HW 5 Solutions

1. P 2.21

(b) $x[n] = h[n] = \alpha^n u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k]\, \alpha^{n-k} u[n-k]$$

$$= \sum_{k=0}^{\infty} \alpha^n u[n-k]$$

if $n < 0 \Rightarrow y[n] = 0$

if $n \geq 0 \Rightarrow y[n] = \sum_{k=0}^{n} \alpha^n = (n+1)\alpha^n.$

Therefore, $y[n] = \begin{cases} (n+1)\alpha^n & \text{if } n \geq 0 \\ 0 & n < 0 \end{cases}$

$= (n+1)\alpha^n u[n]$

(d) $x[n] = u[n] - u[n-5]$

Let $g[n] = u[n] - u[n-6]$

Then $h[n] = g[n-2] + g[n-11]$.

Therefore $y[n] = x[n] * h[n]$

$= x * g[n-2] + x * g[n-11]$

$= w[n-2] + w[n-11]$

where $w[n] = x[n] * g[n]$

Now,

$$w[n] = x[n] * g[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, g[n-k]$$

$$= \sum_{k=0}^{4} \left(u[n-k] - u[n-k-6] \right)$$

$$= \sum_{k=0}^{4} u[n-k] - \sum_{k=0}^{4} u[n-k-6]$$

$$= \begin{cases} \sum_{k=0}^{4} 1 = 5, & n \geq 4 \\ \sum_{k=0}^{n} 1 = n+1, & 0 \leq n \leq 3 \\ 0, & n \leq -1 \end{cases}$$

$$- \begin{cases} 5, & n-6 \geq 4 \iff n \geq 10 \\ n-5, & 0 \leq n-6 \leq 3 \iff 6 \leq n \leq 9 \\ 0, & n-6 \leq -1 \iff n \leq 5 \end{cases}$$

$$= \begin{cases} 0 & n \geq 10 \\ 10-n & 6 \leq n \leq 9 \\ 5 & 4 \leq n \leq 5 \\ n+1 & 0 \leq n \leq 3 \\ 0 & n \leq -1 \end{cases}$$

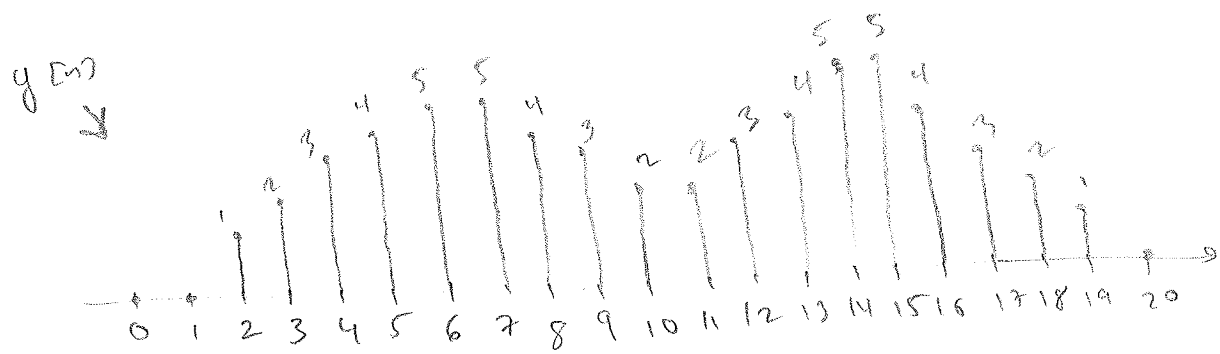
Note that you might find the graphical method easier here.

Therefore

$$y[n] = w[n-2] + w[n-11]$$

$$= \begin{cases} 0, & n \geq 12 \\ 12-n, & 8 \leq n \leq 11 \\ 5, & 6 \leq n \leq 7 \\ n-1, & 2 \leq n \leq 5 \\ 0, & n \leq 1 \end{cases} + \begin{cases} 0, & n \geq 21 \\ 21-n, & 17 \leq n \leq 20 \\ 5, & 15 \leq n \leq 16 \\ n-10, & 11 \leq n \leq 14 \\ 0 & n \leq 10 \end{cases}$$

$$= \begin{cases} 0 & n \geq 21 \\ 21-n & 17 \leq n \leq 20 \\ 5, & 15 \leq n \leq 16 \\ n-10 & 12 \leq n \leq 14 \\ 2 & n = 11 \\ 12-n & 8 \leq n \leq 10 \\ 5, & 6 \leq n \leq 7 \\ n-1, & 2 \leq n \leq 5 \\ 0, & n \leq 1 \end{cases}$$



Note that another way to go about this convolution is by noticing that

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

and therefore

$$y[n] = h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]$$

To find $y[n]$, just add up the $h[n-n_0]$'s above.

