Rigid Motions and Homogeneous Transformations

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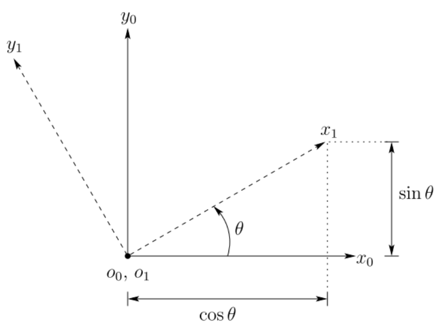
Chapter 2

Homogeneous transformations combine the operations of rotation and translation into a single matrix multiplication.



R_1^0 : Coordinate vectors for the axes of frame $o_1x_1y_1$ with respect to coordinate frame $o_0x_0y_0$:

$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$x_1^0 = \begin{bmatrix} x_1\cdot x_0 \\ x_1\cdot y_0 \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} y_1\cdot x_0 \\ y_1\cdot y_0 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} x_1\cdot x_0 & y_1\cdot x_0 \\ x_1\cdot y_0 & y_1\cdot y_0 \end{bmatrix}$$

Rotation in Three Dimensions

$$R_1^0 = \begin{bmatrix} x_1\cdot x_0 & y_1\cdot x_0 & z_1\cdot x_0 \\ x_1\cdot y_0 & y_1\cdot y_0 & z_1\cdot y_0 \\ x_1\cdot z_0 & y_1\cdot z_0 & z_1\cdot z_0 \end{bmatrix}$$

Remember: If u is parallel to v, $u\cdot v = 1$. Also, if u is perpendicular to v, $u\cdot v = 0$.

- Rotations relative to the current frame (recursively defined); or rotations relative to a fixed frame (usually the world frame).

Summary
We have seen that a rotation matrix, either $R \in SO(3)$ or $R \in SO(2)$, can be interpreted in three distinct ways:

1. It represents a coordinate transformation relating the coordinates of a point p in two different frames.
2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.
3. It is an operator taking a vector and rotating it to a new vector in the same coordinate system.

Table 2.2.1: Properties of the Matrix Group $SO(n)$

- $R \in SO(n)$
- $R^{-1} \in SO(n)$
- $R^{-1} = R^T$
- The columns (and therefore the rows) of R are mutually orthogonal
- Each column (and therefore each row) of R is a unit vector
- $\det R = 1$

Example of Composition of Rotational Transformations:

1. *A rotation of θ about the current x-axis*
2. *A rotation of ϕ about the current z-axis*
3. *A rotation of α about the fixed z-axis*
4. *A rotation of β about the current y-axis*
5. *A rotation of δ about the fixed x-axis*

Example 2.8 — SHV.

Frame 0 → 1 : $Rot_{x_0,\theta}$

Frame 1 → 2 : $Rot_{z_1,\phi}$

Frame 2 → 3 : $Rot_{z_0,\alpha}$ A B C

Frame 3 → 4 : $Rot_{y_3,\beta}$

Frame 4 → 5 : $Rot_{x_0,\delta}$

$R = Rot_{x_0,\delta} \cdot A$

$R = Rot_{x_0,\delta} \cdot B \cdot Rot_{y_3,\beta}$

$R = \{ \delta, \alpha, \theta, \phi, \beta \}$

Chapter 3

- **Forward Kinematics**: Determine the position and orientation of the end effector given the values for the joint variables of the robot. Also known as *configuration kinematics*.
- **Inverse Kinematics**: Determine the values of the joint variables given the end effector's position and orientation.

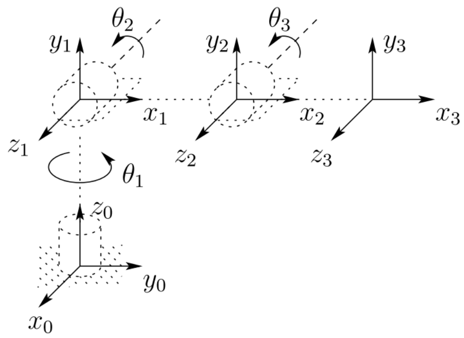
Kinematic Chains

Robot manipulator is composed of a set of links connected together by joints. Each joint has a sigle degee-of-feedom of motion. For instance, the **angle of rotation** for revolute joint and **linear displacement** for a prismatic joint.

- Robot manipulator with n joints have $n+1$ links.
- Joints numbering: 1 to n.
- Links numbering: 0 to n starting from the base.
- Joint i connects link $i-1$ to link i.
- WHen joint i is actuated, link i moves.
- Joint variable q_i: When i correspond to the revolute joint $q_i = \theta_i$. When i correspond to prismatic joint $q_i = d_i$.

$$q_i = \begin{cases} \theta_i & \text{revolute joint} \\ d_i & \text{prismatic joint} \end{cases}$$

Homogeneous Transformation Matrix: Expresses the position and orientation of $o_j x_j y_j z_j$ with respect to $o_i x_i y_i z_i$: (T_j^i).

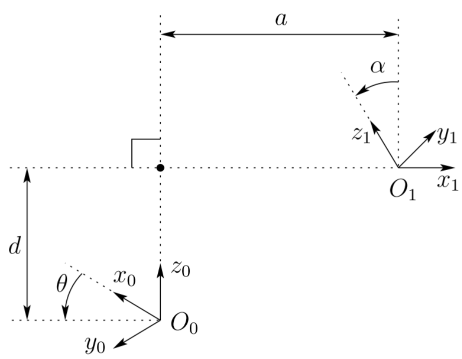


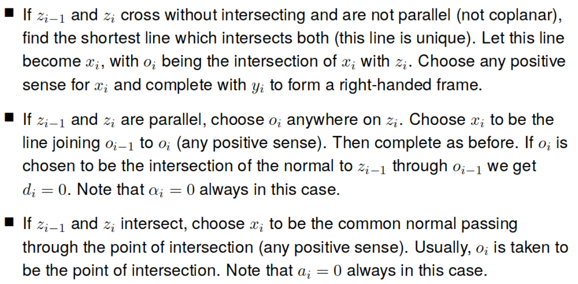
The Denavit-Hartenberg Convention or DH convention

Each Transformation A_i is represented as a product of four basic transformations:

$$\begin{aligned}
A_i &= Rot_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}Rot_{x,\alpha_i} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

where: θ_i, a_i, d_i, α_i correspond to the joint angle, link length, link offset, and link twist.





- If z_{i-1} and z_i cross without intersecting and are not parallel (not coplanar), find the shortest line which intersects both (this line is unique). Let this line become x_i, with o_i being the intersection of x_i with z_i. Choose any positive sense for x_i and complete with y_i to form a right-handed frame.
- If z_{i-1} and z_i are parallel, choose o_i anywhere on z_i. Choose x_i to be the line joining o_{i-1} to o_i (any positive sense). Then complete as before. If o_i is chosen to be the intersection of the normal to z_{i-1} through o_{i-1} we get $d_i = 0$. Note that $\alpha_i = 0$ always in this case.
- If z_{i-1} and z_i intersect, choose x_i to be the common normal passing through the point of intersection (any positive sense). Usually, o_i is taken to be the point of intersection. Note that $a_i = 0$ always in this case.

Figure 1: Hanz Richter's notes.

Inverse Kinematics

Goal: Find the **joint variables** in terms of the end-effector position and orientation.