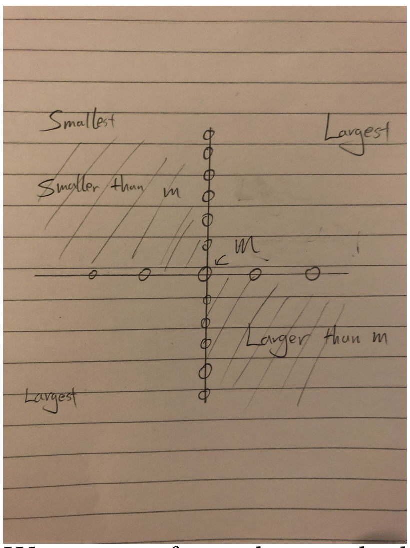
# Question 1:

(a)




We can see from the graph that a quarter of the elements are smaller than m and a quarter of the elements are larger than m.
Therefore, $\frac{n}{4} \leq i \leq \frac{3n}{4}$

(b) Let T(n) be the time needed with QuickSort algorithm on problem of size n.
Step 1: there are n/5 groups and finding median of each takes O(1) time, so Step 1 takes O(n) time
Step 2: We can apply selection algorithm to find the median, so it takes linear, O(n), runtime.
Step 3: Partition takes linear runtime, so O(n) time
Step 4: we have 2 subproblems of size i-1 and size n-i+1, so step 4 takes T(i-1) + T(n-i-1) time

If we let $X_j = 1$ when i = j.
Then $T(n) = \Sigma_{j=\frac{n}{4}}^{j=\frac{3n}{4}} X_j \times (T(i-1) + T(n-i-1)) + O(n)$
$E(T(n)) = \Sigma_{j=\frac{n}{4}}^{j=\frac{3n}{4}} \frac{2}{n} \times (T(i-1) + T(n-i-1)) + O(n)$

(c) The runtime should be $\theta(n \log n)$

$$\begin{aligned} E(T(n)) &= \Sigma_{j=\frac{n}{4}}^{j=\frac{3n}{4}} \frac{2}{n} \times (T(i-1) + T(n-i-1)) + O(n) \\ &= \Sigma_{j=\frac{n}{4}}^{j=\frac{3n}{4}} \frac{2}{n} \times [T(i-1) + T(n-i-1) + O(n)] \end{aligned}$$

we see that average expected runtime is averaged over n/2 cases, each have runtime $T(i-1) + T(n-i-1) + O(n)$ for $\frac{n}{4} \leq i \leq \frac{3n}{4}$

If we solve for $S(n) = S(i-1) + S(n-i-1) + O(n)$ ($\frac{n}{4} \leq i \leq \frac{3n}{4}$),
we will get $S(n) = \theta(n \log n)$
And since we are averaged over n/2 cases of $\theta(n \log n)$ rumtime, the averaged runtime will be $\theta(n \log n)$

## Question 2:

(a) After partitioning and before recursing, RANDSELECT algorithm need to compare the position getting from partitioning and target position. And then choose a subproblem to solve. This step will only cost O(1) runtime since only 1 comparison is needed

(b) QUICKSORT need to recurse on both subproblems while RANDSELECT only need to recurse on one subproblem.

(c) Let $X_j = 1$ if we get position j from partitioning, then
QUICKSORT: $E(Q(n)) = \theta(n) + \frac{1}{n}\Sigma_j E(Q(j-1) + Q(N-j))$
RANDSELECT: $E(R(n)) = \theta(n) + \frac{1}{n}\Sigma_j E(\max\{R(j-1) + R(N-j)\})$

main difference between the two formulas: RANDSELECT only recur on the larger one between R(j-1) and R(n-j)

(d) Guess : $E(Q(n)) \leq a \cdot n$

$$\begin{aligned}
E(Q(n)) &= \theta(n) + \frac{1}{n}\Sigma_{j=1}^{j=n} E(Q(j-1) + Q(n-j)) \\
&\leq bn + \frac{1}{n}\Sigma_{j=1}^{j=n}(a \cdot (j-1) + a \cdot Q(n-j)) \\
&\leq bn + \frac{a}{n}(n * (n-1)) \\
&\leq (a+b)n - a
\end{aligned}$$

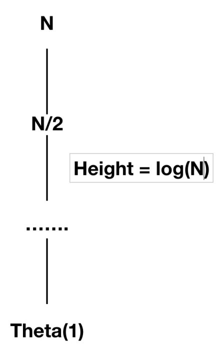
No matter how we choose a, $(a+b)n - a \geq a \cdot n$ at some large n. So our guess is false.

# Question 3:

(a) Prune and incinerate :$P(n) = P(n-1) + 1$
After each cut, the picture is smaller by 1 unit
Guess $P(n) = \theta(n)$ , then $an < P(n) < bn$ for some a,b
Base case: $a < P(2) = 1 < b$
Induction hypothesis: $an \leq P(n) \leq bn$ for all $n\ lek$
Induction Step: $a(k+1) = ak + a \leq P(k+1) = P(k) + 1 \leq bk + b = b(k+1)$
In conclusion, $P(n) = \theta(n)$

Divide and divide again : $D(n) = 2D(n/2) + 1$
After each cut, we have two pieces of picture of half the size
Using master method, we have $n^{\log_b a} = n^{\log_2 2} = n$ and f(n) = 1 = O(n), thus $D(n) = \theta(n)$

Stack them up $S(n) = S(n/2) + 1$
After each cut, the picture in stack have half the size
From the recursion tree below, we see that the height of the tree is log(n).
Thus, the runtime will be $O(logn \cdot 1) = O(logn)$




(b) Stack them up is the best strategy since it has the smallest asymptotic runtime. Both Prune and incinerate and Divide and divide again has linear runtime, so they are both worst strategy.