Jump Diffusion Models with Finite or Infinite Activity

I. Introduction

Many studies in finance have assumed that the underlying asset returns were best described by a continuous, diffusion process such as the Geometric Brownian Motion. [1] Brownian motion has a Gaussian pdf, therefore this assumption cannot generate the fat tails observed in the empirical distributions of returns. For that reason, the literature has advocated the use of jump diffusion processes to explain the behaviour of asset returns. Traditionally, models with jumps in finance have relied on Poisson processes, as in Merton (1976), [2] Ball and Torous (1983) [3] and Bates (1991).[4] These jump-diffusion models allow for a finite number of jumps in a finite time interval, relying on the idea that the Brownian-driven continuous part of the model captures normal asset price variations while the Poisson-driven jump part of the model captures large market movements in response to unexpected information.

More recently, financial models have been proposed that allow for infinitely many jumps in finite time interval. These models can capture both small and frequent jumps, as well as large and infrequent ones. **In section 2 and 3 we will discuss:** Finite Jump Diffusion (JD) processes**:** like the Merton JD and the Kou Process; and Infinite Jump Diffusion processes**:** like the VG model**. In Section 4** we will discuss two statistical tests that are used to discriminate between finite and infinite activity of jumps.

II. Jump Diffusion Models with Finite Activity

These are often used to model rare events such as defaults, credit events, crashes or drawdowns by adding an uncorrelated compound Poisson process to the diffusion of a continuous process (e.g. Arithmetic Brownian Motion). Introducing (random) jumps to a continuous Brownian motion, brings benefits to what concerns a better reproduction of the volatility smile effect and the asymmetric leptokurtic features of returns distributions.[5]

We denote the dynamics of the log price as: $X(t) = \log S(t)$, where $\mathrm{S}(t)$ follows a GBM, while $\mathrm{X}(t)$ evolves according to an ABM. $\mathrm{X}(t)$ is the stochastic variable of interest in this report.[6]

A Jump Diffusion process is a stochastic process $\mathrm{X}(t)$ with independent and stationary increments which is obtained as the sum of an Arithmetic Brownian Motion and an independent compound Poisson process.[7] The compound Poisson process is often used to construct a more flexible process by assigning a specific distribution to the jumps. i.e.

$$X(t) = \overbrace{\mu + \sigma W(t)}^{\text{ABM}} + \overbrace{\sum_{j=1}^{N(t)} Z(j)}^{\text{Compound Poisson}} \tag{1}$$

where, $\mu \in R$, $\sigma > 0$; $W(t)$ is a Brownian Motion; $N(t)$ is a Poisson process and $Z(j)$ is a sequence of i.i.d random variables which are assumed independent of both the Brownian motion and the Poisson process.

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1 See for instance: Chernov, Gallant, Ghysels, & Tauchen, 2003; Christoffersen, Jacobs, Ornthanalai, & Wang, 2008; Duan, Ritchken, & Sun, 2006; Eraker, Johannes, & Polson, 2003, among others.
2 Robert C.Merton (1976): "Option pricing when underlying stock returns are discontinuous";
3 Ball CA, Torous WN (1983): "A Simplified Jump Process for Common Stock Returns";
4 David S. Bates, "Jumps and Stochastic volatility: Exchange Rate Processes Implicity in Deutsche Mark Options"
5 Daniel Wetterau and Joerg Kienitz (2012): "Financial Modelling: Theory, Implementation and Practice with MATLAB Source".
6 Laura Ballotta and Gianluca Fusai (2018): "Tools from Stochastic Analysis for Mathematical Finance: A Gentle Introduction".
7 We can think of the compound Poisson process as follows: i) At time t a jump occurs; ii) When this happens, the Poisson process increases by 1 unit; iii) At the same time, a random draw $Z(j)$ is taken from a given distribution to quantify the jump size and it is summed up to the value of the process at the previous time point.

