Lagrange's Method for Modeling Mechanical Systems

- alternative to free-body diagrams for writing equations of motion
- great for complicated multi-body systems
- comes from ideas about energy conservation

Steps in Lagrange's Method

1) Draw a picture of the system.

2) Determine if the bodies in the system translate, rotate, or both. How many degrees of freedom?

3) Assign of system of generalized coordinates

4) Write down kinematic relationships

5) Compute gravitational potential energy of each mass and potential energy of each spring. U = potential energy

6) Compute translational and rotational kinetic energy for each body. T = kinetic energy

7) Compute the Lagrangian $L = T - U$

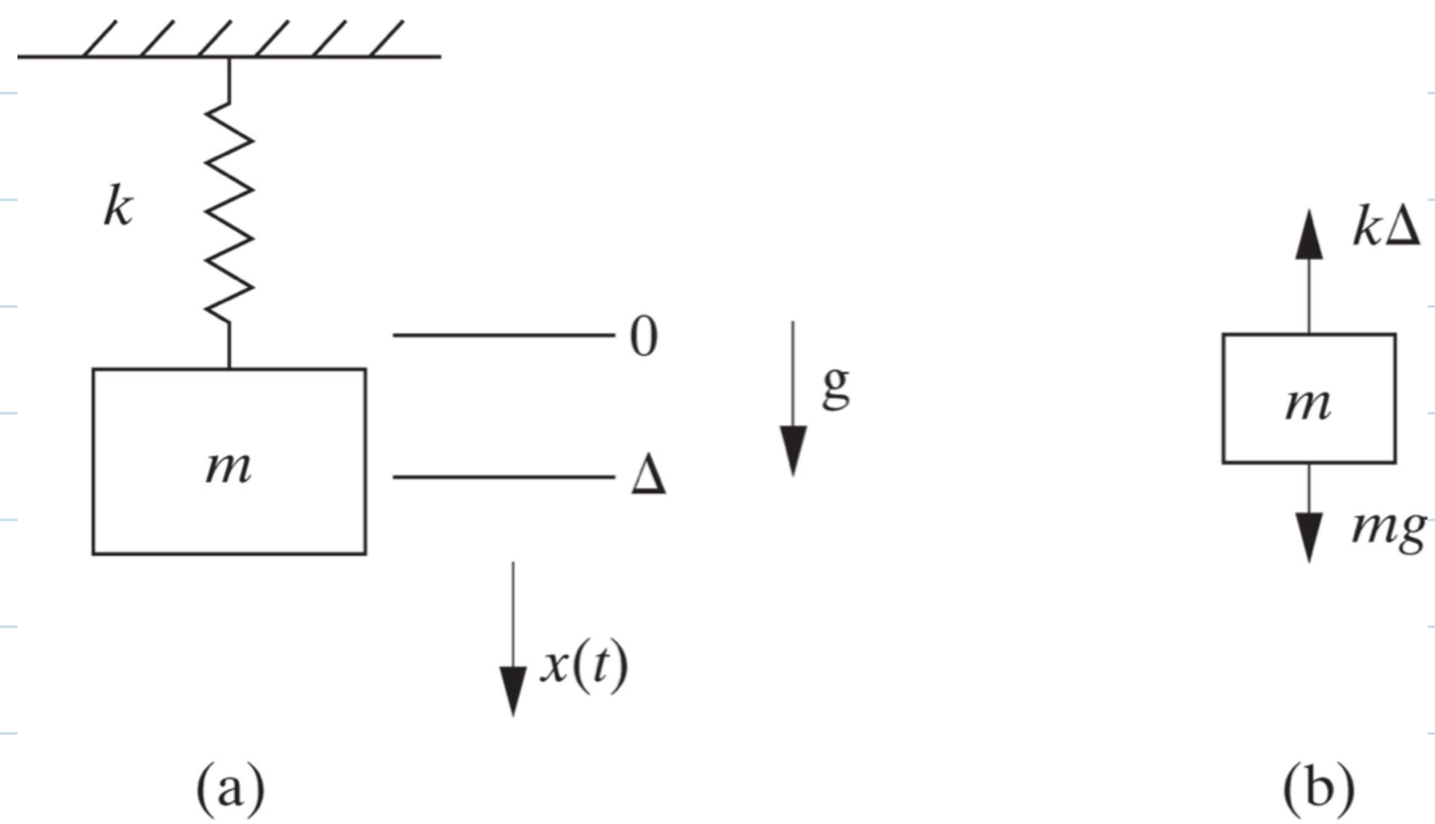
8) For the i^{th} generalized coordinate q_i (e.g. x for translation, θ for rotation)

the related equation of motion is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

F_i is the ith generalized external force.

