 2(H^T H + \lambda I)\hat{u} + 2(x_0^T P^T - \hat{r}^T + \Delta^T) H = 0$$

Question 2

For running cost: $l = e^2 + \lambda P + \lambda_u \Delta u^2$

Power. $P = VI = V\left(\dfrac{V - a\dot{y}/R}{R_m}\right)$ and $V = \dfrac{R_m}{aR}\left(u + R^2 m_{cw} g\right)$ (

First: $P = \dfrac{V^2 - Va\dot{y}/R}{R_m}$ (I)

For $V^2 \Rightarrow$ $V^2 = \dfrac{R_m^2}{(aR)^2}\left(u^2 + 2uR^2 m_{cw} g + R^4 m_{cw}^2 g^2\right)$ (

Replacing II and III to I:

$$P = \frac{R_m}{(aR)^2}\left(u^2 + 2uR^2 m_{cw} g + R^4 m_{cw}^2 g^2\right) - \frac{\dot{y}}{R^2}\left(u + R^2 m_{cw} g\right)$$

$$= \frac{R_m u^2}{a^2 R^2} + \frac{2 R_m u m_{cw} g}{a^2} + \frac{R^2 m_{cw}^2 g^2}{a^2} - \frac{\dot{y} u}{R^2} - \dot{y} m_{cw} g \quad \text{(IV}$$

-11-

Error: $\hat{e} = \hat{y}_f - \hat{y}_{pos} = \hat{y}_f - P_{pos} x_a - H_{pos} \Delta\hat{u}$ (V)

where $x_a = \begin{bmatrix} x \\ u \end{bmatrix}$

Therefore replacing (IV) and (V) to the running cost

$$J = \underbrace{(\hat{y}_f - P_{pos} x_a - H_{pos}\Delta\hat{u})^T(\hat{y}_f - P_{pos} x_a - H_{pos}\Delta\hat{u})}_{\text{Term I}} + \dots$$

$$\underbrace{\lambda\left(\frac{R_m u^2}{a^2 R^2} + \frac{2 R_m u m_{cw} g}{a^2} + \frac{R^2 m_{cw}^2 g^2}{a^2} - \frac{\dot{y} u}{R^2} - \dot{y} m_{cw} g\right)}_{\text{Term II}} + \lambda_u \Delta$$

First, solving Term I:

$$= \left(\hat{y}_f^T - x_a^T P_{pos}^T - \Delta\hat{u}^T H_{pos}^T\right)\left(\hat{y}_f - P_{pos} x_a - H_{pos}\Delta\hat{u}\right)$$

$$= \hat{y}_f^T \hat{y}_f - \hat{y}_f^T P_{pos} x_a - \hat{y}_f^T H_{pos}\Delta\hat{u} - x_a^T P_{pos}^T \hat{y}_f + x_a^T P_{pos}^T P_{pos} x_a$$

$$+ x_a^T P_{pos}^T H_{pos}\Delta\hat{u} - \Delta\hat{u}^T H_{pos}^T \hat{y}_f + \Delta\hat{u}^T H_{pos}^T P_{pos} x_a + \Delta\hat{u}^T H_{pos}^T H_{pos}\Delta\hat{u}$$

Note: $-\hat{y}_f^T H_{pos}\Delta\hat{u} - \Delta\hat{u}^T H_{pos}^T \hat{y}_f = -2\hat{y}_f^T H_{pos}\Delta\hat{u}$

$x_a^T P_{pos}^T H_{pos}\Delta\hat{u} + \Delta\hat{u} H_{pos}^T P_{pos} x_a = 2 x_a^T P_{pos}^T H_{pos}\Delta\hat{u}$

Therefore, Term I can be simplied as:

$$\text{Term I} = \underbrace{(\text{Term I})}_{\text{constants}} + 2\left(x_a^T P_{pos}^T - \hat{y}_t^T\right) H_{pos}\,\Delta\hat{u} + \Delta\hat{u}^T H_{pos}^T H_{pos}\,\Delta\hat{u}$$

