12

Fourier series

We have already discussed, in chapter 4, how complicated functions may be expressed as power series. However, this is not the only way in which a function may be represented as a series, and the subject of this chapter is the expression of functions as a sum of sine and cosine terms. Such a representation is called a *Fourier series*. Unlike Taylor series, a Fourier series can describe functions that are not everywhere continuous and/or differentiable. There are also other advantages in using trigonometric terms. They are easy to differentiate and integrate, their moduli are easily taken and each term contains only one characteristic frequency. This last point is important because, as we shall see later, Fourier series are often used to represent the response of a system to a periodic input, and this response often depends directly on the frequency content of the input. Fourier series are used in a wide variety of such physical situations, including the vibrations of a finite string, the scattering of light by a diffraction grating and the transmission of an input signal by an electronic circuit.

12.1 The Dirichlet conditions

We have already mentioned that Fourier series may be used to represent some functions for which a Taylor series expansion is not possible. The particular conditions that a function $f(x)$ must fulfil in order that it may be expanded as a Fourier series are known as the *Dirichlet conditions*, and may be summarised by the following four points:

(i) the function must be periodic;
(ii) it must be single-valued and continuous, except possibly at a finite number of finite discontinuities;
(iii) it must have only a finite number of maxima and minima within one period;
(iv) the integral over one period of $|f(x)|$ must converge.

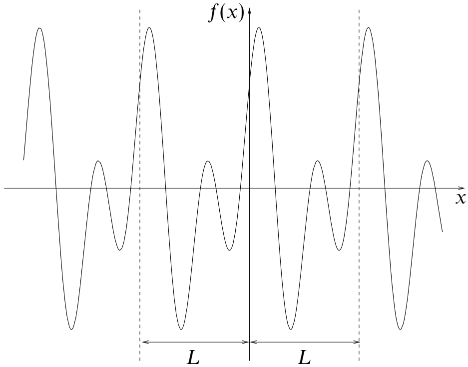

Figure 12.1 An example of a function that may be represented as a Fourier series without modification.

If the above conditions are satisfied then the Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous. The convergence of the Fourier series at points of discontinuity is discussed in section 12.4. The last three Dirichlet conditions are almost always met in real applications, but not all functions are periodic and hence do not fulfil the first condition. It may be possible, however, to represent a non-periodic function as a Fourier series by manipulation of the function into a periodic form. This is discussed in section 12.5. An example of a function that may, without modification, be represented as a Fourier series is shown in figure 12.1.

We have stated without proof that any function that satisfies the Dirichlet conditions may be represented as a Fourier series. Let us now show why this is a plausible statement. We require that any reasonable function (one that satisfies the Dirichlet conditions) can be expressed as a linear sum of sine and cosine terms. We first note that we cannot use just a sum of sine terms since sine, being an odd function (i.e. a function for which $f(-x) = -f(x)$), cannot represent even functions (i.e. functions for which $f(-x) = f(x)$). This is obvious when we try to express a function $f(x)$ that takes a non-zero value at $x = 0$. Clearly, since $\sin nx = 0$ for all values of n, we cannot represent $f(x)$ at $x = 0$ by a sine series. Similarly odd functions cannot be represented by a cosine series since cosine is an even function. Nevertheless, it is possible to represent *all* odd functions by a sine series and *all* even functions by a cosine series. Now, since all functions may be written as the sum of an odd and an even part,

$$\begin{aligned}
f(x) &= \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)] \\
&= f_{\text{even}}(x) + f_{\text{odd}}(x),
\end{aligned}$$

we can write any function as the sum of a sine series and a cosine series.