Example: Mass - spring system with gravity



1) picture - done

2) one translating body

   one DoF

3) x is down, spring is unstretched

   at $x=0$, 0 gravitational potential

   a $x=0$ $\qquad x=q$

4) no fancy kinematic relationships $\quad x=x$

5) $U_{grav} = -mgq \quad U_{spring} = \frac{1}{2}kq^2$

6) $T = \frac{1}{2}m\dot{q}^2$

7) Lagrangian $L = T - U = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} k q^2 + mgq$

8) Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$\frac{d}{dt}(m\dot{q}) + kq - mg = 0 \Rightarrow m\ddot{q} + kq - mg = 0$$

<u>Example</u>: Simple Pendulum

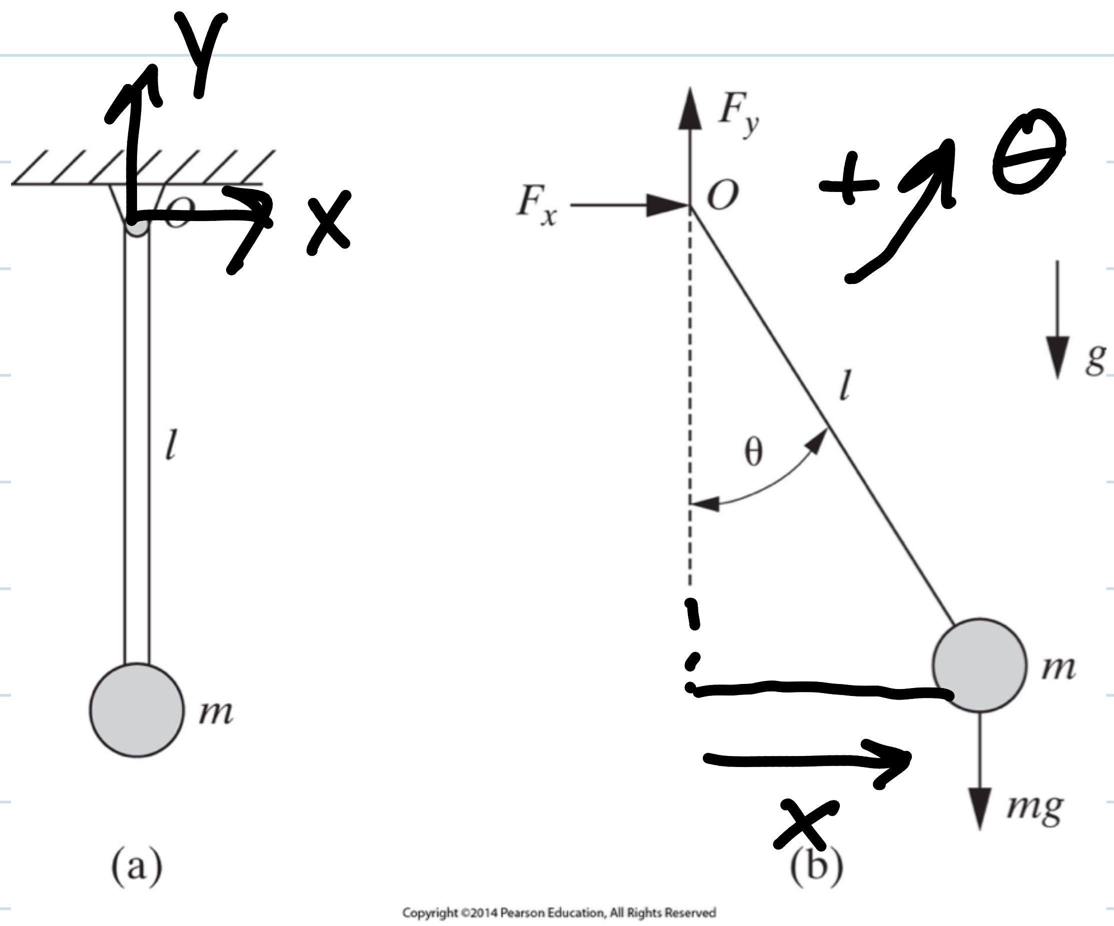
1) picture

2) one translating body

one degree of freedom

3) pick $\theta = q$

zero potential energy at $\theta = 0$



4) Kinematic relationships

$$x = l\sin\theta \qquad y = -l\cos\theta$$

$$\dot{x} = l\dot{\theta}\cos\theta \qquad \dot{y} = l\dot{\theta}\sin\theta$$

5) $U_{grav} = mg(l - l\cos\theta)$

6) $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right)$

$$= \frac{1}{2}ml^2\dot{\theta}^2$$

7) $L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - mg(l - l\cos\theta)$

(2) (1) (4)

8) $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$

(3)

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0 \Longrightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Example: more complex spring-mass system

