Adaptive Safe Control for Driving in Uncertain Environments

Complete Vehicle Model for Simulations

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1 Ground point tire velocities

First, we compute the total ground point velocities at each tire due to the vehicle's motion. We assume a track width W and wheelbase $L = l_f + l_r$

$$v_{\text{Wfl}} = v_{\text{CoG}} - r\left(\frac{W}{2}\cos\beta - l_f \sin\beta\right) \tag{1}$$

$$v_{\text{Wfr}} = v_{\text{CoG}} + r\left(\frac{W}{2}\cos\beta + l_f \sin\beta\right) \tag{2}$$

$$v_{\text{Wrl}} = v_{\text{CoG}} - r\left(\frac{W}{2}\cos\beta + l_r \sin\beta\right) \tag{3}$$

$$v_{\text{Wrr}} = v_{\text{CoG}} + r\left(\frac{W}{2}\cos\beta - l_r \sin\beta\right) \tag{4}$$

Where v_{CoG} is the velocity at the vehicle's CoG, r is the angular velocity about its CoG, and β is the angle between the velocity slip angle at the vehicle's CoG and written as:

$$\beta = \frac{l_r}{L}\tan\delta \tag{5}$$

1.1 Vectorization

Computation of the ground point tire velocities can be SIMD vectorized. Let $\boldsymbol{v}_W \in \mathbb{R}^4$ be the vector of all four tire velocities:

$$\boldsymbol{v}_W = \begin{bmatrix} v_{\text{Wfl}} & v_{\text{Wfr}} & v_{\text{Wrl}} & v_{\text{Wrr}} \end{bmatrix}^T \tag{6}$$

Next, let $\boldsymbol{q}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, $\boldsymbol{q}_2 = \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}^T$, and $\boldsymbol{q}_3 = \begin{bmatrix} l_f & l_f & -l_r & -l_r \end{bmatrix}^T$. With this, we can write down the single instruction to compute $\boldsymbol{v}_W$:

$$\boldsymbol{v}_W = v_{\text{CoG}}\boldsymbol{q}_1 + r\frac{W}{2}\boldsymbol{q}_2 + r\boldsymbol{q}_3 \tag{7}$$

2 Tire Lateral Slip Angle

For computing the slip angle we assume a single-track steering angle δ for the front tires as such:

$$\tan(\delta - \alpha_{\text{fl}}) = \frac{l_f r + v_{\text{CoG}} \sin\beta}{v_{\text{CoG}} \sin\beta} \tag{8}$$

$$\tan(\delta - \alpha_{\text{fr}}) = \frac{l_f r + v_{\text{CoG}} \sin\beta}{v_{\text{CoG}} \sin\beta} \tag{9}$$

$$\tan\alpha_{\text{rl}} = \frac{l_r r - v_{\text{CoG}} \sin\beta}{v_{\text{CoG}} \sin\beta} \tag{10}$$

$$\tan\alpha_{\text{rr}} = \frac{l_r r - v_{\text{CoG}} \sin\beta}{v_{\text{CoG}} \sin\beta} \tag{11}$$

2.1 Vectorization

We start by acknowledging that the lateral tire slip angle *deviations* for each tire are relatively small. We then use small angle approximations to linearize the tangent function. Next, let $\boldsymbol{q}_1 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, $\boldsymbol{q}_2 = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$, $\boldsymbol{q}_3 = \begin{bmatrix}-l_f & -l_f & l_r & l_r\end{bmatrix}^T$, and $\boldsymbol{\alpha} = \begin{bmatrix}\alpha_{\text{fl}} & \alpha_{\text{fr}} & \alpha_{\text{rl}} & \alpha_{\text{rr}}\end{bmatrix}^T$. With this, we can write down the single instruction to compute $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha} = -\boldsymbol{q}_1 + \delta\boldsymbol{q}_2 + \frac{r}{v_{\text{CoG}} \sin\beta}\boldsymbol{q}_3 \tag{12}$$

