Task 1.

$\rho_L = 0{,}10\ kg/m \quad , T = 10N$

$y(0,t) = 0.02\cos(20t)$

a) $\lambda \Rightarrow C = \sqrt{\frac{T}{\rho_L}} \Rightarrow \lambda = \frac{C}{f} \qquad 20 = \omega \Rightarrow f = \frac{\omega}{2\pi}$

$$\lambda = \frac{2\pi \cdot C}{\omega} = \frac{2 \cdot \pi \cdot 10}{20} \Rightarrow \underline{\underline{\lambda = \pi}}\ m$$

b) wave function describing transverse displacement

$x > 0 \quad t > 0$

$y(x,t) = A\cos(\omega t - Kx)$

finding K as $\quad K = \frac{2\pi}{\lambda} = 2 \qquad\qquad K = \frac{2\pi}{f}$

$y(x,t) = 0.02\cos(20t - 2x) \qquad\qquad \omega = f \cdot 2\pi$

$K = \frac{4\pi^2}{\omega}$

c) Power

$$\langle \Pi \rangle_t = \frac{1}{t}\int_0^T W dt = \frac{1}{t}\int_0^T F \cdot v\, dt \qquad\qquad \rho_L C = 1$$

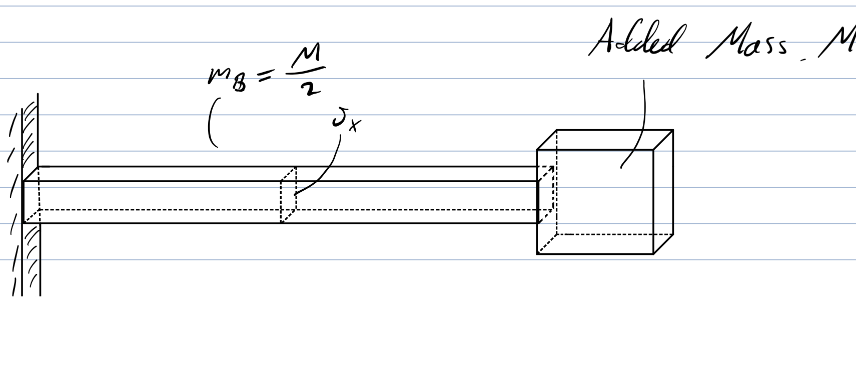
$$v = \frac{\partial F_y}{\partial t} = \frac{\partial}{\partial t}\, 0.02\cos(20t) = \frac{\partial}{\partial t}\, 0.02\, e^{j\omega t} = j0.4\, e^{j\omega t}$$

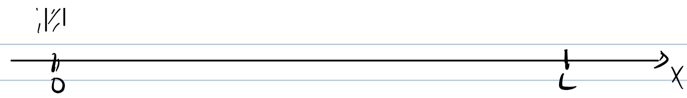
$$Z = \frac{F_z}{v} \Rightarrow F_y = Z \cdot v$$

$$\Rightarrow \langle \Pi \rangle_t = \frac{1}{2} F \cdot v = \frac{1}{2} \cdot Z \cdot v \cdot v = \frac{Z}{2} v^2 = \frac{1}{2} \cdot (0.4)^2 = \underline{\underline{80\ mW}}$$

Task 2

length $= L$, cross-sectional area $= S_x$, fixed at $x=0$, Mass $= M$ at $x=L$





$m_B = \frac{1}{2} M$

a)

$$\xi(x,t) = A e^{j(\omega t - kx)} + B e^{j(\omega t + kx)}$$

For $x = 0$

$$\xi(0,t) = A e^{j(\omega t)} + B e^{j(\omega t)} = 0$$

Because its fixed at $x=0$.

$$A = -B$$

$$\xi(x,t) = A e^{j\omega t}\left(e^{-jkx} - e^{jkx}\right) = C \sin(kx) e^{j\omega t}$$