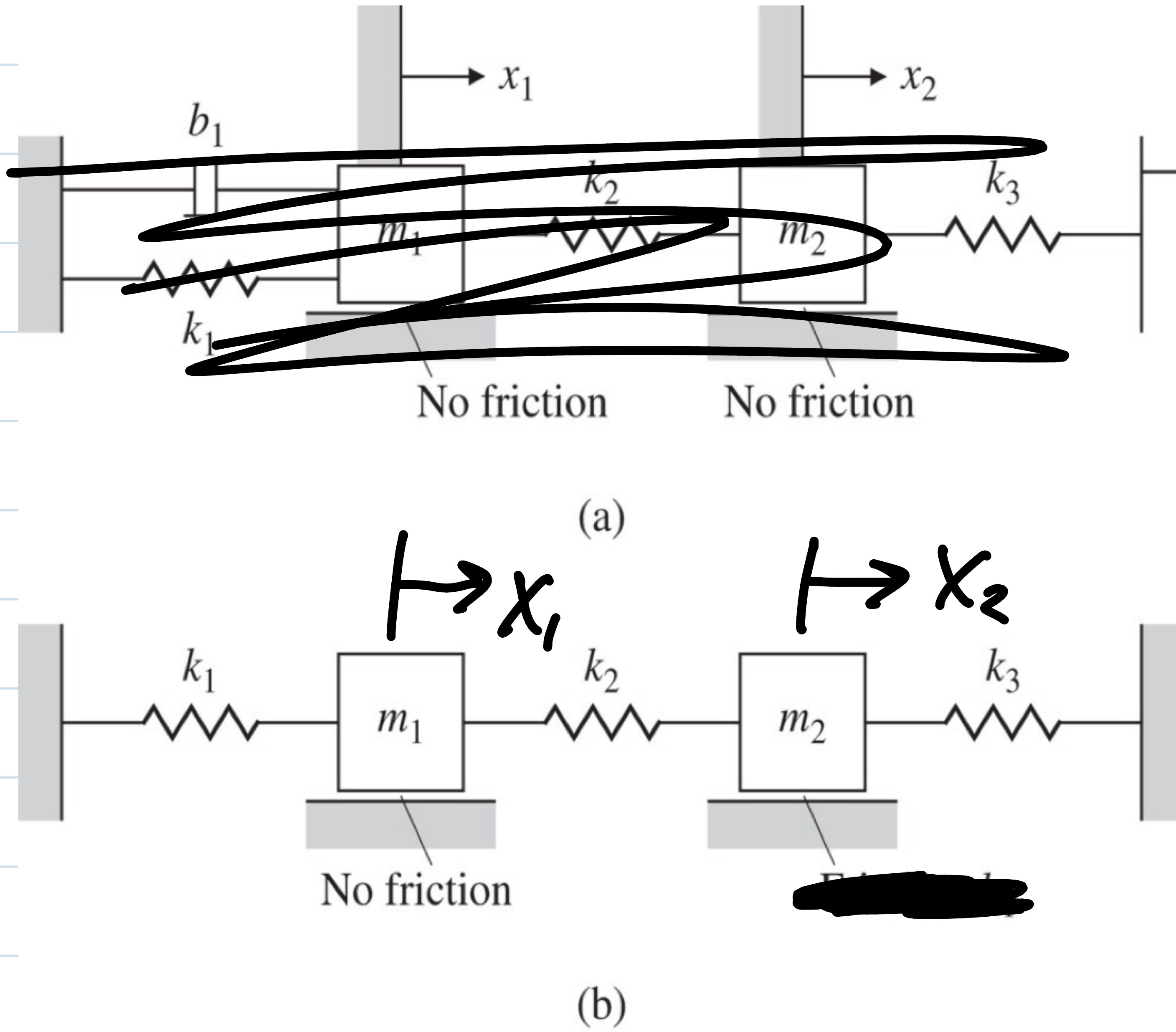
1) picture

2) 2 translating masses

   2 DoF

3) $q = [x_1 \;\; x_2]$

4) no kinematic relationships



5) $U_{spr} = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$

$\quad + \frac{1}{2} k_3 (-x_2)^2$

6) $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

7) $L = T - U = -\frac{1}{2} \underline{k_1 x_1^2} - \frac{1}{2} k_2 (x_2 - x_1)^2 - \frac{1}{2} k_3 x_2^2 + \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

8) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}(m_1\dot{x}_1) - (-k_1x_1 + k_2(x_2 - x_1)) = 0$$

$$m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 = 0 \qquad (1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}(m_2\dot{x}_2) - (-k_2(x_2 - x_1) - k_3x_2) = 0$$

$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 + k_3x_2 = 0 \qquad (2)$$

What about dissipative forces?

