HW #7 Solutions

Q1) P 3.25

(a): $x(t) = \cos(4\pi t) \Rightarrow T = \frac{1}{2}$

We have that

$$a_k = \delta[k-\ell] + \delta[k+\ell] \overset{F.S.}{\longleftrightarrow} 2\cos\left(\ell \frac{2\pi}{T} t\right)$$

where ℓ is a positive integer

$$\Rightarrow \delta[k-1] + \delta[k+1] \overset{F.S.}{\longleftrightarrow} 2\cos(4\pi t)$$

$$\Rightarrow \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \overset{F.S.}{\longleftrightarrow} \cos(4\pi t)$$

(b): $x(t) = \sin(4\pi t) \Rightarrow T = \frac{1}{2}$

You can easily see that $a_k = \delta[k-\ell] - \delta[k+\ell] \overset{F.S.}{\longleftrightarrow} 2j\sin\left(\ell \frac{2\pi}{T} t\right)$

where ℓ is a positive integer

$$\Rightarrow \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1] \overset{F.S.}{\longleftrightarrow} \sin(4\pi t)$$

(c), (d): Let us say $x(t) \overset{F.S.}{\longleftrightarrow} a_k$, $y(t) \overset{F.S.}{\longleftrightarrow} b_k$

$$z(t) = x(t)y(t) \overset{F.S.}{\longleftrightarrow} a_k * b_k = \sum_{k=-\infty}^{\infty} a_\ell b_{k-\ell}$$

$$\Rightarrow z(t) \overset{F.S.}{\longleftrightarrow} \left(\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]\right) * \left(\frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]\right)$$

using convolution properties

$$= \frac{1}{4j}\left\{ \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] - \delta[k-1]*\delta[k+1] + \delta[k+1]*\delta[k-1] - \delta[k+1]*\delta[k+1] \right\}$$

$$= \frac{1}{4j}\left\{\delta[k-2] - \delta[k+2]\right\}$$

$\Rightarrow$ Thus, $z(t) = \frac{1}{2}\sin(8\pi t)$

which matches with $\cos(4\pi t)\sin(4\pi t)$ using the fact that $2\cos\theta\sin\theta = \sin 2\theta$

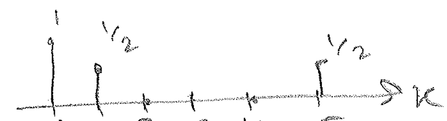
Q2) P 3.30 (a),(b)

We know that

$$x[n]=1 \xleftrightarrow{\mathcal{F.S.}} a_k = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$$

$N=6$

(a) $x[n] = 1 + \cos\left(\frac{2\pi}{6} n\right) \xleftrightarrow{\mathcal{F.S.}}$



$$\mathcal{F.S.}\updownarrow \qquad \mathcal{F.S.}\updownarrow$$

$$\begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases} \qquad \frac{1}{2}\left(\delta[k-1] + \delta[k-5]\right)$$

(b) $y[n] = \sin\left(\frac{2\pi}{6} n + \frac{\pi}{4}\right)$

$$= \frac{1}{2j}\left(e^{j\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)} - e^{-j\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)}\right)$$

$$= \frac{1}{2j} e^{j\frac{\pi}{4}} \cdot e^{j\frac{2\pi}{6}n} - \frac{1}{2j} e^{-j\frac{\pi}{4}} \cdot \underbrace{e^{-j\frac{2\pi}{6}n}}_{e^{j\frac{10\pi}{6}n}}$$

$$= \sum_{k=0}^{5} a_k e^{jk\frac{2\pi}{6}n}$$

$\Rightarrow a_1 = \frac{1}{2j} e^{j\frac{\pi}{4}} \qquad a_5 = -\frac{1}{2j} e^{-j\frac{\pi}{4}}$ otherwise zeros

Q3) P 3.30 (c)

