Question 1: Really Independent

Assume $|V| = n, |E| = m$. Then each node will have degree less than or equal to n

a. Given any set $S \in V$, our verifier will first check that S indeed contains at least k vertices (in O(k) time).
Then for each node it checks that all nodes that is within 2 edge distance is not in S. This will take $O(n^2)$ time for each node since each node has at most (n-1) neighbors. So it is $O(kn^2)$ time in total.
Since we know $k < n$, total runtime is $O(n^3)$ (polytime)

For a yes instance (G, k), our certificate is any really independent set S of size k in G; our verifier will accept (G, k, S). Conversely, if S ⊆ V is such that our verifier accepts (G,k,S) then S is a really independent set of size ≥ k in G and (G, k) is a yes instance.

b. we now want to show that IS $\leq_P$ RIS.
Given an instance of IS (G, k), we construct an instance (G', k') of the Really Independent Set problem as follows.
If $G = (V, E)$, assume $V = \{v_1, v_2, ..., v_n\}$, $E = \{e_1, e_2, ..., e_m\}$

then construct $G' = (V', E')$ with
$V' = V \cup \{w_1, w_2, ..., w_m\}$
$E' = E \cup \{aw_i, w_ib | ab = e_i \in E\} \cup \{w_iw_j | i \neq j \text{ and } i, j \leq m\}$
Put in words, what we do here is add a node at the middle of each edge in graph G and connecting all these nodes.

We now run our black box for Really Independent Set on (G', k) and return the same result it gives.

Claim that if there is independent set of size k in G, then there is really independent set of size k in G.

proof: Assume S is independent set of size k in G. Then every node in S is separated by at least 2 edges. If we take corresponding nodes in G', every nodes will be separated by at least 4 edges. So S is also a really independent set of size k in G'

Claim that if there is really independent set of size k in G, then there is independent set of size k in G

Proof: Assume S is really independent set of size k in G'. Then since all nodes w_i, w_j are neighbors, they can't be separated by 2 edges. Therefore, all nodes in S must be nodes in V. And since nodes in S are separated by 2 edges in G', they can not be adjacent nodes in G. Thus S is also an independent set in G.

Runtime: We are adding in the graph G $|E|$ nodes and $|E| + |E|^2$ edges. Since we know that $|E| < |V|^2 = n^2$, we know that this algorithm will run in $O(n^4)$ time.

In conclusion, we have shown that IS $\leq_P$ RIS

Question 2: Four: Slightly Bigger Than Three

4-SAT $= \{\langle\phi = C_1 \wedge C_2 \wedge ... \wedge C_n\rangle | C_i = (t_{i1} \vee t_{i2} \vee t_{i3} \vee t_{i4}), t_{ij} \in \{x_1, x_2 ..., x_k, \overline{x_1}, ..., \overline{x_k}\}\}$

First, We will show that 4-SAT is NP.
Our certificate will be satisfying assignments A for ϕ. Given A and ϕ, our verifier will check the truth condition of each clause in ϕ and accept only if all clauses are evaluated TRUE. Doing this is in time proportional to the size of ϕ (O(n) time). Conversely if A is a certificate accepted by our verifier, then we know that A is an assignment for ϕ.
Thus we have shown that ϕ is in 4-SAT if and only if there exist a certificate (A) such that our verifier accepts A, ϕ.

We now show that 3-SAT $<_p$ 4-SAT.

For any $\varphi = D_1 \wedge D_2 \wedge ... \wedge D_n$, input to 3-SAT:
1. Introduce a new variable y and Generate new formula:
$\phi = (D_1 \vee y) \wedge (D_1 \vee \overline{y}) \wedge (D_2 \vee y) \wedge (D_2 \vee \overline{y}) \wedge ... \wedge (D_n \vee y) \wedge (D_n \vee \overline{y})$
$\quad = C_{11} \wedge C_{12} \wedge C_{21} \wedge C_{22} \wedge ... \wedge C_{n1} \wedge C_{n2}$
where $C_{i1} = D_i \vee y$ and $C_{i2} = D_i \vee \overline{y}$
2. We now run our black box for 4-SAT on ϕ and return the same result it gives. (This construction is poly-size and poly-time because ϕ has O(n) clauses.)

To finish the proof, we simply need to show that $\varphi \in$ 3-SAT $\Longleftrightarrow \phi \in$ 4-SAT:

$\varphi \in$ 3-SAT $\Rightarrow D_i$ is TRUE for all i
$\Rightarrow (D_i \vee y)$ and $(D_i \vee \overline{y})$ are TRUE for all i
$\Rightarrow C_{ij}$ is TRUE for all ij
$\Rightarrow \phi \in$ 4-SAT

$\phi \in$ 4-SAT $\Rightarrow C_{ij}$ is TRUE for all i
$\Rightarrow (D_i \vee y)$ and $(D_i \vee \overline{y})$ are TRUE for all i
$\Rightarrow D_i$ is TRUE for all i since y and $\overline{y}$ can't be True simultaneously
$\Rightarrow \varphi \in$ 3-SAT