Common choices for the distribution of jumps in financial applications are the Gaussian distribution (Merton Jump Diffusion Model)[8] and the exponential distribution (Kou Process).[9] In the table below we will summarise the rationale and properties of these two models:

	Merton Jump Diffusion Model	**The Kou Process**
Rationale	i) Allows the underlying dynamics to have random jumps and to reproduce more realistic tails behaviour for related log-returns. [10] ii) Jumps can have a large impact on the price of an option, especially when near expiry. Merton's jump-diffusion model with the log-normal distribution correctly produces the volatility smile phenomenon. [11] iii) However, the use of the log-double exponential distribution instead of the log-normal distribution allows the introduction of asymmetric leptokurtic features.	i) This model explains two empirical phenomena—the asymmetric leptokurtic feature, and the volatility smile. [12] The logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponentially distributed. ii) The double exponential distribution is memoryless. This special property explains why the closed-form solutions for various option-pricing problems, are feasible under the double exponential jump-diffusion model while it seems impossible for many other models, including the normal jump-diffusion model (Merton 1976).
Jump-diffusion process and the distribution of $\boldsymbol{X(t)}$	Merton Jump Diffusion Model assumes that jump sizes follow a Gaussian distribution. The process is as follows: $$X(t) = \mu + \sigma W(t) + \sum_{j=1}^{N(t)} Z(j) \quad \boldsymbol{Z \sim N(\mu_z, \sigma_z^2)}$$ where μ_z is the mean of the jump sizes; σ_z^2 is the volatility of the jump sizes. In the Merton model the density function of $X(t)$ can be obtained as the weighted average of an infinite number of Gaussian densities with different means and variances; the weights are computed according to the Poisson distribution: $$f_X(x) = \sum_{j=0}^{\infty} w_j\, n(x, \mu_j, \sigma_j)$$ where $w_j = \frac{e^{-\lambda t}(\lambda t)^j}{j!}$; $\mu_j = \mu + j\mu_z$; $\sigma_j^2 = \sigma^2 + j\sigma_z^2$ and n is the Gaussian PDF.	The density function of exponentially distributed jump sizes is given by: $$f_z(z) = p \cdot \eta_1 e^{-\eta_1 z} 1_{[z \geq 0]} + (1-p) \cdot \eta_2 e^{\eta_2 z} 1_{[z<0]}$$ where: • p: probability of an upward jump. • η_1: parameter of the exponential distribution controlling the upward jumps; therefore, the upward jumps have mean $1/\eta_1$. • η_2: parameter of the exponential distribution controlling the downward jumps; therefore, the upward jumps have mean $1/\eta_2$. The distribution function of $X(t)$ is not available in closed form.
Properties	i) Finite jump diffusion process.[13] ii) Numerically inverting the characteristic function of MJD model we obtain the first two moments of the distribution as follows: $$E[X(t)] = (\mu + \lambda\mu_z)t$$ $$V[X(t)] = \left(\sigma^2 + \lambda(\mu_z^2 + \sigma_z^2)\right)t$$ iii) μ_z: mean of the jump sizes; it controls the sign of the skewness index.[14]	i) Finite jump diffusion process. ii) The characteristic function is known in closed form and it is used to obtain the moments of the log process. In particular, we have: $$E[X(t)] = \left(\mu + \lambda(p/\eta_1 - (1-p)/\eta_2)\right)t$$ $$V[X(t)] = \left(\sigma^2 + 2\lambda(p/\eta_1^2 - (1-p)/\eta_2^2)\right)t$$ iii) The downside of moving to the log-double-exponential distribution is that it uses more parameters than the log-normal distribution. Moreover, many theoretical results are

[8]Robert C.Merton (1976): "Option pricing when underlying stock returns are discontinuous";
[9] S. G. Kou (2002): "A Jump-Diffusion Model for Option Pricing".
[10] Empirically, stock returns tend to have fat tails, which is inconsistent with the Black-Scholes model's assumptions. Merton's Jump-Diffusion Model superimposes a jump component on a diffusion component.
[11] S.Salmi and J.Toivanen: "An Iterative Method for Pricing American Options under Jump-Diffusion Models"
[12] S. G. Kou (2002): "A Jump-Diffusion Model for Option Pricing".
[13] By the definition of Poisson distribution, there cannot be more than one jump per time period. Hence, the Poisson process can only generate a finite number of jumps over a finite time horizon.
[14] Hence, the Merton jump diffusion has a distribution which is skewed to the left if μ_z< 0 and skewed to the right if $\mu_z > 0$

	σ_z: volatility of the jump sizes. The larger this volatility, the larger the excess kurtosis of the process.	only valid for the log-normal distribution.
Simulation	**Step 1**: Simulate the continuous part of the JD diffusion process, i.e., the ABM, on the given time partition. **Step 2:** Simulate the number of jumps occurring between t_j and t_{j+1} as a Poisson distribution with rate of arrival λ i.e., $N \sim Poisson\left(\lambda(t_j, t_{j+1})\right)$. **Step 3:** Generate $Z \sim N(0,1)$; set $J = \mu_z N + \sigma_z \sqrt{N} Z$. **Step 4:** Sum the ABM and J.	**Step 1**: Simulate the continuous part of the JD diffusion process, i.e., the ABM. **Step 2:** Simulate the number of jumps occurring between t_j and t_{j+1} as a Poisson distribution with rate of arrival λ i.e., $N \sim Poisson\left(\lambda(t_j, t_{j+1})\right)$. **Step 3:** Generate $Z \sim \varepsilon(p, \eta_1, \eta_2)$; set $J = \mu_z N + \sigma_z \sqrt{N} Z$. **Step 4:** Sum the ABM and J.

III. Jump Diffusion Models with Infinite Activity: The Variance Gamma Model

The purpose in considering the Variance Gamma model was to provide a model for stock market returns which is practical and empirically more reasonable than the classic approach using Brownian motion.[15] The model allows for fat tails (i.e. the distribution has skewness and excess kurtosis) and retains the assumptions of independent stationary increments. The VG process is obtained by evaluating Brownian motion (with constant drift and volatility) at a random time change given by a gamma distribution. In contrast to traditional Brownian motion, the VG process is a pure jump process with an infinite arrival rate of jumps, but unlike Brownian motion (that also has infinite motion), the process has finite variation and can be written as the difference of two increasing processes, each giving separately the market up and down moves.[16]

There are two ways of expressing the Variance Gamma model: i) as the difference of two Gamma processes; ii) as a time-changed Brownian motion; In this section we will focus on the second approach. A Time Changed Brownian motion is a process of the form: $X(t) = \theta G(t) + \sigma W\left(G(t)\right)$, where $W(t)$ is a Brownian Motion and $G(t)$ is a positive increasing stochastic process. [17]

Using a time changed Brownian motion is mainly motivated by: i) Empirical evidence shows that stock log-returns are Gaussian but only under trade time, rather than standard calendar time; ii) the time change construction recognizes that stock prices are largely driven by news, and the time between one piece of news and the next is random as is its impact.[18]