Let $x[n] \xleftrightarrow{\mathcal{F.S.}} a_k$. $y[n] \xleftrightarrow{\mathcal{F.S.}} b_k$

$$z[n] = x[n]y[n] \xleftrightarrow{\mathcal{F.S.}} a_k \circledast_N b_k = \sum_{\ell=0}^{N-1} a_\ell b_{k-\ell}$$

$$x[n] \xleftrightarrow{\mathcal{F.S.}} a_k = \delta[k] + \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k-5]$$

$$y[n] \xleftrightarrow{\mathcal{F.S.}} b_k = \frac{1}{2j} e^{j\frac{\pi}{4}} \delta[k-1] - \frac{1}{2j} e^{-j\frac{\pi}{4}} \delta[k-5]$$

$$\Rightarrow a_k \circledast_N b_k = \left(\delta[k] + \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k-5]\right) \circledast_N$$

$$\frac{1}{2j}\left(e^{j\frac{\pi}{4}}\delta[k-1] - e^{-j\frac{\pi}{4}}\delta[k-5]\right)$$

We have that $\delta[n-\ell_1] \circledast_N \delta[n-\ell_2] = \delta[n-\mathrm{mod}(\ell_1+\ell_2, N)]$

$$\Rightarrow \quad = \frac{1}{2j}\Big(e^{j\frac{\pi}{4}}\delta[k-1] - e^{-j\frac{\pi}{4}}\delta[k-5] + \frac{1}{2}e^{j\frac{\pi}{4}}\delta[k-2] - \frac{1}{2}e^{-j\frac{\pi}{4}}\underbrace{\delta[k-6]}_{\delta[k]}$$

$$+\frac{1}{2}e^{j\frac{\pi}{4}}\underbrace{\delta[k-6]}_{\delta[k]} - \frac{1}{2}e^{-j\frac{\pi}{4}}\underbrace{\delta[k-10]}_{\delta[k-4]}\Big)$$

$$= \frac{1}{4j}\left(e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}}\right)\delta[k] + \frac{1}{2j}e^{j\frac{\pi}{4}}\delta[k-1] + \frac{1}{4j}e^{j\frac{\pi}{4}}\delta[k-2]$$

$$-\frac{1}{4j}e^{-j\frac{\pi}{4}}\delta[k-4] - \frac{1}{2j}e^{-j\frac{\pi}{4}}\delta[k-5]$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{4}\right)\delta[k] + \frac{1}{2j}\left(e^{j\frac{\pi}{4}}\delta[k-1] - e^{-j\frac{\pi}{4}}\delta[k-5]\right)$$

$$+\frac{1}{4j}\left(e^{j\frac{\pi}{4}}\delta[k-2] - e^{-j\frac{\pi}{4}}\delta[k-4]\right)$$

※ Advanced: $a_k = e^{j\theta}\delta[k-\ell] + e^{-j\theta}\delta[k-(N-\ell)]$

$$\overset{\mathcal{F.S.}}{\longleftrightarrow} 2\cos\left(\ell\frac{2\pi}{N}n + \theta\right)$$

$$a_k = e^{j\theta}\delta[k-\ell] - e^{-j\theta}\delta[k-(N-\ell)]$$

$$\overset{\mathcal{F.S.}}{\longleftrightarrow} 2j\sin\left(\ell\frac{2\pi}{N}n + \theta\right)$$

Please try verifying it by yourself.

$$\overset{\mathcal{F.S.}}{\longleftrightarrow} \quad z[n] = \frac{1}{2}\sin\left(\frac{\pi}{4}\right) + \frac{1}{2j}2j\sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)$$

$$+\frac{1}{4j}2j\sin\left(\frac{4\pi}{6}n + \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}n + \frac{\pi}{4}\right)$$