2. P 2.22

(a) $x(t) = e^{-\alpha t} u(t)$ $\quad h(t) = e^{-\beta t} u(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau} u(\tau) \cdot e^{-\beta(t-\tau)} u(t-\tau)\, d\tau$$

$$= \int_0^t e^{-\beta t} e^{-(\alpha-\beta)\tau}\, d\tau$$

$$= e^{-\beta t} \int_0^t e^{(\beta-\alpha)\tau}\, d\tau$$

* if $\alpha \neq \beta$

$$y(t) = \begin{cases} e^{-\beta t} \left[\dfrac{e^{(\beta-\alpha)\tau}}{\beta-\alpha} \right]_0^t = \dfrac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}, & t > 0 \\ 0 & t \leq 0 \end{cases}$$

* if $\alpha = \beta$

$$y(t) = \begin{cases} t e^{-\beta t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

(c) $h(t) = (-t+1) \cdot (u(t) - u(t-1))$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$= \int_0^1 x(t-\tau)(-\tau+1)\, d\tau$$

For $-0.5 \leq t \leq 0.5$

$$y(t) = \int_0^{t+0.5} (-\tau+1)\, d\tau - \int_{t+0.5}^{1} (-\tau+1)\, d\tau$$

$$= -\frac{(1-\tau)^2}{2} \Big|_0^{t+1/2} + \frac{(1-\tau)^2}{2} \Big|_{t+1/2}^{1}$$

$$= -\frac{(\frac{1}{2}-t)^2}{2} - \left(-\frac{1}{2}\right) - \frac{(\frac{1}{2}-t)^2}{2}$$

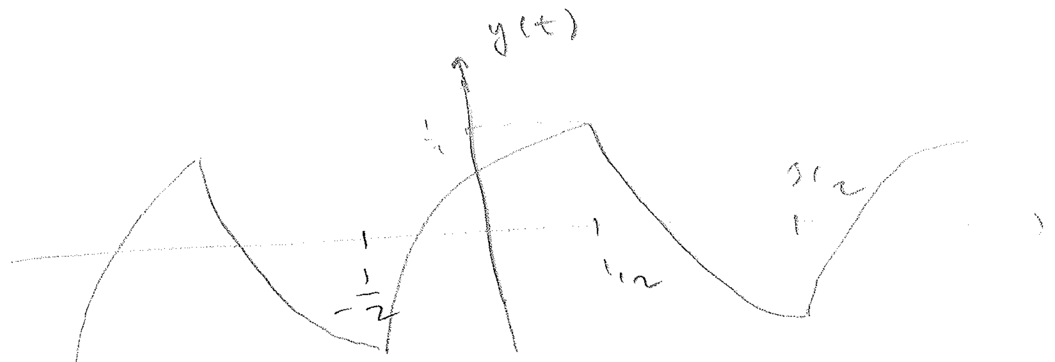
$$= -\left(\frac{1}{2}-t\right)^2 + \frac{1}{2} = -t^2 + t + 1/4$$

For $0.5 \le t \le 1.5$

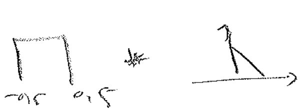
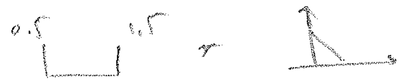
$$y(t) = -\int_0^{t-0.5} (-\tau+1)\,d\tau + \int_{t-0.5}^{1} (-\tau+1)\,d\tau$$

$$= \frac{(\tau-1)^2}{2}\Big|_0^{t-1/2} + -\frac{(1-\tau)^2}{2}\Big|_{t-1/2}^{1}$$

$$= \frac{(t-\frac{3}{2})^2}{2} - \frac{1}{2} + \left\{0 + \frac{(\frac{3}{2}-t)^2}{2}\right\} = \left(t-\frac{3}{2}\right)^2 - \frac{1}{2}$$



Note that we used the fact that the output is also periodic with period 2, as the input since convolution is LTI.

