DE LA RECHERCHE À L’INDUSTRIE

An averaged model for thick spray

Kinetic and hyperbolic equations: modeling, analysis and numerics

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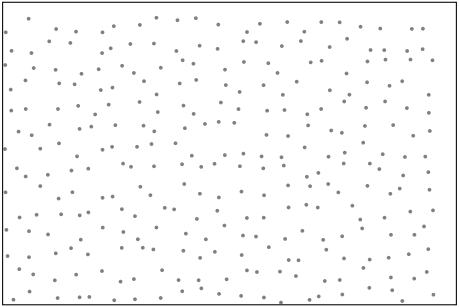
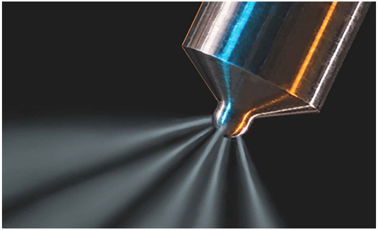
Dispersed phase (droplet or dust) evolving in a continuous phase (gas or incompressible fluid) : their study is a subdomain of the study of multiphase flows



Examples : Clouds, Diesel engines, Medical sprays in the mouth, Nuclear industry, Pharmaceutical industry



(a) Diesel engine fuel injector

(b) Medical spray

- Unknowns for the gas : macroscopic quantities

$$\varrho(t, \boldsymbol{x}) \geq 0, \quad \boldsymbol{u}(t, \boldsymbol{x}) \in \mathbf{R}^3, \quad e(t, \boldsymbol{x}) \geq 0, \quad \alpha(t, x) \in (0, 1].$$

  Usually follow a hyperbolic (compressible Euler equations) or parabolic (Navier-Stokes) equation.

- Unknown for the dispersed phase : kinetic distribution function

$$f(t, \boldsymbol{x}, \boldsymbol{v}) \geq 0$$

  with $\boldsymbol{v}$ the velocity of the droplets.
  Usually follow a Vlasov or Vlasov-Boltzmann equation
  We assume that the particles are monodisperse : all particle have the same radius r.

Various effects can be added in the models :

- Internal energy for the droplets
- Compressibility, rotation of the droplets
- Inelastic collisions and breakup of the droplets
- Chemical reactions

- Models in the context of combustion theory introduced in Williams [1985]
- Classification of sprays O'Rourke [1981]
- Case of thin sprays :
  - Vlasov-Euler :
    - Local-in-time well posedness for strong solution Baranger and Desvillettes [2006], Mathiaud [2010]
    - Global weak solution in 1D with finite energy Cao [2022]
  - Vlasov-Navier-Stokes :
    - Global existence of weak solution on the 3D-torus Boudin, Desvillettes, Grandmont, and Moussa [2009] and the inhomogenous case Choi and Kwon [2015]
    - Large time behavior studied in Choi [2016], Ertzbischoff, Han-Kwan, and Moussa [2021], Han-Kwan, Moussa, and Moyano [2020]
- Thick sprays :
  - Linear stability studied in Buet, Després, and Desvillettes [2022]
  - Boudin, Desvillettes, and Motte [2003].
  - Recent numerical work Benjelloun, Desvillettes, Ghidaglia, and Nielsen [2012]

Compressible Vlasov-Euler. Coupling made through a drag force. The coefficient $D_\star \geq 0$ is a friction coefficient.

$$\begin{cases} \partial_t \varrho + \boldsymbol{\nabla}_x \cdot (\varrho \boldsymbol{u}) = 0 \\ \partial_t (\varrho \boldsymbol{u}) + \boldsymbol{\nabla}_x \cdot (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla}_x p = D_\star \displaystyle\int_{\mathbf{R}^3} (\boldsymbol{v} - \boldsymbol{u}) f \, \mathrm{d}v \\ \partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_v \cdot (\Gamma f) = 0 \\ m_\star \Gamma = -D_\star (\boldsymbol{v} - \boldsymbol{u}) \end{cases}$$

- **Well-posedness**: There exists a unique strong local-in-time solution to thin sprays equations for smooth $(\mathscr{C}_c^1 \cap H^s)$ initial data and well-behaved p Baranger and Desvillettes [2006]
- Extension to the full system with internal energy and collision operator by Mathiaud [2010].