All the terms of a Fourier series are mutually orthogonal, i.e. the integrals, over one period, of the product of any two terms have the following properties:

$$\langle z_r | v_p \rangle = \int_{x_0}^{x_0+L} \sin\left(\frac{2\pi r x}{L}\right) \cos\left(\frac{2\pi p x}{L}\right) dx = 0 \quad \text{for all } r \text{ and } p, \tag{12.1}$$

$$\langle v_r | v_p \rangle = \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi r x}{L}\right) \cos\left(\frac{2\pi p x}{L}\right) dx = \begin{cases} L & \text{for } r = p = 0, \\ \frac{1}{2}L & \text{for } r = p > 0, \\ 0 & \text{for } r \neq p, \end{cases} \tag{12.2}$$

$$\langle z_r | z_p \rangle = \int_{x_0}^{x_0+L} \sin\left(\frac{2\pi r x}{L}\right) \sin\left(\frac{2\pi p x}{L}\right) dx = \begin{cases} 0 & \text{for } r = p = 0, \\ \frac{1}{2}L & \text{for } r = p > 0, \\ 0 & \text{for } r \neq p, \end{cases} \tag{12.3}$$

where r and p are integers greater than or equal to zero; these formulae are easily derived. A full discussion of why it is possible to expand a function as a sum of mutually orthogonal functions is given in chapter 17.

The Fourier series expansion of the function $f(x)$ is conventionally written

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \underbrace{\cos\left(\frac{2\pi r x}{L}\right)}_{v_r(x)} + b_r \underbrace{\sin\left(\frac{2\pi r x}{L}\right)}_{z_r(x)} \right], \tag{12.4}$$

where a_0, a_r, b_r are constants called the *Fourier coefficients*. These coefficients are analogous to those in a power series expansion and the determination of their numerical values is the essential step in writing a function as a Fourier series.

This chapter continues with a discussion of how to find the Fourier coefficients for particular functions. We then discuss simplifications to the general Fourier series that may save considerable effort in calculations. This is followed by the alternative representation of a function as a complex Fourier series, and we conclude with a discussion of Parseval's theorem.

12.2 The Fourier coefficients

We have indicated that a series that satisfies the Dirichlet conditions may be written in the form (12.4). We now consider how to find the Fourier coefficients for any particular function. For a periodic function $f(x)$ of period L we will find that the Fourier coefficients are given by

$$a_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi r x}{L}\right) dx, = \frac{\langle v_r | f \rangle}{\langle v_r | v_r \rangle} \tag{12.5}$$

$$b_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi r x}{L}\right) dx, = \frac{\langle z_r | f \rangle}{\langle z_r | z_r \rangle} \tag{12.6}$$

where x_0 is arbitrary but is often taken as 0 or $-L/2$. The apparently arbitrary factor $\frac{1}{2}$ which appears in the a_0 term in (12.4) is included so that (12.5) may apply for $r = 0$ as well as $r > 0$. The relations (12.5) and (12.6) may be derived as follows.

Suppose the Fourier series expansion of $f(x)$ can be written as in (12.4),

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \underbrace{\cos\left(\frac{2\pi rx}{L}\right)}_{v_r(x)} + b_r \underbrace{\sin\left(\frac{2\pi rx}{L}\right)}_{z_r(x)} \right] = \frac{a_0}{2} + \sum_r \left(a_r v_r(x) + b_r z_r(x)\right)$$

Then, multiplying by $\cos(2\pi px/L)$, integrating over one full period in x and changing the order of the summation and integration, we get

$$\begin{aligned}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi px}{L}\right)dx &= \frac{a_0}{2}\int_{x_0}^{x_0+L}\cos\left(\frac{2\pi px}{L}\right)dx \\ &\quad + \sum_{r=1}^{\infty} a_r \int_{x_0}^{x_0+L}\cos\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi px}{L}\right)dx \\ &\quad + \sum_{r=1}^{\infty} b_r \int_{x_0}^{x_0+L}\sin\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi px}{L}\right)dx.\end{aligned} \tag{12.7}$$

We can now find the Fourier coefficients by considering (12.7) as p takes different values. Using the orthogonality conditions (12.1)–(12.3) of the previous section, we find that when $p = 0$ (12.7) becomes

$$\int_{x_0}^{x_0+L} f(x)dx = \frac{a_0}{2}L. \qquad \langle 1|f\rangle = \frac{a_0}{2}\overbrace{\langle 1|1\rangle}^{L}$$

When $p \neq 0$ the only non-vanishing term on the RHS of (12.7) occurs when $r = p$, and so

$$\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi rx}{L}\right)dx = \frac{a_r}{2}L. \qquad \langle v_r|f\rangle = a_r \underbrace{\langle v_r|v_r\rangle}_{L/2}$$

The other Fourier coefficients b_r may be found by repeating the above process but multiplying by $\sin(2\pi px/L)$ instead of $\cos(2\pi px/L)$ (see exercise 12.2).

