Intelligent Control Systems
EEC 645/745, MCE 693/793

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Spring 2016

# 1 Double Pendulum Lagrange Example

1) Draw a picture
2) Assign a coordinate system

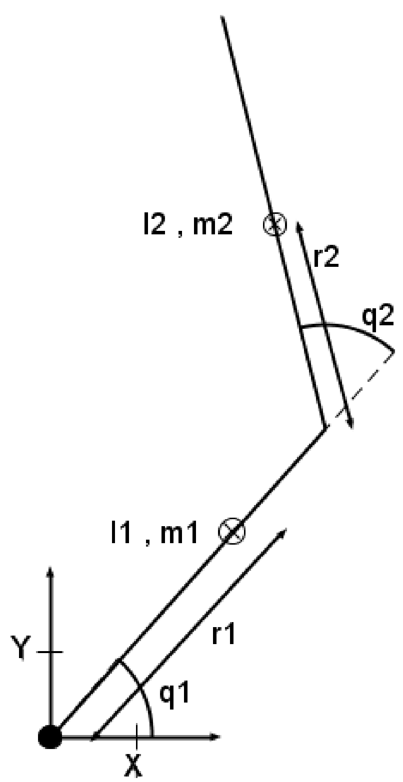


Figure 1: Double Pendulum

3) Determine how many degrees of freedom the system has and assign generalized coordinates $\vec{q}$. $\vec{q} = [q_1, q_2]^T$
4) Compute the potential energy for each element. Define zero gravitational potential energy

when COM of the body is at $y = 0$.

$$x_1 = r_1 cosq_1 \quad y_1 = r_1 sinq_1$$

Potential energy of link 1:

$$U_1 = m_1 g y_1 = m_1 g r_1 sinq_1$$

for link 2:

$$x_2 = l_1 cosq_1 + r_2 cos(q_1 + q_2) \quad y_2 = l_1 sinq_1 + r_2 sin(q_1 + q_2)$$

Potential energy of link 2:

$$U_2 = m_2 g y_2 = m_2 g(l_1 sinq_1 + r_2 sin(q_1 + q_2))$$

Total potential energy:

$$U = U_1 + U_2 = m_1 g r_1 sinq_1 + m_2 g(l_1 sinq_1 + r_2 sin(q_1 + q_2))$$

5) Compute transitional and rotational kinetic energy:

$$\dot{x}_1 = -\dot{q}_1 r_1 sinq_1 \quad \dot{y}_1 = \dot{q}_1 r_1 cosq_1$$

Velocity of COM of link 1:

$${V_1}^2 = \dot{x}_1^2 + \dot{y}_1^2 = \dot{q}_1^2 r_1^2$$

$$T_{1trans} = \frac{1}{2} m_1 V_1^2 = \frac{1}{2} m_1 \dot{q}_1^2 r_1^2$$

$$T_{1rot} = \frac{1}{2} I_1 \dot{q}_1^2$$

$$\dot{x}_2 = -\dot{q}_1 l_1 sinq_1 - (\dot{q}_1 + \dot{q}_2) r_2 sin(q_1 + q_2) \quad \dot{y}_2 = \dot{q}_1 l_1 cosq_1 + (\dot{q}_1 + \dot{q}_2) r_2 cos(q_1 + q_2)$$

${V_2}^2 = \dot{x}_2^2 + \dot{y}_2^2 = \dot{q}_1^2 l_1^2 sin^2 q_1 + (\dot{q}_1^2 + 2\dot{q}_1\dot{q} + \dot{q}_2^2) r_2^2 sin^2(q_1 + q_2) + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2) l_1 r_2 sinq_1 sin(q_1 + q_2) + \dot{q}_1^2 l_1^2 cos^2 q_1 + (\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) r_2^2 cos^2(q_1 + q_2) + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2) l_1 r_2 cosq_1 cos(q_1 + q_2)$

$$= \dot{q}_1^2 l_1^2 + (\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) r_2^2 + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2) l_1 r_2 (sinq_1 sin(q_1 + q_2) + cosq_1 cos(q_1 + q_2))$$

by $cos(u - v) = cosu \times cosv + sinu \times sinv$

$$= \dot{q}_1^2 l_1^2 + (\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) r_2^2 + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2) l_1 r_2 cosq_2$$

