

Chapter 1

Application of the Markov model to Life Insurance

1.1 Traditional Rating of Life Contracts

Before starting with the Markov model, I would like to summarise how traditional calculations using commutation functions are performed. Usually one starts with the probabilities of death and then calculates a decrement table starting with, say, 100000 persons at age 20.

After that one, has to calculate the different commutation functions, which I assume everybody knows by heart. These numbers depend on the persons alive and on the technical interest rate i . Only when you have done this it is (in the classical framework) possible to calculate the necessary premiums. In the following we will look a little bit closer at the calculation of a single premium for an annuity. To do this we need the following commutation functions:

$$D_x = v \times l_x \text{ where } l_x \text{ denotes the number of persons alive at age } x.$$
$$C_x = v \times (l_{x+1} - l_x)$$

Having this formalism it is well known that

$$\ddot{a}_x = \frac{N_x}{D_x}$$

From this example is easily seen that almost all premiums can be calculated by summation and multiplication of commutation functions. Such an approach has its advantages in an environment where calculations have to be performed by hand, or where computers are expensive. Calculation becomes messy if benefits are considered with guarantees or with refunds.

The Markov model here presented offers rating of life contracts without using commutation functions. It starts with calculation of the reserves and uses the involved probabilities directly. In order to see such a calculation let's review the above-mentioned example: We will use ${}_np_x$ to denote the probability of a person aged exactly x surviving for n years.

$$\begin{aligned}\ddot{a}_x &= \sum_{j=0}^{\infty} {}_jp_x \times v^j \\ &= 1 + p_x \times \ddot{a}_{x+1}\end{aligned}$$

The above formula gives us a recursion for the mathematical reserves of the contract. Hence one can calculate the necessary single premiums just by recursion. In order to do this, we need an initial condition, which is in our case $V_\omega = 0$.

The interpretation of the formula is easy: The necessary reserve at age x consists of two parts:

1. The annuity payment, and
2. The necessary reserve at age $x+1$. (These reserves must naturally be discounted.)

It should be pointed out that the calculation does not need any of the commutation functions; only p_x and the discount factor v are used. As a consequence this method does not produce the overheads of traditional methods.

In the following paragraphs the discrete time, discrete state Markov model is introduced and solutions of some concrete problems are offered.

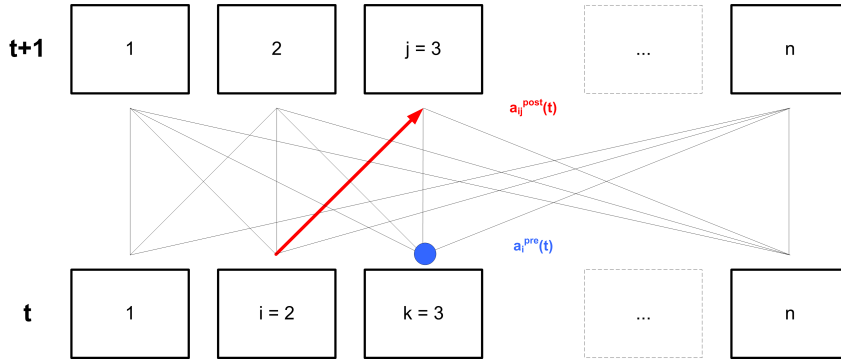
At this point, it is necessary to stress the fact that the following frame work can be used, with some modifications, in an environment with stochastic interest. But as we are limited in space and time we have to restrict ourselves to deterministic constant discount rates.

1.2 Life Insurance considered as Random Cash flows

The starting point of the Markov model is a set of states, which correspond to the different possible conditions of the insured persons. In life insurance the set of states usually consists of alive, dead. The set of states will be denoted by S .

