Tufts UNIVERSITY

PROBABILITY THEORY 1

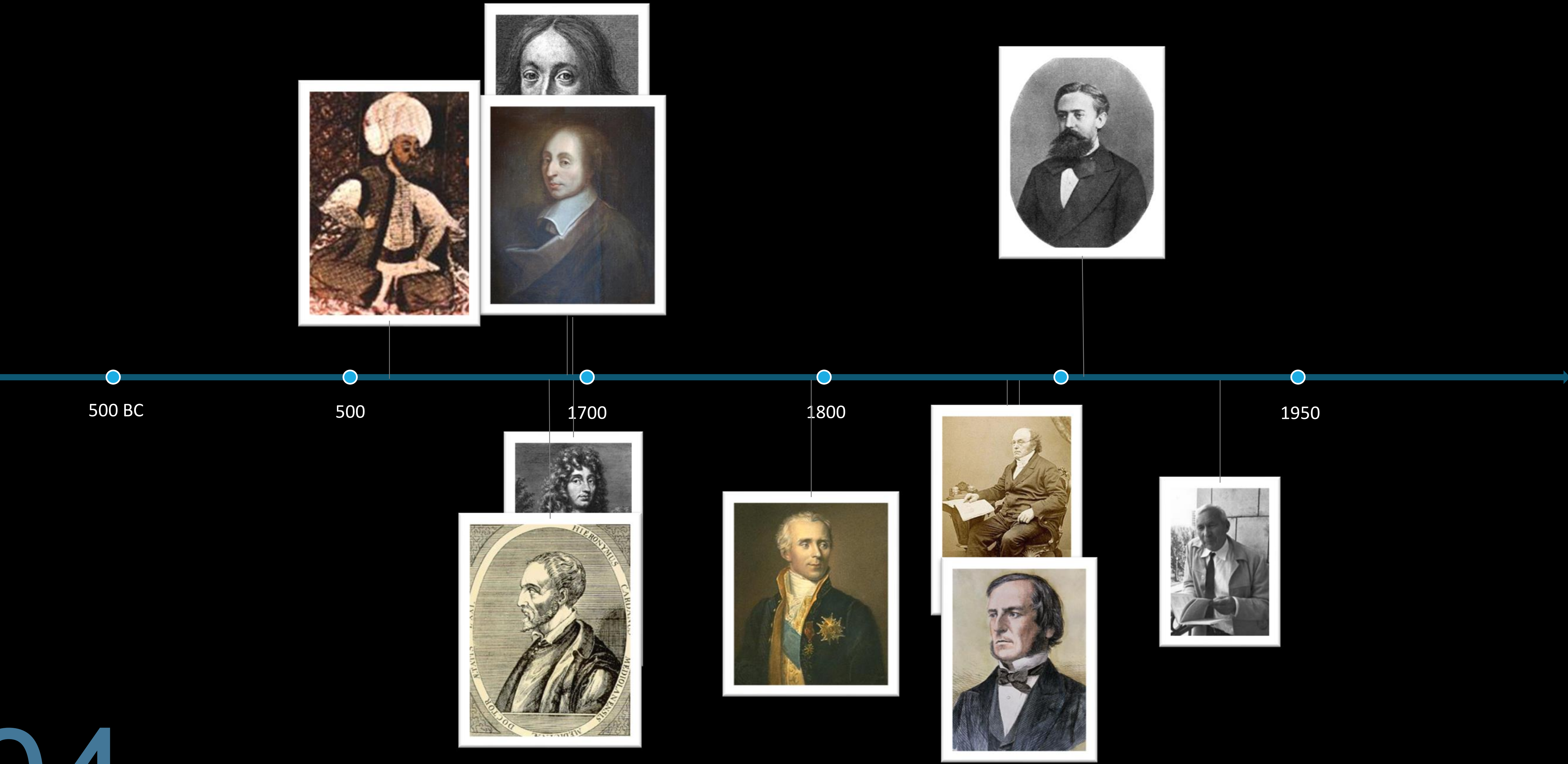
ARTIFICIAL INTELLIGENCE | COMP 131

TODAY ON AI

- Uncertainty
- Probability Theory
- Bayes’ rule
- Questions?