►*Express the square-wave function illustrated in figure 12.2 as a Fourier series.*

Physically this might represent the input to an electrical circuit that switches between a high and a low state with time period T. The square wave may be represented by

$$f(t) = \begin{cases} -1 & \text{for } -\frac{1}{2}T \le t < 0, \\ +1 & \text{for } 0 \le t < \frac{1}{2}T. \end{cases}$$

In deriving the Fourier coefficients, we note firstly that the function is an odd function and so the series will contain only sine terms (this simplification is discussed further in the

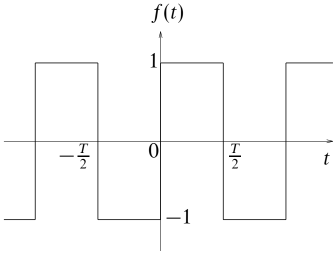

Figure 12.2 A square-wave function.

following section). To evaluate the coefficients in the sine series we use (12.6). Hence

$$\begin{aligned} b_r &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi r t}{T}\right)\,dt \\ &= \frac{4}{T}\int_{0}^{T/2} \sin\left(\frac{2\pi r t}{T}\right)\,dt \\ &= \frac{2}{\pi r}\left[1-(-1)^r\right]. \end{aligned}$$

Thus the sine coefficients are zero if r is even and equal to $4/(\pi r)$ if r is odd. Hence the Fourier series for the square-wave function may be written as

$$f(t) = \frac{4}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \cdots\right), \tag{12.8}$$

where $\omega = 2\pi/T$ is called the *angular frequency*. ◀

12.3 Symmetry considerations

The example in the previous section employed the useful property that since the function to be represented was odd, all the cosine terms of the Fourier series were absent. It is often the case that the function we wish to express as a Fourier series has a particular symmetry, which we can exploit to reduce the calculational labour of evaluating Fourier coefficients. Functions that are symmetric or antisymmetric about the origin (i.e. even and odd functions respectively) admit particularly useful simplifications. Functions that are odd in x have no cosine terms (see section 12.1) and all the a-coefficients are equal to zero. Similarly, functions that are even in x have no sine terms and all the b-coefficients are zero. Since the Fourier series of odd or even functions contain only half the coefficients required for a general periodic function, there is a considerable reduction in the algebra needed to find a Fourier series.

The consequences of symmetry or antisymmetry of the function about the quarter period (i.e. about $L/4$) are a little less obvious. Furthermore, the results are not used as often as those above and the remainder of this section can be omitted on a first reading without loss of continuity. The following argument gives the required results.

Suppose that $f(x)$ has even or odd symmetry about $L/4$, i.e. $f(L/4 - x) = \pm f(x - L/4)$. For convenience, we make the substitution $s = x - L/4$ and hence $f(-s) = \pm f(s)$. We can now see that

$$b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(s)\sin\left(\frac{2\pi rs}{L} + \frac{\pi r}{2}\right) ds,$$

where the limits of integration have been left unaltered since f is, of course, periodic in s as well as in x. If we use the expansion

$$\sin\left(\frac{2\pi rs}{L} + \frac{\pi r}{2}\right) = \sin\left(\frac{2\pi rs}{L}\right)\cos\left(\frac{\pi r}{2}\right) + \cos\left(\frac{2\pi rs}{L}\right)\sin\left(\frac{\pi r}{2}\right),$$

we can immediately see that the trigonometric part of the integrand is an odd function of s if r is even and an even function of s if r is odd. Hence if $f(s)$ is even and r is even then the integral is zero, and if $f(s)$ is odd and r is odd then the integral is zero. Similar results can be derived for the Fourier a-coefficients and we conclude that

(i) if $f(x)$ is even about $L/4$ then $a_{2r+1} = 0$ and $b_{2r} = 0$,
(ii) if $f(x)$ is odd about $L/4$ then $a_{2r} = 0$ and $b_{2r+1} = 0$.

