Jordan Normal Form and Exponential Map

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Outline

Introduction

Block Diagonal Matrix

$$\begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix} \lambda_n & 1 \\ & \lambda_n \end{matrix}} \end{bmatrix}$$

Figure: Jordan Normal Form

Definition

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number of times eigenvalue λ is repeated along the diagonal of Jordan matrix $\mathbf{J}$.

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Generalized eigenvector

Define $\mathbf{N}=A-\lambda\mathbf{I}$. Call $v\neq 0$ a generalised eigenvector with λ for $\mathbf{A}$ if $\mathbf{N}^k\mathbf{v}=0$, for some natural k. If $k=1$, $\mathbf{v}$ is called an eigenvector.

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Defective Matrix

A square matrix which is not diagonalizable. *Example*;

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Where $\lambda=3$, with algebraic multiplicity 2.

Theorem

Definition; the $n \times n$ matrix $\mathbf{J}_{\lambda,n}$ with $\lambda's$ on the diagonal , $1'$s on the super-diagonal and $0'$s elsewhere is called a Jordan block matrix.

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Since in Jordan form there is no basis of eigenvectors , it means that there is a basis of generalized eigenvectors.

Motivation for Jordan block $\mathbf{A} = \mathbf{J}_{\lambda,n}$

λ is the eigenvalue, n is the order of the square matrix.

- **Example:** Simple Jordan Matrix

$$\mathbf{A} = \mathbf{J}_{2,3} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \text{with (Alg m =3) and (Geo m=1)}$$

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$$A_2 e_1 = 0, \qquad A_2 e_2 = e_1, \qquad A_2 e_3 = e_2$$

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Since $(\mathbf{A}_2)^2 e_2 = 0$ and $(\mathbf{A}_2)^3 e_3 = 0$, and so $\mathbf{e}_2$ and $\mathbf{e}_3$ are called generalized eigenvectors. Thus there is a basis of generalized eigenvectors.

Jordan form with Exponential Matrices

Recall the unique solution of the initial value problem

$$\begin{cases} \mathbf{x}' = \mathbf{A}\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

is given by $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0$.

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$$\mathbf{A} = \mathbf{T}\mathbf{J}\mathbf{T}^{-\mathbf{1}}, \quad e^{t\mathbf{A}} = e^{t(\mathbf{T}\mathbf{J}\mathbf{T}^{-\mathbf{1}})} = \mathbf{T}^{-\mathbf{1}}e^{t\mathbf{J}}\mathbf{T}.$$

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$$\mathbf{A} = \mathbf{T}\mathbf{J}\mathbf{T}^{-\mathbf{1}}, \quad e^{t\mathbf{A}} = e^{t(\mathbf{T}\mathbf{J}\mathbf{T}^{-1})} = \mathbf{T}^{-\mathbf{1}}e^{t\mathbf{J}}\mathbf{T}.$$

Knowing that $\mathbf{N} = \mathbf{J} - \lambda\mathbf{I}$, is a nilpotent matrices with nilpotency n, then $e^{t\mathbf{J}} = e^{\lambda_i \mathbf{I}t + \mathbf{N}t}$, where

$$e^{\mathbf{N}t} = \mathbf{I} + t\mathbf{N} + \frac{\mathbf{N}^2 t^2}{2!} + \ldots + \frac{\mathbf{N}^{n-1} t^{n-1}}{(n-1)!}$$

Then

$$e^{t\mathbf{J}} = e^{\lambda_i \mathbf{I} t}\left(\mathbf{I} + t\mathbf{N} + \frac{\mathbf{N}^2 t^2}{2!} + \ldots + \frac{\mathbf{N}^{n-1} t^{n-1}}{(n-1)!}\right)$$

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e^{t\mathbf{J}} &= e^{\lambda_i \mathbf{I} t}\left(\mathbf{I} + t\mathbf{N} + \frac{\mathbf{N}^2 t^2}{2!} + \ldots + \frac{\mathbf{N}^{n-1} t^{n-1}}{(n-1)!}\right) \\
e^{t\mathbf{J}} &= \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}\left(\mathbf{I} + t\mathbf{N} + \frac{\mathbf{N}^2 t^2}{2!} + \ldots + \frac{\mathbf{N}^{n-1} t^{n-1}}{(n-1)!}\right) \\
e^{t\mathbf{A}} &= \mathbf{T}^{-1} e^{t\mathbf{J}} \mathbf{T}
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e^{t\mathbf{J}} &= e^{\lambda_i \mathbf{I} t}\left(\mathbf{I} + t\mathbf{N} + \frac{\mathbf{N}^2 t^2}{2!} + \ldots + \frac{\mathbf{N}^{n-1} t^{n-1}}{(n-1)!}\right) \\
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e^{t\mathbf{A}} &= \mathbf{T}^{-\mathbf{1}} e^{t\mathbf{J}} \mathbf{T} \\
x(t) &= e^{t\mathbf{A}} x_0 = (\mathbf{T}^{-\mathbf{1}} e^{t\mathbf{J}} \mathbf{T}) x_0
\end{aligned}
$$

Where x_0 is the initial value.

- **Example: system of ODE,** Given $\begin{cases} \frac{dy_1}{dt} = \lambda y_1 + y_2 \\ \frac{dy_2}{dt} = \lambda y_2, \end{cases}$

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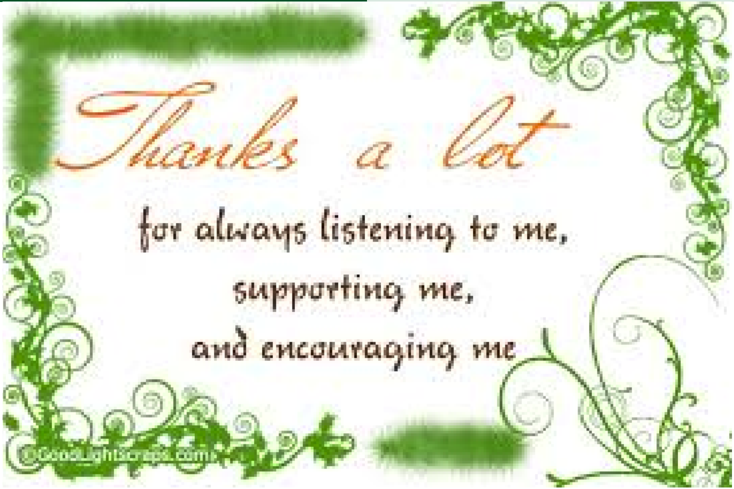
y &= c_2 e^{\lambda t} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_1 e^{\lambda t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}
$$

Conclusion

- No matter whether the matrix generated by the system of ODE is not diagonalizable we can get solution using Jordan norm form.
- Jordan form can be used as tool to generalize the solution for some ordinary differential equation.

References

- Erik,Walhlén(ODE Spring 2011):The Jordan Normal Form
- http://mathworld.wolfram.com/JordanCanonicalForm.html
- https://doi.org/10.2200/S00146ED1V01Y200808MAS002


Thanks a lot
for always listening to me,
supporting me,
and encouraging me
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