HW #4 Solution.

1. P1.27

(a). $y(t) = x(t-2) + x(2-t) = x(t-2) + x(-(t-2))$

① <u>memory</u> : $y(t)$ depends on the past input $x(t-2)$

② <u>time-variant</u> : For input $x'(t) = x(t-t_0)$.

$$\text{output} = x'(t-2) + x'(2-t) = x(t-2-t_0) + x(2-t-t_0)$$
$$= x(t-2-t_0) + x(2-t-t_0)$$
$$\neq y(t-t_0) = x(t-t_0-2) + x(-t+t_0+2)$$

③ <u>Linear</u> : Let $x_1(t) \to y_1(t) = x_1(t-2) + x_1(2-t)$, $x_2(t) \to y_2(t) = x_2(t-2) + x_2(2-t)$

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \alpha_1 x_1(t-2) + \alpha_2 x_2(t-2) + \alpha_1 x_1(2-t) + \alpha_2 x_2(2-t)$$
$$= \alpha_1 \left(x_1(t-2) + x_1(2-t) \right) + \alpha_2 \left(x_2(t-2) + x_2(2-t) \right)$$
$$= \alpha_1 y_1(t) + \alpha_2 y_2(t).$$

④ <u>Not causal</u> : $y(0) = x(-2) + x(2) \Rightarrow$ system depends on the future data.

⑤ <u>Stable</u> : If $|x(t)| < M \ \forall t$, then $|y(t)| \le |x(t-2)| + |x(2-t)| \le 2M \ \forall t$.

(b). $y(t) = \cos(3t)\, x(t)$.

① <u>memoryless</u> : $y(t)$ depends only on present value of $x(t)$.

② <u>time-ivariant</u> : For input $x'(t) = x(t-t_0)$

$$\text{output} = \cos(3t) \cdot x'(t) = \cos(3t)\, x(t-t_0) \neq y(t-t_0) = \cos(3t-3t_0)\, x(t-t_0)$$

③ <u>Linear</u> : $x_1(t) \longrightarrow y_1(t) = \cos(3t)\, x_1(t)$, $x_2(t) \longrightarrow y_2(t) = \cos(3t)\, x_2(t)$

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \cos(3t) \left\{ \alpha_1 x_1(t) + \alpha_2 x_2(t) \right\}$$
$$= \alpha_1 \cos(3t)\, x_1(t) + \alpha_2 \cos(3t)\, x_2(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

④ <u>Causal</u>

⑤ <u>Stable</u> : If $|x(t)| < M \ \forall t$, then $|y(t)| = \underbrace{|\cos(3t)|}_{\le 1 \ \forall t} |x(t)| \le |x(t)| < M$.

(c). $y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$.

① Memory : $y(t)$ depends on $x(-\infty)$ up to $x(t)$.

② Time variant : For input $x'(t) = x(t-t_0)$

$$\text{output} = \int_{-\infty}^{2t} x'(\tau)\,d\tau = \int_{-\infty}^{2t} x(\tau - t_0)\,d\tau \underset{s=\tau-t_0}{=} \int_{-\infty}^{2t-t_0} x(s)\,ds$$

$$\neq y(t-t_0) = \int_{-\infty}^{2t-2t_0} x(\tau)\,d\tau$$

change of variable

③ Linear : Let $x_1(t) \to y_1(t) = \int_{-\infty}^{2t} x_1(\tau)\,d\tau$, $x_2(t) \to y_2(t) = \int_{-\infty}^{2t} x_2(\tau)\,d\tau$.

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \int_{-\infty}^{2t} \alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)\,d\tau$$

$$= \int_{-\infty}^{2t} \alpha_1 x_1(\tau)\,d\tau + \int_{-\infty}^{2t} \alpha_2 x_2(\tau)\,d\tau = \alpha_1 y_1(t) + \alpha_2 y_2(t).$$

④ Not causal : $y(t)$ depends on $x(2t)$.

⑤ Unstable : Let $x(t) = 1 \ \forall t$.

$|y(t)| = \left| \int_{-\infty}^{2t} x(\tau)\,d\tau \right|$. At $t=0$, $y(0) = \left| \int_{-\infty}^{0} d\tau \right| = \infty$

2. P1.28

(a) $y[n] = x[-n]$.

① Memory

② Time-variant : For input $x'[n] = x[n-n_0]$, output $= x'[-n] = x[-n-n_0]$ $\neq y[n-n_0] = x[-n+n_0]$

③ Linear : $x_1[n] \to y_1[n] = x_1[-n]$, $x_2[n] \to y_2[n] = x_2[-n]$,

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \alpha_1 x_1[-n] + \alpha_2 x_2[-n] = \alpha_1 y_1[n] + \alpha_2 y_2[n].$$

④ Not causal : when $n<0$, $y[n]$ depends on future value.

