Exercises for Lecture Course on Numerical Optimization (NUMOPT)
Albert-Ludwigs-Universität Freiburg – Winter Term 2015-2016

Exercise 8: Equality Constrained Optimization

(to be completed during exercise session on Dec 16, 2015 or sent by email to dimitris.kouzoupis@imtek.uni-freiburg.de before Dec 18, 2015)

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Aim of this exercise sheet is to explore the concepts of equality constrained optimization discussed in the lecture.

Exercise Tasks

1. **Constrained optimization with `fmincon`:** Recall the simple equality constrained example discussed in the lecture,

$$\underset{x_1,x_2}{\text{minimize}}\ x_2 \tag{1}$$

$$\text{subject to: } x_1^2 + x_2^2 - 1 = 0 \tag{2}$$

which consists of a linear objective and a nonlinear equality constraint.

   (a) Study the template file `fmincon_example.m` to understand how `fmincon` works. You will need to use it yourself in the next task. Type `help fmincon` in the command window for more details on the function.

   (0 points)

   (b) Derive the first order optimality conditions for this problem. Write a simple Newton method of your choice to find the optimal solution. To which point (or points) does your method converge?

   (2 points)

   (c) Prove that you have found the minimum point using the Second Order Sufficient Conditions.

   (1 point)

2. **Hanging chain, re-revisited:** Let us consider once again the example of the hanging chain that we will now solve with `fmincon`. This time we fix the distance between two adjacent masses to a constant length $L_i = L/(N-1)$. Removing the potential energy of the springs from our objective function we are left with the following optimization problem:

$$\underset{y,z}{\text{minimize}} \quad \sum_{i=1}^{N} m\, g_0\, z_i \tag{3a}$$

$$\text{subject to} \quad (y_1, z_1) \ = \ (-2, 1) \tag{3b}$$

$$(y_N, z_N) = (2, 1) \tag{3c}$$

$$(y_i - y_{i+1})^2 + (z_i - z_{i+1})^2 - L_i^2 = 0, \text{ for } i = 1, \dots N-1. \tag{3d}$$

For your experiments use the following parameter values:

```
param.N  = 23;      % number of masses
param.L  = 5;       % chain length
param.m  = 0.2;     % mass of each mass point
param.g  = 9.81;    % acceleration of gravity
param.xi = [-2 1];  % coordinates [y1, z1] of initial point
param.xf = [ 2 1];  % coordinates [yN, zN] of final point
```

(a) Write a function `[f] = chain_objective(x,param)` that implements the objective and a second function `[C,Ceq] = chain_constraints(x,param)` that implements the $2(N-1)$ nonlinear equality constraints. *Note: You can eliminate from the problem constraints* (3b) *and* (3c)*, similarly to exercise sheet* 6.

(2 points)

(b) Use `fmincon` to solve the equality constrained optimization problem. You may need to tune the options to allow the algorithm to fully converge to an optimal solution. You can illustrate your results with the plotting function `plot_chain(y,z,param)`.

(2 points)

(c) Evaluate the Jacobian of your constraints at the optimal solution (how?) and check whether LICQ holds. *Hint: Use the MATLAB command* `rank`.

(2 points)

(d) Extend your function `[C,Ceq] = chain_constraints(x,param)` to fix a third point of the chain somewhere between the other two, as demonstrated in Figure 1.

(1 point)

(e) Assuming you can fix only one mass point at a time (apart from the edges), fixing which mass points would lead to infeasibility or LICQ violation? Choose a feasible set of constraints for your problem and confirm your intuition numerically.

(2 points)

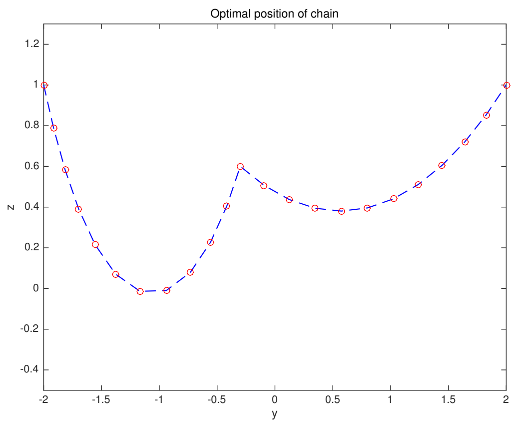

Figure 1: Hanging chain with three fixed masses.

3. **LICQ and Newton method:** Consider the following nonlinear equality constrained optimization problem:

$$\min_{x} \quad f(x)$$
$$s.t. \quad g(x) = 0,$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$ and the linear system associated with the k-*th* iteration of the Newton method:

$$\begin{bmatrix} B & A^T \\ A & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ -\Delta\lambda \end{pmatrix} = - \begin{pmatrix} \nabla f(x^k) - \nabla g(x^k)\lambda \\ g(x^k) \end{pmatrix} \tag{4}$$

with $B := \nabla^2 f(x^k) - \sum_{i=1}^{p} \lambda_i \nabla^2 g_i(x^k)$ and $A = \nabla g(x^k)^\top$.

Prove that if A has full row rank and $Z^T B Z \succ 0$, with the columns of $Z \in \mathbb{R}^{n\times p}$ forming a basis for the null space of A, the iteration matrix in (4) is invertible. *Remark: this provides a sufficient condition under which a search direction can be obtained.*

(2 points)

This sheet gives in total 14 points.