Q4) P 3.34

※ periodic impulse train $\sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{F.S.} a_k = \frac{1}{T} \quad \forall k$

$$\Rightarrow T\sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{F.S.} a_k = 1 \quad \forall K$$

$$\updownarrow F.S.$$

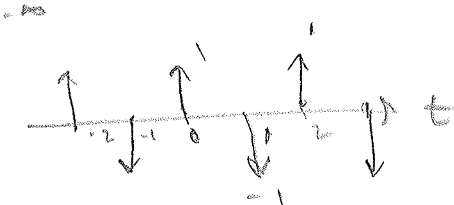
$$\sum_{k=-\infty}^{\infty} (1) e^{jk\frac{2\pi}{T}t}$$

$$\therefore \; T\sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4|\tau|} e^{-j\omega\tau} d\tau = \int_0^{\infty} e^{-4\tau} e^{-j\omega\tau} d\tau + \int_{-\infty}^{0} e^{4\tau} e^{-j\omega\tau} d\tau$$

$$= \frac{1}{4+j\omega} + \frac{1}{4-j\omega} = \frac{8}{16+\omega^2}$$

(b) $x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-n) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(t-k)$

$=$ 

Let $y(t)$ = impulse train with $T=2$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \delta(t-2k)$$

then, we can see that $x(t) = y(t) - y(t-1)$

Note that $y(t) \xleftrightarrow{F.S.} a_k = \frac{1}{2} \quad \forall K$

$$\therefore \; x(t) \xleftrightarrow{F.S.} \frac{1}{2} - \frac{1}{2} e^{-jk\frac{2\pi}{2}\cdot 1} \quad \forall K$$

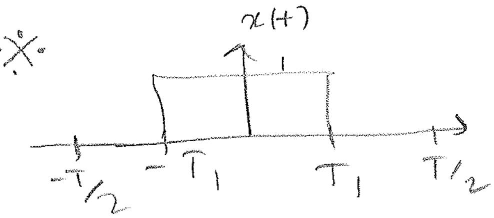
$$= \frac{1}{2} - \frac{1}{2}(-1)^K \quad \forall K$$

Thus, $x(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2} - \frac{1}{2}(-1)^k\right) e^{jk\frac{2\pi}{2}t}$

$\therefore \; y(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2} - \frac{1}{2}(-1)^k\right) H\!\left(jk\frac{2\pi}{2}\right) e^{jk\frac{2\pi}{2}t}$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\left(\frac{1}{2} - \frac{1}{2}(-1)^k\right) \cdot \frac{8}{16+k^2\pi^2}}_{\text{F.S coefficients of } y(t)} \cdot e^{jk\pi t}$$

(c) ※



$\xleftrightarrow{F.S.} \; a_k = \begin{cases} \frac{2T_1}{T} & k=0 \\ \frac{\sin\left(k\frac{2\pi T_1}{T}\right)}{k\pi} & k \neq 0 \end{cases}$

$T_1 = 1/4, \; T = 1$

$$\therefore \; y(t) \xleftrightarrow{F.S} \begin{cases} 1/2 \, H(j,0) = \frac{1}{4} & k=0 \\ \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} H(j2\pi k) = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} \cdot \frac{8}{16+4\pi^2k^2} & k \neq 0 \end{cases}$$

Q5) P 3.37

(a) $h[n] = \left(\frac{1}{2}\right)^{|n|}$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} e^{-j\omega n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}e^{-j\omega}\right)^n + \underbrace{\sum_{n=-\infty}^{0} \left(2e^{-j\omega}\right)^n}_{=\sum_{n=0}^{\infty}\left(\frac{1}{2e^{j\omega}}\right)^n}$$

$$= \frac{1}{2e^{j\omega}}\left(\frac{1}{1-1/2e^{j\omega}}\right) + \frac{1}{1-1/2e^{-j\omega}}$$

$$= \frac{1}{2e^{j\omega}-1} + \frac{2}{1-1/2e^{-j\omega}}$$

※ $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$ is periodic with period 4