Euler-Lagrange equations become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i + \frac{\partial D}{\partial \dot{q}_i}$$

Where D is a dissipation function

- for damping $\quad D = -\tfrac{1}{2} b v^2$

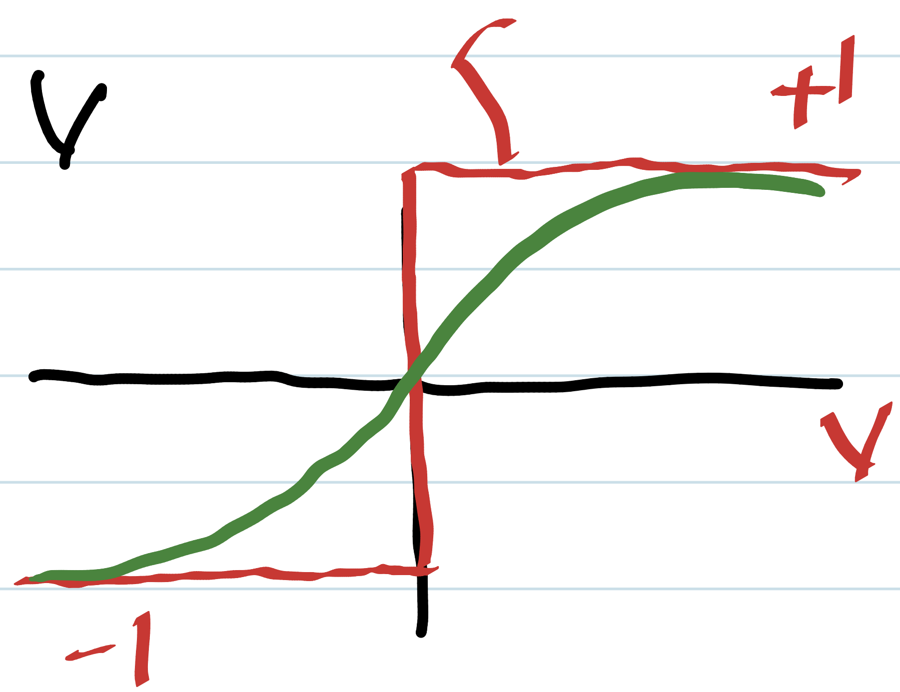
  damping coefficient

- for Coulomb friction $\quad D = -\mu N \,\mathrm{sgn}(v)\, v$

  friction coeff

  normal force

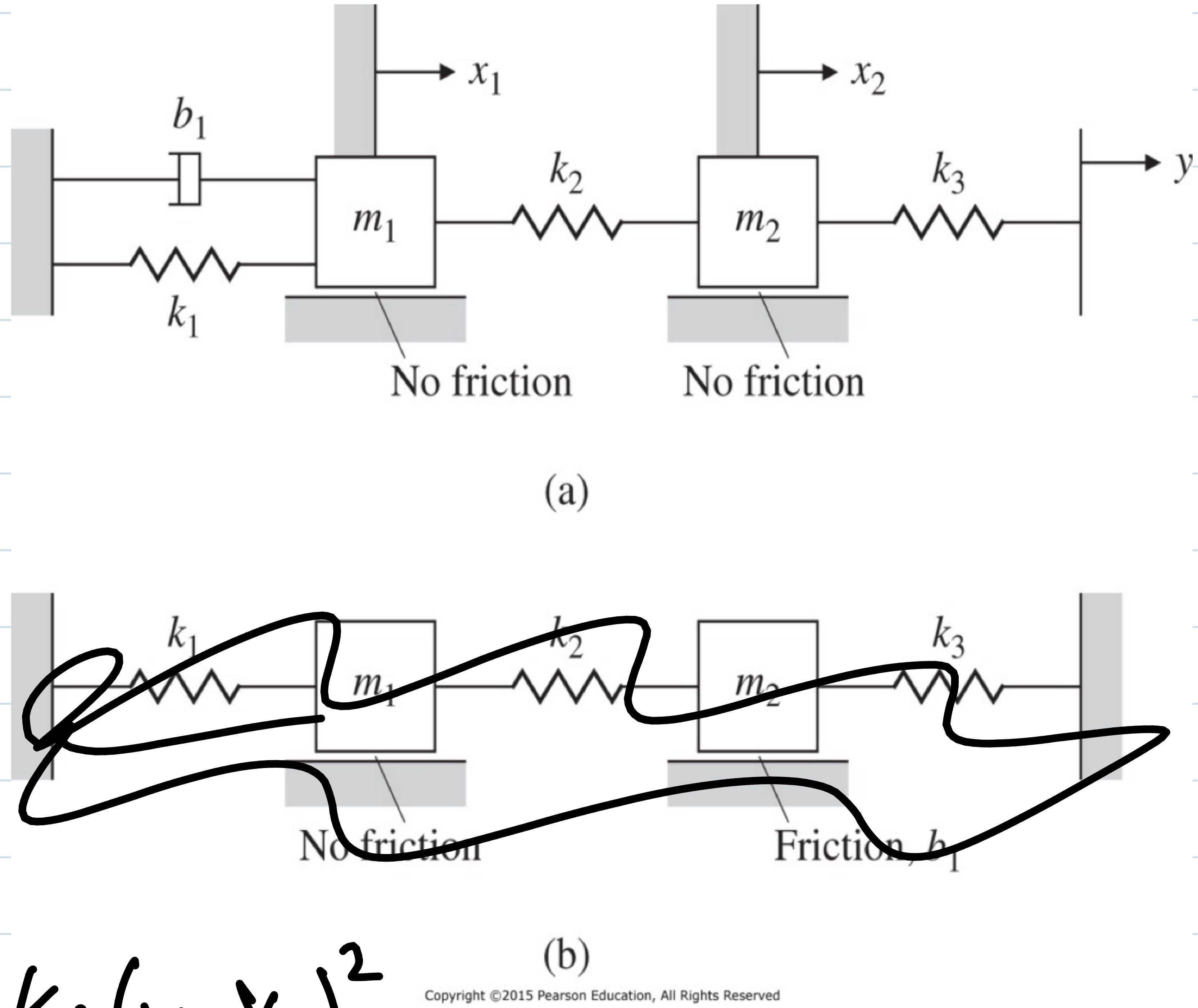
sgn(v)