3 Longitudinal and Lateral Slip

For given tire, with wheel's rotational velocity $v_{\text{R}} = a\omega$ (a is the tire radius), ground point velocity v_{W}, and lateral slip angle α, we calculate the longitudinal (s_L) and lateral (s_S) slip ratios as follows:

$$s_L = \frac{v_{\text{R}} - v_{\text{W}}}{\max\{v_{\text{W}}, v_{\text{R}} \cos\alpha\}} \tag{13}$$

$$s_S = \begin{cases} \tan\alpha & s_L \geq 0 \\ \frac{v_{\text{R}}}{v_{\text{W}}} \sin\alpha & s_L < 0 \end{cases} \tag{14}$$

$$s_{\text{Res}}^2 = s_L^2 + s_S^2 \tag{15}$$

$$\tag{16}$$

Here, we see that s_L and s_S are not Lipschitz continuous due to the max function in s_L and the piecewise definition of s_S. This can lead to stiffness and cause numerical instabilities in the ODE solver. To fix this, we make use of Filippov solutions to introduce approximate continuum to regions of discontinuity. For the max function, we use a conservative smooth max function as follows:

$$\text{softmax}_\kappa(x, y) = \frac{1}{2}\left(x + y + \sqrt{(x - y)^2 + \kappa}\right) \tag{17}$$

where, the hyperparameter κ controls the smoothness of the softmax function. And for the piecewise function, we make use of the sigmoid function as follows:

$$s_S = \tan\alpha\, \sigma\left(\frac{s_L}{\lambda}\right) + \frac{v_{\text{R}}}{v_{\text{W}}} \sin\alpha \left(1 - \sigma\left(\frac{s_L}{\lambda}\right)\right) \tag{18}$$

where, the hyperparameter λ controls the slope of the sigmoid function.

3.1 Vectorization

Let $\boldsymbol{q} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\boldsymbol{\omega} = \begin{bmatrix} \omega_{fl} & \omega_{fr} & \omega_{rl} & \omega_{rr} \end{bmatrix}^T$ be the vectorized wheel angular speeds of each tire, using which we can write down the single instructions for computing lateral and longitudinal slip ratios for all tires:

$$\boldsymbol{s}_L = \frac{\boldsymbol{v}_W - a\boldsymbol{w}}{\text{softmax}_\kappa(\boldsymbol{v}_W, \boldsymbol{w})} \tag{19}$$

$$\boldsymbol{s}_S = \tan \boldsymbol{\alpha}\, \sigma\left(\frac{\boldsymbol{s}_L}{\lambda}\right) + \frac{a\boldsymbol{w}}{\boldsymbol{v}_W} \sin \boldsymbol{\alpha} \left(\boldsymbol{q} - \sigma\left(\frac{\boldsymbol{s}_L}{\lambda}\right)\right) \tag{20}$$

$$\boldsymbol{s}_{\text{Res}}^2 = \boldsymbol{s}_L^2 + \boldsymbol{s}_S^2 \tag{21}$$

Note, that all functional operations shown above are element-wise.

4 Tire Forces

Based on the each tire's longitudinal and tire slip calculations, we can calculate the longitudinal (F_L) and lateral (F_S) forces generated in the respective tire's frame of reference:

$$F_{\text{L rl}} = mg\frac{\mu(s_{\text{Res rl}})}{s_{\text{Res rl}}} \left(s_{\text{L rl}} \cos \alpha_{\text{rl}} + s_{\text{S rl}} \sin \alpha_{\text{rl}}\right) \tag{22}$$

$$F_{\text{S rl}} = mg\frac{\mu(s_{\text{Res rl}})}{s_{\text{Res rl}}} \left(-s_{\text{L rl}} \sin \alpha_{\text{rl}} + s_{\text{S rl}} \cos \alpha_{\text{rl}}\right) \tag{23}$$

$$F_{\text{L rr}} = mg\frac{\mu(s_{\text{Res rr}})}{s_{\text{Res rr}}} \left(s_{\text{L rr}} \cos \alpha_{\text{rr}} + s_{\text{S rr}} \sin \alpha_{\text{rr}}\right) \tag{24}$$

$$F_{\text{S rr}} = mg\frac{\mu(s_{\text{Res rl}})}{s_{\text{Res rr}}} \left(-s_{\text{L rr}} \sin \alpha_{\text{rr}} + s_{\text{S rr}} \cos \alpha_{\text{rr}}\right) \tag{25}$$

$$F_{\text{L fl}} = mg\frac{\mu(s_{\text{Res fl}})}{s_{\text{Res fl}}} \left(s_{\text{L fl}} \cos \alpha_{\text{fl}} + s_{\text{S fl}} \sin \alpha_{\text{fl}}\right) \tag{26}$$

$$F_{\text{S fl}} = mg\frac{\mu(s_{\text{Res fl}})}{s_{\text{Res fl}}} \left(-s_{\text{L fl}} \sin \alpha_{\text{fl}} + s_{\text{S fl}} \cos \alpha_{\text{fl}}\right) \tag{27}$$

$$F_{\text{L fr}} = mg\frac{\mu(s_{\text{Res fr}})}{s_{\text{Res fr}}} \left(s_{\text{L fr}} \cos \alpha_{\text{fr}} + s_{\text{S fr}} \sin \alpha_{\text{fr}}\right) \tag{28}$$

$$F_{\text{S fr}} = mg\frac{\mu(s_{\text{Res fr}})}{s_{\text{Res fr}}} \left(-s_{\text{L fr}} \sin \alpha_{\text{fr}} + s_{\text{S fr}} \cos \alpha_{\text{fr}}\right) \tag{29}$$