Compressible Vlasov-Euler equation. Coupling through a drag force and volume fraction. The coefficient $D_\star \geq 0$ is a friction coefficient. The coefficient $m_\star = \frac{4}{3}\pi r^3$ denotes the mass of the particle if we assume that that have a density equals to 1.

$$
\begin{cases}
\partial_t(\alpha\varrho) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho\boldsymbol{u}) = 0 \\
\partial_t(\alpha\varrho\boldsymbol{u}) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho\boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla}_x p = m_\star \boldsymbol{\nabla}_x p \int_{\mathbf{R}^3} f \,\mathrm{d}v + D_\star \int_{\mathbf{R}^3} (\boldsymbol{v} - \boldsymbol{u}) f \,\mathrm{d}v \\
\partial_t(\alpha\varrho e) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho e\boldsymbol{u}) + p(\partial_t \alpha + \boldsymbol{\nabla}_x \cdot (\alpha\boldsymbol{u})) = D_\star \int_{\mathbf{R}^3} |\boldsymbol{v} - \boldsymbol{u}|^2 f \,\mathrm{d}v \\
\partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_v \cdot (\Gamma f) = 0 \\
\alpha = 1 - m_\star \int_{\mathbf{R}^3} f \,\mathrm{d}v \\
m_\star \Gamma = -m_\star \boldsymbol{\nabla}_x p - D_\star (\boldsymbol{v} - \boldsymbol{u})
\end{cases}
$$

Some mathematical issues :

- Solution blow up in finite time, cannot use usual weak solution to describe shocks because of the term $\boldsymbol{\nabla}_x p \cdot \boldsymbol{\nabla}_v f$ in the Vlasov equation.
- Well-posedness result, even locally-in-time, is still lacking.

Motivation to modify the model, with a regularisation based on convolution.

$$\begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho\boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho\boldsymbol{u}) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla}_x p = m_\star \boldsymbol{\nabla}_x p \displaystyle\int_{\mathbf{R}^3} f \,\mathrm{d}v + D_\star \int_{\mathbf{R}^3} (\boldsymbol{v}-\boldsymbol{u}) f \,\mathrm{d}v \\ \partial_t(\alpha\varrho e) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho e\boldsymbol{u}) + p(\partial_t\alpha + \boldsymbol{\nabla}_x \cdot (\alpha\boldsymbol{u})) = D_\star \displaystyle\int_{\mathbf{R}^3} |\boldsymbol{v}-\boldsymbol{u}|^2 f \,\mathrm{d}v \\ \partial_t f + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_v \cdot (\Gamma f) = 0 \\ \alpha = 1 - m_\star \displaystyle\int_{\mathbf{R}^3} f \,\mathrm{d}v \\ m_\star \Gamma = -m_\star \boldsymbol{\nabla}_x p - D_\star(\boldsymbol{v}-\boldsymbol{u}) \end{cases}$$

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Consider a single spherical particle of radius r living inside a gas. Neglecting the friction, the gas acts on the particles with the force :

$$m_\star \Gamma = -\int_{\mathbf{S}^2} p\boldsymbol{n}\,\mathrm{d}x$$

which is rewritten as

$$m_\star \Gamma = -\int_{\mathbf{S}^3} \boldsymbol{\nabla}_x p\,\mathrm{d}x = -m_\star \int_{\mathbf{R}^3} w(\boldsymbol{x} - \cdot)\boldsymbol{\nabla}_x p\,\mathrm{d}x = m_\star (w \star \boldsymbol{\nabla}_x p)$$

with the convolution kernel

$$w(\boldsymbol{y}) = \frac{1}{m_\star}\mathbf{1}_{|\boldsymbol{y}|<r}(\boldsymbol{y}).$$

$m_\star = \frac{4}{3}\pi r^3$.