⑤ Stable

(b) $y[n] = x[n-2] - 2x[n-8]$

① Memory

② Time-Invariant : For input $x'[n]=x[n-n_0]$,

$$\text{output} = x'[n-2]-2x'[n-8] = x[n-2-n_0]-2x[n-8-n_0]$$
$$= y[n-n_0] = x[n-n_0-2]-2x[n-n_0-8].$$

③ Linear

④ Causal.

⑤ Stable : If $|x[n]|<M$ $\forall$ integer n,

$$|y[n]| = |x[n-2]-2x[n-8]| < 3M \quad \forall \text{ integer } n.$$

(c). $y[n] = n\,x[n]$.

① Memoryless.

② Time-variant : For input $x'[n]=x[n-n_0]$,

$$\text{output} = n\,x'[n] = n\,x[n-n_0] \neq y[n-n_0] = (n-n_0)\,x[n-n_0].$$

③ Linear : $x_1[n] \to y_1[n] = n\,x_1[n]$, $x_2[n] \to y_2[n] = n\,x_2[n]$.

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \to n(\alpha_1 x_1[n] + \alpha_2 x_2[n]) = \alpha_1 y_1[n] + \alpha_2 y_2[n].$$

④ Causal

⑤ Unstable

3. P.1.28.

(e). $y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n=0 \\ x[n+1] & n \leq -1 \end{cases}$.

① Memory.

② Time-variant : For input $x'[n]=x[n-n_0]$,

$$\text{output} = \begin{cases} x[n-n_0] & n\geq 1 \\ 0 & n=0 \\ x[n+1-n_0] & n\leq -1 \end{cases} \neq y[n-n_0]$$

③ Linear

④ Not Causal : $y[n]$ depends on $x[n+1]$ for $n\leq -1$.

⑤ Stable

(f). $y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n] & n \leq -1 \end{cases} = x[n] - x[n]\delta[n] = x[n](1-\delta[n])$.

① Memoryless

② Time-variant : For $x'[n] = x[n-n_0]$,

output $= x'[n](1-\delta[n]) = x[n-n_0](1-\delta[n])$

$\neq y[n-n_0] = x[n-n_0](1-\delta[n-n_0])$.

③ Linear

④ Causal

⑤ Stable

(g). $y[n] = x[4n+1]$.

① Memory.

② Time-variant : For input $x'[n] = x[n-n_0]$,

output $= x'[4n+1] = x[4n+1-n_0] \neq y[n-n_0] = x[4n-4n_0+1]$.

③ Linear

④ Not Causal

⑤ Stable

4. (a) Time-invariant $\Rightarrow$ $\delta[n-1] \longrightarrow y[n] = 2^{n-1}e^{-j(n-1)}u[n-2]$.

(b). $x[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$

Linear $\Rightarrow$ $\delta[n] + \delta[n-1] \rightarrow y[n] = 2^n e^{-jn}u[n-1] + 2^{n-1}e^{-j(n-1)}u[n-2]$.

5. P.1.30.

(a). $y(t) = x(t-4)$ is invertible

$y(t) \longrightarrow \boxed{\text{shift 4 to the left}} \longrightarrow x(t)$. $\quad (x(t) = y(t+4))$

(b). $y(t) = \cos(x(t))$ is not invertible,

$$\left.\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x(t) + 2\pi \end{aligned}\right) \longrightarrow \left.\begin{aligned} y_1(t) &= \cos(x(t)) \\ y_2(t) &= \cos(x(t)+2\pi) = \cos(x(t)) \end{aligned}\right) \text{ identical}$$

(c). $y[n] = n\,x[n]$ is <u>not invertible</u>,

since $x_1[n] = 10\cdot\delta[n]$, $x_2[n] = -10\cdot\delta[n]$ $\Rightarrow$ $\left.\begin{array}{l} y_1[n] = 0 \\ y_2[n] = 0 \end{array}\right)$ identical.

(e): $y[n] = \begin{cases} x[n-1] & n \ge 1 \\ 0 & n = 0 \\ x[n] & n \le -1 \end{cases}$ is <u>invertible</u>.

since we have inverse system $y[n] \rightarrow \boxed{} \rightarrow x[n] = \begin{cases} y[n+1] & n \ge 0 \\ y[n] & n \le -1 \end{cases}$

6. P.1.30

(f) $y[n] = x[n]\,x[n-1]$ is <u>not invertible</u>

since $x_1[n] = x[n]$, $x_2[n] = -x[n]$ gives the same result.

(g). $y[n] = x[1-n]$ is <u>invertible</u>

$y[n] \rightarrow \boxed{\text{flip to y-axis} \ \& \ \text{shift 1 to right}} \rightarrow x[n] = y[-(n-1)]$

(j). $y(t) = \dfrac{d\,x(t)}{dt}$ is <u>not invertible</u>

because if $x(t) = \text{constant}$, $y(t) = 0$.

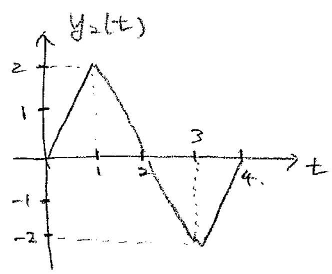
(n). $y[n] = \begin{cases} x[n/2] & \text{even} \\ 0 & \text{odd} \end{cases}$ is <u>invertible</u>

Since we have inverse system : $x[n] = y[2n]$

17. P.1.31

(a). $x_2(t) = x_1(t) - x_1(t-2)$

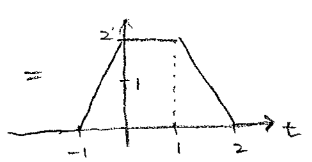
LTI $\Rightarrow$ $y_2(t) = y_1(t) - y_1(t-2) =$



(b). $x_3(t) = x_1(t+1) + x_1(t)$

LTI $\Rightarrow$ $y_3(t) = y_1(t+1) + y_1(t)$

$=$



8. P.1.43

(a). $x(t) \longrightarrow y(t)$.

Time Invariant $\Rightarrow$. $x(t+T) \longrightarrow y(t+T)$.

Since $x(t) = x(t+T)\ \forall t$ from the periodic $x(t)$ with period T,

$\therefore$ basically we input the same signal to the system.

thus, the outputs are exactly the same $\Rightarrow y(t) = y(t+T)\ \forall t$.

$\therefore$ $y(t)$ is periodic with T.

\. DT case.

$x[n] \longrightarrow y[n]$.

Time Invariant $\Rightarrow x[n+N] \rightarrow y[n+N]$

Since $x[n] = x[n+N]\ \forall$ integer n, this implies $y[n] = y[n+N]\ \forall$ integer n.

(b). No need to turn in. But, you can try, e.g.,
time-invariant system of $y(t) = x(t) - x(t-1)$

9. No need to turn in

$$\xrightarrow{x(t)} \boxed{H_1} \xrightarrow{z(t)} \boxed{H_2} \longrightarrow y(t).$$

(a). H_1 : LTI, H_2 : LTI. $\Rightarrow$ $H_2(H_1(\cdot))$ is also LTI?

Yes. ① Linear. let $y_1(t) = H_2(H_1(x_1(t)))$
$y_2(t) = H_2(H_1(x_2(t)))$.

From H_1, H_2 LTIs.

$$H_2(H_1(\alpha x_1(t) + \alpha x_2(t))) = H_2(\alpha_1 H_1(x_1(t)) + \alpha_2 H_1(x_2(t)))$$
$$= \alpha_1 H_2(H_1(x_1(t))) + \alpha_2 H_2(H_1(x_2(t))) = \alpha_1 y_1(t) + \alpha_2 y_2(t).$$

② Time-Invariant.

$$H_2(H_1(x(t-t_0))) = H_2(z(t-t_0)) = y(t-t_0).$$

(b). H_1 : non-linear , H_2 : non-linear , $H_2(H_1(\cdot))$ is non-linear ?

FALSE.

Let H_1 : $z(t) = t\,x(t)$, $t > 0$.
H_2 : $y(t) = \frac{2z(t)}{t}$, $t > 0$.
} 2 Non-linear system.

$\Rightarrow$ $H_2(H_1(\cdot))$: $y(t) = \frac{2t\,x(t)}{t} = 2x(t)$, $t > 0$.

$\therefore$ $H_2(H_1(\cdot))$ is linear.

(c). $x[n] \xrightarrow{\text{Sys 1}} w[n] \xrightarrow{\text{Sys 2}} z[n] \xrightarrow{\text{Sys 3}} y[n]$.

$$\Rightarrow y[n] = z[2n] = w[2n] + \frac{1}{2}w[2n-1] + \frac{1}{4}w[2n-2].$$

$$= x[n] + 0 + \frac{1}{4}x[n-1].$$

※ Sys 1 : $w[n] = \begin{cases} x[n/2] & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

Sys 2 : $z[n] = w[n] + \frac{1}{2}w[n-1] + \frac{1}{4}w[n-2]$.

Sys 3 : $y[n] = z[2n]$.

$\therefore$ $y[n] = x[n] + \frac{1}{4}x[n-1]$.

overall interconnected system is LTI.