2022Annuity

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Types of Insurance we want to look at

- Immediate payout annuity, prenumerando $\ddot{a}_x$
- Immediate payout annuity, postnumerando a_x
- defferred annuity, prenumerando $_{m}|\ddot{a}_x$
- temporariy annuity, prenumerando $\ddot{a}_{x:n}$

What are the respective Random Variables - Immediate payout annuity, prenumerando $Y = \sum_{k=0}^{\infty} \nVdash_{K \geq k} \times v^k$ - Immediate payout annuity, postnumerando $Y = \sum_{k=1}^{\infty} \nVdash_{K \geq k} \times v^k$ - defferred annuity, prenumerando $Y = \sum_{k=m}^{\infty} \nVdash_{K \geq k} \times v^k$ - temporariy annuity, prenumerando $Y = \sum_{k=0}^{n-1} \nVdash_{K \geq k} \times v^k$

What do we want to calculate for each type of insurance

- Definition
- Expected Value
- Expected Value expressed with Commutation Functions
- Recusion between different years

$\ddot{a}_x$

- Definition: $Y = \sum_{k=0}^{\infty} \nVdash_{K \geq k} \times v^k$
- Expected Value: $\ddot{a}_x = \mathbb{E}[Y] = \sum_{k=0}^{\infty} \mathbb{E}[\nVdash_{K \geq k} \times v^k] = \sum_{k=0}^{\infty} {}_k p_x \times v^k$

$$
\begin{aligned}
Y &= \sum_{k=0}^{\infty} \nVdash_{K \geq k} \times v^k && (1) \\
&= \ddot{a}_K && (2) \\
&= 1 + v + \ldots v^K && (3) \\
&= \frac{1 - v^{K+1}}{1 - v} && (4)
\end{aligned}
$$

- Expected Value: $\ddot{a}_x = \frac{1 - \mathbb{E}[v^{K+1}]}{1-v} = \frac{1 - A_x}{1-v}$

- Expected Value expressed with Commutation Functions

$$\begin{aligned}
\ddot{a}_x &= \sum_{k=0}^{\infty} v^k {}_k p_x && (5)\\
&= \sum_{k=0}^{\infty} v^k \frac{l_{x+k}}{l_x} && (6)\\
&= \sum_{k=0}^{\infty} \frac{v^{x+k}}{v^x} \frac{l_{x+k}}{l_x} && (7)\\
&= \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} && (8)\\
&= \frac{N_x}{D_x} && (9)
\end{aligned}$$

- Recusion between different years: Need to relate A_x and A_{x+1}: You need ${}_k p_x = p_x \times {}_{k-1} p_{x+1}$

$$\begin{aligned}
\ddot{a}_x &= \mathbb{E}[Y] && (10)\\
&= \sum_{k=0}^{\infty} {}_k p_x v^k && (11)\\
&= (\sum_{k=0}^{0} + \sum_{k=1}^{\infty}) {}_k p_x v^k && (12)\\
&= 1 + \sum_{k=1}^{\infty} {}_k p_x v^k && (13)\\
&= 1 + p_x \times v \times \ddot{a}_{x+1} && (14)
\end{aligned}$$

Hence

$$\ddot{a}_x = 1 + p_x \times v \times \ddot{a}_{x+1} \tag{15}$$

${}_n E_x$

- Definition: $Z = v^n \times \mathbb{1}_{K \geq n}$
- Expected Value: ${}_n E_x = \mathbb{E}[Z] = {}_n p_x v^n$
- Expected Value expressed with Commutation Functions

Hence

$${}_n E_x = p_x \times v \times {}_{n-1} E_{x+1} \tag{16}$$

In consequence we have

$$\begin{aligned} \ddot{a}_x &= \sum_{k=0}^{\infty} {}_k p_x v^x && (17) \\ &= \sum_{k=0}^{\infty} {}_k E_x && (18) \end{aligned}$$

a_x

- Definition: $Y = \sum_{k=1}^{\infty} \nVdash_{K \geq k} \times v^k$
- Expected Value: $a_x = \mathbb{E}[Y] = \sum_{k=1}^{\infty} \mathbb{E}[\nVdash_{K \geq k} \times v^k] = \sum_{k=1}^{\infty} {}_k p_x \times v^k$

$$\begin{aligned} Y &= \sum_{k=1}^{\infty} \nVdash_{K \geq k} \times v^k && (19) \\ &= \ddot{a}_K && (20) \\ &= v + \ldots v^K && (21) \\ &= \frac{1 - v^{K+1}}{1 - v} - 1 && (22) \end{aligned}$$