Another way to go about this problem is by convolving , we also get from this the convolution of  by LTI properties. Then add the outputs for a couple of shifts to get it over a period of 2.

3. P.2.28

(b) $h[n] = (0.8)^n u[n+2] \Rightarrow h[-1] \neq 0, h[-2] \neq 0 \Rightarrow$ Not causal

$$\sum_{n=-\infty}^{\infty} |(0.8)^n u[n+2]| = \sum_{n=-2}^{\infty} (0.8)^n \underset{\boxed{k=n+3}}{=} \sum_{k=1}^{\infty} (0.8)^{k-3}$$

$$= (0.8)^{-2} \sum_{k=1}^{\infty} (0.8)^{k-1}$$

$$= (0.8)^{-2} \frac{1}{1-0.8} < \infty \Rightarrow \text{Stable}$$

(d) $h[n] = 5^n u[3-n] \Rightarrow$ Not causal

$$\sum_{n=-\infty}^{\infty} |5^n u[3-n]| = \sum_{n=-\infty}^{3} 5^n \underset{\boxed{k=4-n}}{=} \sum_{k=1}^{\infty} 5^{4-k}$$

$$= 5^4 \left(\frac{1}{5}\right) \sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^{k-1}$$

$$= 5^3 \cdot \frac{1}{1-1/5} < \infty \Rightarrow \text{Stable}$$

(f) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[1-n] \Rightarrow$ Not causal

$$\sum_{n=-\infty}^{\infty} |h[n]| \leq \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] + \sum_{n=-\infty}^{\infty} (1.01)^n u[1-n]$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=-\infty}^{1} (1.01)^n$$

$$= \frac{1}{1-1/2} + \sum_{n=-1}^{\infty} (1.01)^{-n}$$

$$= 2 + (1.01) + \sum_{n=0}^{\infty} (1.01)^{-n}$$

$$= 3.01 + \frac{1}{1-1/1.01} < \infty \Rightarrow \text{Stable}$$

(g) $h[n] = n\left(\frac{1}{3}\right)^n u[n-1] \Rightarrow$ Causal

$$\sum_{n=-\infty}^{\infty} \left| n\left(\tfrac{1}{3}\right)^n u[n-1] \right| = \sum_{n=1}^{\infty} n\left(\tfrac{1}{3}\right)^n = \frac{1}{3}\sum_{n=1}^{\infty} n\left(\tfrac{1}{3}\right)^{n-1}$$

$$= \frac{1}{3}\,\frac{1}{\left(1-\frac{1}{3}\right)^2} < \infty \Rightarrow \text{Stable}$$

Note that
$$\sum_{n=1}^{\infty} n r^{n-1} = \frac{1}{(1-r)^2} \quad \text{if } |r|<1$$

4. P 2.29

(c) $h(t) = e^{-2t} u(t+50) \Rightarrow$ Not causal

$$\int_{-\infty}^{\infty} |e^{-2t} u(t+50)|\,dt = \int_{-50}^{\infty} e^{-2t}\,dt$$

$$= \frac{e^{-2t}}{-2}\Big|_{-50}^{\infty} = 0 + \frac{e^{100}}{2} < \infty \Rightarrow \text{Stable}$$

(e) $h(t) = e^{-6|t|} \Rightarrow$ Not Causal

$$\int_{-\infty}^{\infty} |e^{-6|t|}|\,dt = \int_{0}^{\infty} e^{-6t}\,dt + \int_{-\infty}^{0} e^{6t}\,dt$$

$$= \frac{e^{-6t}}{-6}\Big|_{0}^{\infty} + \frac{e^{6t}}{6}\Big|_{-\infty}^{0} = \frac{1}{6}+\frac{1}{6} = \frac{1}{3} < \infty$$