generalize dissipative force $F_{d_i} = \dfrac{\partial \Delta}{\partial \hat{q}_i}$

Example:

1) picture



2) 2 translating bodies

2 DoF

3) $q = [x_1 \quad x_2]$

4) $X_1 = x_1 \qquad X_2 = x_2$

5) $U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (y - x_2)^2$

6) $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

7) $L = T - U = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \left[\frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (y - x_2)^2 \right]$

8) $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = \frac{\partial D}{\partial \dot{x}_1}$ $\quad D = -\frac{1}{2} b_1 \dot{x}_1^2$

$\frac{d}{dt} (m_1 \dot{x}_1) + k_1 x_1 - k_2 (x_2 - x_1) = -b_1 \dot{x}_1$

$m_1 \ddot{x}_1 + k_1 x_1 + k_2 x_1 - k_2 x_2 + b_1 \dot{x}_1 = 0$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = \frac{\partial D}{\partial \dot{x}_2}$

$\frac{d}{dt} (m_2 \dot{x}_2) + k_2 (x_2 - x_1) - k_3 (y - x_2) = 0$

$m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 + k_3 x_2 - k_3 y = 0$

Double pendulum Lagrange example

1) draw a picture
2) assign coordinates x, y 2 DoF
3) $q = [q_1 \;\; q_2]$
4) Kinematic relationships

$x_1 = r_1 \cos q_1 \qquad y_1 = r_1 \sin q_1$

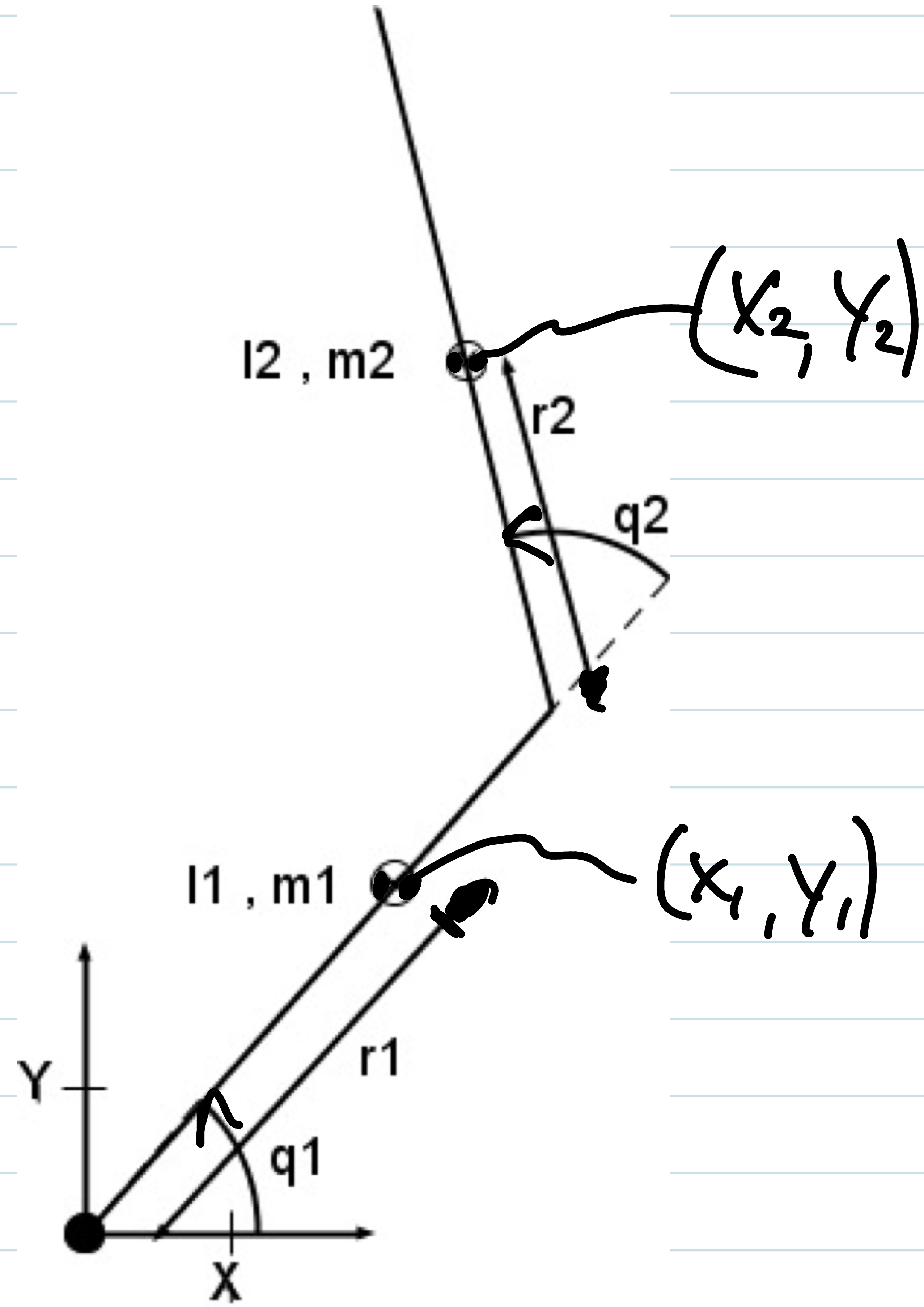
$\dot{x}_1 = -\dot{q}_1 r_1 \sin q_1 \qquad \dot{y}_1 = \dot{q}_1 r_1 \cos q_1$