We reintroduce the friction in the force term and modify the equation in a way that preserves the global conservation properties.
Considering additionally that the modifications should be kept to a minimum, we are led to propose the following system, using the notation $\langle\cdot\rangle = w \star \cdot$:

(1)

$$\begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho\boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho\boldsymbol{u}) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla}_x p = m_\star \boldsymbol{\nabla}_x p \displaystyle\int_{\mathbf{R}^3} \langle f\rangle \,\mathrm{d}v + D_\star \int_{\mathbf{R}^3} (\boldsymbol{v}-\boldsymbol{u}) f \,\mathrm{d}v \\ \partial_t(\alpha\varrho e) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho e\boldsymbol{u}) + p(\partial_t\alpha + \boldsymbol{\nabla}_x \cdot (\alpha\boldsymbol{u})) = D_\star \displaystyle\int_{\mathbf{R}^3} |\boldsymbol{v}-\boldsymbol{u}|^2 f \,\mathrm{d}v \\ \partial_t f + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_v \cdot (\Gamma f) = 0 \\ \alpha = 1 - m_\star \displaystyle\int_{\mathbf{R}^3} \langle f\rangle \,\mathrm{d}v \\ m_\star\Gamma = -m_\star\langle\boldsymbol{\nabla}_x p\rangle - D_\star(\boldsymbol{v}-\boldsymbol{u}) \end{cases}$$

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Proposition

The system (1) is conservative in total mass, total momentum and total energy.

- It is obvious for the mass of the fluid and the mass of the particles

$$\partial_t(\alpha\varrho) + \boldsymbol{\nabla}_x \cdot (\alpha\varrho\boldsymbol{u}) = 0, \quad \partial_t f + \boldsymbol{\nabla}_x \cdot (\boldsymbol{v} f) + \boldsymbol{\nabla}_v \cdot (\Gamma f) = 0$$

- For the total momentum, we write

$$m_\star \partial_t \int_{\mathbb{R}^3} f\boldsymbol{v}\,\mathrm{d}v + m_\star \boldsymbol{\nabla}_x \cdot \int_{\mathbb{R}^3} f\boldsymbol{v}\otimes\boldsymbol{v}\,\mathrm{d}v = -m_\star\langle\boldsymbol{\nabla}p\rangle \int_{\mathbb{R}^3} f\,\mathrm{d}v - D_\star \int_{\mathbb{R}^3} (\boldsymbol{v}-\boldsymbol{u}) f\,\mathrm{d}v.$$

Summing with the momentum equation of the fluid :

$$\partial_t\left(\alpha\varrho\boldsymbol{u} + m_\star \int_{\mathbb{R}^3} f\boldsymbol{v}\,\mathrm{d}v\right) + \boldsymbol{\nabla}_x \cdot \left(\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u} + m_\star \int_{\mathbb{R}^3} f\boldsymbol{v}\otimes\boldsymbol{v}\,\mathrm{d}v\right) + \boldsymbol{\nabla}_x p$$
$$= m_\star\left(\boldsymbol{\nabla}_x p \int_{\mathbb{R}^3} \langle f\rangle\,\mathrm{d}v - \langle\boldsymbol{\nabla}_x p\rangle \int_{\mathbb{R}^3} f\,\mathrm{d}v\right).$$

and we have $\int_{\mathbb{R}^3}\left(\boldsymbol{\nabla}p\int_{\mathbb{R}^3}\langle f\rangle\,\mathrm{d}v - \langle\boldsymbol{\nabla}p\rangle\int_{\mathbb{R}^3} f\,\mathrm{d}v\right)\,\mathrm{d}x = 0.$