SECTION 01

Uncertainty



04

Brief history of Probability theory

Emergence and Uncertainty in Gestaltism.

- There are many **situations** where uncertainty arises:
  - When you travel, you reason about the possibility of delays
  - When an insurance company offers a policy, it has calculated the risk that one will claim
  - When you play a game, one cannot be certain what the other player will do
  - A medical expert system that diagnoses disease has to deal with the results of tests that are sometimes incorrect

- Agents which can reason about the effects of **uncertainty** should do better than those that don’t.

- How should uncertainty be **represented**?

Let's say we are planning a trip. We need to go to the airport. Will **leaving for the airport t minutes before the flight** (A_t) get me there on time?

- Partially observable state (state of the road, other drivers' plans, etc.)
- Noisy sensors (traffic reports, etc.)
- Uncertainty in action outcomes (flat tire, etc.)
- Complexity of the model (predicting traffic is hard, etc.)

- Given the following beliefs, which action do I chose?

$$P(A_{25m}|all\_known\_variables) = 0.04$$

$$P(A_{90m}|all\_known\_variables) = 0.70$$

$$P(A_{2h}|all\_known\_variables) = 0.95$$

$$P(A_{24h}|all\_known\_variables) = 0.9999$$

- **Decision theory** combines the agent's beliefs (Probability Theory) and desires (**Utility Theory**), defining the best action as the one that maximizes expected utility

SECTION 02

Probability Theory

- Probability assertions summarize the effect of:
  - **Laziness**: failure to enumerate exceptions, and qualifications of actions, etc.
  - **Theoretical ignorance**: complex models, etc.
  - **Practical ignorance**: lack of relevant facts, initial conditions, etc.

- **Bayesian** or **Subjective** probability relates propositions to one's own state of knowledge
- Probabilities do assert a **belief** and **not facts**
- Probabilities of propositions change with new evidence

$$P(A_{25} | \mathbf{no\_reported\_accident}) = 0.80$$

$$P(A_{25} | \mathrm{no\_reported\_accident}, \mathbf{5AM}) = 0.90$$

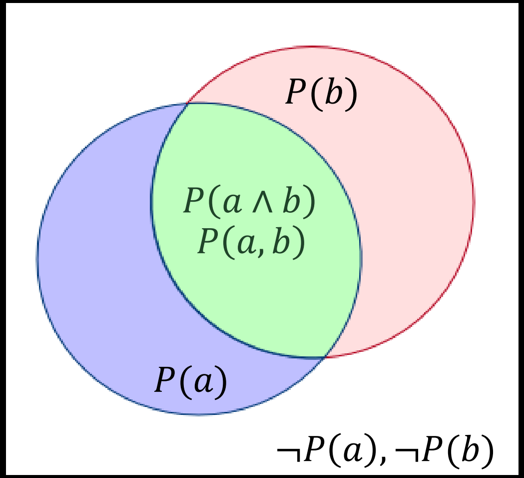
- A **random variable** is some event of the world that is uncertain:
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to the airport?

- Like **variables** in a **CSP**, random variables have a **domain** D:
  - **Propositional** or **Boolean**: R in $D = \{\mathbf{T}, \mathbf{F}\}$ (often write as {r, ¬r})
  - **Multi-valued**: T in $D = \{\text{hot}, \text{cold}\}$
  - **Discrete** or **continuous** (**finite** or **infinite**): $D = [0, \infty)$

The values of the domain for a random variable are called **outcomes**.

With each outcome $x \in D$ of A, we associate a number $P(A = x)$ that measures the probability or degree of belief that the event will occur. $P(A = x)$ or $P(x)$ is called **probability.**

1. $\forall x\colon\ 0 \leq P(A = x) \leq 1 \; and \; \sum_x P(A = x) = 1$
2. $P(\textbf{True}) = 1, P(\textbf{False}) = 0$
3. $\neg P(a) = 1 - P(a)$
4. $\forall\, a, b \in D\colon\ P(a \vee b) = P(a) + P(b) - P(a \wedge b)$



- Each variable's value has an associated probability called **prior** that corresponds to the **belief** prior to the arrival of any evidence.

