Regularization Methods for Nucleon-Nucleon Effective Field Theory

James V. Steele and R. J. Furnstahl

Department of Physics

The Ohio State University, Columbus, OH 43210

(February, 1998)

Abstract

Attempts to apply effective field theory (EFT) methods to nonrelativistic nucleon-nucleon (NN) scattering have raised questions about the nature and limitations of an EFT expansion when used nonperturbatively. We discuss the characteristics of a meaningful EFT analysis and compare them with traditional approaches to NN scattering. A key feature of an EFT treatment is a systematic expansion in powers of momentum, which we demonstrate using an error analysis introduced by Lepage. A clear graphical determination of the radius of convergence for the momentum expansion is also obtained. We use these techniques to compare cutoff regularization, two forms of dimensional regularization, and the dibaryon approach, using a simple model for illustration. The naturalness of the parameters and predictions for bound-state energies are also shown.

I. INTRODUCTION

The study of nuclear forces and nucleon-nucleon (NN) scattering has a long history [1]. Numerous models have evolved that provide superb reproductions of data over large energy ranges with only a few parameters per scattering channel. Nevertheless, the models do not provide reliable error estimates, and their connection to the symmetries and dynamics of the underlying theory of the strong interaction, quantum chromodynamics (QCD), is obscure.

Recently, many papers have addressed the application of effective field theory (EFT) methods to NN scattering [2–14]. The hope is that such an analysis will yield a systematic, controlled expansion and bring new understanding to how various processes involving nucleons relate to each other [3]. This requires a nonperturbative treatment of the EFT to properly account for the interaction between two or more heavy nucleons [4].

There are currently many disagreements in the literature about the nature and limitations of an EFT expansion in this case [5–9,11]. Regularization is required to handle divergences that arise, but the results are said to depend on the regularization scheme used (see Table I) and the size of the scattering length involved. More generally, it is claimed that the behavior and predictive power expected from a true effective field theory is not exhibited by every regularization method when applied nonperturbatively [5,6]. However, others claim that results are independent of the regularization method when proper power counting is applied [3,10].

Our goal here is to clarify the important features of a nonperturbative EFT. We can do this most clearly by making a side-by-side comparison of the regularization schemes in Table I using the error analysis advocated by Lepage [6]. This comparison clearly illustrates which schemes behave like a true EFT. Those familiar with the successful phenomenological models for NN scattering such as the Reid [15], Bonn [1], and Paris [1] potentials, which already reproduce the data well, may wonder why there is a need to reformulate the problem in terms of an EFT. We use the Reid potential as an example to compare conventional and EFT approaches.

We start by reviewing the expected behavior and radius of convergence of an EFT and outline the various regularization schemes in Section II. We then reiterate Lepage's discussion on how to analyze effective field theory behavior in terms of error plots [6] in Section III. We also show how to numerically extend his approach beyond second order in the momentum expansion and how to determine the radius of convergence graphically from the error plots. This procedure allows us to compare different regularization methods on the same footing.

Our experience with a variety of different models suggests that the issues mentioned above are generic. Therefore we use the simple delta-shell potential without pion contributions as a

TABLE I. Regularization schemes and their abbreviations used throughout this paper.

name	regularization scheme
CR[G]	Cutoff regularization with gaussian weighting in the potential [6]
$\mathrm{DR}[\overline{\mathrm{MS}}]$	Dimensional regularization with modified minimal subtraction [7]
DR[PDS]	Dimensional regularization with power divergence subtraction [12]
dibaryon	Additional low-energy degree of freedom associated with a bound or nearly bound
	state $[9,10]$.

convenient "laboratory" for comparing regularization methods in Section IV. We find that all of the regularization schemes considered in this paper have a proper radius of convergence except for dimensional regularization with modified minimal subtraction $(DR[\overline{MS}])$ when applied as in Ref. [7]. Furthermore, a good radius of convergence is linked to the naturalness of the constants from the effective lagrangian and the ability to make reliable error estimates.

An effective field theory should provide a low-energy realization of the S-matrix. This implies that after fitting the potential to a given order by the (scattering) phase shifts, estimates of the bound state energies should have similar error scaling. We present results for the bound state in Section IV and our conclusions in Section V.

II. DEFINING AN EFFECTIVE FIELD THEORY

The strong interaction at low energy can be described by an effective lagrangian of hadrons, whose form is constrained by the symmetries of the underlying theory, QCD. Restricting the characteristic momentum p of the interaction to be less than a scale Λ implies that physics at larger mass scales, such as from the exchange of heavier particles, is not resolved. This separates the physics into a long-distance part, given by the light dynamical fields in the effective lagrangian, and a short-distance part consisting of the heavy degrees of freedom and entering only through renormalization of effective lagrangian couplings. An exception is the nucleon, which due to baryon number conservation can be present as a heavy source.

The pion is the lightest of the hadrons and a prime candidate to be included as a dynamical degree of freedom. Its Goldstone nature leads to a systematic treatment, known as chiral perturbation theory [16], that has been thoroughly studied. No systematic treatment is known for non-Goldstone particles such as the ρ meson. Its exclusion from the effective lagrangian sets the scale of new, underlying physics $\Lambda \sim m_{\rho}$.

We stress that the effects of the short-distance physics on the long-distance (low-momentum) observables can be reproduced by generic terms in the effective lagrangian, organized in powers of derivatives over Λ , or a momentum expansion [17]. The coefficients of these terms summarize the remnants of the short-distance physics. In the future, these renormalized coefficients could in principle be calculated on the lattice. Until then, they must be fixed by existing data.

As each additional coefficient in the lagrangian is fixed, the error in the S-matrix should improve by a power of p^2/Λ^2 [6]. (If the interaction contains single powers of momentum coupled, for example, to an external field, the improvement may be by powers of p/Λ .) This identification of the accuracy with powers of Λ , called power counting, implies that the calculation of every observable from the same EFT will give the same systematic momentum dependence on the error, as long as the corresponding effective operator for each observable is determined to the same order in Λ . Furthermore, the dimensionless coefficients of the terms in the effective lagrangian are typically of order unity (natural) when the cutoff is chosen on the order of the physics not contained in the lagrangian. All the above features are favorable for systematic predictions and an assessment of the corrections.

At best, this expansion breaks down when the momenta in the processes involved are comparable to Λ , at which point the short-distance physics begins to be resolved. This point is referred to as the radius of convergence of the EFT. We will see below (e.g., the second

plot of Fig. 1) that this also has a clear graphical interpretation as a convergence of the error involved when an increasing number of orders in p/Λ are taken into account in the effective field theory.

Systems consisting of two (or more) nucleons with momenta well below the scale Λ can be treated nonrelativistically. Interactions between the nucleons lead to infrared divergences that require a summation to all orders of a certain class of diagrams [4]. This can be achieved by solving the Schrödinger equation or, equivalently, the Lippmann-Schwinger equation

$$T = V + VG_0T (1)$$

The short-distance effective lagrangian is schematically mapped to a coordinate-space potential consisting of highly singular terms with delta functions and derivatives of delta functions. Only after specifying a regularization scheme is the effective potential V defined. The energy dependence of V can be traded for three-momentum dependence by using the equations of motion [18].

The different ways the potential can be regularized lead to different candidate effective field theories. In perturbative applications the predictions of an EFT are independent of the regularization method. However, this result has not been established for nonperturbative applications; indeed, some doubts have been raised about the equivalence [5,11].

Cutoff regularization is a physically intuitive method for dealing with divergences in which momenta greater than the scale of new physics Λ are explicitly suppressed. This can be implemented by regulating the momentum integrals [2] or by regulating the potential itself [6]. We focus on the latter approach and simplify our discussion by concentrating on S-wave scattering. We assume the long-distance physics is properly taken into account in the potential and focus only on the short-distance EFT potential. After defining a regularized delta-function: $\delta_a^3(\mathbf{r}) \equiv e^{-r^2/2a^2}/(2\pi)^{3/2}a^3$, the S-wave potential can be taken as

$$\langle \mathbf{r}'|V_{\text{CR[G]}}|\mathbf{r}\rangle = 4\pi a^2 \delta^3(\mathbf{r} - \mathbf{r}') \left(c \,\delta_a^3(\mathbf{r}) + \frac{1}{2} d \,a^2 \,\nabla^2 \delta_a^3(\mathbf{r}) + \frac{1}{4} e \,a^4 \,\nabla^4 \delta_a^3(\mathbf{r}) + \dots \right),\tag{2}$$

which is equivalent to including a gaussian suppression of the transferred momentum $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, as seen from the Fourier transform:

$$\langle \mathbf{p}' | V_{\text{CR[G]}} | \mathbf{p} \rangle \equiv V_{\text{CR[G]}}(\mathbf{p}, \mathbf{p}') = 4\pi a^2 \left(c - \frac{1}{2} d \mathbf{q}^2 a^2 + \frac{1}{4} e \mathbf{q}^4 a^4 + \ldots \right) e^{-\mathbf{q}^2 a^2 / 2} .$$
 (3)

We will refer to this method as Cutoff Regularization with Gaussian weighting, or CR[G] for short. The coefficients c, d, \ldots , are dimensionless. The factor of $1/4\pi$ picked up by each additional term in the Born series requires a 4π to be factored out in order to render these coefficients natural [4], *i.e.* of order unity, as we discuss further in Section IV. The coefficients are determined order-by-order from matching to the available data. To fit the first n constants in the potential requires at least n points of data. This data should be taken at as low momentum as feasible to minimize the contribution from terms of $\mathcal{O}(\mathbf{q}^{2n}a^{2n})$ that have been omitted.

We need to make a few comments on the simplicity of Eqs. (2) and (3). First, terms that do not contribute to S-wave states such as $\mathbf{p} \cdot \mathbf{p}'$ are included in the potential (as can be seen by expanding the \mathbf{q}^2 term for example) to simplify the position-space expression that we use to evaluate the amplitude [6]. However, the regularized delta-function mixes

these terms and they end up contributing to S-wave scattering at higher orders in the a^2 expansion. The net effect to the order we are working is to shift the constant e by a natural amount. Since we are only interested in the naturalness of the constants in this paper and not their exact value, we will not discuss the effect of such contributions below.

Second, terms such as $d' \nabla \delta_a^3(\mathbf{r}) \cdot \nabla$ and $e' a^4 (\nabla^2 \delta_a^3(\mathbf{r}) \nabla^2 + h.c.)$ are omitted from Eq. (2). These terms, when applied with on-shell solutions of the Lippmann-Schwinger equation, only serve to modify the coefficients already taken into account in Eq. (2) [19]. Since we fit at most three constants below, we can take d' = e' = 0 without any loss of generality to our discussion here. Extending the analysis off-shell and to channels other than S-wave requires a more careful analysis of these contributions.

One advantage to the cutoff method is that an increase of the momentum cutoff 1/a beyond the scale for new physics Λ is signaled by unnaturally large coefficients in Eq. (3) and degraded EFT estimates for the scattering amplitude. This behavior was illustrated by Lepage [6] and we have verified his results in our analysis. It provides a way to determine Λ for an unknown potential or data and suggests $1/a \sim \Lambda$ as a natural value for the cutoff. This choice ensures that all the physics treated correctly at low-momentum is still taken into account, but the higher-momentum physics is suppressed. Note that while the cutoff is roughly fixed by the physics, it is not fine-tuned to data as is done for the coefficients in $V_{\text{CR[G]}}$ [6]. Instead, the cutoff is a generic parameter of the EFT that dictates the radius of convergence and naturalness of the constants.