- For the energy equation, it is less obvious and we use the fact that

$$\alpha = 1 - m_\star \int_{\mathbf{R}^3} \langle f \rangle \,\mathrm{d}v$$

The total energy equation writes, with $E = \frac{\boldsymbol{u}^2}{2} + e$,

$$\begin{aligned}
&\partial_t \left(\alpha \varrho E + m_\star \int_{\mathbb{R}^3} f \frac{|\boldsymbol{v}|^2}{2} \,\mathrm{d}v \right) + \boldsymbol{\nabla}_x \cdot \left(\alpha \varrho \boldsymbol{u} E + m_\star \int_{\mathbb{R}^3} f \boldsymbol{v} \frac{|\boldsymbol{v}|^2}{2} \,\mathrm{d}v \right) \\
&= -\boldsymbol{\nabla}_x \cdot (\alpha \boldsymbol{u} p) - m_\star \boldsymbol{\nabla}_x \cdot \left(p \int_{\mathbb{R}^3} \langle f \rangle \boldsymbol{v} \,\mathrm{d}v \right) \\
&+ m_\star \left(\boldsymbol{\nabla}_x p \int_{\mathbb{R}^3} \langle f \rangle \boldsymbol{v} \,\mathrm{d}v - \langle \boldsymbol{\nabla}_x p \rangle \int_{\mathbb{R}^3} f \boldsymbol{v} \,\mathrm{d}v \right).
\end{aligned}$$

Again we have that $\int_{\mathbb{R}^3} \left(\boldsymbol{\nabla} p \int_{\mathbb{R}^3} \langle f \rangle \boldsymbol{v} \,\mathrm{d}v - \langle \boldsymbol{\nabla} p \rangle \int_{\mathbb{R}^3} f \boldsymbol{v} \,\mathrm{d}v \right) \,\mathrm{d}x = 0$, In the case where p and e follow a perfect gas law, we have

Proposition

Formally, one has the entropy inequality

$$\partial_t(\alpha\varrho S) + \boldsymbol{\nabla}\cdot(\alpha\varrho S\boldsymbol{u}) = \frac{D_\star}{T}\int_{\mathbf{R}^3}|\boldsymbol{v}-\boldsymbol{u}|^2 f\,\mathrm{d}v \geq 0.$$

with $S = C_v \ln(e\varrho^{\gamma-1})$, $T > 0$ is the temperature of the gas.

Recall that

$$\alpha = 1 - m_\star \int_{\mathbf{R}^3} \langle f \rangle \, \mathrm{d}v.$$

Proposition

Under boundedness assumptions of the initial data, and assumptions on the regularity of velocity variables. If $\alpha(0,x) \in (0,1]$ *then* $\alpha(t,x) \in (0,1]$ *for all time* $t > 0$ *for smooth solutions.*

We work on the barotropic version of (1), setting $m_\star = 1$ and $D_\star = 0$ for simplicity

$$(2)\qquad \begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla}p = \boldsymbol{\nabla}p\displaystyle\int_{\mathbb{R}^3}\langle f\rangle\,\mathrm{d}v \\ \partial_t f + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f - \langle\boldsymbol{\nabla}_x p\rangle\cdot\boldsymbol{\nabla}_v f = 0 \\ \alpha = 1 - \displaystyle\int_{\mathbb{R}^3}\langle f\rangle\,\mathrm{d}v \\ p = p(\varrho) = \varrho^\gamma, \quad \gamma > 1. \end{cases}$$

Denoting $V_M(0) = \sup_{(\boldsymbol{x},\boldsymbol{v})\in\mathbf{R}^3\times\mathbf{R}^3, f(0,\boldsymbol{x},\boldsymbol{v})>0} |\boldsymbol{v}|$

Theorem (VF, C. Buet, B. Després)