The second point which originates from the life contract has to do with the so-called contractual functions which depend on the states and the time. Hence the structure of a generalised life contract can be thought of:

Contractual situation between time t and time $t + 1$



From the above diagram it can be seen that a finite number of states is considered, and that for

each transition $i \rightarrow j$ two different sums are paid, namely $a_{ij}^{\text{Post}}(t)$ at the end of the considered time interval and $a_i^{\text{Pre}}(t)$ at the beginning of it. It is clear that the value of the payment stream $a_{ij}^{\text{Post}}(t)$ has to be discounted by v in order to be compatible with $a_i^{\text{Pre}}(t)$. Probably it is worth remarking that the use of the two payment streams $a_i^{\text{Pre}}(t)$ and $a_{ij}^{\text{Post}}(t)$ eases the solution of things like payments during the year and the distinction between lump

sums (generally payable at the end of the period) and annuities (at the beginning). Finally it must be said that premiums payable to the insurer can (not must (!)) be considered as benefits with the opposite sign.

Until now we have defined the sums which are payable if a certain insured event occurs. Now there has to be a probability law in order to rate the different transitions. In the following we denote by $p_{ij}(t, t+1)$ the probability of transition at time t from state $i \rightarrow j$. Hence in the language of the above diagram there is one transition probability assigned to each line between two states.

So summarising a Markov life insurance model consists of the following:

S	A finite state space (set).
$((p_{ij}(t))_{(i,j) \in S^2})_{t \in (1,2,\dots,\omega)}$	The transition probabilities describing the Markov chain X_t on S .
$((a_i^{\text{Pre}}(t))_{i \in S})_{t \in (1,2,\dots,\omega)}$	The prenumerando benefits relating, paying at the beginning of the corresponding period.
$((a_{ij}^{\text{Post}}(t))_{(i,j) \in S^2})_{t \in (1,2,\dots,\omega)}$	The postnumerando benefits relating, paying at the end of the corresponding period, if a transition $i \rightarrow j$ happens.
$((v_i(t))_{i \in S})_{t \in (1,2,\dots,\omega)}$	The yearly discount rate from $[t, t+1[$. We have $v_t = \sum_{j \in S} I_j(t) v_i(t)$.

1.3 Reserves, Recursion and Premiums

One of the most important quantities in actuarial science is the prospective reserve, as the insurer must have this amount of money for each policy. Therefore the concept of the prospective reserve is known to all actuaries. It is defined to be the present value of the future cash flow A given the information at present. Formally we write

$$V_j^+(t, A) := \mathbb{E}[V(t, A \times \chi_{[t, \infty)}) \mid X_t = j],$$

(where j denotes the state at time t). This notation tells us, that the reserve depends heavily on the state of the policy.

In the context of the above we have

$$\begin{aligned} \Delta A(t) &= \sum_{j \in S} I_j(t) \times a_i^{\text{Pre}}(t) + \sum_{(i,j) \in S \times S} \Delta N_{ij}(t) \times a_{ij}^{\text{Pre}}(t), \\ A(t) &= \sum_{k \leq t} \Delta A(k), \\ \Delta V(t, A) &= v(t) \Delta A(t), \\ &= v(t) \left[\sum_{j \in S} I_j(t) \times a_i^{\text{Pre}}(t) + \sum_{(i,j) \in S \times S} \Delta N_{ij}(t) \times a_{ij}^{\text{Pre}}(t) \right], \\ v(t) &= \prod_{\tau \leq t} \left[\sum_{j \in S} I_j(\tau) \times v_j(\tau) \right]. \end{aligned}$$

The direct calculation of the necessary reserves for the different states is not too easy if you consider a general time continuous Markov model. An advantage of this model is the existence of a powerful backwards recursion. The following formula (Thiele difference equation) allows the recursive calculation of the necessary reserves and hence of the necessary single premiums:

$$V_i^+(t) = a_i^{\text{Pre}}(t) + \sum_{j \in S} v_i(t) p_{ij}(t) \{a_{ij}^{\text{Post}}(t) + V_j^+(t+1)\}. \quad (1.1)$$

The interpretation of the formula is almost the same as in the trivial example at the beginning. In principle the present reserve consists of payments due to the different possible transitions and the discounted values of the future necessary reserves. It can be seen that the above recursion uses only the different benefits, the probabilities and the discount factor. In order to calculate the reserve for a certain age one

has to do a backwards recursion starting at the expiration date of the policy. For annuities this is usually the age ω when everybody has died. Starting the recursion it is necessary to have boundary conditions, which depend on the payment stream at the expiration date. Usually the boundary conditions are taken to be zero for all reserves. It should be pointed out that one has to do this recursion for the reserves of all states simultaneously.