- A **distribution** is a table of probability values:

$P(Temperature)$

T	**P**
hot	0.5
cold	0.5

$P(Weather)$

W	**P**
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

- We can create events out of **combinations** of the outcomes of **random variables**

- An **atomic event** is a complete specification of the values of the random variables of interest:
  - Example: if our world consists of only **two** Boolean random variables, then the world has a **four possible atomic events**:

$$
\begin{aligned}
Toothache &= \textbf{true} \wedge Cavity = \textbf{true} \\
Toothache &= \textbf{true} \wedge Cavity = \textbf{false} \\
Toothache &= \textbf{false} \wedge Cavity = \textbf{true} \\
Toothache &= \textbf{false} \wedge Cavity = \textbf{false}
\end{aligned}
$$

- The set of **all possible atomic events** has two properties:
  - It is **mutually exhaustive** (nothing else can happen)
  - It is **mutually exclusive** (only one of the four can happen at one time)

- An **atomic event** (or **event**) is a set E of outcomes:

$$X_1, X_2, \dots, X_n \quad P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) \quad P(x_1, x_2, \dots, x_n)$$

- We can use atomic events to calculate the probability of any new combination:
  - Probability that it's $\text{hot} \wedge \text{sunny}$?
  - Probability that it's hot?
  - Probability that it's $\text{hot} \vee \text{sunny}$?

A **joint distribution** ($\wedge$ distribution) over a set of random variables is a table that specifies a probability for each assignment (or **outcome**) when the **random variables happen all at the same time**:

- Must respect the following rules:

$$P(x_1, x_2, \dots, x_n) \geq 0$$

$$\sum_{(x_1, x_2, \dots, x_n)} P(x_1, x_2, \dots, x_n) = 1$$

T	W	$P(T, W)$
hot	sun	0.4
hot	$rain$	0.1
$cold$	sun	0.2
$cold$	$rain$	0.3

- A **probabilistic model** is a joint distribution over a set of random variables

Typically, the events we care about are **partial assignments**, like $P(T = \text{hot})$. **Marginal distributions** are sub-tables of joint distributions in which some variables have been eliminated:

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

$P(Temperature, Weather)$

$\boldsymbol{T}$	$\boldsymbol{W}$	$\boldsymbol{P(T,W)}$
hot	sun	0.4
hot	$rain$	0.1
$cold$	sun	0.2
$cold$	$rain$	0.3

$$\forall t \in T : P(t) = \sum_{w=\{sun, rain\}} P(t, w)$$

$$\forall w \in W : P(w) = \sum_{t=\{hot, cold\}} P(t, w)$$

$P(Temperature)$

$\boldsymbol{T}$	$\boldsymbol{P}$
hot	0.5
$cold$	0.5

$P(Weather)$

$\boldsymbol{W}$	$\boldsymbol{P}$
sun	0.6
$rain$	0.4

A **conditional probability** expresses the likelihood that one event a will occur if b occurs:

$$P(a \mid b)$$

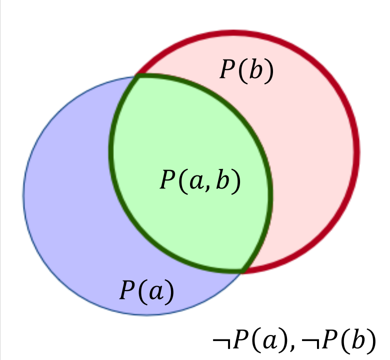
$$P(Cavity = \mathbf{T}) = 0.2$$

$$P(Toothace = \mathbf{T} \mid Cavity = \mathbf{T}) = 0.6$$

- So conditional probabilities reflect the fact that **some events** make other **events more** (or **less**) **likely**

- If one event doesn't affect the likelihood of another event, they are said to be independent and therefore: $$P(a \mid b) = P(a)$$

- Conditional probability and joint probability are related: $$\boldsymbol{P(a \mid b) = \frac{P(a, b)}{P(b)}}$$



Let's assume we have the following probabilistic model:

$\boldsymbol{T}$	$\boldsymbol{W}$	$\boldsymbol{P(T,W)}$
hot	sun	0.4
hot	$rain$	0.1
$cold$	sun	0.2
$cold$	$rain$	0.3

$$P(W = sun|T = cold) = \frac{P(T = cold, W = sun)}{P(T = cold)}$$

$$P(T = cold, W = sun) = \mathbf{0.2}$$

$$P(T = cold) = P(T = cold, W = sun) + P(T = cold, W = rain)$$
$$P(T = cold) = 0.2 + 0.3 = \mathbf{0.5}$$

$$P(W = sun|T = cold) = \frac{0.2}{0.5} = \mathbf{0.4}$$

- How we can work out the likelihood of two events occurring together given their **prior** and **conditional probabilities**? We use the **product rule**:

$$P(a, b) = P(a \mid b)\, P(b) = P(b \mid a) P(a)$$

- So in our example:

$$\begin{aligned} P(Tootache, Cavity) &= P(Tootache \mid Cavity)\, P(Cavity) \\ &= P(Cavity \mid Toothache)\, P(Toothache) \end{aligned}$$

- Unfortunately, this doesn't answer the question:
  "I have toothache. Do I have a cavity?"

SECTION 03

Bayes’ rule

- We can rearrange the two parts of the product rule:

$$P(a,b) = P(a \mid b)\,P(b) = P(b \mid a)P(a)$$

$$P(a \mid b)\,P(b) = P(b \mid a)P(a)$$

- **Bayes' rule** states that: $P(a \mid b) = \dfrac{P(b \mid a)}{P(b)}P(a)$

- Then Bayes' rule is a **relationship cause-effect** that allows us to use a probability model to infer the likelihood of the hidden cause:

$$\underbrace{\overbrace{P(cause \mid observation)}^{\text{POSTERIOR BELIEF}}}_{\text{HYPOTHESIS}} = \frac{\overbrace{P(observation \mid cause)}^{\text{CONDITIONAL BELIEF}}}{\underbrace{P(observation)}_{\text{EVIDENCE}}}\overbrace{P(cause)}^{\text{PRIOR BELIEF}}$$

- We can think about some events as being **hidden** causes: not necessarily directly observed (i.e. a cavity)

- Observations **must have arisen** because of **one** of the hypothesised causes. We **cannot** reason directly about causes we have not imagined.

- Sometimes is **easier** to model how **likely observable effects** are given **hidden causes**: $P(tootcache \mid cavity)$

- In fact good models of $P(observation \mid cause)$ are **often available** to us in real domains (i.e. medical diagnosis)

- If we know $P(observation \mid cause)$ for **every cause**, we can avoid having to know $P(observation)$:

$$P(cause \mid effect) = \frac{P(effect \mid cause)\, P(cause)}{P(effect)} = \frac{P(effect \mid cause)\, P(cause)}{\sum_{\forall c \in Causes} P(effect \mid c)\, P(c)}$$

- Sometimes it's harder to find out $P(effect \mid cause)$ for **all causes** independently than it is simply to find out $P(effect)$

Suppose a **doctor** knows that:

- **Meningitis** causes a **stiff neck** in 50% of cases: $P(s \mid m) = 0.5$
- She also knows that the **probability** in the general population of someone having a **stiff neck** at **any time** is $1/20$: $P(s) = 0.05$
- She also has to know the incidence of meningitis in the population is $1/50{,}000$: $P(m) = 0.00002$
- Using Bayes' rule, she can calculate the **probability** the patient has meningitis:

$$P(m \mid s) = \frac{P(s \mid m)\, P(m)}{P(s)} = \frac{0.5 \times 0.00002}{0.05} = 0.0002 = 1/5000$$

- Why wouldn’t the doctor be better off if she just knew the **likelihood of meningitis** given a **stiff neck**? I.e. information in the diagnostic direction from symptoms to causes?

- Because **diagnostic knowledge** is often more fragile than **causal knowledge**

- Suppose there was a meningitis epidemic? The rate of meningitis goes up 20 times within a group:

$$P(m \mid s) = \frac{P(s \mid m)\, P(m)}{P(s)} = \frac{0.5 \times \boxed{0.0004}}{0.05} = 0.004 = 1/250$$

- The **conditional belief** $P(s|m)$ is unaffected by the change in $P(m)$, whereas the diagnostic model $P(m|s) = 1/5000$ is now **completely wrong**.

PRACTICE

Exercises from the textbook:

any exercise of chapter 13

READINGS

Chapters 13

QUESTIONS ?

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