Some disadvantages of the cutoff method are that the nonperturbative solution of the Lippmann-Schwinger equation requires numerical techniques and does not transparently exhibit simple power counting. Furthermore, both chiral and gauge symmetries are broken with a cutoff and require additional counterterms to restore them.

An alternative approach is to use dimensional regularization, which preserves the symmetries of the underlying theory and analytically can be shown to have simple power counting. This can be implemented for on-shell solutions of Eq. (1) by using the potential

$$V(\mathbf{p}, \mathbf{p}') = \frac{4\pi}{\Lambda_s^2} \left(c - d \frac{\mathbf{p}^2 + \mathbf{p}'^2}{2\Lambda_s^2} + e \frac{(\mathbf{p}^2 + \mathbf{p}'^2)^2}{4\Lambda_s^4} + \ldots \right) , \tag{4}$$

and using the modified minimal subtraction prescription $DR[\overline{MS}]$ [7] or power divergence subtraction DR[PDS] [12] on the momentum integrals. Here Λ_s is a scale introduced to make the coefficients dimensionless. Since $DR[\overline{MS}]$ has no scale (such as a cutoff) associated with the divergences, the natural size for Λ_s after fitting to the data is dictated by physical low-energy scales such as the scattering length a_s and effective range r_e . As shown in Refs. [7,8], keeping the nonperturbative amplitude to all orders in p^2 , the momentum expansion is in powers of $p^2a_sr_e/2$, which breaks down at very small momentum when the scattering length is large. This is a critical issue, since the NN system is known experimentally to have an almost bound state in the 1S_0 channel and a weakly bound state in the 3S_1 channel (the deuteron). These are reflected in a large scattering length corresponding to momentum scales of about 35 MeV and 90 MeV respectively.

One way to fix this pathology of $DR[\overline{MS}]$ is to introduce a low-energy degree of freedom, referred to as the dibaryon, into the lagrangian to parameterize the rapid energy dependence

of the amplitude from the bound or almost bound state [9].¹ Taking the large scattering length explicitly into account removes it from the residual momentum expansion and improves the radius of convergence. Another solution is to use DR[PDS], which allows an arbitrary scale μ to be included in the subtraction. This can be shown to give a momentum expansion in $p^2r_e/2(\mu-1/a_s)$, which is well behaved for large scattering length if an appropriate μ is chosen [12]. With the power counting scheme advocated in [12], DR[PDS] can be shown to produce a μ independent result. Since $\mu = 0$ corresponds to DR[$\overline{\rm MS}$], this means a specific power counting scheme cures the problems of dimensional regularization as noted in [3,10]. For comparison, all references to DR[$\overline{\rm MS}$] below refer to the naive solution of the Lippmann-Schwinger equation, and DR[PDS] will refer to this modified power counting scheme. All three dimensional regularization techniques can be solved analytically for short-range potentials.

Finally, we contrast conventional nonrelativistic NN phenomenology to the effective field theory approach. The most important difference is that while the phenomenological potentials incorporate basic pion physics such as one-pion exchange, none of them has a systematic and complete inclusion of the long-distance Goldstone boson physics. But what about the systematic incorporation of short-range physics, as we consider here? If we take a Reid-style potential [15] as an example we can see similarities. The original Reid potential consists of a sum of Yukawa interactions, with fixed masses chosen as integer multiples of the pion mass, and coefficients that can be varied to fit the data. In general for S-waves,

$$V_{\text{Reid}}(\mathbf{p}, \mathbf{p}') = \frac{c_1}{\mathbf{q}^2 + m_1^2} + \frac{c_2}{\mathbf{q}^2 + m_2^2} + \dots,$$
 (5)

we have a series of Yukawa potentials that can be viewed as regularized delta functions with different regulator masses m_i . Since we deal only with the short range interactions, we are interested in setting these masses on the order of the scale of underlying physics instead of m_{π} . If the terms in Eq. (5) are combined, they generate a Taylor expansion in \mathbf{q}^2 multiplied by a function that suppresses large momentum, similar to the effective field theory potential Eq. (3). However, since each term has a different mass, a clear separation scale Λ is not identified.

Furthermore, the conventional procedure for fitting the constants has very different consequences. Reid sought the best global fit, varying the constants to minimize the χ^2 fit to the data over the entire range of momentum considered. This means he used a weighting based only on the errors in the data. Although this optimizes the overall agreement with experiment, it neglects the fact emphasized in effective field theories that a theoretical error of order p^{2n}/Λ^{2n} is also present in the short-distance physics.

The systematics of an EFT are obscured or lost completely unless this theoretical error is also taken into account in the weighting of the fit. The ability to relate observables and processes to each other depends on the ordering of corrections in a well-defined momentum expansion. Furthermore, only in such a context would constants larger or smaller

¹This can also be shown to arise from a certain resummation of terms in any regularization scheme [3,10]. For $DR[\overline{MS}]$, however, it is a requirement for a proper radius of convergence as we will see in Section IV.

than expected be a signal of new physics or symmetry constraints when matching to QCD. Therefore, the predictability of the effective field theory is intimately connected with a consistent treatment of the error, which requires both the empirical and theoretical errors to be taken into account. This is not considered in many EFT analyses and so we summarize the philosophy of effective field theory in the next section and show how to organize the analysis to reap the full benefits.