- Expected Value: $a_x = \frac{1 - \mathbb{E}[v^{K+1}]}{1-v} - 1 = \frac{1 - A_x}{1-v} - 1 = \frac{v - A_x}{1-v}$
- Expected Value expressed with Commutation Functions

$$\begin{aligned} a_x &= \sum_{k=1}^{\infty} v^k {}_k p_x && (23) \\ &= \sum_{k=1}^{\infty} v^k \frac{l_{x+k}}{l_x} && (24) \\ &= \sum_{k=1}^{\infty} \frac{v^{x+k}}{v^x} \frac{l_{x+k}}{l_x} && (25) \\ &= \sum_{k=1}^{\infty} \frac{D_{x+k}}{D_x} && (26) \\ &= \frac{N_{x+1}}{D_x} && (27) \end{aligned}$$

- Recusion between different years: Need to relate A_x and A_{x+1}: You need ${}_kp_x = p_x \times {}_{k-1}p_{x+1}$

$$\begin{aligned}
a_x &= \mathbb{E}[Y] && (28)\\
&= \sum_{k=1}^{\infty} {}_kp_x v^k && (29)\\
&= (\sum_{k=0}^{0} + \sum_{k=1}^{\infty}) {}_kp_x v^k && (30)\\
&= \sum_{k=1}^{\infty} {}_kp_x v^k && (31)\\
&= p_x \times v \times \ddot{a}_{x+1} && (32)\\
&= p_x \times v \times (1 + a_{x+1}) && (33)
\end{aligned}$$

Hence

$$a_x = p_x \times v \times (1 + a_{x+1}) \tag{34}$$

${}_{m|}\ddot{a}_x$

- Definition: $Y = \sum_{k=m}^{\infty} \nVdash_{K \geq k} \times v^k$
- Expected Value: ${}_{m|}\ddot{a}_x = \mathbb{E}[Y] = \sum_{k=m}^{\infty} \mathbb{E}[\nVdash_{K \geq k} \times v^k] = \sum_{k=m}^{\infty} {}_kp_x \times v^k$
- Expected Value expressed with Commutation Functions

$$\begin{aligned}
{}_{m|}\ddot{a}_x &= \sum_{k=m}^{\infty} v^k {}_kp_x && (35)\\
&= \sum_{k=m}^{\infty} v^k \frac{l_{x+k}}{l_x} && (36)\\
&= \sum_{k=m}^{\infty} \frac{v^{x+k}}{v^x} \frac{l_{x+k}}{l_x} && (37)\\
&= \sum_{k=m}^{\infty} \frac{D_{x+k}}{D_x} && (38)\\
&= \frac{N_{x+m}}{D_x} && (39)\\
&= \frac{N_{x+m}}{D_x} \frac{D_{x+m}}{D_{x+m}} && (40)\\
&= {}_mE_x \, \ddot{a}_{x+m} && (41)
\end{aligned}$$

- Recusion between different years: Need to relate A_x and A_{x+1}: You need ${}_kp_x = p_x \times {}_{k-1}p_{x+1}$.

For $m > 1$

$$
\begin{aligned}
{}_{m}|\ddot{a}_x &= \mathbb{E}[Y] && (42)\\
&= \sum_{k=m}^{\infty} {}_{k}p_x v^k && (43)\\
&= (\sum_{k=0}^{0} + \sum_{k=m}^{\infty}) {}_{k}p_x v^k && (44)\\
&= \sum_{k=1}^{\infty} {}_{k}p_x v^k && (45)\\
&= p_x \times v \times {}_{m-1}|\ddot{a}_{x+1} && (46)
\end{aligned}
$$

