We have been given $a_i^{pre}(t)$ and $a_{ij}^{post}(t)$, and a Markov Chain X_t. We assume a constant discount factor $v = \frac{1}{1+i}$.

What is the Cash flow induced

We define $I_i(t) = \chi_{X_t = i}$. Which mean that you would get the following annuity at time t: $\sum_{i \in S} a_i^{pre}(t) \times I_i(t)$. Which transition (death) benefit would you get at time t? You get $a_{ij}^{post}(t)$ if you are in state i at time t and in state j at $t+1$? ie $a_{ij}^{post}(t) \times I_i(t) \times I_j(t+1)$. In sum for the death benefit $\sum_{i,j \in S^2} a_{ij}^{post}(t) \times I_i(t) \times I_j(t+1)$.

So the cash flow at times t can be calculated as

$$A(t) = \sum_{i \in S} a_i^{pre}(t) \times I_i(t) + \sum_{i,j \in S^2} a_{ij}^{post}(t) \times I_i(t) \times I_j(t+1)$$
$$\tilde{A}(t) = \sum_{i \in S} a_i^{pre}(t) \times I_i(t) + \sum_{i,j \in S^2} a_{ij}^{post}(t) \times I_i(t) \times I_j(t+1) \times v$$

What is the value of this insurance cash flow?

We define the **Mathematical Reserve** as

$$\begin{aligned} V_j(t) &= \mathbb{E}[PV\, of\, future\, CF \,|\, X_t = j] \\ &= \mathbb{E}[\sum_{\tau=0}^{\infty} v^\tau \tilde{A}(t+\tau) |\, X_t = j] \end{aligned}$$

In order to calculate the mathematical reserve you can substitute $\tilde{A}$ in the formula below and ultimately what you need to calculate (keeping in mind the linearity of the $\mathbb{E}$ functional as the following quantities:

$$\mathbb{E}[I_i(t+\tau) | X_t = j] = p_{ji}(t, t+\tau)$$
$$\mathbb{E}[I_i(t+\tau) \times I_k(t+\tau+1) | X_t = j] = ?$$

How do we do this?

$$\mathbb{E}[I_i(t+\tau) \times I_k(t+\tau+1) | X_t = j] = P[X_{t+\tau+1} = k, X_{t+\tau} = i | X_t = j]$$

Now we do the same as in the proof of the Chapman-Kolmogorov-Equation]

$$\begin{aligned} \mathbb{E}[I_i(t+\tau) \times I_k(t+\tau+1) | X_t = j] &= P[X_{t+\tau+1} = k, X_{t+\tau} = i | X_t = j] \\ &= \frac{P[X_{t+\tau+1} = k, X_{t+\tau} = i, X_t = j]}{P[X_t = j]} \\ &= \frac{P[X_{t+\tau+1} = k, X_{t+\tau} = i, X_t = j]}{P[X_t = j]} \times \frac{P[X_t = j, X_{t+tau} = i]}{P[X_t = j, X_{t+tau} = i]} \\ &= P[X_{t+\tau} = i | X_t = j] \times P[X_{t+\tau+1} = k | X_t = j, X_{t+\tau} = i] \\ &= p_{ji}(t, t+\tau) \times p_{ik}(t+\tau, t+\tau+1) \end{aligned}$$

If we put now all things together we can can calcuate the mathematical reserves as follows

$$\begin{aligned} V_j(t) &= \mathbb{E}[\sum_{\tau=0}^{\infty} v^\tau \tilde{A}(t+\tau) |\, X_t = j] \\ &= \sum_{\tau=0}^{\infty} v^\tau \left(\sum_{i \in S} a_i^{pre}(t+\tau) \times p_{ji}(t, t+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+\tau) p_{ji}(t, t+\tau) \times p_{ik}(t+\tau, t+\tau+1) \times v \right) \end{aligned}$$