III. PHILOSOPHY OF EFT ANALYSIS

In this section, we describe how to analyze and compare the different regularization schemes in a manner commensurate with the features of an effective field theory outlined in the previous section². The potential is not a measurable quantity, so instead we must use a scattering observable such as the phase shift $\delta(p)$ to determine the constants. Inserting the potential Eq. (4) into the Lippmann-Schwinger equation Eq. (1) and specifying a regularization scheme, we can solve for the amplitude and determine the phase shift by one of the following two equivalent relations

$$T(p) = -\frac{4\pi}{Mp} e^{i\delta(p)} \sin \delta(p) , \qquad -\frac{4\pi}{M} \frac{1}{T(p)} = p \cot \delta(p) - ip .$$
 (6)

In the left plot of Fig. 1, we show the results from using CR[G] to fit one, two, and three constants in Eq. (3), as compared to the exact S-wave phase-shift for the delta-shell potential [20], which models the underlying physics and is discussed in detail in Section IV. A first glance at the plot shows the approximation to the phase shift improves as more constants in the potential are fit. However, a second look shows that it is very difficult to gather any quantitative information from the plot. At what point the curve deviates enough to be considered inaccurate is not clear, and the radius of convergence of the EFT expansion is completely obscure.

It is therefore more informative to plot not the phase shift itself, but the *error* in the phase shift: $|\delta_{\text{eff}} - \delta_{\text{true}}|$ [6]. If the effective field theory follows proper power counting, then this error should improve by two powers of momentum as each additional coefficient in the potential is fixed [see Eq. (4)]. However, a simple calculation shows that every term in the momentum expansion of the phase shift contains the scattering length [7]. This means that the error in the phase shift could be numerically sensitive to a large scattering length and contaminate the power counting.

This is not an inherent problem. We avoid this issue with no loss of generality by plotting the error of $p \cot \delta(p)$ instead:

$$\Delta[p \cot \delta(p)] \equiv |p \cot \delta_{\text{eff}}(p) - p \cot \delta_{\text{true}}(p)|. \tag{7}$$

Since a large scattering length is synonymous with a near bound state (or pole) in the amplitude, Eq. (6) shows that this pole is cleanly mapped only to the first constant in a momentum expansion of $p \cot \delta(p)$. It is known from conventional scattering theory that

²This analysis builds on the comprehensive discussion in Ref. [6].

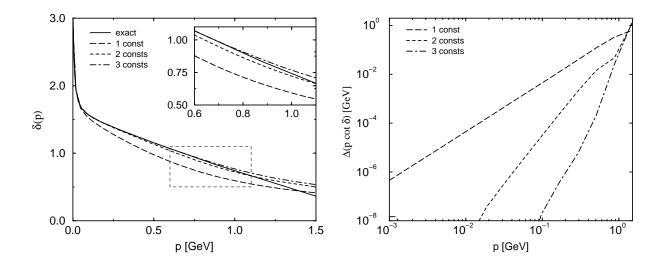


FIG. 1. The phase shift $\delta(p)$ (left) and the error in $p \cot \delta(p)$ (right), each plotted as a function of p for the delta-shell potential with a weakly bound state, as discussed in Section IV. The solid line is the exact result and the dashed lines show the CR[G] fit for one, two, and three constants.

this combination has a well defined expansion in p^2 for short-range potentials known as the "effective range expansion"

$$p \cot \delta(p) = -\frac{1}{a_s} + \frac{1}{2}r_e p^2 + v_2 p^4 + \dots , \qquad (8)$$

which defines the scattering length a_s and effective range r_e . When long-range potentials are included, Eq. (8) is only valid at low momentum or is even inapplicable. In this case, one must define a modified effective range expansion [21]. Since the effective theory contains the same long-distance physics as the true underlying theory, Eq. (7) can be modified to have a clean momentum expansion [19]. For short range potentials as considered here, it suffices to use Eq. (7).

This now gives a rigorous way to fit the coefficients in the effective potential $V(\mathbf{p}, \mathbf{p}')$ to data. After evaluating the combination $p \cot \delta_{\text{eff}}(p)$ in the effective theory, we subtract it from the true result (either data or an exact solution to a model problem) and fit the difference,

$$\Delta p \cot \delta(p) = \alpha + \beta \frac{p^2}{\Lambda^2} + \gamma \frac{p^4}{\Lambda^4} + \dots , \qquad (9)$$

to a polynomial in p^2/Λ^2 to as high an order as possible. The convergence rate of Eq. (9) determines the radius of convergence of the EFT, so we use this observable in our analysis below. The use of other observables, such as the bound-state energy, is discussed in Section IV.

Using a spread of momentum near zero, the polynomial fit should be weighted with both the expected theoretical error in momentum and any additional experimental noise. The resulting coefficients $\alpha, \beta, \gamma, \ldots$ are then minimized with respect to variations in the effective

potential constants c, d, e, \ldots using an optimization code. In practice, this method is more robust and numerically stable than matching the values of $p \cot \delta(p)$ at discrete points to fix the constants. This allows us to extend the analysis of Lepage [6] beyond second order as shown in the next section. We also note that such a procedure is needed when matching to data even when the EFT observables can be calculated analytically.

The number of coefficients that can be minimized is given by the number of constants retained in the effective potential. We used DPOLFT from package SLATEC [22] to find the polynomial fit and MINF [23], which is based on the Powell method, to carry out the minimization. Normal accuracies in minimization using double precision numbers with this method are 10^{-15} or better.

Plotting the error in $p \cot \delta(p)$ as a function of p on a log-log plot, we expect a straight line with slope given by the dominant (lowest) power of p/Λ in the error [6]. As we include more constants, the slope in this error should increase, signifying the removal of higher powers of p/Λ . The second plot of Fig. 1 clearly demonstrates the order-by-order improvement in the amplitude as more constants are added to the effective potential. With one constant the slope of the error is two $[i.e., \mathcal{O}(p^2/\Lambda^2)]$ and increases by two with each additional constant.

