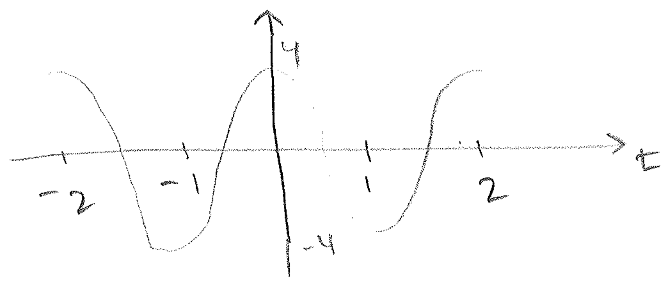
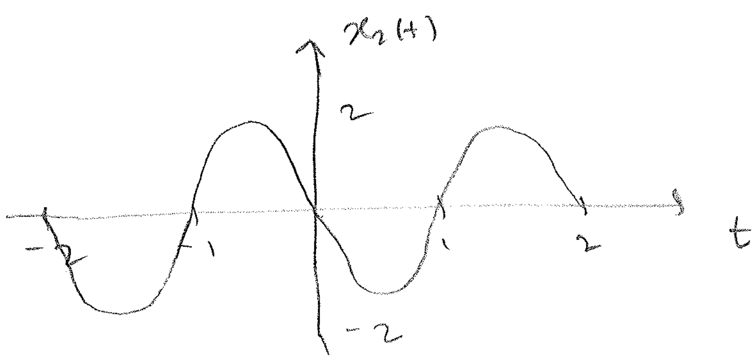
HW #6 Solutions

Questions : 1, 2, 3, 4, 5, 7, 8, 11

1. ① $x_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t} = 2e^{j\pi t} + e^{-j\pi t} = 4\cos(\pi t)$



$x_2(t) = j\left(e^{j\pi t} - e^{-j\pi t}\right) = j \cdot 2j\sin(\pi t) = -2\sin(\pi t)$



② $y(t) = x_1(t) + x_2(t)$

$$a_k = \frac{1}{2}\int_T y_1(t) e^{-jk\pi t}\, dt = \frac{1}{2}\int_T x_1(t) e^{-jk\pi t}\, dt + \frac{1}{2}\int_T x_2(t) e^{-jk\pi t}\, dt$$

= F.S. coeff. of $x_1(t)$ + F.S. coeff. of $x_2(t)$

$$= \begin{cases} 2+j & k=1 \\ 2-j & k=-1 \\ 0 & \text{otherwise} \end{cases}$$

2. ① $a_k = \delta[k-3] + \delta[k+3]$ $\quad T = 10$

$$x(t) = e^{j3\frac{2\pi}{10}t} + e^{-j3\frac{2\pi}{10}t} = 2\cos\left(\frac{3\pi}{5}t\right)$$

② $a_k = e^{-2|k|}$

$$x(t) = \sum_{k=-\infty}^{\infty} e^{-2|k|} e^{jk\frac{2\pi}{10}t} = \sum_{k=0}^{\infty} e^{-2k} e^{jk\frac{2\pi}{10}t} + \sum_{k=-1}^{-\infty} e^{2k} e^{jk\frac{2\pi}{10}t}$$

$$= \sum_{k=0}^{\infty} \underbrace{\left(e^{-2} \cdot e^{j\frac{\pi}{5}t}\right)^k}_{\left|e^{-2} \cdot e^{j\frac{\pi}{5}t}\right| < 1} + \sum_{k=1}^{\infty} \underbrace{\left(e^{-2} \cdot e^{-j\frac{\pi}{5}t}\right)^k}_{\left|e^{-2} \cdot e^{-j\frac{\pi}{5}t}\right| < 1}$$

$$= \frac{1}{1 - e^{-2}e^{j\frac{\pi}{5}t}} + \frac{e^{-2}e^{-j\frac{\pi}{5}t}}{1 - e^{-2}e^{-j\frac{\pi}{5}t}}$$

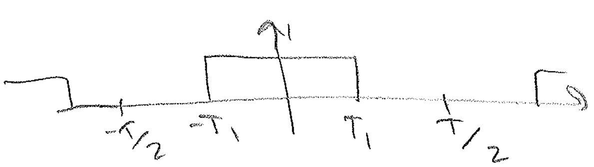
③ when $T = \sqrt{3}$,

$$x(t) = 2\cos(2\sqrt{3}\,\pi t)$$

④ when $T = \sqrt{3}$,

$$x(t) = \frac{1}{1 - e^{-2}e^{j\frac{2\pi}{\sqrt{3}}t}} + \frac{e^{-2} \cdot e^{-j\frac{2\pi}{\sqrt{3}}t}}{1 - e^{-2}e^{-j\frac{2\pi}{\sqrt{3}}t}}$$