$\overset{F.S.}{\Longleftrightarrow} \quad a_k = \frac{1}{4} \quad \forall\, k = \langle 4 \rangle$

$\Rightarrow \quad x[n] = \sum_{k=\langle 4 \rangle} \frac{1}{4} e^{jk\frac{2\pi}{4}n}$

Thus, $y[n] = \sum_{k=\langle 4 \rangle} \frac{1}{4} H(e^{jk\frac{2\pi}{4}}) e^{jk\frac{2\pi}{4}n}$

$\therefore \quad y[n] \overset{F.S.}{\Longleftrightarrow} b_k = \frac{1}{4} H(e^{jk\frac{2\pi}{4}}) \quad \forall\, k = \langle 4 \rangle$

where $H(e^{j\omega}) = \frac{1}{2e^{j\omega}-1} + \frac{2}{2-e^{j\omega}}$

(b) $x[n] = \begin{cases} 1 & n = 0, \pm 1 \\ 0 & n = \pm 2, \pm 3 \end{cases}$, periodic with period 6

$a_k = \frac{1}{6} \sum_{n=\langle 6 \rangle} x[n] e^{-jk\frac{2\pi}{6}n} = \frac{1}{6}\left(e^{jk\frac{2\pi}{6}} + 1 + e^{-jk\frac{2\pi}{6}}\right)$

$= \frac{1}{6}\left(1 + 2\cos\left(k\frac{2\pi}{6}\right)\right)$

$\therefore \quad y[n] \overset{F.S.}{\Longleftrightarrow} b_k = a_k \cdot H(e^{jk\frac{2\pi}{6}}) \quad \forall\, k = \langle 6 \rangle$

Q6) P3.38

$$h[n] = \begin{cases} 1 & 0 \leq n \leq 2 \\ -1 & -2 \leq n \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$= -e^{j\omega 2} - e^{j\omega} + 1 + e^{-j\omega} + e^{-j\omega 2}$$

$$= 1 - 2j\sin(\omega) - 2j\sin(2\omega)$$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k] = \sum_{k=\langle 4 \rangle} \frac{1}{4} e^{jk\frac{2\pi}{4}n} \quad \text{--- see Q5}$$

$$\therefore \quad y[n] = \sum_{k=\langle 4 \rangle} \frac{1}{4} H\left(e^{jk\frac{2\pi}{4}}\right) e^{jk\frac{2\pi}{4}n}$$

$$\overset{F.S}{\Longleftrightarrow} \quad b_k = \frac{1}{4} H\left(e^{jk\frac{2\pi}{4}}\right) \quad \forall\, k = \langle 4 \rangle$$

$$\therefore \quad b_k = \frac{1}{4}\left(1 - 2j\sin\left(\frac{k\pi}{2}\right) - 2j\underbrace{\sin(k\pi)}_{=0\ \forall k}\right)$$

$$= \frac{1}{4}\left(1 - 2j\sin\left(\frac{k\pi}{2}\right)\right) \quad \forall\, k = \langle 4 \rangle$$

Q7) P 3.39

input $x[n]$ with period $N=3$ can be represented by $x[n] = \sum_{k=\langle 3\rangle} a_k e^{jk\frac{2\pi}{3}n}$

$$x[n] \xrightarrow[\text{LTI}]{h[n]} y[n] = \sum_{k=\langle 3\rangle} a_k \underline{H(e^{jk\frac{2\pi}{3}})} e^{jk\frac{2\pi}{3}n}$$

※ $H(e^{j\omega}) = 1$ if $|\omega| \le \frac{\pi}{8}$

Thus, $y[n] \overset{F.S.}{\longleftrightarrow} b_k = a_k H(e^{jk\frac{2\pi}{3}})$

but $H(e^{jk\frac{2\pi}{3}}) = 1$ when $k=0$

$= 0$ when $k=1,2$

∴ $y[n]$ has only one non-zero F.S. coefficients.