$\Rightarrow$ Stable

(g) $h(t) = \left(2e^{-t} - e^{\frac{t-100}{100}}\right) u(t) \Rightarrow$ Causal

$$= \left(2e^{-t} - e^{t/100}\cdot\tfrac{1}{e}\right) u(t)$$

$$2e^{-t} = \frac{1}{e} e^{t/100} \Rightarrow 2e = e^{\frac{101}{100}t} \Rightarrow \ln 2 + 1 = \frac{101}{100} t$$

$$\Rightarrow t^* = \frac{100}{101}(\ln 2 + 1)$$

That is, $t > t^*$, $2e^{-t} < e^{t/100} \cdot \frac{1}{e}$

Therefore,

$$\int_{-\infty}^{\infty} |h(t)|\, dt \geq \int_0^{t^*} 2e^{-t} - e^{t/100} \cdot \frac{1}{e}\, dt + \int_{t^*}^{\infty} e^{t/100} \cdot \frac{1}{e} - 2e^{-t}\, dt$$

But $\int_{t^*}^{\infty} e^{t/100} \frac{1}{e}\, dt = \infty \Rightarrow$ <u>unstable</u>

5. P 2.40

(a) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau$.

Then, $h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau - 2)\, d\tau$

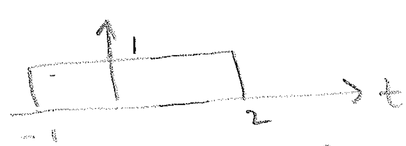
$$= \begin{cases} e^{-(t-2)} & t > \tau \\ 0 & t < \tau \end{cases} = e^{-(t-2)} u(t-2)$$

Or, you can solve it by

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau-2)\, d\tau \underset{s=\tau-2}{=} \int_{-\infty}^{t-2} e^{-(t-s-2)} x(s)\, ds$$

$$= \int_{-\infty}^{\infty} x(s)\, e^{-(t-s-2)} u(-(s-(t-2)))\, ds$$

$$= \int_{-\infty}^{\infty} x(s) \underbrace{e^{-(t-s-2)} u(t-s-2)}_{h(t-s)}\, ds \Rightarrow h(t) = e^{-(t-2)} u(t-2)$$

(b) Now, $x(t) =$ 

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau = \int_{-1}^{2} e^{-(t-\tau-2)} u(t-\tau-2)\, d\tau$$

- If $t-2<-1 \Rightarrow t<1$, then $y(t)=0$.
- If $-1 \le t-2 < 2 \Rightarrow 1 \le t < 4$,

$$y(t)=e^{2-t}\int_{-1}^{t-2} e^{\tau} d\tau = e^{2-t}\left[e^{\tau}\right]_{-1}^{t-2}$$

$$= e^{2-t}\left(e^{t-2}-e^{-1}\right) = 1-e^{1-t}$$

- If $t-2 \ge 2 \Rightarrow t \ge 4$

$$y(t)= e^{2-t}\int_{-1}^{2} e^{\tau} d\tau = e^{2-t}\left(e^{2}-1/e\right).$$

6. ① $x[n]=\delta[n]$ then $y[n]=h[n]$.

That is, $h[n]=\delta[n]-0.5h[n-1]$

$h[0]=1-0.5h[-1]=1$ ($h[-1]=0$ since initial condition $y[-1]=0$)

$h[1]=0-0.5h[0]=-0.5$

$h[2]=0-0.5h[1]=(-0.5)^2$

$\vdots$

$h[n]=(-0.5)^n u[n]$

② $y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ when $x[n]=2^{-n}u[n]$

$$= \sum_{k=0}^{\infty} 2^{-k}(-0.5)^{n-k}u[n-k]$$

$$y[5]=\sum_{k=-\infty}^{\infty} 2^{-k}(-0.5)^{5-k}u[5-k]$$

$$= (-0.5)^5\sum_{k=0}^{5}(-1)^{-k}=0$$

③ $y[n] = \sum_{k=0}^{\infty} 2^{-k} (-0.5)^{n-k} u[n-k]$

if $n < 0$ $\quad y[n] = 0$

if $n \geq 0$ $\quad y[n] = (-0.5)^n \sum_{k=0}^{n} (-1)^k$

There fore

$$y[n] = \begin{cases} 0 & n < 0 \\ (-0.5)^n & n \geq 0 \text{ \& even} \\ 0 & n \geq 0 \text{ \& odd} \end{cases}$$