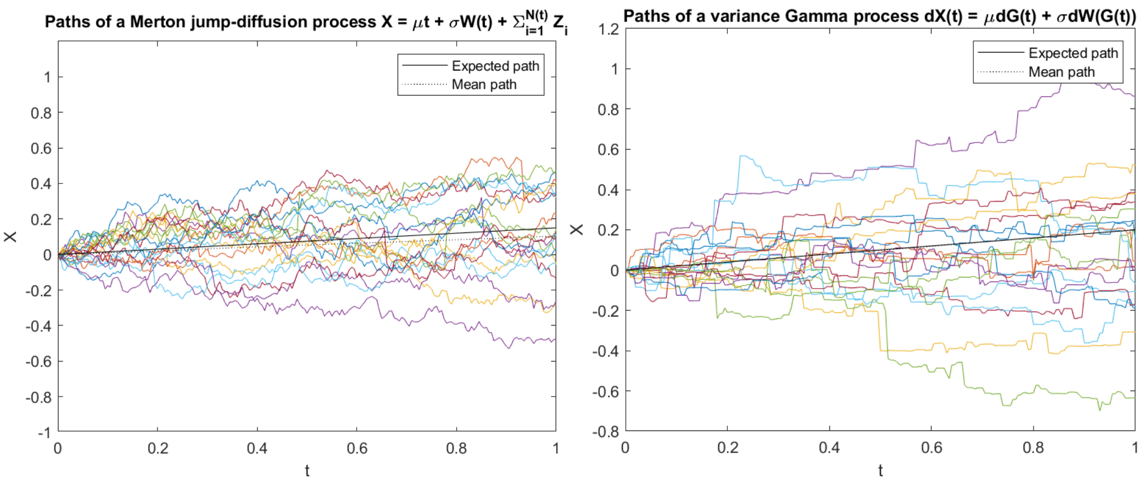
Properties of the VG process

- The parameters of the Gamma process are chosen so that $E[G(t)] = t$, i.e., the process chosen as random clock is an unbiased representation of calendar time. Let us assume that $G(t)$ is a Gamma process with parameters $\alpha = \lambda = k^{-1}$, for any positive constant k, so that $E[G(t)] = t$ and $Var[G(t)] = kt$. Assume $G(t)$ is independent of $W(t)$. Then $X(t)$ is a VG process.
- The VG process has infinite activity, i.e. it is characterized by an infinite number of jumps in any finite time period.
- The first moment of the VG process is $E[X(t)] = \theta t$, whereas the second moment is $V[X(t)] = (\sigma^2 + \sigma^2 k)t$

The simulated paths of the Variance Gamma process and the Merton JD model are presented in Figure 1 below. We observe that both MJD and time-changed Brownian motions are characterized by trajectories formed by many little movements with large, rare movements interspersed; however, whilst in the case of MJD process the small increments have Gaussian distribution, in the case of time-changed Brownian motion this distribution has skewness and excess kurtosis.

[15]Dilip B. Madan and Eugene Seneta (1990): "The Variance Gamma (V.G.) Model for Share Market Returns".
[16] Carr et al. (1998): "The Variance Gamma Process and Option Pricing"
[17] Laura Ballotta and Gianluca Fusai (2018): "Tools from Stochastic Analysis for Mathematical Finance: A Gentle Introduction".
[18] Ibid.

Figure 1: Simulated paths of Variance Gamma and Merton JD model



Source: Matlab source code provided as part of the "Numerical Methods" Module, UCL (2021).

IV. Two statistical tests to discriminate between finite or infinite activity of jumps

In this section we will present two statistical procedures to discriminate between the finite and infinite activity of jumps, while allowing in both cases for the presence of a continuous component in the model. When implemented on high-frequency stock returns, both tests point toward the presence of infinite-activity jumps in the data. This section is based on the findings of Aït-Sahalia and Jacod (2011): "Testing whether jumps have finite or infinite activity".

Model Setup

The authors consider a univariate process $X(t)$ (i.e. the log of an asset price) which is observed on a fixed time interval $[0, T]$, at discretely and regularly spaced time intervals $i\Delta_n$ (typically measured in seconds in a high-frequency context).

Assuming that the observed path has jumps, **the aim is to test whether there is a 'finite activity' or an 'infinite activity' for the jump component of $X(t)$.** This is done by providing asymptotic testing procedures, as the time lag Δ_n between successive observations goes to 0. To do that, first it is assumed that the process $X(t)$ is an Itô semimartingale. One may then write $X(t)$ as:

$$X(t) = X(0) + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW_s + \underbrace{\int_0^t \int x 1_{|x|<1} (\mu - \nu)(ds, dx)}_{\text{small jumps}} + \underbrace{\int_0^t \int x 1_{|x|>1}\, \mu(ds, dx)}_{\text{large jumps}}$$

where, $W(t)$ is a standard Wiener process. Ultimately, the question about the finite or infinite degree of activity of jumps is a question about the behaviour of the compensator ν near 0. There are always a finite number of big jumps. The question is whether there are a finite or infinite number of small jumps. This is controlled by the behaviour of ν near 0.