All the above results follow automatically when the Fourier coefficients are evaluated in any particular case, but prior knowledge of them will often enable some coefficients to be set equal to zero on inspection and so substantially reduce the computational labour. As an example, the square-wave function shown in figure 12.2 is (i) an odd function of t, so that all $a_r = 0$, and (ii) even about the point $t = T/4$, so that $b_{2r} = 0$. Thus we can say immediately that only sine terms of odd harmonics will be present and therefore will need to be calculated; this is confirmed in the expansion (12.8).

12.4 Discontinuous functions

The Fourier series expansion usually works well for functions that are discontinuous in the required range. However, the series itself does not produce a discontinuous function and we state without proof that the value of the expanded $f(x)$ at a discontinuity will be half-way between the upper and lower values. Expressing this more mathematically, at a point of finite discontinuity, x_d, the Fourier series converges to

$$\tfrac{1}{2}\lim_{\epsilon\to 0}[f(x_d + \epsilon) + f(x_d - \epsilon)].$$

At a discontinuity, the Fourier series representation of the function will overshoot its value. Although as more terms are included the overshoot moves in position

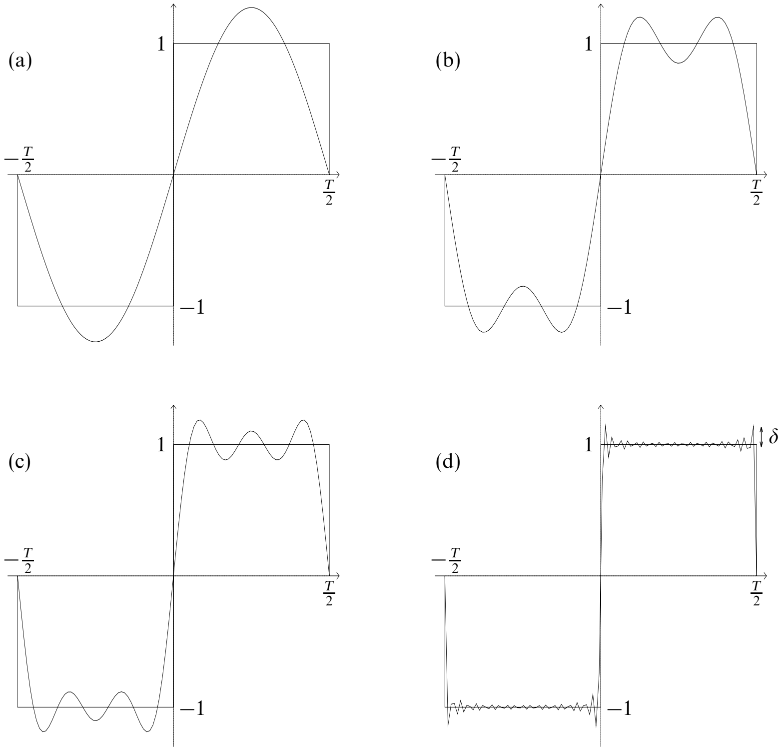

Figure 12.3 The convergence of a Fourier series expansion of a square-wave function, including (a) one term, (b) two terms, (c) three terms and (d) 20 terms. The overshoot δ is shown in (d).

arbitrarily close to the discontinuity, it never disappears even in the limit of an infinite number of terms. This behaviour is known as *Gibbs' phenomenon*. A full discussion is not pursued here but suffice it to say that the size of the overshoot is proportional to the magnitude of the discontinuity.

►*Find the value to which the Fourier series of the square-wave function discussed in section 12.2 converges at $t = 0$.*

It can be seen that the function is discontinuous at $t = 0$ and, by the above rule, we expect the series to converge to a value half-way between the upper and lower values, in other words to converge to zero in this case. Considering the Fourier series of this function, (12.8), we see that all the terms are zero and hence the Fourier series converges to zero as expected. The Gibbs phenomenon for the square-wave function is shown in figure 12.3. ◄

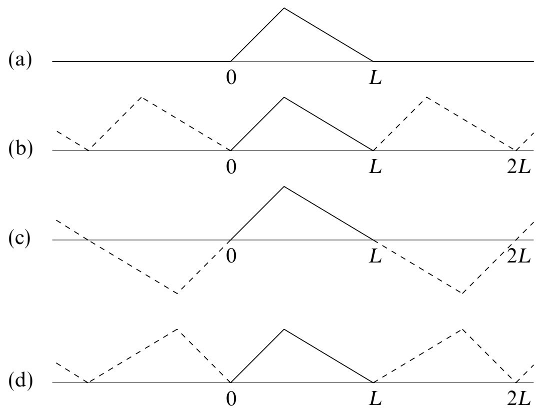

Figure 12.4 Possible periodic extensions of a function.