For $x = L$ we have an attached mass. The force on a free end is zero, but gives us

$$F_x(L,t) = E S_x \left(\frac{\partial \xi}{\partial x}\right) = E S_x k C \cos(kx) e^{j\omega t}$$

Transferred into the attached mass
Force on mass using N. 2nd law

$$F = m \cdot a$$

$$F_{x,Mass} = M \cdot \frac{\partial^2 \xi}{\partial t^2} = M \omega^2 \sin(kL) e^{j\omega t}$$

$$F_{x,mass} = F_{x,Bar}$$

$$M \omega^2 \sin(kL) e^{j\omega t} = E S_x k C \cos(kL) e^{j\omega t}$$

$$\frac{\cancel{C} \cdot \sin(kL)}{\cancel{C} \cdot \cos(kL)} = \frac{E S_x k \, \cancel{e^{j\omega t}}}{M \omega^2 \cancel{e^{j\omega t}}}$$

$$\tan(kL) = \frac{E S_x k}{M \omega^2} \qquad \omega_n = c_B k_n$$

$$\tan(kL) = \frac{E S_x \cancel{k_n}}{M c_B^2 k_n^{\cancel{2}}} \qquad c_B = \sqrt{\frac{E}{\rho}} \Rightarrow E = c_B^2 \cdot \rho$$

$$\tan(kL) = \frac{\cancel{c_B^2} \cdot \rho \cdot S_x}{M \cancel{c_B^2} k_n}$$

$$\tan(kL) = \frac{\rho S_x}{M k_n} \qquad \rho S_x \cdot L = m_B, \quad | \cdot L$$

$$L \tan(kL) = \frac{\rho S_x \cdot L}{M k_n} = \frac{m_B}{M k_n}$$

$$L k_n \cdot \tan(kL) = \frac{m_B}{M}, \qquad M = 2 m_B$$

$$L k_n \cdot \tan(kL) = \frac{1}{2}$$

Solving using Wolframalpha using $x = L k_n$

$$x = \pm 0.653, \pm 3.292, \pm 6.362$$

Inserting into $f_n = \frac{x}{2\pi} \frac{C_B}{L}$ we get

$$f_1 = 0.1 \frac{C_B}{L}, f_2 = 0.52 \frac{C_B}{L}, f_3 = 1.01 \frac{C_B}{L}$$

b) $K = E S_x / L$ $\quad \frac{M}{m_B} = 2 m_B, M = 2 \rho S_x L$

$$f_0 = \frac{1}{2\pi}\left(\frac{K}{M}\right)^{1/2} = \frac{1}{2\pi}\left(\frac{E \cancel{S_x}}{L} \cdot \frac{1}{2\rho \cancel{S_x} L}\right)^{1/2}$$

$$= \frac{1}{2\pi}\left(\frac{E}{2\rho L^2}\right)^{1/2} = \frac{1}{2\pi}\left(\frac{1}{2L^2} \cdot \frac{E}{\rho}\right)^{1/2}, \sqrt{\frac{E}{\rho}} = C_B$$

$$= \frac{1}{2\pi}\left(\frac{C_B^2}{2L^2}\right)^{\frac{1}{2}} = \frac{1}{2\sqrt{2}\pi} \cdot \frac{C_B}{L}$$

$$f_0 = \frac{1}{\pi 2 \cdot \sqrt{2}} \cdot \frac{C_B}{L} = 0.113 \frac{C_B}{L}$$

c) Taylor expansion for $\tan(x)$

$$\tan x \cong x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cancel{\frac{17}{315}x^7}$$

$$k_n L \tan(k_n L) \cong k_n L\left(k_n L + \frac{1}{3}(k_n L)^3 + \frac{2}{15}(k_n L)^5\right)$$

$$\approx \frac{m_B}{M + m_B}$$

$$\Rightarrow f_0 = \frac{C_B}{2\pi L}\sqrt{\frac{m_B}{M + m_B}}$$

Could also be done using $\cot x$.