To keep the solution as nonparametric as possible, and to avoid specifying the structure of the volatility or of the jumps, the authors use several assumptions.[19]

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Assumption 1: The processes b_t, σ_t, and $\int (x^2 \wedge 1) F_t(dx)$ *are* pre-locally bounded.
Assumption 2: The drift process b_t, is càdlàg, and the volatility process σ_t is an Itô semimartingale satisfying Assumption 1.
Assumption 3: The Lévy measure $F_t = F_t(\omega, dx)$, is of the form that satisfies equations (5-8) in the paper.
Assumption 4: The previous assumption is designed for the test for which the null is "finite activity." For the symmetric test, the assumption needed is stronger.[19] We have Assumption 3 with $\gamma t = \beta$ [a constant in (0, 2)], and $\delta t = 0$.
Assumption 5 and Assumption 6: specifies locally boundedness assumptions, such that the Lévy measure F_t is of the form given in Equations (21-23) of the paper.

Two tests: the finite activity null hypothesis vs. the infinite activity null hypothesis

The goal is to employ two testing procedures, one where the null hypothesis is finite jump activity and the symmetric problem where the null hypothesis is infinite jump activity. They can be noted as:

- $\boldsymbol{Test\ 1}$: $H_0: \Omega_T^f$ vs $H_0: \Omega_T^i$
- $\boldsymbol{Test\ 2}$: $H_0: \Omega_T^i$ vs $H_0: \Omega_T^f$ (5)

where $\Omega_T^f = \{\omega\colon t \to X_\omega(t)$ has finite jumps in $[0, T]\}$ and $\Omega_T^i = \{\omega\colon t \to X_\omega(t)$ has infinite jumps in $[0, T]\}$.

	Test 1: Finite activity null hypothesis	Test 2: Infinite activity null hypothesis
Definition of the test statistic	Under the null hypothesis of finite-activity jumps, the test statistic is estimated as a *truncated power variation in the following way:* $$\boldsymbol{S_n} = \frac{\boldsymbol{B}_{(p,u_n,k\Delta_n)_T}}{\boldsymbol{B}_{(p,u_n,\Delta_n)_T}}$$ where, p, is the power order and it is a positive integer; k is a positive integer; u_n is a sequence of positive numbers, which will serve as thresholds for truncating the increments when necessary and will go to 0 as the sampling frequency increase and for any p >0 the truncated power variation $B_{(p,u_n,\Delta_n)_t}$, is given by Equation (32) in the paper. [20]	We cannot simply use the same test statistic S_n as before for the second testing problem. [21] In order to design a test statistic which is model-free under the null of infinite activity, we choose three real values $\gamma > 1$ and $p' > p > 2$, and then define a family of test statistics as follows: $$\boldsymbol{S'_n} = \frac{\boldsymbol{B}_{(p',\gamma u_n,\Delta_n)_T}\boldsymbol{B}_{(p,u_n,\Delta_n)_T}}{\boldsymbol{B}_{(p',u_n,\Delta_n)_T}\boldsymbol{B}_{(p,\gamma u_n,\Delta_n)_T}}$$ Unlike the previous statistic S_n, we now use different powers $p\ and\ p'$ and different levels of truncation u_n and γu_n.
Limiting behaviour of the test statistic	The limiting behaviour of the statistic S_n, in terms of convergence in probability is given by **Theorem 1,** which shows that the statistic S_n behaves differently depending upon whether the number of jumps is finite or not: i) **if the number of jumps is finite,** then for $\rho < 1/2$, we have $S_n \to k^{p/2-1}$. Intuitively, if the number of jumps is finite, then at some point along the asymptotic the truncation eliminates jumps, and the residual behaviour of the truncated power variation is driven by the continuous part of the semimartingale. ii) **If the jumps have infinite** activity for $\rho < \rho_1(p)$ we have $S_n \to 1$. The asymptotic behaviour of the truncated power variation is driven by the small jumps, whether the Brownian motion is present or not, and the truncation rate matters.	The limiting behaviour of the statistic S'_n, in terms of convergence in probability is given by **Theorem 4,** which shows that: i) **if the number of jumps is finite**, the test statistic $S'_n \to 1$ and to a value different from 1 under the null hypothesis of infinite activity. Indeed, for finite jump activity, the behaviour of each one of the four truncated power variations in S'_n is driven by the continuous part of the semimartingale. ii) **if jumps have infinite activity** $S'_n \to \gamma^{(p'-p)}$, then the small jumps are the ones that matter and the truncation level becomes material, producing four terms that all tend to zero but at different orders.
Critical Region	Theorem 1 implies that for Test 1 a reasonable critical region is $C_n = S_n < c_n$ for $c_n\ in\ the\ interval\ (1, k^{p/2-1})$. The asymptotic level and power are respectively 0 and 1 if the model satisfies *Assumption 5.*	Theorem 4 implies that a reasonable critical region is $C_n = S'_n < c_n$ and if $c_n = c$,, is in the interval $(1, \gamma^{p'-p})$, the asymptotic level and power are, respectively, 0 and 1.