Remark: With this formula we can also calculate the expected cash flows at time t as follows:

$$\mathbb{E}[A(t+\tau) \,|\, X_t = j] = \sum_{i \in S} a_i^{pre}(t+\tau) \times p_{ji}(t, t+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+\tau) p_{ji}(t, t+\tau) \times p_{ik}(t+\tau, t+\tau+1)$$

Thiele Difference Equation this is the relationship between the mathematical reserves between times t and $t+1$. The relationship is as follows:

$$V_j(t) = a_j^{pre}(t) + v \sum_{k \in S} p_{jk}(t, t+1) \times \left(a_{jk}^{post}(t) + V_k(k+1) \right)$$

To prove this equation we split the time-sum into $\tau = 0$ and the rest. For $\tau = 0$ we get

$$a_j^{pre}(t) + v \sum_{k \in S} p_{jk}(t, t+1) \times a_{jk}^{post}(t)$$

as per above.

In a second step we need to consider (**NEW AND IMPROVED FORMULA**:)

$$\begin{aligned} &\sum_{\tau=1}^{\infty} v^\tau \left(\sum_{i \in S} a_i^{pre}(t+\tau) \times p_{ji}(t, t+\tau) + \sum_{(i,k) \in S^2} a_{ik}^{post}(t+\tau) p_{ji}(t, t+\tau) \times p_{ik}(t+\tau, t+\tau+1) \times v \right) \\ &= \sum_{\tau=1}^{\infty} v^\tau \sum_{i \in S} p_{ji}(t, t+\tau) \times \left(a_i^{pre}(t+\tau) + \sum_{k \in S} a_{ik}^{post}(t+\tau) p_{ik}(t+\tau, t+\tau+1) \times v \right) \end{aligned}$$

We can now calculate the quantity $p_{ji}(t, t+\tau)$ as follows, by means of the Chapman-Kolmorgorov equation

$$p_{ji}(t, t+\tau) = \sum_{l \in S} p_{jl}(t, t+1) \times p_{li}(t+1, t+\tau)$$

$$\begin{aligned} &\sum_{\tau=1}^{\infty} v^\tau \left(\sum_{i \in S} a_i^{pre}(t+\tau) \times p_{ji}(t, t+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+\tau) p_{ji}(t, t+\tau) \times p_{ik}(t+\tau, t+\tau+1) \times v \right) \\ &= \sum_{\tau=1}^{\infty} v^\tau p_{ji}(t, t+\tau) \times \left(\sum_{i \in S} a_i^{pre}(t+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+\tau) p_{ik}(t+\tau, t+\tau+1) \times v \right) \\ &= \sum_{\tau=1}^{\infty} v^\tau \left(\sum_{l \in S} p_{jl}(t, t+1) \times p_{li}(t+1, t+\tau) \right) \times \left(\sum_{i \in S} a_i^{pre}(t+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+\tau) p_{ik}(t+\tau, t+\tau+1) \times v \right) \\ &= \sum_{l \in S} p_{jl}(t, t+1) \times v \times \left(\sum_{\tau=0}^{\infty} v^\tau p_{li}(t+1, t+1+\tau) \times \left(\sum_{i \in S} a_i^{pre}(t+1+\tau) + \sum_{i,j \in S^2} a_{ij}^{post}(t+1+\tau) p_{ik}(t+1+\tau, t+1+\tau+1) \times v \right) \right) \\ &= \sum_{l \in S} p_{jl}(t, t+1) \times v \times V_l(t+1) \end{aligned}$$

Remarks:

1) For one life we have a recursion of reals $A_x = q_x \times v + p_x \times v \times A_{x+1}$. In case of MR of a Markov model we have a recursion of vectors.

2) To solve it one needs boundary conditions as per before with $V_j(\omega) = 0 \, \forall j \in S$

3) Thiele Difference Equations leads to the same results as for the classical life insurance we have seen.

In [ ]: