```
<< "~\\P-rec.m"
```

```
<< "~\\fastZeil.m"
```

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
(**************the first proof **************)
```

```
(*********an*********)
```

In[110]:=

```
a[n_] := Sum[(Binomial[n - 1, k] Binomial[n + k, k])^2 / (n (4 k^2 - 1)), {k, 0, n - 1}];
```

求和 二项式系数 二项式系数

In[10]:=

```
rec = Zb[(Binomial[n - 1, k] Binomial[n + k, k])^2 / (n (4 k^2 - 1)), {k, 0, n - 1}, n, 3]
```

二项式系数 二项式系数

If `-1 + n' is a natural number, then:

Out[10]=

$$\{n^3 (1+n) (5+2n) \mathrm{SUM}[n] - (1+n)(5+2n)(62+191n+152n^2+35n^3)\mathrm{SUM}[1+n] + (2+n)(1+2n)(88+224n+163n^2+35n^3)\mathrm{SUM}[2+n] - (2+n)(3+n)^3(1+2n)\mathrm{SUM}[3+n] == 0\}$$

In[88]:=

```
L = n^3 (1 + n) (5 + 2 n) - (1 + n) (5 + 2 n) (62 + 191 n + 152 n^2 + 35 n^3) N +
    (2 + n) (1 + 2 n) (88 + 224 n + 163 n^2 + 35 n^3) N^2 - (2 + n) (3 + n)^3 (1 + 2 n) N^3;
```

数值运算 数值运算 数值运

In[152]:=

```
rLogBound[L, n, N, {-1, 1, 9, 61, 587, 7575}, 1, 4]
```

数值运算

$$17+12\sqrt{2}-\frac{1}{n^2}-\frac{3(17+12\sqrt{2})(-264+37\sqrt{2})}{64n^2}-\frac{9(17+12\sqrt{2})}{2n} <= a\_\{n+1\}/a\_n <=$$
$$17+12\sqrt{2}+\frac{1}{n^2}-\frac{3(17+12\sqrt{2})(-264+37\sqrt{2})}{64n^2}-\frac{9(17+12\sqrt{2})}{2n} \quad \text{for } n>=5$$

a_n preserves the bounds for n>=605

the bounds hold for n>=607

Out[152]=

```
{True, 607}
```

In[107]:=

$$h1[n\_] := 17+12\sqrt{2}-\frac{9(17+12\sqrt{2})}{2n}-\frac{3(17+12\sqrt{2})(-264+37\sqrt{2})}{64n^2};$$

In[176]:=

$$v[n\_] := -\frac{153/2+54\sqrt{2}}{n}+\frac{675/4+7617\sqrt{2}\big/64}{n^2} \;//\; \mathrm{FullSimplify}$$

完全简化

In[169]:=

```
a0 = 17 + 12 √2 ;
N0 = 607;
m = 2;
k0 = 2;
```

In[177]:=

$$\text{Reduce}\left[n \geq 607 \,\&\&\, a0 + v[n] + 1/n\text{^}m < a0 \left(\frac{n}{n+1}\right)\text{^}k0\right]$$

约化

Out[177]=

$n \geq 607$

In[112]:=

$$\text{Reduce}\left[n \geq 607 \,\&\&\, a[607] \leq \left(17 + 12\sqrt{2}\right)\text{^}607\right]$$

约化

Out[112]=

$n \geq 607$

In[174]:=

$$\frac{v[n] + \frac{1}{n\text{^}m}}{a0\,(n+1)} - \frac{v[n-1] - \frac{1}{(n-1)\text{^}m}}{a0\,(n-1)} + \frac{\left(v[n-1] - \frac{1}{(n-1)\text{^}m}\right)\text{^}2}{a0\text{^}2\,(n-1)} - \frac{2\,k0\,\text{Log}[n]}{n\,(n\text{^}2-1)}$$

Out[174]=

$$-\frac{-\frac{1}{(-1+n)^2} - \frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,(-1+n)^2} - \frac{5\left(17+12\sqrt{2}\right)}{2\,(-1+n)}}{\left(17+12\sqrt{2}\right)(-1+n)} + \frac{\left(-\frac{1}{(-1+n)^2} - \frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,(-1+n)^2} - \frac{5\left(17+12\sqrt{2}\right)}{2\,(-1+n)}\right)^2}{\left(17+12\sqrt{2}\right)^2(-1+n)} + \frac{\frac{1}{n^2} - \frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,n^2} - \frac{5\left(17+12\sqrt{2}\right)}{2\,n}}{\left(17+12\sqrt{2}\right)(1+n)} - \frac{4\,\text{Log}[n]}{n\left(-1+n^2\right)}$$

In[175]:=

```
Reduce[n ≥ 607 && % < 0]
```

约化

Out[175]=

$n \geq 607$

```
(********bn**********)
```

In[154]:=

```
Zb[(3 k^2 + 3 k + 1) (Binomial[n - 1, k] Binomial[n + k, k]) ^2 / n^3, {k, 0, n - 1}, n, 3]
```

二项式系数　二项式系数

```
If `-1 + n' is a natural number, then:
```

Out[154]=

$$\left\{n^3\,(1+n)\,(5+2n)\left(11+12n+3n^2\right)\left(25+24n+6n^2\right)\text{SUM}[n] - (1+n)\,(5+2n)\left(3076+21646\,n+59512\,n^2+82777\,n^3+64134\,n^4+28137\,n^5+6552\,n^6+630\,n^7\right)\text{SUM}[1+n] + (2+n)\,(1+2n)\left(5072+30640\,n+73445\,n^2+93469\,n^3+68751\,n^4+29271\,n^5+6678\,n^6+630\,n^7\right)\text{SUM}[2+n] - (2+n)\,(3+n)^3\,(1+2n)\left(2+6n+3n^2\right)\left(7+12n+6n^2\right)\text{SUM}[3+n] == 0\right\}$$

In[155]:=

```
L2 = n^3 (1 + n) (5 + 2 n) (11 + 12 n + 3 n^2) (25 + 24 n + 6 n^2) - (1 + n) (5 + 2 n)
    (3076 + 21646 n + 59512 n^2 + 82777 n^3 + 64134 n^4 + 28137 n^5 + 6552 n^6 + 630 n^7) N +
   (2 + n) (1 + 2 n) (5072 + 30640 n + 73445 n^2 + 93469 n^3 + 68751 n^4 + 29271 n^5 + 6678 n^6 + 630 n^7)
    N^2 - (2 + n) (3 + n)^3 (1 + 2 n) (2 + 6 n + 3 n^2) (7 + 12 n + 6 n^2) N^3;
```

数值运算 数值运算 数值运算

In[156]:=

```
rLogBound[L2, n, N, {1, 8, 87, 1334, 25045}, 1, 4]
```

数值运算

$$17+12\sqrt{2}-\frac{1}{n^2}-\frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,n^2}-\frac{5\left(17+12\sqrt{2}\right)}{2\,n}\text{<=a\_\{n+1\}/a\_n<=}$$

$$17+12\sqrt{2}+\frac{1}{n^2}-\frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,n^2}-\frac{5\left(17+12\sqrt{2}\right)}{2\,n}\text{ for n>=3}$$

a_n preserves the bounds for n>=208

the bounds hold for n>=210

Out[156]=

```
{True, 210}
```

In[157]:=

$$\text{v[n\_] :=}-\frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,n^2}-\frac{5\left(17+12\sqrt{2}\right)}{2\,n};$$

In[162]:=

```
a0 = 17 + 12 √2;
N0 = 210;
m = 2;
k0 = 2;
```

In[180]:=

$$\text{Reduce}\left[n \geq 210\;\&\&\;a0+v[n]+1/n^{\wedge}m < a0\left(\frac{n}{n+1}\right)^{\wedge}k0\right]$$

约化

Out[180]=

```
n ≥ 210
```

In[179]:=

$$\frac{v[n]+\frac{1}{n^{\wedge}m}}{a0\,(n+1)}-\frac{v[n-1]-\frac{1}{(n-1)^{\wedge}m}}{a0\,(n-1)}+\frac{\left(v[n-1]-\frac{1}{(n-1)^{\wedge}m}\right)^{\wedge}2}{a0^{\wedge}2\,(n-1)}-\frac{2\,k0\,\text{Log}[n]}{n\,(n^{\wedge}2-1)};$$

In[168]:=