[20] This test computes the truncated power variations at two different frequencies in the numerator and denominator, but otherwise uses the same power p and truncation level u_n.

[21] The reason is that, while the distribution of S_n is model-free under the null hypothesis of the first testing problem, it is no longer model-free under the null hypothesis of the second testing problem: its distribution when jumps have infinite activity depends upon the degree of jump activity, β. While it is possible to estimate β consistently [see Aït-Sahalia and Jacod (2009a)], it would be preferable to construct a statistic whose implementation does not require a preliminary estimate of the degree of jump activity.

Implementation	i) To implement this test in practice, we simply need to compute truncated power variations for various powers and at two different sampling frequencies ($\Delta_n$and $\mathrm{k}\Delta_n$). ii) we do not need to estimate any aspect of the dynamics of the X, such as its drift or volatility processes or its jump measure. In that sense, the test statistic is non-parametric, or model-free.	i) As was the case for Test 1, the asymptotic distribution of S'_n under the null is again model-free ii) Implementing Test 2 simply requires the computation of truncated power variations B of order $p, 2p\ , p + p'\ and\ 2p$ ' at truncation levels u_n and γu_n iii) No other aspects of the dynamics of the X process, such as its degree of jump activity β for instance, need to be estimated.

<u>Discussion of findings based on simulations and empirical results</u>

Simulation results[22] based on the test statistic S_n and S'_n for various levels of truncation α[23] show that:

- For both tests, higher values of the jump scale parameter make the resulting model approximate the behaviour of the infinite-activity jump component more closely, while for lower scale parameters and low values of α, the behaviour is determined by the continuous component.
  - It is observed that for very small values of α, the behaviour of $\mathrm{S_n}\ and\ \mathrm{S'_n}$tracks that of the Brownian component <u>since the truncation effectively eliminates the finite-activity jumps.</u>
  - As α increases, the curves then start reversing course and tend to 1 instead, <u>the limit driven by the infinite-activity jump process</u>.

Implementing the two test statistics to real data, shows the following:[24]

- Based on Theorem 1, $S_n \to k^{p/2-1}$ under the null of finite activity, and $S_n \to 1$, under the alternative of infinite activity. As shown in Figure 1, S_n is close to 1, which l**eads us to reject the null hypothesis of finite activity.** A consistent result is obtained when using the test statistic S'_n which leads us to not reject the null hypothesis of infinite jump activity.[25] To summarize, the answer from both tests appears indicative of infinite jump activity in those data.
- Finally, the convergence of S_n to 1 as the sampling interval decreases, indicates that the null hypothesis of finite activity is rejected when high-frequency data (of the order of seconds) are used. On the other hand, using longer sampling intervals (of the order of minutes) makes it impossible to reject the null hypothesis of finite activity using S_n. This is consistent with the results obtained for the second test statistic S'_n.