7. $y(t) = \frac{1}{15} \int_{t-15}^{t} x(s)\, ds$

① Linear :

$$\frac{1}{15} \int_{t-15}^{t} \alpha_1 x_1(s) + \alpha_2 x_2(s)\, ds$$

$$= \frac{\alpha_1}{15} \int_{t-15}^{t} x_1(s)\, ds + \frac{\alpha_2}{15} \int_{t-15}^{t} x_2(s)\, ds$$

$$= \alpha_1 y_1(s) + \alpha_2 y_2(s) .$$

Time - Invariant : Let $x'(t) = x(t-t_0)$

$$\frac{1}{15} \int_{t-15}^{t} x'(s)\, ds = \frac{1}{15} \int_{t-15}^{t} x(s-t_0)\, ds = \frac{1}{15} \int_{t-t_0-15}^{t-t_0} x(s)\, ds$$

$$= y(t-t_0)$$

② $x(t) = \delta(t) \longrightarrow \boxed{\text{SYS}} \longrightarrow y(t) = h(t)$

$$h(t) = \frac{1}{15} \int_{t-15}^{t} \delta(s)\, ds = \frac{1}{15} \quad \text{if} \quad 0 \leq t \leq 15$$

(3) Memory : since $y(t)$ depends on $x(t-15)$

Stable : $\int_{-\infty}^{\infty} |h(t)|\, dt = \int_0^{15} \frac{1}{15}\, dt = 1 < \infty$

Causal : $h(t) = 0 \quad \forall \quad t < 0$.

(4) Let $x(t) = \cos(2\pi t)$

$$y(t) = \frac{1}{15} \int_{t-15}^{t} \cos(2\pi s)\, ds = \frac{1}{15} \left[\frac{\sin(2\pi s)}{2\pi} \right]_{t-15}^{t}$$

$$= \frac{1}{15 \cdot 2\pi} \left(\sin 2\pi t - \sin(2\pi t - 30\pi) \right) = 0$$

Note that the moving average system considered here is from $t-15$ to t.

But 15 is a multiple of the period of $x(t) = \cos(2\pi t)$

$\Rightarrow$ the moving average of $x(t)$ from current t to the past $t-15$ will always be zero.

(5) Not invertible : $x_1(t) = \cos(2\pi t)$, $x_2(t) = \sin(2\pi t)$ have outputs $y_1(t) = y_2(t) = 0$.

8. $\frac{d}{dt} y(t) = -y(t) + x(t)$

① $x(t) = \delta(t)$ $\quad\quad y(t) = e^{-t} u(t)$

$$\frac{d}{dt} e^{-t} u(t) = \begin{cases} -e^{-t} & t > 0 \\ \delta(t) & t = 0 \end{cases} = \underbrace{-e^{-t} u(t)}_{-y(t)} + \underbrace{\delta(t)}_{x(t)}$$

or
$$\frac{d}{dt} e^{-t} u(t) = \left(\frac{d}{dt} e^{-t}\right) u(t) + e^{-t}\left(\frac{d\, u(t)}{dt}\right)$$

$$= -e^{-t} u(t) + e^{-(0)} \delta(t)$$

$$= -e^{-t} u(t) + \delta(t)$$

② $h(t) = e^{-t} u(t)$ $\quad$ since $x(t) = \delta(t) \xrightarrow{sys} y(t) = h(t)$

9. (a) $|x_i|^2 = x_i \cdot x_i = 1 \quad \forall i$

$x_i \cdot x_j = 0 \quad \forall i \neq j$

(b) $$x = \left(\frac{21\sqrt{2} + 6\sqrt{3} + 4\sqrt{6}}{60}, \frac{-21\sqrt{2} + 6\sqrt{3} + 4\sqrt{6}}{60}, \frac{3\sqrt{3} - 4\sqrt{6}}{30}\right)$$

(c) $(0.7, 0.3, 0.4) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$

$$x_1 (0.7, 0.3, 0.4) = \alpha_1 \underbrace{(x_1 \cdot x_1)}_{1} + \alpha_2 \underbrace{(x_1 \cdot x_2)}_{0} + \alpha_3 \underbrace{(x_1 \cdot x_3)}_{0}$$

$$= \alpha_1$$

$x_2 (0.7, 0.3, 0.4) = \alpha_2$

$x_3 (0.7, 0.3, 0.4) = \alpha_3$

$$\boxed{\begin{aligned} \alpha_1 &= \frac{\sqrt{2}}{5} \\ \alpha_2 &= \frac{14\sqrt{3}}{15} \\ \alpha_3 &= \sqrt{6}/30 \end{aligned}}$$

(d) Using orthonormal basis,

we can express any arbitrary signal vector

x of size 3 by

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

We can know the output of any signal

by only knowing the output to x_1, x_2, x_3

if the system is linear.