$x_2 = l_1 \cos q_1 + r_2 \cos(q_1 + q_2)$

$\dot{x}_2 = -\dot{q}_1 l_1 \sin q_1 - (\dot{q}_1 + \dot{q}_2) r_2 \sin(q_1 + q_2)$

$y_2 = l_1 \sin q_1 + r_2 \sin(q_1 + q_2)$

$\dot{y}_2 = \dot{q}_1 l_1 \cos q_1 + (\dot{q}_1 + \dot{q}_2) r_2 \cos(q_1 + q_2)$



5) Potential energy: Define zero potential energy when CoM of body is at $y=0$

Link 1 potential energy $U_1 = m_1 g y_1 = m_1 g r_1 \sin q_1$

$$U_2 = m_2 g y_2 = m_2 g (l_1 \sin q_1 + r_2 \sin(q_1+q_2))$$

6) Compute translational and rotational kinetic energy

$$T_{1trans} = \frac{1}{2} m_1 V_1^2 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2} m_1 \left[\dot{q}_1^2 r_1^2 \sin^2 q_1 + \dot{q}_1^2 r_1^2 \cos^2 q_1 \right]$$

$$\boxed{\dot{x}_1 = -\dot{q}_1 r_1 \sin q_1 \qquad \dot{y}_1 = \dot{q}_1 r_1 \cos q_1}$$

$$= \frac{1}{2} m_1 \dot{q}_1^2 r_1^2$$

$T_{1rot} = \frac{1}{2} I_1 \dot{q}_1^2$ $\qquad$ I_1 is moment of inertia about CoM

$$\dot{x}_2 = -\dot{q}_1 l_1 \sin q_1 - (\dot{q}_1 + \dot{q}_2) r_2 \sin(q_1 + q_2)$$

$$\dot{y}_2 = \dot{q}_1 l_1 \cos q_1 + (\dot{q}_1 + \dot{q}_2) r_2 \cos(q_1 + q_2)$$

$$T_{2trans} = \frac{1}{2} m_2 V_2^2 = \frac{1}{2} m_2 \Big[\dot{q}_1^2 l_1^2 + (\dot{q}_1^2 + 2\dot{q}_1 \dot{q}_2 + \dot{q}_2^2) r_2^2 + 2\dot{q}_1(\dot{q}_1 + \dot{q}_2) \ldots$$