Then we convert the tire forces into the vehicle's frame as follows:

$$F_{\text{X fl}} = F_{\text{L fl}} \cos \delta - F_{\text{S fl}} \sin \delta \tag{30}$$

$$F_{\text{Y fl}} = F_{\text{S fl}} \cos \delta + F_{\text{L fl}} \sin \delta \tag{31}$$

$$F_{\text{X fr}} = F_{\text{L fr}} \cos \delta - F_{\text{S fr}} \sin \delta \tag{32}$$

$$F_{\text{Y fr}} = F_{\text{S fr}} \cos \delta + F_{\text{L fr}} \sin \delta \tag{33}$$

$$F_{\text{X rl}} = F_{\text{L rl}} \tag{34}$$

$$F_{\text{Y rl}} = F_{\text{S rl}} \tag{35}$$

$$F_{\text{X rr}} = F_{\text{L rr}} \tag{36}$$

$$F_{\text{Y rr}} = F_{\text{S rr}} \tag{37}$$

With this, we have obtained the Lateral and Longitudinal tire forces for each of the four vehicle's tires (rl, rr, fl, fr). Note that vectorizing the tire forces is trivial given the previous efforts at vectorizing the slip ratios and ground-point velocities.

5 Drivetrain Model

We assume an open drivetrain with equal torque distribution and linear dynamics, as shown below:

$$J_\omega \dot{\omega}_{\text{fl}} = -aF_{\text{L fl}} + \frac{1}{4}T_e \tag{38}$$

$$J_\omega \dot{\omega}_{\text{fr}} = -aF_{\text{L fr}} + \frac{1}{4}T_e \tag{39}$$

$$J_\omega \dot{\omega}_{\text{rl}} = -aF_{\text{L rl}} + \frac{1}{4}T_e \tag{40}$$

$$J_\omega \dot{\omega}_{\text{rr}} = -aF_{\text{L rr}} + \frac{1}{4}T_e \tag{41}$$

Where, J_ω is the wheel inertia, a is the wheel radius, and τ_ω is the driving torque from the engine/motor.

6 Complete Vehicle Model

We can now write down the complete vehicle dynamics as follows:

$$
\begin{aligned}
\dot{x}_{\mathrm{CoG}} &= v_x \cos\psi - v_y \sin\psi && (42)\\
\dot{y}_{\mathrm{CoG}} &= v_x \sin\psi + v_y \cos\psi && (43)\\
\dot{\psi} &= r && (44)\\
m\dot{v_x} &= m v_y r + F_{\mathrm{X\ fl}} + \\
&\qquad F_{\mathrm{X\ fr}} + F_{\mathrm{X\ rl}} + F_{\mathrm{X\ rr}} && (45)\\
m\dot{v_y} &= -m v_y r + F_{\mathrm{Y\ fl}} + \\
&\qquad F_{\mathrm{Y\ fr}} + F_{\mathrm{Y\ rl}} + F_{\mathrm{Y\ rr}} && (46)\\
I_z \dot{r} &= l_f \left(F_{\mathrm{Y\ fl}} + F_{\mathrm{Y\ fr}}\right) \\
&\qquad - l_r \left(F_{\mathrm{Y\ rl}} + F_{\mathrm{Y\ rr}}\right) \\
&\qquad + \frac{W}{2}\left(-F_{\mathrm{X\ rl}} + F_{\mathrm{X\ rr}}\right) \\
&\qquad + \frac{W}{2}\left(-F_{\mathrm{X\ fl}} + F_{\mathrm{X\ fr}}\right) && (47)\\
J_\omega \dot{\omega}_{\mathrm{fl}} &= -a F_{\mathrm{L\ fl}} + \frac{1}{4} T_e && (48)\\
J_\omega \dot{\omega}_{\mathrm{fr}} &= -a F_{\mathrm{L\ fr}} + \frac{1}{4} T_e && (49)\\
J_\omega \dot{\omega}_{\mathrm{rl}} &= -a F_{\mathrm{L\ rl}} + \frac{1}{4} T_e && (50)\\
J_\omega \dot{\omega}_{\mathrm{rr}} &= -a F_{\mathrm{L\ rr}} + \frac{1}{4} T_e && (51)\\
\dot{\delta} &= \dot{\delta} && (52)
\end{aligned}
$$