We consider $\gamma > 1$, $s \in \mathbf{N}$ *such that* $s \geq 3$. *Let* $(\varrho_0, \varrho_0 \boldsymbol{u}_0) : \mathbf{R}^3 \to \Omega$ *with* Ω *relatively compact open set of* $(0, +\infty) \times \mathbf{R}^3$ *and* $f_0 : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}_+$ *satisfying*

$$\varrho_0 - 1 \in H^s(\mathbf{R}^3), \quad \boldsymbol{u}_0 \in H^s(\mathbf{R}^3), \quad f_0 \in \mathscr{C}^1_c(\mathbf{R}^3 \times \mathbf{R}^3) \cap H^s(\mathbf{R}^3 \times \mathbf{R}^3)$$

such that $\|f_0\|_{L^\infty} < \frac{1}{2^4 \|w\|_{L^1} V_M(0)^3}$.
Then, one can find $T > 0$ *such that there exists a solution of* (2) $(\varrho, \varrho\boldsymbol{u}, f)$ *belonging to* $\mathscr{C}^1([0,T] \times \mathbf{R}^3, \Omega') \times \mathscr{C}^1_c([0,T] \times \mathbf{R}^3 \times \mathbf{R}^3, \mathbf{R}_+)$ *with* $\overline{\Omega} \subset \Omega'$ *relatively compact in* $(0, +\infty) \times \mathbf{R}^3$. *Moreover this solution is unique.*

Principle of the proof :

- Use theory of symmetrisable hyperbolic system Majda [1984] and theory of characteristics for the control of H^s norm of f and its support like in Baranger and Desvillettes [2006]

Question : How to symmetrize hyperbolic part of the system ?
Idea : Rearrange the equations in the hyperbolic part and treat the terms containing derivatives of α as source terms

$$\partial_t \mathbf{U} + \boldsymbol{\nabla} \cdot \mathbf{F}(\mathbf{U}) = b(\mathbf{U}, f)$$

with $\mathbf{U} = \begin{pmatrix} \varrho \\ \varrho \boldsymbol{u} \end{pmatrix}$, $\mathbf{F}(\mathbf{U}) = \begin{pmatrix} \varrho \boldsymbol{u} \\ \varrho \boldsymbol{u} \otimes \boldsymbol{u} + p\mathrm{Id} \end{pmatrix}$ and

$$b(\mathbf{U}, f) = \begin{pmatrix} \dfrac{\varrho \boldsymbol{u} \cdot \boldsymbol{\nabla}_x \int_{\mathbf{R}^3} \langle f \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f \rangle \,\mathrm{d}v} - \dfrac{\varrho \boldsymbol{\nabla}_x \cdot \int_{\mathbf{R}^3} \langle f \rangle \boldsymbol{v} \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f \rangle \,\mathrm{d}v} \\ \dfrac{\varrho \boldsymbol{u} \otimes \boldsymbol{u} \int_{\mathbf{R}^3} \boldsymbol{\nabla}_x \langle f \rangle \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f \rangle \,\mathrm{d}v} - \dfrac{\varrho \boldsymbol{u} \boldsymbol{\nabla}_x \cdot \int_{\mathbf{R}^3} \langle f \rangle \boldsymbol{v} \,\mathrm{d}v}{1 - \int_{\mathbf{R}^3} \langle f \rangle \,\mathrm{d}v} \end{pmatrix}.$$

It is then equivalent to construct a solution to

$$\begin{cases} S(\mathbf{U})\partial_t \mathbf{U} + \sum_{i=1}^{3} (SA_i)(\mathbf{U})\partial_{x_i}\mathbf{U} &= S(\mathbf{U})b(\mathbf{U}, f), \\ \partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f - \langle \boldsymbol{\nabla}_x p \rangle \cdot \boldsymbol{\nabla}_v f &= 0, \end{cases}$$

with

$$S(\mathbf{U}) = \begin{pmatrix} p'(\varrho) + |\boldsymbol{u}|^2 & -\boldsymbol{u}^T \\ -\boldsymbol{u} & \mathrm{Id} \end{pmatrix},$$

and $A_i = \partial_{\mathbf{U}} \mathbf{F}_i$ for $1 \le i \le 3$.