After the calculation of the different reserves one can naturally determine the corresponding necessary single premiums by the principle of equivalence.

We want to end this section with a short proof of the above mentioned Thiele recursion:

We know that $A(t) = \sum_{k \leq t} \Delta A(k)$ and also that

$$\Delta V(t, A) = v(t) \left[\sum_{j \in S} I_j(t) \times a_i^{\text{Pre}}(t) + \sum_{(i,j) \in S \times S} \Delta N_{ij}(t) \times a_{ij}^{\text{Pre}}(t) \right].$$

Hence we have

$$\begin{aligned} V_i^+(t) &= \frac{1}{v(t)} \mathbb{E} \left[\sum_{\tau=t}^{\infty} v(\tau) \times \Delta A(\tau) \mid X_t = i \right] \\ &= \frac{1}{v(t)} \mathbb{E} \left[\sum_{j \in S} I_j(t+1) \times \sum_{\tau=t}^{\infty} v(\tau) \times \Delta A(\tau) \mid X_t = i \right], \end{aligned}$$

remarking that $\sum_{j \in S} I_j(t+1) = 1$. If we now consider all the terms in $\Delta A(t)$ for a given $I_j(t+1)$ for $j \in S$, it becomes obvious that the Markov chain changes from $i \rightarrow j$ and in consequence only $N_{ik}(t)$ increases by one for $k = j$. If we furthermore use the projection property and the linearity of the conditional expected value and the fact that $\mathbb{E}[I_j(t+1) \mid X_t = i] = p_{ij}(t, t+1)$, together with the Markov property, we get the formula if we split $V_i^+(t)$ as follows:

$$\begin{aligned} V_i^+(t) &= \frac{1}{v(t)} \mathbb{E} \left[\sum_{\tau=t}^{\infty} v(\tau) \times \Delta A(\tau) \mid X_t = i \right] \\ &= \frac{1}{v(t)} \mathbb{E} \left[\left\{ \sum_{\tau=t}^t + \sum_{\tau=t+1}^{\infty} \right\} v(\tau) \times \Delta A(\tau) \mid X_t = i \right]. \end{aligned}$$

Doing this decomposition we get for the first part:

$$\text{Part}_1 = a_i^{Pre}(t) + \sum_{j \in S} v_i(t) p_{ij}(t) a_{ij}^{Post}(t),$$

and for the second:

$$\text{Part}_2 = \sum_{j \in S} v_i(t) p_{ij}(t) V_j^+(t+1).$$

Adding the two parts together we get the desired result:

$$V_i^+(t) = a_i^{Pre}(t) + \sum_{j \in S} v_i(t) p_{ij}(t) \{a_{ij}^{Post}(t) + V_j^+(t+1)\}.$$

More concretely we have

$$\begin{aligned} V_i^+(t) &= \frac{1}{v(t)} \mathbb{E} \left[\sum_{j \in S} I_j(t+1) \times \sum_{\tau=t}^{\infty} v(\tau) \times \Delta A(\tau) \mid X_t = i \right] \\ &= a_i^{Pre}(t) + \sum_{j \in S} \mathbb{E} \left[I_j(t+1) \times \sum_{\tau=t}^{\infty} \frac{v(\tau)}{v(t)} \times \Delta A(\tau) \mid X_t = i \right] \\ &= a_i^{Pre}(t) + \sum_{j \in S} \mathbb{E} \left[I_j(t+1) v_i(t) \left\{ a_{ij}^{Post} + \right. \right. \\ &\quad \left. \left. + \mathbb{E} \left[\sum_{\tau=t+1}^{\infty} \frac{v(\tau)}{v(t+1)} \times \Delta A(\tau) \mid X_t = i, X_{t+1} = j \right] \right\} \mid X_t = i \right] \\ &= a_i^{Pre}(t) + \sum_{j \in S} v_i(t) p_{ij}(t) \{a_{ij}^{Post}(t) + V_j^+(t+1)\}. \end{aligned}$$

We remark that this section can only be a short introduction to this topic and we refer to [?] for a more extensive discussion.

List of Figures