```
<< "~\\P-rec.m"
```

In[2]:=
```
<< RISC`fastZeil`
```

Fast Zeilberger Package version 3.61
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In[3]:=
```
(**************The first proof **************)
```

In[4]:=
```
(*********an*********)
```

In[5]:=
```
a[n_] := Sum[(Binomial[n - 1, k] Binomial[n + k, k])^2 / (n (4 k^2 - 1)), {k, 0, n - 1}];
```
求和 二项式系数 二项式系数

In[6]:=
```
rec = Zb[(Binomial[n - 1, k] Binomial[n + k, k])^2 / (n (4 k^2 - 1)), {k, 0, n - 1}, n, 3]
```
二项式系数 二项式系数

```
If `-1 + n' is a natural number, then:
```

Out[6]= $\{n^3 (1+n) (5+2n) \mathrm{SUM}[n] - (1+n)(5+2n)\left(62+191n+152n^2+35n^3\right)\mathrm{SUM}[1+n] + (2+n)(1+2n)\left(88+224n+163n^2+35n^3\right)\mathrm{SUM}[2+n] - (2+n)(3+n)^3(1+2n)\mathrm{SUM}[3+n] == 0\}$

In[7]:=
```
L = n^3 (1 + n) (5 + 2 n) - (1 + n) (5 + 2 n) (62 + 191 n + 152 n^2 + 35 n^3) N +
    (2 + n) (1 + 2 n) (88 + 224 n + 163 n^2 + 35 n^3) N^2 - (2 + n) (3 + n)^3 (1 + 2 n) N^3;
```
数值运算 数值运算 数值运

In[8]:=
```
rLogBound[L, n, N, {-1, 1, 9, 61, 587, 7575}, 1, 4]
```
数值运算

$$17+12\sqrt{2}-\frac{1}{n^2}-\frac{3\left(17+12\sqrt{2}\right)\left(-264+37\sqrt{2}\right)}{64\,n^2}-\frac{9\left(17+12\sqrt{2}\right)}{2\,n} <= a\_\{n+1\}/a\_n <= 17+12\sqrt{2}+\frac{1}{n^2}-\frac{3\left(17+12\sqrt{2}\right)\left(-264+37\sqrt{2}\right)}{64\,n^2}-\frac{9\left(17+12\sqrt{2}\right)}{2\,n} \text{ for } n>=5$$

```
a_n preserves the bounds for n>=605
the bounds hold for n>=607
```

Out[8]= {True, 607}

In[9]:= $v[n\_] := -\frac{\frac{153}{2}+54\sqrt{2}}{n}+\frac{\frac{675}{4}+\frac{7617\sqrt{2}}{64}}{n^2}$ // FullSimplify
完全简化

In[10]:=
```
(***Check for the third and forth conditions***)
```
校验

In[11]:=
```
a0 = 17 + 12 √2;
N0 = 607;
m = 2;
k0 = 3;
```

In[15]:= Reduce$\left[n \geq 607 \,\&\&\, a0 + v[n] + 1/n\char`^m < a0 \left(\frac{n}{n+1}\right)\char`^k0\right]$
约化

Out[15]=
n ≥ 607

In[16]:= Reduce[n ≥ 607 && a[N0] ≤ (a0)^N0]
约化

Out[16]=
n ≥ 607

In[17]:= $$\frac{v[n] + \frac{1}{n\char`^m}}{a0\ (n+1)} - \frac{v[n-1] - \frac{1}{(n-1)\char`^m}}{a0\ (n-1)} + \frac{\left(v[n-1] - \frac{1}{(n-1)\char`^m}\right)\char`^2}{a0\char`^2\ (n-1)} - \frac{2\ k0\ \mathrm{Log}[n]}{n\ (n\char`^2-1)};$$

In[18]:= Reduce[n ≥ 607 && % < 0]
约化

Out[18]=
n ≥ 607

In[19]:= (********bn**********)

In[20]:= Zb[(3 k^2 + 3 k + 1) (Binomial[n - 1, k] Binomial[n + k, k])^2 / n^3, {k, 0, n - 1}, n, 3]
二项式系数 二项式系数

If `-1 + n' is a natural number, then:

Out[20]=
$$\{n^3\ (1+n)\ (5+2n)\ (11+12n+3n^2)\ (25+24n+6n^2)\ \mathrm{SUM}[n] - (1+n)\ (5+2n)\ (3076+21646n+59512n^2+82777n^3+64134n^4+28137n^5+6552n^6+630n^7)\ \mathrm{SUM}[1+n] + (2+n)\ (1+2n)\ (5072+30640n+73445n^2+93469n^3+68751n^4+29271n^5+6678n^6+630n^7)\ \mathrm{SUM}[2+n] - (2+n)\ (3+n)^3\ (1+2n)\ (2+6n+3n^2)\ (7+12n+6n^2)\ \mathrm{SUM}[3+n] == 0\}$$