$$\ldots l_1 r_2 \cos q_2 \Big]$$

$$T_{2rot} = \frac{1}{2} I_2 (\dot{q}_2 + \dot{q}_1)^2$$

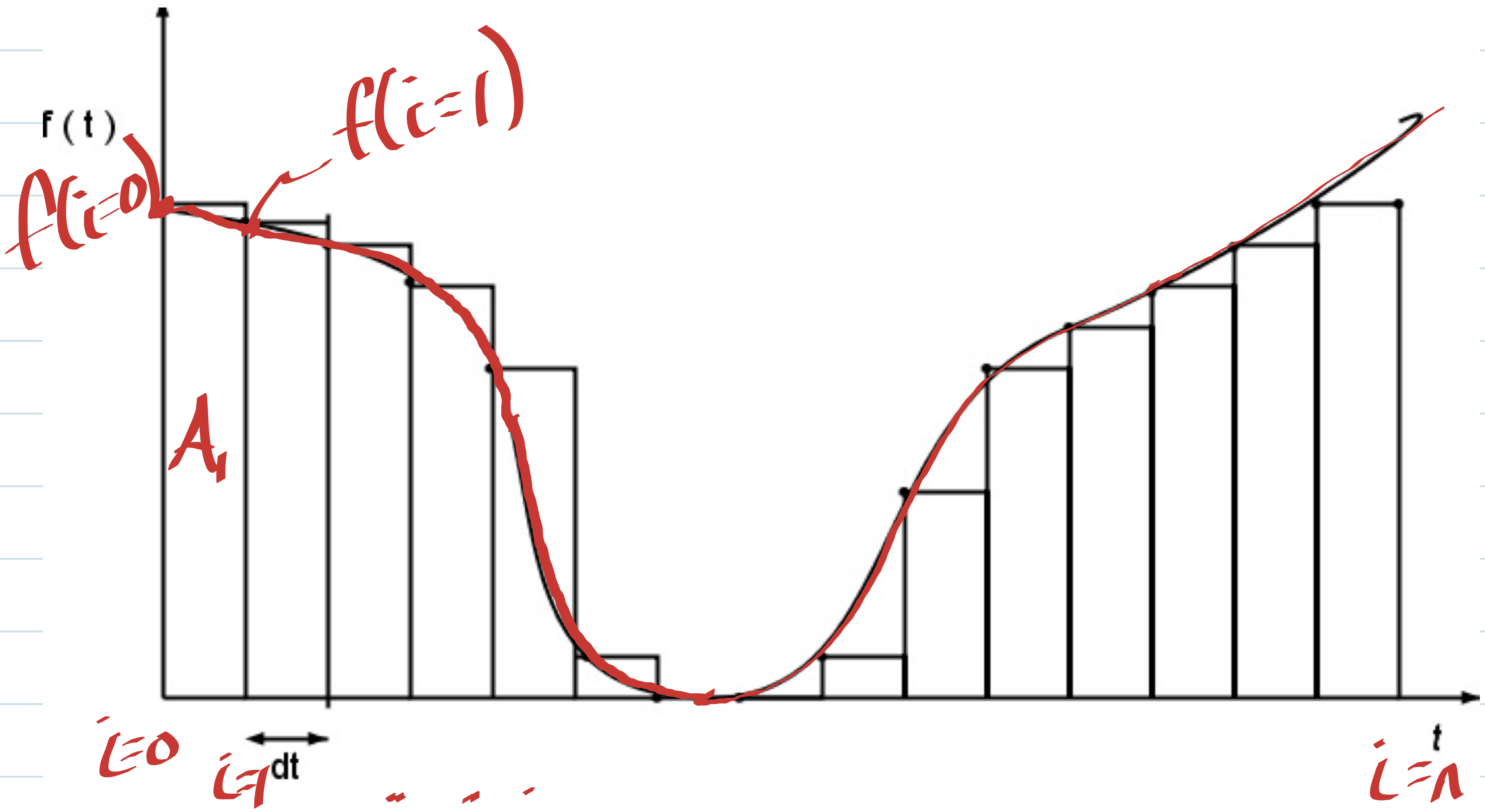
$$T_{tot} = T_{1trans} + T_{1rot} + T_{2trans} + T_{2rot}$$

7) $L = T - U$

8) have Matlab symbolic math compute partial derivatives for Euler-Lagrange eqs.

How to solve the Euler-Lagrange equations

First how do you integrate a function numerically?