$\ddot{a}_{x:n}$

- Definition: $Y = \sum_{k=0}^{n-1} \nVdash_{K \geq k} \times v^k$
- Expected Value: $\ddot{a}_{x:n} = \mathbb{E}[Y] = \sum_{k=0}^{n-1} \mathbb{E}[\nVdash_{K \geq k} \times v^k] = \sum_{k=0}^{n-1} {}_{k}p_x \times v^k$
- Expected Value expressed with Commutation Functions

$$
\begin{aligned}
\ddot{a}_{x:n} &= \sum_{k=0}^{n-1} v^k {}_{k}p_x && (47)\\
&= \sum_{k=0}^{n-1} v^k \frac{l_{x+k}}{l_x} && (48)\\
&= \sum_{k=0}^{n-1} \frac{v^{x+k}}{v^x} \frac{l_{x+k}}{l_x} && (49)\\
&= \sum_{k=0}^{n-1} \frac{D_{x+k}}{D_x} && (50)\\
&= \frac{N_x - N_{x+n}}{D_x} && (51)
\end{aligned}
$$

- Recusion between different years: Need to relate A_x and A_{x+1}: You need ${}_{k}p_x = p_x \times {}_{k-1}p_{x+1}$

$$
\begin{aligned}
\ddot{a}_{x:n} &= \mathbb{E}[Y] && (52)\\
&= \sum_{k=0}^{n-1} {}_{k}p_x v^k && (53)\\
&= (\sum_{k=0}^{0} + \sum_{k=1}^{n-1}) {}_{k}p_x v^k && (54)\\
&= 1 + \sum_{k=1}^{n-1} {}_{k}p_x v^k && (55)\\
&= 1 + p_x \times v \times \ddot{a}_{x+1:n-1} && (56)
\end{aligned}
$$

Hence

$$\ddot{a}_{x:n} = 1 + p_x \times v \times \ddot{a}_{x+1:n-1} \tag{57}$$

General Annuity

We are entitled to get an annuity $r_k \in \mathbb{R}$ if we are alive at this point in time. Given the current age is x we need to look at $Y = \sum_{k=0}^{\infty} r_{x+k} \nVdash_{K \geq k} \times v^k$ - Expected Value: $\ddot{a}_x^{gen} = \mathbb{E}[Y] = \sum_{k=0}^{\infty} r_{x+k} \mathbb{E}[\nVdash_{K \geq k} \times v^k] = \sum_{k=0}^{\infty} r_{x+k\,k}p_x \times v^k$

- Expected Value expressed with Commutation Functions

$$\begin{aligned}
\ddot{a}_x^{gen} &= \sum_{k=0}^{\infty} r_{x+k} v^k {}_kp_x \qquad (58)\\
&= \sum_{k=0}^{\infty} r_{x+k} v^k \frac{l_{x+k}}{l_x} \qquad (59)\\
&= \sum_{k=0}^{\infty} r_{x+k} \frac{v^{x+k}}{v^x} \frac{l_{x+k}}{l_x} \qquad (60)\\
&= \sum_{k=0}^{\infty} \frac{r_{x+k}\, D_{x+k}}{D_x} \qquad (61)\\
& \qquad (62)
\end{aligned}$$

- Recusion between different years: Need to relate A_x and A_{x+1}: You need ${}_kp_x = p_x \times {}_{k-1}p_{x+1}$

$$\begin{aligned}
\ddot{a}_x^{gen} &= \mathbb{E}[Y] \qquad (63)\\
&= \sum_{k=0}^{\infty} r_{x+k}\, {}_kp_x v^k \qquad (64)\\
&= (\sum_{k=0}^{0} + \sum_{k=1}^{\infty}) r_{x+k k}p_x v^k \qquad (65)\\
&= r_x + \sum_{k=1}^{\infty} {}_kp_x v^k \qquad (66)\\
&= r_x + p_x \times v \times \ddot{a}_{x+1}^{gen} \qquad (67)
\end{aligned}$$

Hence

$$\ddot{a}_x^{gen} = r_x + p_x \times v \times \ddot{a}_{x+1}^{gen} \tag{68}$$

with boundary condition $\ddot{a}_{\omega+1}^{gen} = 0$

Example

$r_{x+k} = (1 + \alpha)^k$

$$
\begin{aligned}
\ddot{a}_x^{gen} &= \sum_{k=0}^{\infty}(1+\alpha)^k v^k {}_kp_x && (69)\\
&= \sum_{k=0}^{\infty}(\frac{1+\alpha}{1+i})^k {}_kp_x && (70)\\
&= \frac{\tilde{N}_x}{\tilde{D}_x} && (71)
\end{aligned}
$$

$$
\frac{1+\alpha}{1+i} = \frac{1}{1+\tilde{i}} \qquad (72)
$$

Using Taylor approximation $\tilde{i} \approx i - \alpha$

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