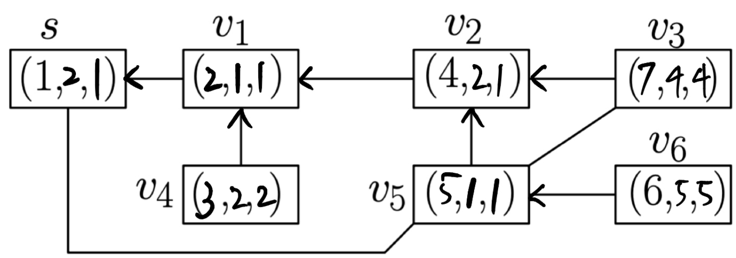
Question 1:

a.b.



c. cut vertices are v1 and v5
v1 is a cut vertex because $v4.\beta = v1.dics = 2$ and v1 is the parent of v4
v5 is a cut vertex because $v6.\beta = v5.dics = 5$ and v5 is the parent of v6

Question 2:

(a) A simple method would be run Prim's algorithm on G_2.
Runtime = O((n+1) + (m+n)log(n+1)) = O(n + (m+n)logn)
Correctness is guaranteed by the correctness of Prim's algorithm.

(b) A faster algorithm would be:
step1: Find all edges adjacent to v
step2: Find the lightest weight edge adjacent to v, call this edge e.
Then $T_2 = T_1 + e$ would be a MST for G_2

Runtime: step 1 will go through all edges of G_2, so it will take O(m+n) time. step 2 chooses the lightest weight edge out of O(n) edges, so it takes O(n) time. So this algorithm takes O(m+n) time in total.

Correctness: Firstly, since G_1 is a subgraph of G_2, we know that the MST for G_1 would be a subgraph of MST of G_2. By cut lemma, we know that the lightest weight edge adjacent to v will be in MST (and no other edges adjacent to v). And in lecture we have shown that if T_1 is a subset of MST and e is an edge to add, then $T_2 = T_1 + e$ would also be a subset of MST. And since T_2 already span edges of G_2, it is a MST for G_2

Question 3:

We could solve this problem by using Kruskal's algorithm with one small change. Since all the edges' weight are integers in the range 1 to k, we can now use a counting sort to sort the edges. And since this step is the bottleneck in the Kruskal's algorithm, we could speed up Kruskal's algorithm by doing this.

The runtime for counting sort is O(k+n), while in the original Kruskal's algorithm, we need O(mlogm) rumtime

So, the runtime is now O(n + (k+n) + nlogn + m) = O(nlogn + m)