<u>Task 3</u>

$$Z(x,y) = Z_1(x,y) + Z_2(x,y)$$

a) Assuming time-harmonic vibration

$$Z(x,y,t) = Z(x,y)\, e^{j\omega t}$$

Love-Kirchoffs plate model is as follows

$$\nabla^4 z + \frac{\rho h}{D}\frac{\partial^2 z}{\partial t^2} = 0$$

Inserting into L-K

$$\nabla^4 z(x,y) - \omega^2 \frac{\rho h}{D} z(x,y) = 0 \quad , \quad \omega^2 \frac{\rho h}{D} = \gamma^4$$

$$z(x,y)\left(\nabla^4 - \gamma^4\right) = 0 = z(x,y)\left(\nabla^2 - \gamma^2\right)\left(\nabla^2 + \gamma^2\right)$$

This is satisfied for

$$\begin{cases} z(x,y)(\nabla^2 - \gamma^2) = 0 \\ z(x,y)(\nabla^2 + \gamma^2) = 0 \end{cases}$$

We transfer to polar and separate $z(x,y)$ into $R(r)\,\Theta(\theta)$

$$z(r,\theta) = C_{m,n} J_m(\gamma r) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}$$

To find the modified Bessel functions we use $\gamma = j\gamma$ to get $I_m(r)$ and $K_m(r)$ that are equivalent to the hyperbolic functions sinh and cosh.

This gives 4 solutions for the 4th order equation.

$$z(r,\theta) = C_{m,n} J_n(\gamma r) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} + I_n K_m(r) \begin{Bmatrix} \cosh\theta \\ \sinh\theta \end{Bmatrix}$$

b)

$$z(r,\theta,t) = A J_m(\gamma_{m,n} r) + B I_m(\gamma_{m,n} r) \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} e^{j\omega t}$$

Modified Bessel: $\quad y(x) = C I_x(x) + D K_m(x)$

$$r = 0, \; D = 0 \Rightarrow R_1(r) = A J_x(r)$$

B.C then gives:

Clamped end : $z(r=a) = 0$

Centr symmetry : $\frac{\partial z}{\partial r}(r=0) = 0$

$z(r=a) = 0$

$$A J_m(\gamma_{m,n}, a) + B I_m(\gamma_{m,n}, a) = 0$$

$$A J_m(\gamma_{m,n}, a) = -B I_m(\gamma_{m,n}, a)$$

$$\frac{\partial z}{\partial r}(r=0)=0:$$

$$A\frac{\partial}{\partial r}J_{m,n}(\gamma_{m,n}\,r)\Big|_{r=0}+B\frac{\partial}{\partial r}I_{m,n}(\gamma_{m,n}\,r)\Big|_{r=0}=0$$

Using $\frac{\partial J_0(x)}{\partial x}=-J_1(x)$

For $m=0$

$$\Rightarrow A(-\gamma_{0,n})J_1(\gamma_{0,n}\,0)+B\gamma_{0,n}I_1(\gamma_{0,n}\,0)=0$$

For $r=a$ we get

$$\frac{-BI_0(\gamma_{0,n}\,a)}{BI_1(\gamma_{0,n}\,a)}=\frac{AJ_0(\gamma_{0,n}\,a)}{AJ_1(\gamma_{0,n}\,a)}$$

$$\Rightarrow\quad\frac{-I_0(\gamma_{0,n}\,a)}{I_1(\gamma_{0,n}\,a)}=\frac{J_0(\gamma_{0,n}\,a)}{J_1(\gamma_{0,n}\,a)}$$

c) using 6.92 in U.A

$$f_{m,n}=\frac{(\gamma_{m,n}\,a)^2}{2\pi a\sqrt{12(1-\nu^2)}}C_B t$$

$$f_{0,1}=6554.3\,Hz$$