3. From Example 3.8, we have that



$$\xrightarrow{F.S.} \quad \hat{a}_k = \frac{\sin\left(k\frac{2\pi T_1}{T}\right)}{k\pi}, \text{ when } k \neq 0$$

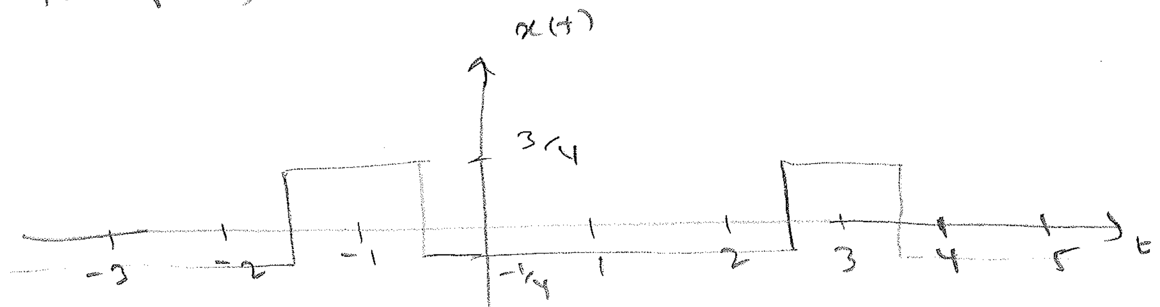
$$\hat{a}_0 = \frac{2T_1}{T}$$

(a) $$a_k = \begin{cases} 0 & k = 0 \\ \dfrac{(j)^k \sin\left(\frac{k\pi}{4}\right)}{k\pi} = \left(e^{j\frac{\pi}{2}}\right)^k \dfrac{\sin k\pi/4}{k\pi} = \left(e^{jk\frac{2\pi}{4}1}\right) \cdot \dfrac{\sin\left(\frac{k2\pi \cdot \frac{1}{2}}{4}\right)}{k\pi} \end{cases}$$

Then, we can see that $T_1 = \frac{1}{2}$ and shifting by -1,

and dc-offseting by $-\frac{1}{4}$, i.e., $x(t) = \hat{x}(t+1) - \frac{1}{4}$

Therefore,

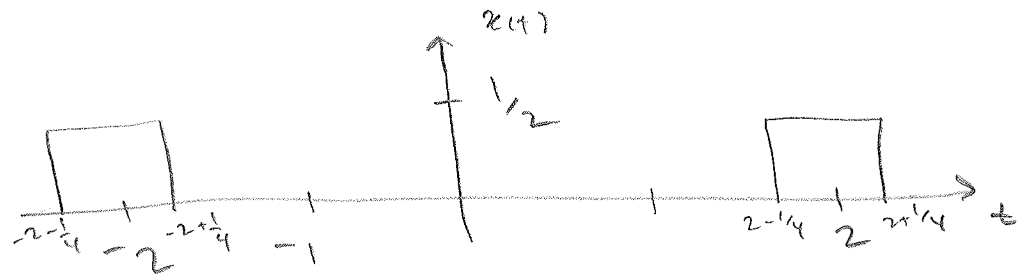


(b) $a_k = \begin{cases} 1/16 \\ (-1)^k \dfrac{\sin k\pi/8}{2k\pi} \end{cases} = \dfrac{1}{2} (e^{\pm j\pi})^k \dfrac{\sin \frac{k\pi}{8}}{k\pi}$

$= \dfrac{1}{2} \left(e^{jk\frac{2\pi}{4}(\pm 2)}\right) \cdot \dfrac{\sin\left(\frac{k \cdot 2\pi \cdot 1/4}{4}\right)}{k\pi}$