This plot also gives a clear graphical interpretation of the radius of convergence of the EFT as the point where the lines of error converge. The second plot of Fig. 1 shows that $\Lambda \sim 1\,\text{GeV}$. For this purpose, the error plot is much more informative than plots of the phase shift itself. Once the momentum is on the order of the cutoff, the effective theory breaks down as the short-distance physics is resolved. Only by inclusion of physics at and above the scale Λ can the effective field theory be taken beyond this point. We will discuss further consequences of this type of analysis in the next section.

IV. ILLUSTRATION WITH THE DELTA-SHELL POTENTIAL

In this section, we illustrate the points made above by using EFT techniques with the different regularization methods to systematically describe the "unknown" short-range physics of a specific example. We could use NN scattering data, but for our purposes it is more convenient to use an exactly solvable potential to serve as data in order to have a clean understanding of what features are important. The delta-shell potential has been used in the past to simulate the large scattering length found in NN scattering [20]. Kaplan used this potential to illustrate the benefits of the dibaryon approach [9]. He found upon considering NN scattering that the inclusion of long-distance pion physics did not change the conclusions. This agrees with the experience of other authors [5,7] that the addition of pions as long-range interactions does not affect the properties of the short-range expansion. The delta-shell potential is therefore a sufficient model for our purposes here.

The potential can be written in terms of the nucleon mass M, the coupling g, and the range of the potential r_0 . This short-range potential represents the new physics of our underlying theory, and so we take $r_0 = 1/\Lambda$ below,

$$V_{\text{true}}(r) = -g \frac{\Lambda}{M} \delta\left(r - \frac{1}{\Lambda}\right). \tag{10}$$

It has exactly one bound state for $g \ge 1$ and no bound states for g < 1. Scattering with $p > \Lambda$ probes the details of the potential, so we expect Λ to be the radius of convergence

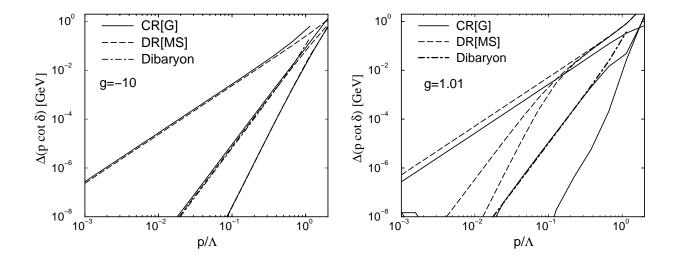


FIG. 2. The error in $p \cot \delta(p)$ plotted as a function of p/Λ for a small scattering length without a bound state g = -10 and for a large scattering length with a bound state g = 1.01.

of a well-tuned EFT. The scattering length becomes very large for g near 1, whereas the effective range (and the rest of the terms in the effective range expansion) are of natural size for all g:

$$a_s = \frac{g}{g-1} \frac{1}{\Lambda} , \qquad r_e = \frac{2(g+1)}{3g} \frac{1}{\Lambda} .$$
 (11)

The scattering length in the ${}^{1}S_{0}$ channel of NN-scattering can be modeled by choosing $(g, \Lambda) = (0.99, m_{\rho})$. This potential Eq. (10) with different g's will be the "laboratory" from which we compare the different regularization schemes. Results for actual measured data and the inclusion of pions will be discussed in a future publication [19].

We will first gain some intuition by graphically reproducing Kaplan's result that $DR[\overline{MS}]$ has a small radius of convergence if the scattering length is large. From Eq. (11), we see that choosing g = 0.99 gives a scattering length one hundred times larger than choosing g = -10. At the same time, we investigate the effect from the presence of a bound state by taking g = 1.01. The effective potential is given by Eq. (4) with the mass scale Λ_s associated with the inverse delta-shell radius and the prescription of using $DR[\overline{MS}]$ on all divergent integrals.

As mentioned above, the momentum expansion for $DR[\overline{MS}]$ can be shown analytically to be $p^2a_sr_e/2$ [7,9]. This implies using Eq. (11) that the radius of convergence for g=1.01 and 0.99 should be roughly 1/10 that of the g=-10 case. Fixing the constants in the potential Eq. (4) by matching $p \cot \delta(p)$ as outlined in the previous section, we produce the results in Fig. 2. The dashed lines show the $DR[\overline{MS}]$ results for one, two, and three constants respectively. Indeed all three lines converge to $p/\Lambda \sim 1$ for g=-10 and $p/\Lambda \sim 0.1$ for g=1.01. The results for g=0.99 fall on top of the g=1.01 results and are therefore not shown. This implies the presence of a bound state as opposed to an almost bound state does not matter, but the size of the scattering length does. It would be difficult to draw these conclusions had we only plotted the phase shift itself (the left plot of Fig. 1). Of course, we

TABLE II. Effective potential for g = -10 (small scattering length) to three different orders for different regularization schemes. $\Lambda a = 1$ for CR[G] and $\mu = \Lambda$ for DR[PDS].

	$\mathrm{DR}[\overline{\mathrm{MS}}]$			DR[PDS]			CR[G]		
	c	d	e	c	d	e	c	d	e
$\mathcal{O}(p^2/\Lambda^2)$	0.758	_	_	8.33			1.49		
$\mathcal{O}(p^4/\Lambda^4)$	0.758	-0.206	_	8.33	-25.0		2.67	-1.04	
$\mathcal{O}(p^6/\Lambda^6)$	0.758	-0.206	0.0672	8.33	-25.0	38.2	2.67	-1.58	3.75

TABLE III. Effective potential for g = 1.01 (large scattering length) to three different orders for different regularization schemes. $\Lambda a = 1$ for CR[G] and $\mu = \Lambda$ for DR[PDS].