$$T_{2trans} = \frac{1}{2} m_2 V_2^2 = \frac{1}{2} m_2 [\dot{q}_1^2 l_1^2 + (\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) r_2^2 + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2) l_1 r_2 cosq_2]$$

$$T_{2rot} = \frac{1}{2} I_2 (\dot{q}_2 + \dot{q}_1)^2$$

$$T_{tot} = T_{1trans} + T_{2trans} + T_{1rot} + T_{2rot}$$

$$T = \frac{1}{2} m_1 \dot{q}_1^2 r_1^2 + \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} m_2 [\dot{q}_1^2 l_1^2 + (\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) r_2^2 + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2) l_1 r_2 cosq_2] + \frac{1}{2} I_2 (\dot{q}_2 + \dot{q}_1)^2$$

6) Compute the Lagrangian

$$L = T - U$$

$L = \frac{1}{2}m_1\dot{q}_1^2 r_1^2 + \frac{1}{2}I_1\dot{q}_1^2 + \frac{1}{2}m_2[\dot{q}_1^2 l_1^2 + (\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2)r_2^2 + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2)l_1 r_2 cosq_2] + \frac{1}{2}I_2(\dot{q}_2 + \dot{q}_1)^2 - m_1 g r_1 sinq_1 - m_2 g(l_1 sinq_1 + r_2 sin(q_1 + q_2))$

7) Write down the equations of motion for each $q_i$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i \quad F = [\tau_1, \tau_2]^T$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = \tau_1$$

$$\frac{\partial L}{\partial \dot{q}_1} = m_1\dot{q}_1 r_1^2 + I_1\dot{q}_1 + m_2\dot{q}_1 l_1^2 + m_2\dot{q}_1 r_2^2 + m_2\dot{q}_2 r_2^2 + 2m_2\dot{q}_2 l_1 r_2 + m_2\dot{q}_1 l_1 r_2 cosq_2 cosq_2$$

$$\frac{\partial L}{\partial q_1} = -m_1 g r_1 cosq_1 - m_2 g l_1 cosq_1 - m_2 g r_2 cos(q_1 + q_2)$$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = \ddot{q}_1(m_1 r_1^2 + I_1 + m_2 l_1^2 + m_2 r_2^2 + 2m_2 l_1 r_2 cosq_2) + m_2\ddot{q}_2 r_2^2 - 2m_2\dot{q}_1 l_1 r_2 sinq_2 + gcosq_1(m_1 r_1 + m_2 l_1) + m_2 g r_2 cos(q1 + q2) = \tau_1$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = \tau_2$$

$$\frac{\partial L}{\partial \dot{q}_2} = m_2\dot{q}_1 r_2^2 + m_2\dot{q}_2 r_2^2 + m_2\dot{q}_1 l_1 r_2 cosq_2 + I_2\dot{q}_2$$

$$\frac{\partial L}{\partial q_2} = -m_2\dot{q}_1(\dot{q}_1 + \dot{q}_2)l_1 r_2 sinq_2 - m_2 g r_2 cos(q_1 + q_2)$$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = \tau_2 = m_2\ddot{q}_1 r_2^2 + m_2\ddot{q}_2 r_2^2 + m_2\ddot{q}_1 l_1 r_2 cosq_2 - m_2\dot{q}_1\dot{q}_2 l_1 r_2 sinq_2 + I_2\ddot{q}_2 + m_2\dot{q}_1(\dot{q}_1 + \dot{q}_2)l_1 r_2 sinq_2 + m_2 g r_2 cos(q_1 + q_2) = \tau_2$

8) Put this in matrix form

$$M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$$

## 2 Euler Integration

How to integrate a function (definition of integral). Integral is the area under the curve.