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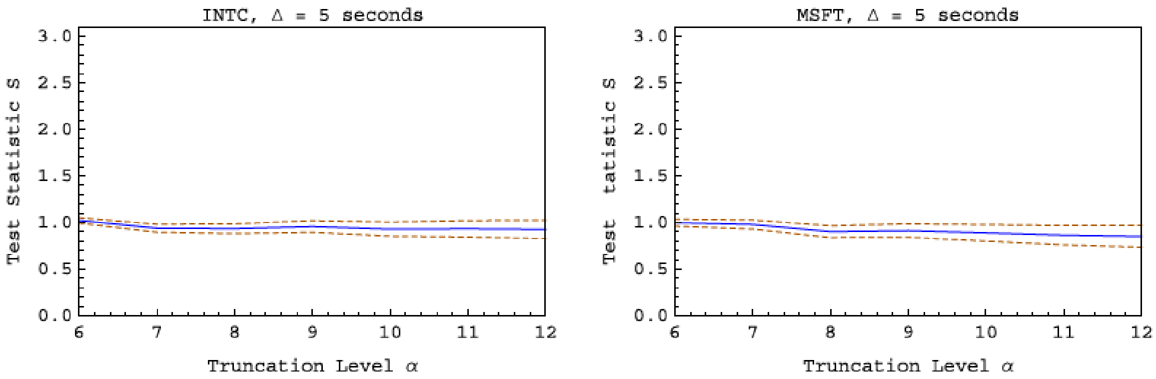
[22] The data generating process for the simulations is the stochastic volatility model $dX(t) = \sigma_t dW_t + \theta dY_t$ with $\sigma_t = v_t^{1/2}$ and $dv_t = \chi(\eta - v_t)dt + \xi v_t^{1/2} dB_t + dJ_t$. J_t is a compound Poisson jump process with jumps that are uniformly distributed and the jump process Y_t is either a β-stable process with β= 1, that is, a Cauchy process (which has infinite activity) or a compound Poisson process (which has finite activity).

[23] α - denotes the threshold expressed as the number of (normalized) standard deviations of the Brownian part.

[24] Empirical data consist of all transactions recorded during the year 2006 on two of the most actively traded stocks, Intel (INTC) and Microsoft (MSFT).

[25] Based on Theorem 4 under the null of infinite activity $S_n' \to \gamma^{(p'-p)}$ and $S_n' \to 1$ under the alternative of finite activity. activity. The authors find that S_n' is close to the predicted value $\gamma^{(p'-p)}$ which leads us to not reject the null hypothesis of infinite activity.

Figure 2: Test statistic Sn employed on empirical data



Source: Figure 11 in Aït-Sahalia and Jacod (2011): "Testing whether jumps have finite or infinite activity".

Interpretation of results and comparison with other studies

When implemented on high-frequency stock returns, both tests point toward the presence of infinitely active jumps in the data, while lower frequency data tend to be more compatible with finite jump activity. This makes sense intuitively because while finite jumps reflect news-related shocks; a series of small jumps for infinite activity models correspond to price moves that are significant- a common phenomenon in a high frequency trading context. [26] In practice, the tests presented in this section need a lot of data to be effective, i.e., a high sampling frequency and we also need to consider the implications of microstructure noise. This has the important consequence that the two statistics are no longer able to distinguish between the two hypotheses of finite or infinite activity when noise dominates

The empirical results in this paper are in line with the empirical results of a companion paper, Aït-Sahalia and Jacod (2009a), [27] which contains an extension to Itô semimartingales of the classical Blumenthal–Getoor index β This parameter β takes values between 0 and 2 and plays the role of a "degree of jump activity" for infinitely active jump processes. If the estimator of β is found to be "high" in its range [0, 2], as it is the case in the empirical findings of Aït-Sahalia and Jacod (2009a), it is a strong evidence against finite activity.

[26] IMF (2018): "Global Financial Stability Report, October 2018 A Decade After the Global Financial Crisis: Are We Safer?"
[27] Aït-Sahalia, Yacine, and Jean Jacod (2009a): "Estimating the Degree of Activity of Jumps in High Frequency Data." Annals of Statistics 37 (5A): 2202–44.