	$\mathrm{DR}[\overline{\mathrm{MS}}]$			DR[PDS]			CR[G]		
	c	d	e	c	d	e	c	d	e
$\overline{\mathcal{O}(p^2/\Lambda^2)}$	84.2		_	-0.842	_		-1.42		
$\mathcal{O}(p^4/\Lambda^4)$	84.2	-5.64×10^3		-0.842	-0.564		-1.42 -0.946	0.842	
$\mathcal{O}(p^6/\Lambda^6)$	84.2	-5.64×10^3	3.78×10^{5}	-0.842	-0.564	-0.152	-0.937	0.618	-0.181

can also show these results analytically for this simple model, but the analysis applies much more generally, when part or all of the calculation is done numerically.

The constants c, d, and e are given in Tables II and III. Their values are calculated numerically to at least 8 digits to produce the accuracy of Fig. 2 and agree with the analytical values. This serves as a check on our numerical routines. The correlation between the naturalness of the constants and the radius of convergence is apparent. The constants for the g = 1.01 case are extremely large, reflecting the breakdown of the effective field theory for DR $\overline{\rm [MS]}$ much below the expected scale Λ .

One way to fix this behavior is to introduce the dibaryon [8–10]. This takes the large scattering length into account by explicitly introducing a low-energy s-channel degree of freedom into the effective lagrangian. For g > 0 or g < -1, the potential can be written as

$$V_{\text{dibaryon}}(\mathbf{p}, \mathbf{p}') = C - \frac{y^2}{E + \Delta}; \qquad C = \frac{2\pi}{M\Lambda}, \quad y^2 = \frac{3\pi\Lambda}{M^2} \frac{1+g}{g}, \quad \Delta = \frac{3\Lambda^2}{2M} \frac{1-g}{g}, \quad (12)$$

with $E = p^2/M$ always kept on-shell. Since there seem to be three constants fit in Eq. (12), one might think the dibaryon will have an error of $\mathcal{O}(p^6/\Lambda^6)$. However, the dibaryon amplitude is only matched to second order in the momentum when deriving the relations in Eq. (12) [9] and indeed shows an error of $\mathcal{O}(p^4/\Lambda^4)$ for the two values of g in Fig. 2 (plotted as the dot-dashed line). The slope and magnitude of the error do not depend on the scattering length, as expected.

We next repeat the calculations using the cutoff regularization method CR[G] with $1/a = \Lambda$. There are various ways to numerically solve the Schrödinger equation with a cutoff, but we have found the following procedure to be particularly efficient and numerically robust. First, the variable phase method is used to solve for the phase shift. This is a differential equation,

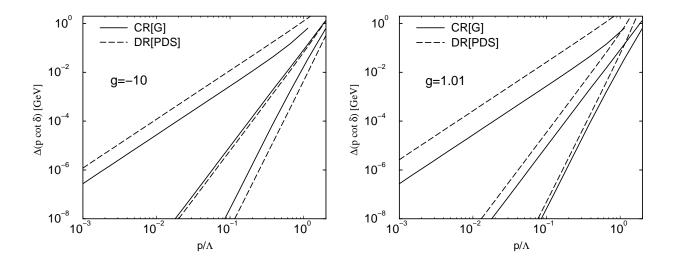


FIG. 3. The error in $p \cot \delta(p)$ plotted as a function of p/Λ for a small scattering length without a bound state g = -10 and for a large scattering length with a bound state g = 1.01.

$$\delta'(r) = -\frac{M}{p} V(r) \sin^2(pr + \delta(r)) \quad \text{with } \delta(0) = 0 , \qquad (13)$$

which expresses the change in the phase shift as the potential is built up from zero at r=0 to its full value at $r=\infty$. The boundary condition ensures that the full phase shift given by $\delta(\infty)$ is zero in the absence of a potential and defines the otherwise ambiguous multiple of π in the phase shift. We use the routine ODE from package ODE [22] to solve the differential equation. To obtain the accuracy shown in the plots, we needed to do a weighted polynomial fit of $\Delta p \cot \delta(p)$ up to $p/\Lambda = 0.1$.

The solid lines in Fig. 2 show that CR[G] does work regardless of the scattering length. In fact, with $\Lambda \sim 1/a$, the result is just as good as the dibaryon for the same number of constants. The values for the constants are given in Tables II and III, showing they are all natural for both q's (although the third constant is somewhat small for q = 1.01).

Note that as more constants are fixed, the lower-order constants are modified. This occurs because even after truncating the potential in Eq. (3) to a given order, it still contains all orders in p^2 from the gaussian factor. The nonperturbative solution of the Lippmann-Schwinger equation therefore can generate terms of any order in p^2/Λ^2 . The amplitude itself is matched to the true result order-by-order in the momentum so the power counting of the potential is destroyed. This consequence of the cutoff regularization is not necessarily relevant since the potential is not an observable. It is interesting to note, however, that these modifications are relatively small. This is also true when adding the long distance physics of the pion in fitting to actual NN data [19].

In summary, Fig. 2 shows that all regularization schemes considered so far produce useful effective field theories for a small scattering length, but $DR[\overline{MS}]$ fails for large scattering length. A failure of the power counting in powers of p/Λ is reflected in unnatural constants in the potential.

We now focus on the most recently proposed regularization scheme, DR[PDS] [12]. The

potential is given by Eq. (4), but with a further subtraction of an arbitrary mass scale $\mu/4\pi$ from the linearly divergent integrals. This term mimics the behavior of the cutoff method, although still allowing a simple enough form to be solved analytically. An additional prescription compared to the DR[$\overline{\rm MS}$] case is an expansion of observables to the same order in p^2 as the potential Eq. (4). If this is not done, the results are μ dependent, with $\mu=0$ reproducing DR[$\overline{\rm MS}$] and μ larger than the nucleon mass approaching the CR[G] result in Fig. 2.