$T_1 = 1/4$ and shifting by 2, coefficient $1/2$

Therefore, $x(t) = \frac{1}{2}\hat{x}(t \pm 2)$



4. P. 3.23

(c) $a_k = \begin{cases} jk & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$

$$x(t) = \sum_{k=-2}^{2} jk\, e^{jk\frac{2\pi}{4}t} = j \sum_{k=-2}^{2} k \cdot e^{jk\frac{\pi}{2}t}$$

$$= j\left(-2e^{-j\pi t} - e^{-j\frac{\pi}{2}t} + e^{j\frac{\pi}{2}t} + 2e^{j\pi t}\right)$$

$$= 2j\left(e^{j\pi t} - e^{-j\pi t}\right) + j\left(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}\right)$$

$$= 2j \cdot 2j\sin(\pi t) + j \cdot 2j\sin\left(\frac{\pi}{2}t\right)$$

$$= -4\sin(\pi t) - 2\sin\left(\frac{\pi}{2}t\right)$$

(d) $a_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{4}t} = \sum_{k=-\infty}^{\infty} 1 \cdot e^{j(2k)\frac{\pi}{2}t} + 2 \cdot e^{j(2k+1)\frac{\pi}{2}t}$$

$$= \sum_{k=-\infty}^{\infty} e^{jk\pi t} + 2e^{jk\pi t} \cdot e^{j\frac{\pi}{2}t}$$

$$= \sum_{k=-\infty}^{\infty} e^{jk\pi t}\left(1 + 2e^{j\frac{\pi}{2}t}\right)$$

$$= \left(1 + 2e^{j\frac{\pi}{2}t}\right) \underbrace{\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{2}t}}$$

This is a F.S. representation of a period-2 signal when it's F.S. coefficients are all 1

But $\displaystyle\sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{F.S.} \frac{1}{T} \quad \forall k$

$$\sum_{k=-\infty}^{\infty} \delta(t-2k) \xleftrightarrow{F.S.} \frac{1}{2} \quad \forall k$$

$$2\sum_{k=-\infty}^{\infty} \delta(t-2k) \xleftrightarrow{F.S} 1 \quad \forall k$$

Therefore,

$$x(t) = \left(1+2e^{j\frac{\pi}{2}t}\right) \cdot 2\sum_{k=-\infty}^{\infty} \delta(t-2k)$$

$$= 2\left(1+2e^{j\frac{\pi}{2}t}\right) \cdot \sum_{k=-\infty}^{\infty} \delta(t-2k)$$

5. P.3.24

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases} \quad \text{periodic with } T=2$$

(a) $a_0 = \frac{1}{2}\int_0^2 x(t)\,dt = \frac{1}{2}$

(b) $$\frac{d}{dt}x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2 \end{cases} \quad \text{periodic with } T=2$$

$$\hat{a}_k = \frac{1}{2}\int_0^2 \left(\frac{d}{dt}x(t)\right) e^{-jk\pi t}\,dt \qquad \hat{a}_0 = 0$$

$$= \frac{1}{2}\int_0^1 e^{-jk\pi t}\,dt - \frac{1}{2}\int_1^2 e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-jk\pi}-1}{-jk\pi} - \frac{e^{-jk2\pi}-e^{-jk\pi}}{-jk\pi}\right]$$

$$= \frac{1-(-1)^k}{jk\pi}, \quad k \ne 0$$

(c) $$x(t) \xleftrightarrow{F.S.} a_k$$

$$\frac{d}{dt}x(t) \longleftrightarrow jk\frac{2\pi}{T}a_k$$

$$\therefore\ a_k = \begin{cases} 1/2 & k=0 \\ \dfrac{(-1)^k-1}{k^2\pi^2} & k \ne 0 \end{cases}$$

7.

P 3.28

(1) P. 3.28 (a) with fig. P. 3.28 (b)

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \quad , N = 6$$

$$= \frac{1}{6}\left(1 + e^{-jk\frac{\pi}{3}} + \underbrace{e^{-jk\frac{2\pi}{3}}}_{e^{jk\frac{\pi}{3}}} + \underbrace{e^{-jk\pi}}_{(-1)^k}\right)$$

$$= \frac{1}{6}\left(1 + (-1)^k + 2\cos\left(\pi/_3\, k\right)\right)$$

Since a_k is real for all k, $\angle a_k = 0 \ \forall k$ ← "for all"

(2) P. 3.28 (c)

$N = 4$. $\quad a_k = \frac{1}{4}\sum_{n=0}^{3}\left(1 - \sin\frac{\pi n}{4}\right) e^{-jk\frac{\pi}{2}n}$

$$= \frac{1}{4}\left(1 + \left(1 - \frac{\sqrt{2}}{2}\right) e^{-jk\frac{\pi}{2}} + 0 + \left(1-\frac{\sqrt{2}}{2}\right)\underbrace{e^{-jk\frac{3\pi}{2}}}_{e^{jk\frac{\pi}{2}}}\right)$$

$$= \frac{1}{4} + 2\left(1 - \frac{\sqrt{2}}{2}\right)\cos\left(k\frac{\pi}{2}\right)$$

Since a_k is real $\forall k$, $\angle a_k = 0 \ \forall k$.

