Erli Cai

Question 1: Totally

a. Rice Theorem does not apply.
Suppose we have have two Turing Machines M_1, M_2 with M_1 rejects all strings and M_2 loops on all strings, then $L(M_1) = L(M_2)$.
M_1 will halt on all input, while M_2 does not halt on any input
Thus this is not a property of language and Rice's theorem does not apply.

b.
First we show there exist function f from H to TOT,
$f =$ "on input $\langle M, x\rangle$:

1. compute/construct a TM $\hat{M}$ such that $\langle M, x\rangle \in H \iff \langle\hat{M}\rangle \in TOT$:
   $\hat{M} =$" on input y:
   1. Ignore y for now
   2. Simulate M on x;
      1. If M halts on x, $\hat{M}$ *accepts* y;
      2. If M does not halt on x, send $\hat{M}$ into a loop state."
2. Return $\langle\hat{M}\rangle$. "

We claim that M halts on x if and only if $\hat{M}$ halts on all input

$$\begin{aligned}\langle M, x\rangle \in H &\Rightarrow M \text{ accepts } x \\ &\Rightarrow M \text{ halts on x} \\ &\Rightarrow \hat{M} \text{ accepts all input y} \\ &\Rightarrow \langle\hat{M}\rangle \in \mathsf{TOT}\end{aligned}$$

$$\begin{aligned}\langle M, x\rangle \notin \mathsf{A}_{TM} &\Rightarrow M \text{ does not accept } x \\ &\Rightarrow M \text{ does not halt on input } x \\ &\Rightarrow \hat{M} \text{ loops on all input } y \\ &\Rightarrow \langle\hat{M}\rangle \notin \mathsf{TOT}\ .\end{aligned}$$

Thus, we have shown that $H \leq_m TOT$.

Question 2: Not Totally

$\overline{TOT} = \{\langle M \rangle \mid M$ is a TM and M doesn't halt on some inputs$\}$.
First we show there exist function f from H to $\overline{TOT}$,
$f =$ "on input $\langle M, x \rangle$:

1. compute/construct a TM $\hat{M}$ such that $\langle M, x \rangle \in H \iff \langle \hat{M} \rangle \in \overline{TOT}$:

   $\hat{M} =$" on input y:
   1. Ignore y for now
   2. Simulate M on x;
      1. If M halts on x, send $\hat{M}$ into a loop state;
      2. If M does not halt on x, $\hat{M}$ *accepts* y."
2. Return $\langle \hat{M} \rangle$. "

We claim that $\langle M, x \rangle \in H$ if and only if $\langle \hat{M} \rangle \in \overline{TOT}$

$$\begin{aligned} \langle M, x \rangle \in H &\Rightarrow M \text{ accepts } x \\ &\Rightarrow M \text{ halts on x} \\ &\Rightarrow \hat{M} \text{ loop on all input y} \\ &\Rightarrow \langle \hat{M} \rangle \in \overline{\mathsf{TOT}} \end{aligned} \qquad \begin{aligned} \langle M, x \rangle \notin \mathsf{A}_{TM} &\Rightarrow M \text{ does not accept } x \\ &\Rightarrow M \text{ does not halt on x} \\ &\Rightarrow \hat{M} \text{ accept all input } y \\ &\Rightarrow \langle \hat{M} \rangle \notin \overline{\mathsf{TOT}}\ . \end{aligned}$$

Thus, we have shown that $H \leq_m \overline{TOT}$.

Question 3: Differences

a. $A \setminus A'$ is decidable.
Since A, A' are decidable, there exists M_1, M_2 such that $L(M_1) = A, L(M_2) = A'$.
Construct M_3 in the following way:
M_3 on input x:
Simulate M_2 on x,
if M_2 accepts x, then reject;
If M_2 rejects x, then simulate M_1 on x, if M_1 accepts x, then accept, if M_1 rejects x, then reject

By construction $L(M_3) = A \setminus A'$ Since M_1, M_2 are decidable, we are guaranteed that M_1, M_2 will halt. So M_3 will halt. So $A \setminus A'$ is decidable.

b. $A \setminus B$ is not even recognisable.
We will show it by contradiction:
Let M_1 be a Turing machine that decides A.
For the sake of contradiction, suppose we have M_3 that recognise $A \setminus B$.
Then construct M_2 in the following way
M_2 on input x:
Simulate M_1 on x, if reject, then reject.
If M_1 accept x, then simulate M_3 on x, if M_3 accept x, accept x.
By construction M_2 will accept $\overline{B}$.

Since both B and $\overline{B}$ are recognisable, we know that B is decidable (Contradiction)
Therefore, our original assumption ($A \setminus B$ is recognisable) is False

c. $B \setminus A$ is recognisable but not decidable.
Given A, B there exists M_1, M_2 such that $L(M_1) = A, L(M_2) = B'$.
Construct M_3 in the following way
M_3 on input x:
Simulate M_2 on x,
if M_2 rejects x, then reject;
If M_2 accepts x, then simulate M_1 on x, if M_1 accepts x, then reject, if M_1 rejects x, then accept

By construction $L(M_3) = B \setminus A$ Since B is s recognisable but not decidable, It might loop when we simulate M_2 on x.So M_3 may not halt and $B \setminus A$ is recognisable but not decidable.