In[21]:= $$L2 = n^3\ (1+n)\ (5+2n)\ (11+12n+3n^2)\ (25+24n+6n^2) - (1+n)\ (5+2n)\ (3076+21646n+59512n^2+82777n^3+64134n^4+28137n^5+6552n^6+630n^7)\ N + (2+n)\ (1+2n)\ (5072+30640n+73445n^2+93469n^3+68751n^4+29271n^5+6678n^6+630n^7)\ N\char`^2 - (2+n)\ (3+n)^3\ (1+2n)\ (2+6n+3n^2)\ (7+12n+6n^2)\ N\char`^3;$$
数值运算 数值运算 数值运算

In[22]:= rLogBound[L2, n, N, {1, 8, 87, 1334, 25045}, 1, 4]
数值运算

$$17+12\sqrt{2} - \frac{1}{n^2} - \frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,n^2} - \frac{5\left(17+12\sqrt{2}\right)}{2\,n} \text{<=a\_\{n+1\}/a\_n<=}$$
$$17+12\sqrt{2} + \frac{1}{n^2} - \frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,n^2} - \frac{5\left(17+12\sqrt{2}\right)}{2\,n} \text{ for n>=3}$$

a_n preserves the bounds for n>=208

the bounds hold for n>=210

Out[22]=
{True, 210}

In[23]:= $v[n\_] := -\frac{5\left(-56+3\sqrt{2}\right)\left(17+12\sqrt{2}\right)}{64\,n^2}-\frac{5\left(17+12\sqrt{2}\right)}{2\,n};$

In[24]:= (***Check for the third and forth conditions***)
校验

In[25]:= $a0 = 17 + 12\sqrt{2};$
N0 = 210;
m = 2;
k0 = 2;

In[29]:= $\text{Reduce}\left[n \geq 210 \,\&\&\, a0 + v[n] + 1/n\text{^}m < a0\left(\frac{n}{n+1}\right)\text{^}k0\right]$
约化

Out[29]=
$n \geq 210$

In[30]:= $\frac{v[n]+\frac{1}{n\text{^}m}}{a0\,(n+1)}-\frac{v[n-1]-\frac{1}{(n-1)\text{^}m}}{a0\,(n-1)}+\frac{\left(v[n-1]-\frac{1}{(n-1)\text{^}m}\right)\text{^}2}{a0\text{^}2\,(n-1)}-\frac{2\,k0\,\text{Log}[n]}{n\,(n\text{^}2-1)};$

In[31]:= Reduce[n ≥ 210 && % < 0]
约化

Out[31]=
$n \geq 210$

In[32]:= (*******The analytic proof*******)

In[33]:= $v1[n\_] := \frac{35\,n\text{^}3-152\,n\text{^}2+191\,n-62}{n\text{^}3};$

$v2[n\_] := -\frac{(n-2)\,(2\,n-1)\,\left(-88+224\,n-163\,n^2+35\,n^3\right)}{n^3\,(n-1)\,(2\,n-5)};$

$v3[n\_] := \frac{(n-3)\text{^}3\,(n-2)\,(2\,n-1)}{(n-1)\,n\text{^}3\,(2\,n-5)};$

In[36]:= $f1[n\_] := -\frac{\left(17+12\sqrt{2}\right)\left(-256\,n\text{^}3+2304\,n\text{^}2-8352\,n+16\,245+444\sqrt{2}\,n-3108\sqrt{2}\right)}{32\,(2\,n-3)\text{^}3};$

In[37]:= $\text{Reduce}\left[5 \leq n \leq 8 \,\&\&\, f1[n] < \frac{a[n]}{a[n-1]} < f1[n+1],\ \text{Integers}\right]$
约化 整数域

Out[37]=
n == 5 || n == 6 || n == 7 || n == 8

In[38]:= Reduce[n ≥ 9 && v1[n] > 0 && v2[n] < 0 && v3[n] > 0]
约化

Out[38]=
$n \geq 9$

In[39]:= m =.;

In[40]:=
```
v1[m + 1] + v2[m + 1] / f1[m] + v3[m + 1] / (f1[m] × f1[m + 1]) - f1[m + 1] // Together //
  Simplify
```
归并
化简

Out[40]=

$$\left(3\left(-73\,728\left(-83\,799+59\,212\sqrt{2}\right)+5\left(502\,133\,292\,723+366\,827\,211\,896\sqrt{2}\right)m+\left(3\,052\,263\,728\,061+1\,835\,372\,121\,892\sqrt{2}\right)m^2-\left(2\,279\,562\,847\,107+1\,503\,787\,352\,356\sqrt{2}\right)m^3+5\left(-287\,325\,208\,461+36\,350\,404\,796\sqrt{2}\right)m^4+4\left(929\,405\,738\,616+32\,491\,830\,311\sqrt{2}\right)m^5-32\left(68\,228\,120\,367+15\,627\,572\,572\sqrt{2}\right)m^6+896\left(-1\,004\,638\,368+1\,683\,587\,995\sqrt{2}\right)m^7-4096\left(-482\,576\,565+400\,488\,752\sqrt{2}\right)m^8+32\,768\left(-33\,486\,000+25\,037\,047\sqrt{2}\right)m^9-1\,048\,576\left(-251\,304+182\,629\sqrt{2}\right)m^{10}+50\,331\,648\left(-433+312\sqrt{2}\right)m^{11}\right)\right)\Big/\left(32\,m\left(-1+m+2m^2\right)^3\left(-16\,245+3108\sqrt{2}+\left(8352-444\sqrt{2}\right)m-2304\,m^2+256\,m^3\right)\left(-9941+2664\sqrt{2}+\left(4512-444\sqrt{2}\right)m-1536\,m^2+256\,m^3\right)\right)$$

In[41]:=
```
Reduce[m ≥ 8 && % > 0]
```
约化

Out[41]=

$m \geq 8$

In[42]:=
```
v1[m + 1] + v2[m + 1] / f1[m + 1] + v3[m + 1] / (f1[m - 1] × f1[m]) - f1[m + 2] // Together //
  Simplify
```
归并
化简

Out[42]=

$$-\left(\left(3\left(-8192\left(-288\,514\,018\,851+204\,008\,263\,232\sqrt{2}\right)-15\left(6\,469\,360\,245\,898\,319+4\,710\,072\,175\,177\,132\sqrt{2}\right)m+3\left(-9\,562\,320\,700\,816\,943+5\,527\,981\,801\,765\,060\sqrt{2}\right)m^2+\left(151\,471\,648\,919\,042\,367+119\,022\,696\,596\,491\,516\sqrt{2}\right)m^3+\left(33\,078\,383\,364\,943\,425-142\,665\,973\,544\,180\,252\sqrt{2}\right)m^4-18\left(4\,407\,885\,831\,621\,453+1\,178\,992\,653\,595\,244\sqrt{2}\right)m^5+32\left(-4\,341\,739\,585\,757\,190+6\,771\,915\,067\,570\,289\sqrt{2}\right)m^6-384\left(-429\,616\,233\,978\,975+445\,923\,007\,564\,748\sqrt{2}\right)m^7-1536\left(-53\,260\,927\,427\,289+36\,078\,409\,622\,762\sqrt{2}\right)m^8+2048\left(-118\,012\,102\,844\,181+88\,519\,282\,763\,117\sqrt{2}\right)m^9-16\,384\left(-11\,146\,916\,791\,311+8\,271\,095\,875\,304\sqrt{2}\right)m^{10}+65\,536\left(-1\,117\,423\,186\,344+821\,957\,138\,051\sqrt{2}\right)m^{11}-2\,097\,152\left(-8\,706\,194\,787+6\,362\,019\,076\sqrt{2}\right)m^{12}+16\,777\,216\left(-166\,429\,368+121\,070\,497\sqrt{2}\right)m^{13}-536\,870\,912\left(-462\,624+335\,269\sqrt{2}\right)m^{14}+25\,769\,803\,776\left(-433+312\sqrt{2}\right)m^{15}\right)\right)\Big/\left(32\,m\,(1+m)^3\,(-3+2m)\,(1+2m)^3\left(-27\,157+3552\sqrt{2}+\left(13\,728-444\sqrt{2}\right)m-3072\,m^2+256\,m^3\right)\left(-16\,245+3108\sqrt{2}+\left(8352-444\sqrt{2}\right)m-2304\,m^2+256\,m^3\right)\left(-9941+2664\sqrt{2}+\left(4512-444\sqrt{2}\right)m-1536\,m^2+256\,m^3\right)\right)\right)$$

In[43]:=
```
Reduce[m ≥ 8 && % < 0]
```
约化

Out[43]=

$m \geq 8$