Since we are only dealing with a short-range potential, the DR[PDS] prescription reproduces the effective range expansion Eq. (8) by construction. The DR[PDS] results in Fig. 3 are therefore μ independent, although the constants still depend on μ . We take $\mu = \Lambda$ to compare with CR[G]. This produces natural constants for g = 1.01 but somewhat unnatural ones for g = -10 as seen in Table II. This is an accidental consequence of the momentum expansion being in powers of $p^2 r_e/2(\mu - 1/a_s)$, so that for g = -10 and $\mu = \Lambda$ [see Eq. (11)] the denominator is nearly zero. The best result in this case occurs for $\mu = 0$, which reproduces the DR[MS] result and has natural constants. This implies that the scale μ is not functionally equivalent to the cutoff 1/a in CR[G] since it does not always signal the onset of new physics at the scale Λ . However, the μ independence of the scattering length shows this prescription for power counting produces a satisfatory radius of convergence, even for DR[MS] ($\mu = 0$) [3,10,12].

For both large and small scattering length, DR[PDS] does quite well, with a radius of convergence $p/\Lambda \sim 1$. The CR[G] result is better for one constant since the cutoff generates an effective range r_e close to the true result. Overall, DR[PDS] is a convenient method to produce reasonable analytical results, and depending on the problem at hand either CR[G] or DR[PDS] may be suitable. One should note, however, that only DR[PDS] provides a strict diagram by diagram power counting [12].

Finally, we return to the Reid potential [15]. The original Reid analysis used a global fit and only one adjustable parameter in each channel, for a result with approximately the same error (roughly a few percent of the data) at all momenta. However, we can apply the EFT fitting procedure instead. If we use Yukawa masses comparable to Λ , we anticipate similar results to CR[G]. Indeed, if a low-momentum fit is done to the constant c_1 , the error plot is similar to the CR[G] result with one constant (Fig. 2). Adding a second short-range Yukawa does as well as CR[G] with two constants, since the Yukawas play off each other to allow the next order error in \mathbf{q}^2 to be removed. This interplay becomes increasingly complex at higher orders. Furthermore, the additional mass scales obscure (or smear out) the role of Λ as a scale that separates the known from the unknown physics in effective field theories. Traditional NN potentials such as Reid are well suited for global fits. Systematic predictions with controlled error estimates are more properly analyzed using an effective field theory.

We now turn to an investigation of the binding energy. If the EFT is truly reproducing the S-matrix of the underlying theory order-by-order in a momentum expansion, it should reproduce the binding energies and other observables to the same order of accuracy as the phase shifts. We therefore use the binding energy prediction as a consistency check for our candidate effective field theories.

We have already fit the potentials to a given order by the scattering phase shifts above, and we use these potentials without adjustment to solve for the binding energy. This can be done analytically for the DR schemes by finding the poles in the scattering amplitude.

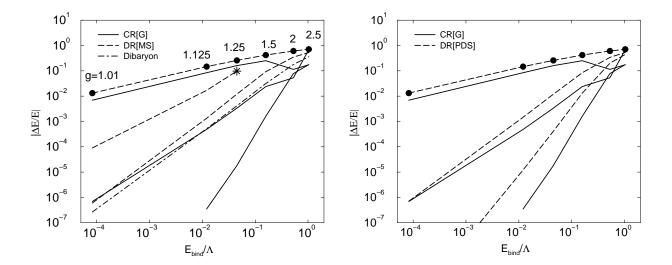


FIG. 4. The error in the binding energy for the candidate effective field theories for representative values of g from 1.01 to 2.5. The star signifies the absence of real binding energies for $DR[\overline{MS}]$ with two constants at the values of g > 1.25 considered.

The exact binding energy E_{bind} for the delta-shell potential is given by solving the equation

$$\frac{1}{g} = \frac{1 - e^{-2\eta}}{2\eta}, \qquad \eta = \frac{\sqrt{ME_{\text{bind}}}}{\Lambda}. \tag{14}$$

There is only one bound state to predict in the delta-shell potential, and if it is shallow enough, even the effective range expansion with the values of a_s and r_e can determine its value. A better test is to increase g until $E_{\rm bind}$ is large and on the order of Λ , and use the EFT to determine the accuracy of the binding energy prediction as a function of this variation. If a true radius of convergence is present, the effective field theory should break down for $E_{\rm bind}/\Lambda \sim 1$. The binding energy is 6.48×10^{-2} MeV for g=1.01 but quickly increases to 812 MeV for g=2.5.

We plot the relative error in the binding energy in Fig. 4. Both CR[G] and DR[PDS] show the clear power counting behavior and proper radius of convergence expected from a true effective field theory. This gives a graphical verification that the errors in the binding energy really do follow power counting rules. We have checked that the same behavior is seen when plotted as a function of the average momentum $\sqrt{\langle p^2 \rangle}/\Lambda$. The dibaryon result also follows the expected error scaling. In contrast, the deficiencies of DR[MS] regularization seen for the phase shifts are manifested here as binding energies that do not follow the EFT error scaling, improving to a lesser degree than expected. In addition, for values of g > 1.25 with two constants, the DR[MS] S-matrix shows no bound state with a real energy.

Therefore, our results show that most regularization procedures demonstrate the characteristics of a systematic predictive effective field theory. The fit of more and more constants in the effective potential improves the predictive power order-by-order in the momentum expansion. The radius of convergence of the EFT is independent of the scattering length and is given by the scale where new physics enters.

V. CONCLUSIONS

Attempts to apply effective field theory methods to the nonrelativistic nucleon-nucleon scattering problem and then to the nuclear many-body problem have been stalled because of controversies concerning the nature and limitations of an EFT expansion when used nonperturbatively. The familiar dictum that "calculated observables are independent of the regularization method" has been questioned in this context. To help resolve these issues we have made a direct comparison of the various regularization approaches.