10. Recall:

If $x(t) = e^{j\omega t} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ \text{sys} \\ h(t)\end{matrix}} \longrightarrow y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$

$$= e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau}_{H(j\omega)}$$

$$= H(j\omega)\, x(t)$$

$h(t) = 3^{-t} u(t)$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau = \int_0^{\infty} 3^{-\tau} e^{-j\omega\tau}\, d\tau$$

$$= \left[\frac{3^{-\tau} e^{-j\omega\tau}}{\ln(3^{-1} e^{-j\omega})} \right]_0^{\infty}$$

$$= 0 - \frac{1}{\ln(3^{-1} e^{-j\omega})} = -\frac{1}{\ln\frac{1}{3} - j\omega}$$

$$= \frac{1}{\ln 3 + j\omega}$$

Therefore,

$$y(t) = x(t) \cdot H(j\omega) \Rightarrow e^{j3t}\, H(j3) = \frac{e^{j(3t)}}{\ln 3 + j3}$$

11. Let $x(t) = e^{-jt} + 2e^{2jt} + 3e^{j2\sqrt{2}t}$

$$= \underbrace{e^{j(-t)}}_{x_1(t)} + 2\underbrace{e^{j(2t)}}_{x_2(t)} + 3\underbrace{e^{j(2\sqrt{2}t)}}_{x_3(t)}$$

$$x_1(t) \longrightarrow \boxed{\begin{matrix}LTI\\ h(t)\end{matrix}} \longrightarrow y_1(t) = x_1(t) \cdot H(-j)$$

$$x_2(t) \longrightarrow \boxed{\begin{matrix}LTI\\ h(t)\end{matrix}} \longrightarrow y_2(t) = x_2(t) \cdot H(j2)$$

$$x_3(t) \longrightarrow \boxed{\begin{matrix}LTI\\ h(t)\end{matrix}} \longrightarrow y_3(t) = x_3(t) \cdot H(j2\sqrt{2})$$

$$x(t) = x_1(t) + 2x_2(t) + 3x_3(t)$$

$$\longrightarrow \boxed{\begin{matrix}LTI\\ h(t)\end{matrix}} \longrightarrow y(t) = y_1(t) + 2y_2(t) + 3y_3(t)$$

Therefore,

$$y(t) = H(-j)\,x_1(t) + 2 \cdot H(j2) \cdot x_2(t) + 3H(j2\sqrt{2})\,x_3(t)$$

$$= \frac{e^{j(-t)}}{\ln 3 - j} + \frac{2e^{j(2t)}}{\ln 3 + 2j} + \frac{3e^{j(2\sqrt{2}t)}}{\ln 3 + 2\sqrt{2}j}$$

12. P 3.21

Fourier series of periodic T signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\left(\frac{2\pi}{T}\right)t}.$$

If $x(t)$ is real-valued, its Fourier series coefficients satisfy

$$a_k = a^*_{-k} \quad \forall \, k$$

Therefore, $$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \, e^{jk\left(\frac{2\pi}{T}\right)t} + \sum_{k=-1}^{\infty} a_k \, e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k \, e^{jk\left(\frac{2\pi}{T}\right)t} + a_k^* e^{-jk\left(\frac{2\pi}{T}\right)t} \right)$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{ a_k \, e^{jk\left(\frac{2\pi}{T}\right)t} \right\}$$

Now, $T = 8$ $\quad a_1 = a_{-1}^* = j \quad a_5 = a_{-5} = 2$

$$x(t) = 0 + \underbrace{2\,\mathrm{Re}\left\{ j e^{j\left(\frac{2\pi}{8}\right)t} \right\}}_{\mathrm{Re}\left\{ e^{j\left(\frac{2\pi}{8}t + \frac{\pi}{2}\right)} \right\}} + 2\,\mathrm{Re}\left\{ 2 e^{j5\left(\frac{2\pi}{8}\right)t} \right\}$$

$$= 2\cos\left(\frac{2\pi}{8}t + \frac{\pi}{2}\right) + 4\cos\left(\frac{10\pi}{8}t\right)$$