For "Term II", note we have the difflerention of y: $\dot{y}$, which correspond to the velocity state. So this can be written as:

$$\hat{y}_{vel}^T = x_a^T P_{vel}^T + \Delta\hat{u}^T H_{vel}^T$$

or

$$\hat{y}_{vel} = P_{vel}\,x_a + H_{vel}\,\Delta\hat{u}$$

Additionaly, we can write $\hat{u}$ as

$$\hat{u} = P_u x_a + H_u \Delta\hat{u}$$

Before simplify the "term II" lets solve some of the subterms:

$$\begin{aligned}
\hat{u}^T u &= (x_a^T P_u^T + \Delta\hat{u}^T H_u^T)(P_u x_a + H_u \Delta\hat{u}) \\
&= x_a^T P_u^T P_u x_a + x_a^T P_u^T H_u \Delta\hat{u} + \Delta\hat{u}^T H_u^T P_u x_a + \Delta\hat{u}^T H_u^T H_u \Delta\hat{u} \\
&= \underbrace{x_a^T P_u^T P_u x_a}_{\text{constonts}} + 2 x_a^T P_u^T H_u \Delta\hat{u} + \Delta\hat{u}^T H_u^T H_u \Delta\hat{u}
\end{aligned}$$

"Note we have a constant term here and others terms are in function of Δu. Since we dont need the constants values for J, for next derivation it will be not performed"

For $$\frac{2 R_m m_{cw} g}{a^2}\hat{u} = \frac{2 R_m m_{cw} g}{a^2}\left(P_u x_a + H_u \Delta\hat{u}\right)$$

constont

For $$\frac{\hat{y}_{vel}^T \hat{u}}{R^2} = \frac{1}{R^2}\left(x_a^T P_{vel}^T + \Delta\hat{u}^T H_{vel}^T\right)\left(P_u x_a + H_u \Delta\hat{u}\right)$$

$$= \frac{1}{R^2}\left(\overbrace{x_a^T P_{vel}^T P_u x_a}^{\text{constant}} + x_a^T P_{vel}^T H_u \Delta\hat{u} + \Delta\hat{u}^T H_{vel}^T P_u x_a + \Delta\hat{u}^T H_{vel}^T H_u \Delta\hat{u} \right)$$

For $m_{cw}\, g\, \hat{y}_{vel} = m_{cw}\, g \left(\overbrace{P_{vel} x_a}^{\text{constant}} + H_{vel} \Delta\hat{u} \right)$

Therefore, the second term can be written as:

$$\text{Term } \underline{II} = \underbrace{(\text{Term } I)}_{\text{constants}} + \lambda \Bigg[\frac{R_m}{a^2 R^2} \cdot \left(2 x_a^T P_u^T H_u \Delta\hat{u} + \Delta\hat{u}^T H_u^T H_u \Delta\hat{u} \right) + \cdots$$

$$+ \frac{2 R_m m_{cw}\, g H_u \Delta\hat{u}}{a^2} - \frac{x_a^T P_{vel}^T H_u \Delta\hat{u}}{R^2} - \frac{x_a^T P_u^T H_{vel} \Delta\hat{u}}{R^2} - \frac{\Delta\hat{u}^T H_{vel}^T H_u \Delta\hat{u}}{R^2} + \cdots$$

$$- m_{cw}\, g H_{vel} \Delta\hat{u} \Bigg]$$

Therefore, cost function can be written as:

$$J = J_0 + \text{Term } I + \text{Term } II + \lambda_u \Delta u^T \Delta u$$

$$\boxed{\begin{aligned} J = J_0 &+ \Delta\hat{u}^T \left(H_{pos}^T H_{pos} + \frac{\lambda R_m}{a^2 R^2} H_u^T H_u - \frac{H_{vel}^T H_u}{R^2} + \lambda_u I \right) \Delta\hat{u} + \cdots \\ &+ \Bigg[2 \left(x_a^T P_{pos}^T - \hat{y}_r^T \right) H_{pos} + \frac{2 \lambda R_m}{a^2 R^2} x_a^T P_u^T H_u \Delta\hat{u} + \frac{2 R_m m_{cw}\, g}{a^2} H_u \lambda + \cdots \\ &- \frac{\lambda x_a^T P_{vel}^T H_u}{R^2} - \frac{\lambda x_a^T P_u^T H_{vel}}{R^2} - \lambda m_{cw}\, g H_{vel} \Bigg] \Delta\hat{u} \end{aligned}}$$