To integrate I need to add together the areas of the rectangles

$A_1 = \Delta t\, f(i=0)$

$A_2 = \Delta t\, f(i=1) \qquad A_{12} = A_1 + f(i=1)\Delta t$

$\vdots \qquad\qquad\qquad\qquad A_{123} = A_{12} + f(i=2)\Delta t$

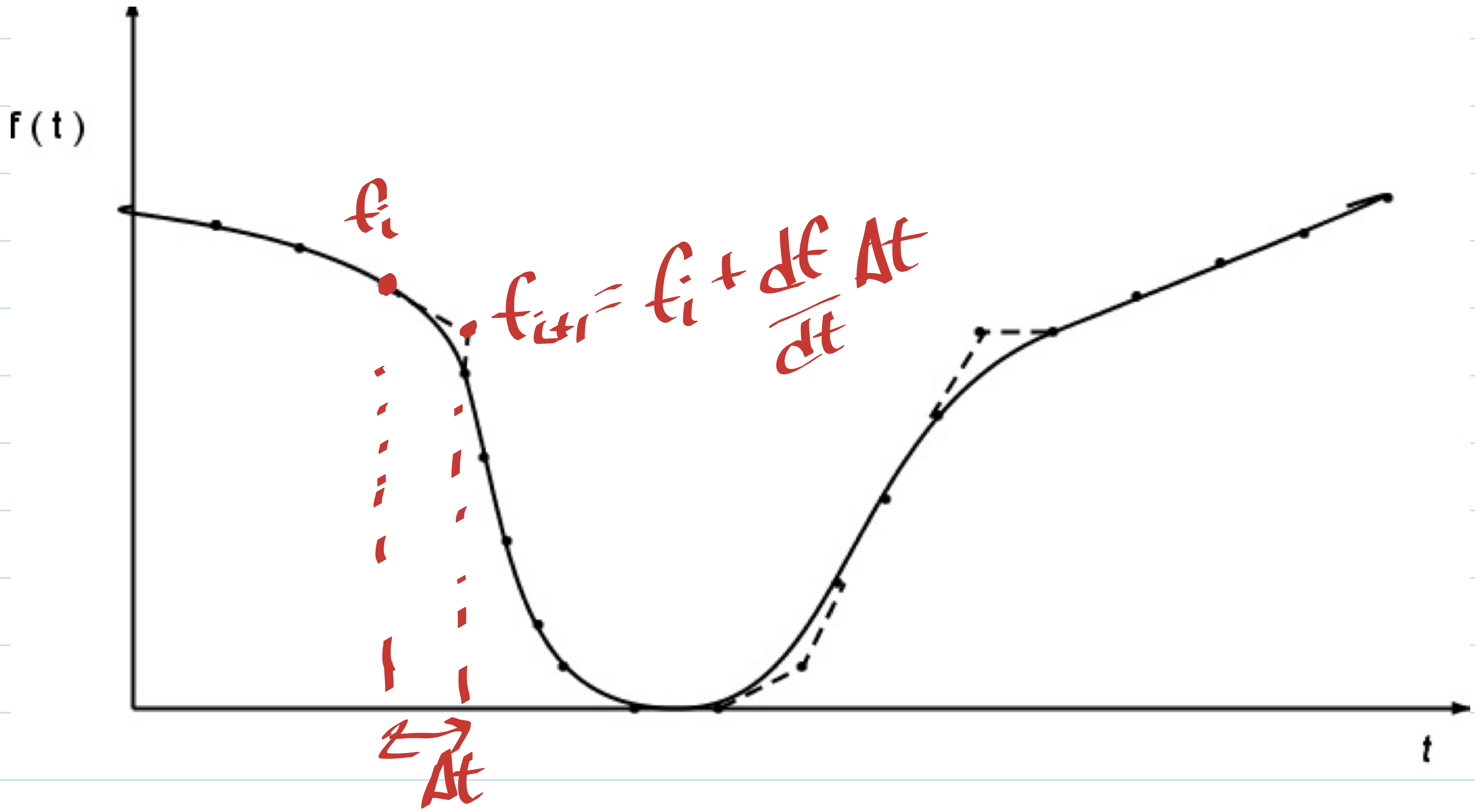
$A_n = \Delta t\, f(i=n-1) \qquad A_{1,2\ldots n} = A_{1\ldots n-1}) + f(i=n-1)\Delta t$

From the definition of an integral

$$\frac{df(t_i)}{dt} = \lim_{\Delta t \to 0} \frac{f(t_{i+1}) - f(t_i)}{\Delta t}$$

$$\Rightarrow f(t_{i+1}) \approx f(t_i) + \frac{df(t_i)}{dt}\Delta t$$

Euler integration



$$f(t_{i+1}) \approx f(t_i) + \frac{df(t_i)}{dt}\Delta t$$

Say you have a 2nd-order differential eqn

$$\ddot{x} + c_1\dot{x} + c_2 x = 0$$

How to integrate this numerically?

1) Solve algebraically for $\ddot{x}$

$$\ddot{x} = -c_1\dot{x} - c_2 x$$

2) start with initial conditions $\dot{x}(0) = a \quad x(0) = b$

3) Compute $\ddot{x}(0) = -c_1\dot{x}(0) - c_2 x(0) = -c_1 a - c_2 b$ $\leftarrow$

4) take an Euler integration step

$$x(\Delta t) = x(0) + \dot{x}(0)\Delta t = b + a\Delta t$$

$$\dot{x}(\Delta t) = \dot{x}(0) + \ddot{x}(0)\Delta t = a + (-c_1 a - c_2 b)\Delta t$$

5) Iterate

$$x_{i+1} = x_i + \dot{x}_i \Delta t$$

$$\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i \Delta t \qquad \ddot{x}_i = -c_1\dot{x}_i - c_2 x_i$$