8. P. 3,29 $N=8$

(a) $a_k = \cos\left(\frac{k\pi}{4}\right) + \sin\left(\frac{3k\pi}{4}\right)$

$x[n] = N\left(\delta[n-\ell] + \delta[n-(N-\ell)]\right)$ where $1 \le \ell \le N-1$

$$\xleftrightarrow{F.S.} \quad a_k = \frac{1}{N}\sum_{n=0}^{N-1}\left(N\delta[n-\ell] + N\delta[n-(N-\ell)]\right)e^{-jk\frac{2\pi}{N}n}$$

$$= e^{-jk\frac{2\pi}{N}\ell} + \underline{e^{-jk\frac{2\pi}{N}(N-\ell)}} \rightarrow \underbrace{e^{-jk\frac{2\pi}{N}n}}_{=1} \cdot e^{jk\frac{2\pi}{N}\ell}$$

$$= e^{-jk\frac{2\pi}{N}\ell} + e^{jk\frac{2\pi}{N}\ell} = 2\cos\left(k\frac{2\pi}{N}\ell\right)$$

$$\Rightarrow \frac{N}{2}\left(\delta[n-\ell] + \delta[n-(N-\ell)]\right) \xleftrightarrow{F.S.} \cos\left(k\frac{2\pi}{N}\ell\right)$$

where $1 \le \ell \le N-1$

11. P 3,33 $h(t) = e^{-4t}u(t)$

$$x(t) \xrightarrow[h(t)]{LTI} y(t) \text{ satisfies}$$

$$\frac{d}{dt}y(t) + 4y(t) = x(t).$$

(a) we know that

$$x(t) = e^{j\omega t} \xrightarrow[h(t)]{LTI} y(t) = \underbrace{e^{j\omega t}}_{x(t)} \cdot \underbrace{H(j\omega)}_{=\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau}$$

let $x(t) = \cos(2\pi t)$ with $T=1$

$$= \frac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t}\right)$$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4\tau}u(\tau)e^{-j\omega\tau}d\tau = \int_0^{\infty} e^{(-4-j\omega)\tau}d\tau$$

$$= \left[\frac{e^{-(4+j\omega)\tau}}{-4-j\omega} \right]_0^\infty = \frac{1}{4+j\omega}$$

$$\therefore \quad y(t) = \frac{1}{2} e^{j2\pi t} \cdot \frac{1}{4+j2\pi} + \frac{1}{2} e^{-j2\pi t} \cdot \frac{1}{4-j2\pi}$$

$$= \frac{1}{8+j4\pi} e^{j2\pi t} + \frac{1}{8-j4\pi} e^{-j2\pi t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{1}t}$$

$$\therefore \quad a_1 = \frac{1}{8+j4\pi}, \quad a_{-1} = \frac{1}{8-j4\pi}$$

otherwise zeros.

(b) $x(t) = \sin 4\pi t + \cos(6\pi t + \pi/4)$

$$= \frac{1}{2j}\left(e^{j4\pi t} - e^{-j4\pi t}\right) + \frac{1}{2}\left(e^{j(6\pi t + \pi/4)} + e^{-j(6\pi t + \pi/4)}\right)$$

$$= \frac{1}{2j} e^{j4\pi t} - \frac{1}{2j} e^{-j4\pi t} + \frac{1}{2} e^{j\frac{\pi}{4}} e^{j6\pi t} + \frac{1}{2} e^{-j\frac{\pi}{4}} e^{-j6\pi t}$$

$$\Rightarrow y(t) = \underbrace{\frac{1}{2j}\left(\frac{1}{4+j4\pi}\right)}_{a_2} e^{j4\pi t} - \underbrace{\frac{1}{2j}\left(\frac{1}{4-j4\pi}\right)}_{a_{-2}} e^{-j4\pi t}$$

$$+ \underbrace{\frac{1}{2} e^{j\frac{\pi}{4}} \cdot \left(\frac{1}{4+j6\pi}\right)}_{a_3} e^{j6\pi t} + \underbrace{\frac{1}{2} e^{-j\frac{\pi}{4}} \left(\frac{1}{4-j6\pi}\right)}_{a_{-3}} e^{-j6\pi t}$$