The proof proceeds via a classical iteration scheme.

- We first work with smooth and compactly supported initial data

$$\varrho_0 - 1 \in \mathcal{D}(\mathbb{R}^3), \quad \boldsymbol{u}_0 \in \mathcal{D}(\mathbb{R}^3), \quad f_0 \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^3).$$

  Later, we will use a mollification process to prove the case of all initial data in H^s.

- We ask that f_0 is small, more precisely, we assume

$$\|f_0\|_{L^\infty} < \frac{1}{2^4 \|w\|_{L^1} V_M(0)^3}.$$

- We note Ω a relatively compact open set of $(0, +\infty) \times \mathbb{R}^3$ such that $\mathbf{U}_0 := \begin{pmatrix} \varrho_0 \\ \varrho_0 \boldsymbol{u}_0 \end{pmatrix} \in \Omega$.

We define the approximation process as follows :

- For $k=0$, one sets $\theta_0=+\infty$ and $(\mathbf{U}^0(t), f^0(t)) = (\mathbf{U}_0, f_0)$. Then, given the quantities θ_k, $\mathbf{U}^k$ and f^k, one defines $(\mathbf{U}^{k+1}, f^{k+1})$ on $[0,\theta_k)$ as the solution of the linear system

$$(3) \qquad S(\mathbf{U}^k)\partial_t \mathbf{U}^{k+1} + \sum_{i=1}^{3}(SA_i)(\mathbf{U}^k)\partial_{x_i}\mathbf{U}^{k+1} = S(\mathbf{U}^k)b(\mathbf{U}^k, f^k),$$

$$\mathbf{U}^{k+1}(0,\boldsymbol{x}) = \mathbf{U}_0(\boldsymbol{x}),$$

$$(4) \qquad \partial_t f^{k+1} + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f^{k+1} - \langle \boldsymbol{\nabla} p^k\rangle \cdot \boldsymbol{\nabla}_v f^{k+1} = 0,$$

$$f^{k+1}(0,\boldsymbol{x},\boldsymbol{v}) = f_0(\boldsymbol{x},\boldsymbol{v}).$$

- We chose Ω' a relatively compact open subset of $(0,+\infty)\times\mathbf{R}^3$ such that $\overline{\Omega}\subset\Omega'$.
- The equation (3) is linear in $\mathbf{U}^{k+1}$, symmetric and has smooth coefficients on $[0,\theta_k)$, therefore it admits a smooth solution on $[0,\theta_k)$.
- For the equation (4), the characteristic method gives us a solution.

$$(5) \qquad S(\mathbf{U}^k)\partial_t \mathbf{U}^{k+1} + \sum_{i=1}^{3} (SA_i)(\mathbf{U}^k)\partial_{x_i}\mathbf{U}^{k+1} = S(\mathbf{U}^k) b(\mathbf{U}^k, f^k),$$

$$\mathbf{U}^{k+1}(0,\boldsymbol{x}) = \mathbf{U}_0(\boldsymbol{x}),$$

$$(6) \qquad \partial_t f^{k+1} + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f^{k+1} - \langle \boldsymbol{\nabla} p^k \rangle \cdot \boldsymbol{\nabla}_v f^{k+1} = 0,$$

$$f^{k+1}(0,\boldsymbol{x},\boldsymbol{v}) = f_0(\boldsymbol{x},\boldsymbol{v}).$$

- Using the fact that $\mathbf{U}_0 \in \Omega$, from the Sobolev embedding for $s > 3/2+1$, $H^s(\mathbf{R}^3) \hookrightarrow L^\infty(\mathbf{R}^3)$, $\exists R > 0$, if $\|\mathbf{U} - \mathbf{U}_0\|_{H^s} \leq R \Rightarrow \mathbf{U} \in \Omega'$. (Recall $\overline{\Omega} \subset \Omega'$).
- Then, one defines θ_{k+1} as the supremum of times $\theta < \theta_k$ such that $\sup\limits_{t\in[0,\theta]} \|\mathbf{U}^{k+1} - \mathbf{U}_0\|_{H^s}(t) \leq R$.
- Finally, $\theta_{k+1} > 0$ since $\mathbf{U}^{k+1}$ is smooth, and $\mathbf{U}_0 \in \Omega$. The sequence $(\mathbf{U}^k, f^k)_{k\in\mathbf{N}}$ is then well-defined.