```
Reduce[n ≥ 210 && % < 0]
```

约化

Out[168]=

```
n ≥ 210
```

In[181]:=

```
(*******The analytic proof*******)
```

In[192]:=

```
v1[n_] := (35 n^3 - 152 n^2 + 191 n - 62)/(n^3);
v2[n_] := -((n - 2) (2 n - 1) (-88 + 224 n - 163 n^2 + 35 n^3))/(n^3 (n - 1) (2 n - 5));
v3[n_] := ((n - 3)^3 (n - 2) (2 n - 1))/((n - 1) n^3 (2 n - 5));
```

$$v1[n\_] := \frac{35 n^3 - 152 n^2 + 191 n - 62}{n^3};\quad v2[n\_] := -\frac{(n-2)(2n-1)\left(-88+224n-163n^2+35n^3\right)}{n^3 (n-1)(2n-5)};\quad v3[n\_] := \frac{(n-3)^3 (n-2)(2n-1)}{(n-1) n^3 (2n-5)};$$

In[202]:=

$$f1[n\_] := -\frac{\left(17+12\sqrt{2}\right)\left(-256 n^3+2304 n^2-8352 n+16245+444\sqrt{2}\, n-3108\sqrt{2}\right)}{32 (2n-3)^3};$$

In[190]:=

$$\text{Reduce}\left[5 \le n \le 8\ \&\&\ f1[n] < \frac{a[n]}{a[n-1]} < f1[n+1],\ \text{Integers}\right]$$

约化 整数域

Out[190]=

n == 5 || n == 6 || n == 7 || n == 8

In[200]:=

Reduce[n ≥ 9 && v1[n] > 0 && v2[n] < 0 && v3[n] > 0]

约化

Out[200]=

n ≥ 9

In[204]:=

m =.;

In[209]:=

v1[m + 1] + v2[m + 1] / f1[m] + v3[m + 1] / (f1[m] × f1[m + 1]) - f1[m + 1] // Together // Simplify

归并 化简

Out[209]=

$$\Big(3\Big(-73728\left(-83799+59212\sqrt{2}\right)+5\left(502133292723+366827211896\sqrt{2}\right)m+\left(3052263728061+1835372121892\sqrt{2}\right)m^2-\left(2279562847107+1503787352356\sqrt{2}\right)m^3+5\left(-287325208461+36350404796\sqrt{2}\right)m^4+4\left(929405738616+32491830311\sqrt{2}\right)m^5-32\left(68228120367+15627572572\sqrt{2}\right)m^6+896\left(-1004638368+1683587995\sqrt{2}\right)m^7-4096\left(-482576565+400488752\sqrt{2}\right)m^8+32768\left(-33486000+25037047\sqrt{2}\right)m^9-1048576\left(-251304+182629\sqrt{2}\right)m^{10}+50331648\left(-433+312\sqrt{2}\right)m^{11}\Big)\Big)\Big/\Big(32m\left(-1+m+2m^2\right)^3\left(-16245+3108\sqrt{2}+\left(8352-444\sqrt{2}\right)m-2304m^2+256m^3\right)\left(-9941+2664\sqrt{2}+\left(4512-444\sqrt{2}\right)m-1536m^2+256m^3\right)\Big)$$

In[210]:=

Reduce[m ≥ 8 && % > 0]

约化

Out[210]=

m ≥ 8

In[217]:=

```
v1[m + 1] + v2[m + 1] / f1[m + 1] + v3[m + 1] / (f1[m - 1] × f1[m]) - f1[m + 2] // Together //
                                                                                       归并
 Simplify
 化简
```

Out[217]=

$$-\Big(\Big(3\,\Big(-8192\,\big(-288\,514\,018\,851+204\,008\,263\,232\,\sqrt{2}\big)-15\,\big(6\,469\,360\,245\,898\,319+4\,710\,072\,175\,177\,132\,\sqrt{2}\big)\,m+3\,\big(-9\,562\,320\,700\,816\,943+5\,527\,981\,801\,765\,060\,\sqrt{2}\big)\,m^2+\big(151\,471\,648\,919\,042\,367+119\,022\,696\,596\,491\,516\,\sqrt{2}\big)\,m^3+\big(33\,078\,383\,364\,943\,425-142\,665\,973\,544\,180\,252\,\sqrt{2}\big)\,m^4-18\,\big(4\,407\,885\,831\,621\,453+1\,178\,992\,653\,595\,244\,\sqrt{2}\big)\,m^5+32\,\big(-4\,341\,739\,585\,757\,190+6\,771\,915\,067\,570\,289\,\sqrt{2}\big)\,m^6-384\,\big(-429\,616\,233\,978\,975+445\,923\,007\,564\,748\,\sqrt{2}\big)\,m^7-1536\,\big(-53\,260\,927\,427\,289+36\,078\,409\,622\,762\,\sqrt{2}\big)\,m^8+2048\,\big(-118\,012\,102\,844\,181+88\,519\,282\,763\,117\,\sqrt{2}\big)\,m^9-16\,384\,\big(-11\,146\,916\,791\,311+8\,271\,095\,875\,304\,\sqrt{2}\big)\,m^{10}+65\,536\,\big(-1\,117\,423\,186\,344+821\,957\,138\,051\,\sqrt{2}\big)\,m^{11}-2\,097\,152\,\big(-8\,706\,194\,787+6\,362\,019\,076\,\sqrt{2}\big)\,m^{12}+16\,777\,216\,\big(-166\,429\,368+121\,070\,497\,\sqrt{2}\big)\,m^{13}-536\,870\,912\,\big(-462\,624+335\,269\,\sqrt{2}\big)\,m^{14}+25\,769\,803\,776\,\big(-433+312\,\sqrt{2}\big)\,m^{15}\Big)\Big)\Big/\Big(32\,m\,(1+m)^3\,(-3+2\,m)\,(1+2\,m)^3\,\big(-27\,157+3552\,\sqrt{2}+\big(13\,728-444\,\sqrt{2}\big)\,m-3072\,m^2+256\,m^3\big)\,\big(-16\,245+3108\,\sqrt{2}+\big(8352-444\,\sqrt{2}\big)\,m-2304\,m^2+256\,m^3\big)\,\big(-9941+2664\,\sqrt{2}+\big(4512-444\,\sqrt{2}\big)\,m-1536\,m^2+256\,m^3\big)\Big)\Big)$$

In[218]:=

```
Reduce[m ≥ 8 && % < 0]
约化
```

Out[218]=

$m \geq 8$