13. (c) $a_k = \frac{1}{3}\int_{-2}^{1} x(t)\, e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt$

$$= \frac{1}{3}\int_{-2}^{0} (2+t)\, e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt + \frac{1}{3}\int_{0}^{1} (2-2t)\, e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt$$

$$= \underbrace{\frac{2}{3}\int_{-2}^{1} e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt}_{①} + \underbrace{\frac{1}{3}\int_{-2}^{0} t\, e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt}_{②}$$

$$- \underbrace{\frac{2}{3}\int_{0}^{1} t\, e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt}_{③}$$

① $$\int_{-2}^{1} e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt = \left[\frac{e^{-jk\left(\frac{2\pi}{3}\right)t}}{-jk\left(\frac{2\pi}{3}\right)}\right]_{-2}^{1}$$

$$= \frac{e^{-jk\left(\frac{2\pi}{3}\right)}}{-jk\left(\frac{2\pi}{3}\right)} - \frac{e^{jk\left(\frac{4\pi}{3}\right)}}{-jk\left(\frac{2\pi}{3}\right)}$$

② $$\int_{-2}^{0} t\, e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt = \left[\frac{t\, e^{-jk\left(\frac{2\pi}{3}\right)t}}{-jk\frac{2\pi}{3}}\right]_{-2}^{0} + \frac{1}{jk\left(\frac{2\pi}{3}\right)}\int_{-2}^{0} e^{-jk\left(\frac{2\pi}{3}\right)t}\, dt$$

$$= -\frac{2\, e^{jk\left(\frac{4\pi}{3}\right)}}{jk\left(\frac{2\pi}{3}\right)} + \frac{1 - e^{jk\left(\frac{4\pi}{3}\right)}}{k^2\left(\frac{2\pi}{3}\right)^2}$$

③ $\int_0^1 t\, e^{-jk(\frac{2\pi}{3})t}\, dt = -\frac{e^{-jk(\frac{2\pi}{3})}}{jk(\frac{2\pi}{3})} + \frac{e^{-jk(\frac{2\pi}{3})} - 1}{k^2(\frac{2\pi}{3})^2}$

Therefore,

$$a_k = \frac{2}{3}\left(\frac{e^{jk(\frac{4\pi}{3})} - e^{-jk(\frac{2\pi}{3})}}{jk(\frac{2\pi}{3})}\right)$$

$$+ \frac{1}{3}\left(-\frac{2e^{jk(\frac{4\pi}{3})}}{jk(\frac{2\pi}{3})} + \frac{1 - e^{jk(\frac{4\pi}{3})}}{k^2(\frac{2\pi}{3})^2}\right)$$

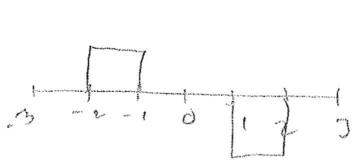
$$+ \frac{2}{3}\left(\frac{e^{-jk(\frac{2\pi}{3})}}{jk(\frac{2\pi}{3})} - \frac{e^{-jk(\frac{2\pi}{3})} - 1}{k^2(\frac{2\pi}{3})^2}\right)$$

$$= \frac{9 - 3e^{-jk(\frac{4\pi}{3})} - 6e^{-jk(\frac{2\pi}{3})}}{4\pi^2 k^2} \quad \forall\, k \neq 0$$

when $k=0$, $a_0 = 1$

(d) $a_k = \frac{1}{2}\int_{-0.5}^{1.5} (\delta(t) - 2\delta(t-1))\, e^{-jk\pi t}\, dt$

$= \frac{1}{2} - e^{-jk\pi} = \frac{1}{2} - (e^{-j\pi})^k = \frac{1}{2} - (-1)^k$