- We need more control on the solution
- We restrict the life-span of the solution and we define T_k as the supremum of times $T \in [0, \theta_k)$ such that

$$\sup_{t\in[0,T]} \|\mathbf{U}^k - \mathbf{U}_0\|_{H^s}(t) \leq R$$

$$\forall t \in [0,T], \quad X_M^k(t) \leq 2X_M(0),$$

$$\forall t \in [0,T], \quad V_M^k(t) \leq 2V_M(0)$$

$$\sup_{t\in[0,T]} \left\| \int_{\mathbb{R}^3} \langle f^k \rangle \, \mathrm{d}v \right\|_{L^\infty} (t) \leq 2^4 \|w\|_{L^1} V_M(0)^3 \|f_0\|_{L^\infty}.$$

and such that $T_{k+1} \leq T_k$.

With

$$X_M^k(t) = \sup_{(\boldsymbol{x},\boldsymbol{v})\in\mathbf{R}^3\times\mathbf{R}^3, f^k(t,\boldsymbol{x},\boldsymbol{v})>0} |\boldsymbol{x}|, \quad V_M^k(t) = \sup_{(\boldsymbol{x},\boldsymbol{v})\in\mathbf{R}^3\times\mathbf{R}^3, f^k(t,\boldsymbol{x},\boldsymbol{v})>0} |\boldsymbol{v}|.$$

The proof relies on two results

Proposition

There exists $T_\star > 0$ *which depends only upon* Ω, Ω', s, $\mathbf{U}_0$, w *and* f_0 *such that, for all* $k \in \mathbf{N}$, $T_k \geq T_\star$.

Proposition

One can find $T_{\star\star} \in (0, T_\star)$ *such that, for* $k \geq 2$,

$$\|\mathbf{U}^{k+1} - \mathbf{U}^k\|_{L^2,T_{\star\star}} \leq \frac{1}{4}\|\mathbf{U}^k - \mathbf{U}^{k-1}\|_{L^2,T_{\star\star}} + \frac{1}{4}\|\mathbf{U}^{k-1} - \mathbf{U}^{k-2}\|_{L^2,T_{\star\star}},$$

$$\|f^k - f^{k-1}\|_{L^2,T_{\star\star}} \leq C(f_0, \Omega', \gamma)\mathrm{TV}(w)\|\mathbf{U}^{k-1} - \mathbf{U}^{k-2}\|_{L^2,T_{\star\star}}.$$

We can then pass to the limit. The sequence $(\mathbf{U}^k, f^k)$ converges in $L^\infty(0, T_{\star,\star}; L^2(\mathbf{R}^3))$ towards some $(\mathbf{U}, f)$. We prove then that $(\mathbf{U}, f)$ is a solution.

Some comments

- We cannot use this idea on the original thick sprays equations.
- The convolution allows us to avoid loss of regularity, with inequalities of the form

$$\|\boldsymbol{\nabla}\langle f\rangle\| \leq \|f\| \mathrm{TV}(w).$$

- It is easy to extend to theorem to the case with friction $D_\star \neq 0$.
- The result could be extended to the case with internal energy and collision operator. Polydispersion could certainly be taken into account.
- Probably no hope to obtain (for general initial data) global smooth (H^s) solutions for the system.

- Improve the proof without the restriction on the L^∞-norm of f_0.
- Work more on the numerical side : Write a scheme that preserves a maximum principle for α.

Preprint available on HAL :

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