We have applied the error analysis suggested by Lepage [6] to a model problem to compare cutoff regularization, two forms of dimensional regularization, and the dibaryon approach in the context of nonperturbative, nonrelativistic effective field theories. This analysis focuses on a key signature of EFT behavior: the systematic scaling of errors with momentum or energy. We summarize some points made by Lepage about applying cutoff effective field theory [6]:

- The cutoff potential is not an observable and is not amenable to power counting. Individual constants can change as higher orders are taken into account, but predictions for observables are still systematically improved.
- Fits to data should be weighted by both the uncertainty in the data and the expected theoretical error from power counting. This applies to any regularization scheme.
- The cutoff should not be taken to infinity but only roughly adjusted to minimize the error, which identifies the resolution scale of the underlying short-distance physics. Using the cutoff as a fine-tuned parameter is not as effective as the introduction of a new low-energy constant in the potential.

Our results verify these points. We also observed that natural coefficients are correlated with an optimal radius of convergence.

New numerical procedures set forth in this paper allow us to work to third order and beyond in the EFT expansion, which is necessary to obtain a clear graphical determination of the radius of convergence for a given observable. Such an analysis is required for a systematic fit to data regardless of the regularization scheme.

We find that all of the regularization methods except for dimensional regularization with modified minimal subtraction are consistent with basic features expected from a useful effective field theory:

- Each additional order in the potential leads to a systematic improvement in the amplitude.
- The radius of convergence for this improvement, when optimized, is dictated by the scale of new physics.
- Other observables are predicted with the same accuracy as the amplitude at each successive improvement.

Our results are consistent with the analysis of van Kolck that, with proper resummations, any effective field theory for short-range interactions is equivalent to an effective range expansion to the same order [3].

The CR[G] and DR[PDS] regularization schemes are each suitable for developing effective field theories of many-nucleon systems. In future work we will use both schemes in extending our fitting procedure and error analysis to NN-scattering (including pions and other channels) and then to nuclear matter.

ACKNOWLEDGMENTS

We thank T. D. Cohen, D. B. Kaplan, G. P. Lepage, and R. J. Perry for useful discussions. This work was supported in part by the National Science Foundation under Grants No. PHY–9511923 and PHY–9258270.

REFERENCES

- H. A. Bethe, Ann. Rev. Nucl. Sci. 21 (1971) 93;
 R. Machleidt, "The Meson Theory of Nuclear Forces and Nuclear Structure," in Advances in Nuclear Physics, Vol. 19. Eds. J.W. Negele and E. Vogt, Plenum (1989), and references therein.
- [2] U. van Kolck, Ph.D. thesis, U. Texas, Aug. 1993; C. Ordonez, L. Ray, and U. van Kolck, Phys. Rev. Lett. 72 (1994) 1982; Phys. Rev. C 53 (1996) 2086.
- [3] U. van Kolck, "Nucleon-Nucleon Interaction and Isospin Violation," hep-ph/9711222.
- [4] S. Weinberg, Phys. Lett. B **251** (1990) 288; Nucl. Phys. B **363** (1991) 3.
- [5] S. R. Beane, T. D. Cohen, and D. R. Phillips, "The Potential of Effective Field Theory in NN Scattering," nucl-th/9709062; D. R. Phillips and T. D. Cohen, Phys. Lett. B 390 (1997) 7; D. R. Phillips, S. R. Beane, and T. D. Cohen, "Nonperturbative Regularization and Renormalization: Simple Examples from Nonrelativistic Quantum Mechanics," hep-th/9706070; K. A. Scaldeferri, D. R. Phillips, C.-W. Kao, and T. D. Cohen, Phys. Rev. C56 (1997) 679.
- [6] G. P. Lepage, "How to Renormalize the Schrodinger Equation," nucl-th/9706029.
- [7] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. B 478 (1996) 629.
- [8] M. Luke and A. V. Manohar, Phys. Rev. D **55** (1997) 4129.
- [9] D. B. Kaplan, Nucl. Phys. B **494** (1997) 471.
- [10] P. F. Bedaque and U. van Kolck, "Nucleon-Deuteron Scattering from an Effective Field Theory," nucl-th/9710073; P. F. Bedaque, H.-W. Hammer, and U. van Kolck, "Effective Theory for Neutron-Deuteron Scattering: Energy Dependence," nucl-th/9802057.
- [11] K. G. Richardson, M. C. Birse, and J. A. McGovern, "Renormalization and Power Counting in Effective Field Theories for Nucleon-Nucleon Scattering," hep-ph/9708435.
- [12] D. B. Kaplan, M. J. Savage, and M. B. Wise, "A New Expansion for Nucleon-Nucleon Interactions," nucl-th/9801034; "Two-nucleon Systems from Effective Field Theory," nucl-th/9802075.
- [13] T.-S. Park, K. Kubodera, D-P. Min, and M. Rho, "Effective Field Theory for Low-Energy Two-Nucleon Systems," hep-ph/9711463.
- [14] J. Gegelia, "Chiral Perturbation Theory Approach to NN Scattering Problem," nucl-th/9802038.
- [15] R. V. Reid, Ann. Phys. **50** (1968) 411.
- [16] J. Gasser and H. Leutwyler, Ann. Phys. NY, 158 (1984) 142; Nucl. Phys. B250 (1985) 465, 517, 539.
- [17] S. Weinberg, Physica 96A (1979) 327.
- [18] H. Georgi, Nucl. Phys. B **361** (1991) 339.
- [19] J. V. Steele and R. J. Furnstahl, in preparation.
- [20] K. Gottfried, Quantum Mechanics, (Benjamin, Reading, MA, 1979).
- [21] J. R. Bergervoet *et al.*, in *Quarks and Nuclear Structure*, ed. K. Bleuler, (Springer-Verlag, New York, 1984).
- [22] These numerical routines can be found online at the Guide to Available Math Software website http://gams.nist.gov/
- [23] M. J. D. Powell, Comp. J. **7** 155 (1964); W. I. Zangwill, Comp. J. **10** (1967) 293.