14. (e) $a_k = \frac{1}{6}\int_{-3}^{3}$ 

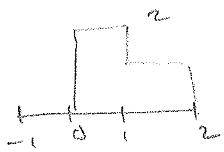
$e^{-j\frac{2\pi}{6}t}\, dt$

$= \underbrace{\frac{1}{6}\int_{-2}^{-1} e^{-jk\frac{\pi}{3}t}\, dt}_{} - \frac{1}{6}\int_1^2 e^{-jk\frac{\pi}{3}t}\, dt$

" $\int_1^2 e^{jk\frac{\pi}{3}t}\, dt$ by change of variables

$= \frac{1}{6}\int_1^2 e^{jk\frac{\pi}{3}t} - e^{-jk\frac{\pi}{3}t}\, dt = \frac{2j}{6}\int_1^2 \sin(\frac{\pi}{3}kt)\, dt$

$$= \begin{cases} -j/3 \left[\dfrac{\cos(\pi/3\, k t)}{\pi/3\, k} \right]_1^2 = \dfrac{-j}{\pi k} \left(\cos\left(\dfrac{2\pi}{3} k\right) - \cos\left(\dfrac{\pi}{3} k\right) \right) & \forall\, k \neq 0 \\ 0 & \text{when } k = 0 \end{cases}$$

(f) $a_k = \dfrac{1}{3} \displaystyle\int_{-1}^{2}$  $e^{-jk\frac{2\pi}{3}t}\, dt$

$$= \frac{1}{3} \int_0^1 2\, e^{-jk\frac{2\pi}{3}t}\, dt + \frac{1}{3} \int_1^2 e^{-jk\frac{2\pi}{3}t}\, dt$$

$$= \frac{2}{3} \left(\frac{1}{-jk\frac{2\pi}{3}} \right) \left[e^{-jk\frac{2\pi}{3}t} \right]_0^1 + \frac{1}{3} \left(\frac{1}{-jk\frac{2\pi}{3}} \right) \cdot \left[e^{-jk\frac{2\pi}{3}t} \right]_1^2$$

$$= \frac{e^{-jk\frac{2\pi}{3}} - 1}{-jk\pi} + \frac{\frac{1}{2}\left(e^{-jk\frac{4\pi}{3}} - e^{-jk\frac{2\pi}{3}} \right)}{-jk\pi}$$

$$= \begin{cases} \dfrac{1 - \frac{1}{2} e^{-jk\frac{2\pi}{3}} - \frac{1}{2} e^{-jk\frac{4\pi}{3}}}{jk\pi} & \forall\, k \neq 0 \\ 1 & \text{when } k = 0 \end{cases}$$

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(b) $a_k = \dfrac{1}{2} \displaystyle\int_{-1}^{1} e^{-t} e^{-jk\pi t}\, dt = \dfrac{1}{2} \int_{-1}^{1} e^{-(1+jk\pi)t}\, dt$

$$= \frac{1}{2} \cdot \frac{1}{-(1+jk\pi)} \left[e^{-(1+jk\pi)t} \right]_{-1}^{1}$$

$$= \begin{cases} \dfrac{-1}{2(1+jk\pi)} \left(\dfrac{(-1)^k}{e} - e\,(-1)^k \right) & \forall\, k \neq 0 \\ \dfrac{1}{2}\left(e - \dfrac{1}{e} \right) & \text{when } k = 0 \end{cases}$$

(c) $a_k = \frac{1}{4}\int_0^2 \sin(\pi t)\, e^{-jk\frac{2\pi}{4}t}\, dt$

$$= \frac{1}{4}\int_0^2 \left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right) e^{-jk\frac{\pi}{2}t}\, dt$$

$$= \frac{-j}{8}\int_0^2 e^{(j\pi - jk\frac{\pi}{2})t}\, dt + \frac{j}{8}\int_0^2 e^{(-j\pi - jk\frac{\pi}{2})t}\, dt$$

$$= -\frac{1}{8}\cdot\frac{1}{\pi - k\frac{\pi}{2}}\cdot\left(e^{j2\pi - jk\pi} - 1\right)$$

$$-\frac{1}{8}\cdot\frac{1}{\pi + k\frac{\pi}{2}}\left(e^{-j2\pi - jk\pi} - 1\right)$$

$$= -\frac{1}{8\pi}\left((-1)^k - 1\right)\cdot\left(\frac{2}{1 - k^2/4}\right) = \frac{(-1)^k - 1}{4\pi(k^2/4 - 1)}$$