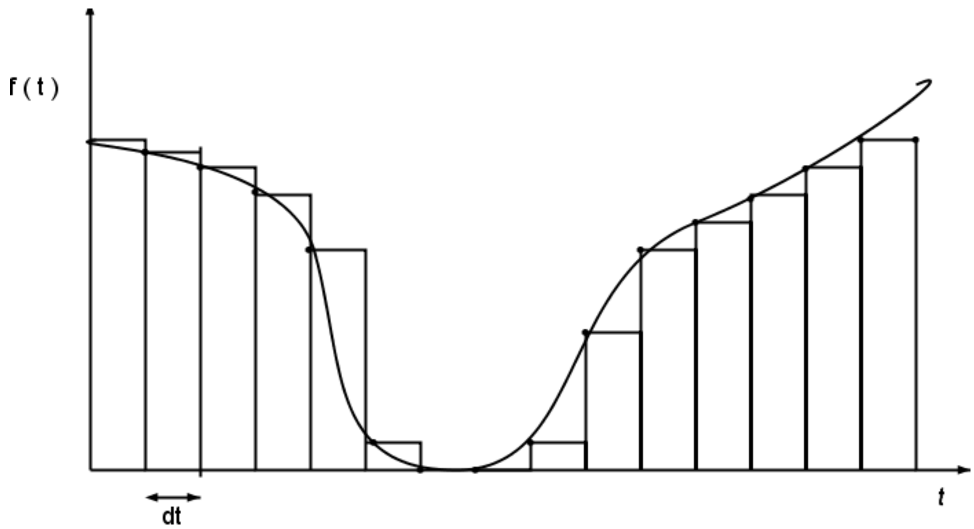


Figure 2: Integral

To integrate I want to add the areas of the rectangles.
$area_1 = f(i=0)\Delta t$
$area_2 = f(i=1)\Delta t \quad area_{1+2} = area_1 + f(i=1)\Delta t$
$area_3 = f(i=2)\Delta t \quad area_{1+2+3} = area_{1+2} + f(i=2)\Delta t$
.
.
.
$area_n = f(i=n-1)\Delta t \quad area_{1+2+..+n} = area_{1..n-1} + f(n-1)\Delta t$
start with $\dot{f}(t)$

$$f(t=i) = f(t=i-1) + \dot{f}(t=i-1)\Delta t$$

or from the definition of the derivative

$$\frac{df(t_i)}{dt} = \lim_{\Delta t \to 0} \frac{f(t_{i+1}) - f(t_i)}{\Delta t} \Rightarrow f(t_{i+1}) \approx f(t_i) + \frac{df(t_i)}{dt}\Delta t$$

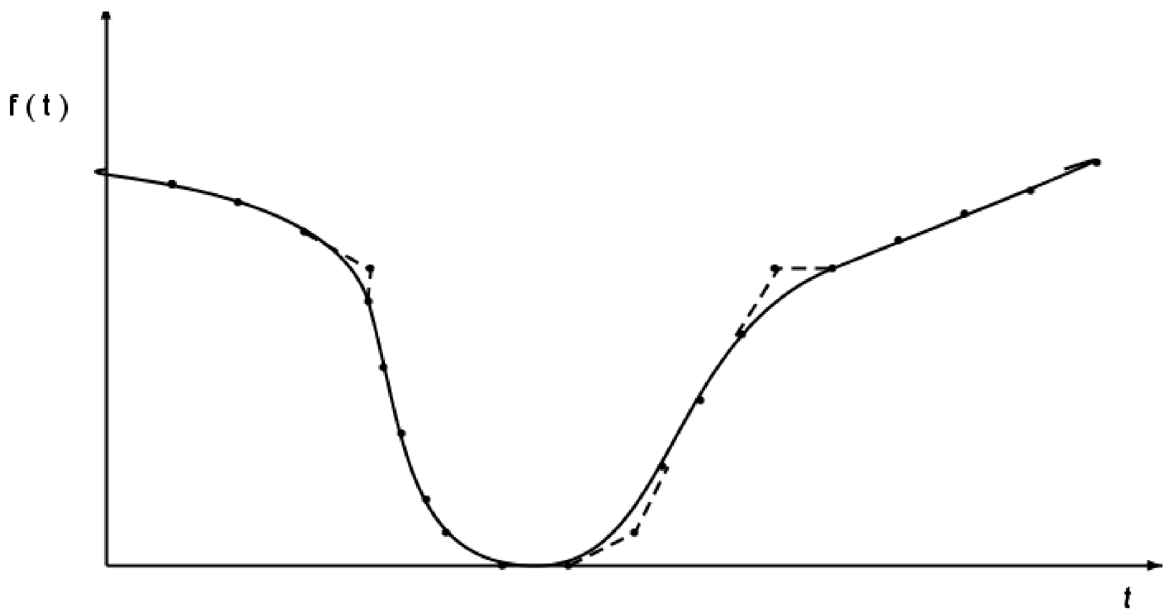


Figure 3: Derivative