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Question 1:

We can't find an algorithm with better runtime

Assume we have an algorithm A for this problem. Since this is a comparison based algorithm, we can consider the decision tree T_A for this algorithm. Each comparison between a and b will have one of the three result, $a > b$, $a < b$ and $a = b$. Thus, each node in decision tree will have 3 children.
And we know that there are n different possible outcomes
Therefore, we will have a tree with height at least $log_3 n$

In class, we have lemma stating that for any algorithm A, consider its decision tree representation T_A. The runtime of A is at least the height of T_A
Therefore, the running time is $\Omega(log_3 n)$.
And in recitation, we have running time is $O(log_3 n)$
In conclusion, A has runtime $\theta(log_3 n)$

Therefore, we can't find an algorithm with better runtime

Question 2:

Let X = number of update
Let $X_i = 1$ when i then number in the list is the largest in first i numbers.
In a list of size k, each number in the list has probability $1/k$ to be the largest in the list.
Now suppose we have a list of length n. This applies to first k numbers in the list thus k-th number has $1/k$ probability to the largest up that number.
Thus, there is $1/k$ probability to update on reading k-th number.
Thus $E(X_i) = \frac{1}{i}$
Therefore, E(X) = $\Sigma_{i=1}^{i=n} \frac{1}{i}$
And since $log_e i + 1 > \Sigma_{i=1}^{i=n} \frac{1}{i} > log_e i$, thus $E(X) = \Theta(\log i)$

Question 3:

(a) Let X = number of battles will occur
$X_i = 1$ if there is fight on island i
Then $X = \Sigma X_i$

$$\begin{aligned} E(X_i) &= P(\text{Two or more vikings on island i}) \\ &= 1 - \text{ P(no viking on island i)} - \text{P(exactly 1 viking on island i)} \\ &= 1 - (\frac{n-1}{n})^k - \frac{k}{n} \times (\frac{n-1}{n})^{k-1} \end{aligned}$$

$E(X) = E(\Sigma_{i=1}^{i=n} X_i) = \Sigma_{i=1}^{i=n} E(X_i) = n \times (1 - (\frac{n-1}{n})^k - \frac{k}{n} \times (\frac{n-1}{n})^{k-1})$

(b) Let Y = number of islands visited by vikings
In recitation we have $E(Y) = n(1 - (\frac{n-1}{n})^k)$
In case when there is only one island (k=1),
we have $E(X) = n \times (1 - (\frac{n-1}{n})^1 - \frac{1}{n} \times (\frac{n-1}{n})^{1-1})) = 0$
and $E(Y) = n \times (1 - (\frac{n-1}{n})^1) = 1$

In the case when there are 400 Vikings and 100 islands(n=100,k=400)
$E(X) = 100 \times (1 - (\frac{100-1}{100})^{400} - \frac{400}{100} \times (\frac{100-1}{100})^{400-1}) = 90.952$
$E(Y) = 100 \times (1 - (\frac{100-1}{100})^{400}) = 98.20$