12.5 Non-periodic functions

We have already mentioned that a Fourier representation may sometimes be used for non-periodic functions. If we wish to find the Fourier series of a non-periodic function only within a fixed range then we may *continue* the function outside the range so as to make it periodic. The Fourier series of this periodic function would then correctly represent the non-periodic function in the desired range. Since we are often at liberty to extend the function in a number of ways, we can sometimes make it odd or even and so reduce the calculation required. Figure 12.4(b) shows the simplest extension to the function shown in figure 12.4(a). However, this extension has no particular symmetry. Figures 12.4(c), (d) show extensions as odd and even functions respectively with the benefit that only sine or cosine terms appear in the resulting Fourier series. We note that these last two extensions give a function of period $2L$.

In view of the result of section 12.4, it must be added that the continuation must not be discontinuous at the end-points of the interval of interest; if it is the series will not converge to the required value there. This requirement that the series converges appropriately may reduce the choice of continuations. This is discussed further at the end of the following example.

►*Find the Fourier series of* $f(x) = x^2$ *for* $0 < x \leq 2$.

We must first make the function periodic. We do this by extending the range of interest to $-2 < x \leq 2$ in such a way that $f(x) = f(-x)$ and then letting $f(x + 4k) = f(x)$, where k is any integer. This is shown in figure 12.5. Now we have an even function of period 4. The Fourier series will faithfully represent $f(x)$ in the range, $-2 < x \leq 2$, although not outside it. Firstly we note that since we have made the specified function even in x by extending

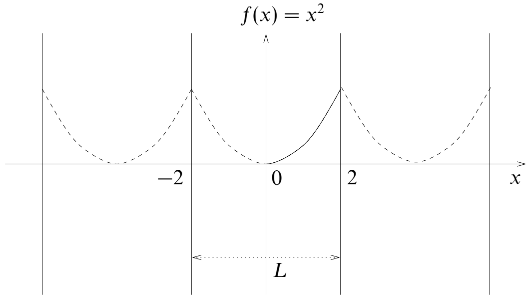

Figure 12.5 $f(x) = x^2$, $0 < x \leq 2$, with the range extended to give periodicity.

the range, all the coefficients b_r will be zero. Now we apply (12.5) and (12.6) with $L = 4$ to determine the remaining coefficients:

$$a_r = \frac{2}{4}\int_{-2}^{2} x^2 \cos\left(\frac{2\pi r x}{4}\right) dx = \frac{4}{4}\int_{0}^{2} x^2 \cos\left(\frac{\pi r x}{2}\right) dx,$$

where the second equality holds because the function is even in x. Thus

$$\begin{aligned}
a_r &= \left[\frac{2}{\pi r}x^2 \sin\left(\frac{\pi r x}{2}\right)\right]_0^2 - \frac{4}{\pi r}\int_0^2 x \sin\left(\frac{\pi r x}{2}\right) dx \\
&= \frac{8}{\pi^2 r^2}\left[x \cos\left(\frac{\pi r x}{2}\right)\right]_0^2 - \frac{8}{\pi^2 r^2}\int_0^2 \cos\left(\frac{\pi r x}{2}\right) dx \\
&= \frac{16}{\pi^2 r^2}\cos \pi r \\
&= \frac{16}{\pi^2 r^2}(-1)^r.
\end{aligned}$$

Since this expression for a_r has r^2 in its denominator, to evaluate a_0 we must return to the original definition,

$$a_r = \frac{2}{4}\int_{-2}^{2} f(x) \cos\left(\frac{\pi r x}{2}\right) dx.$$

From this we obtain

$$a_0 = \frac{2}{4}\int_{-2}^{2} x^2\, dx = \frac{4}{4}\int_{0}^{2} x^2\, dx = \frac{8}{3}.$$

The final expression for $f(x)$ is then

$$x^2 = \frac{4}{3} + 16\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi^2 r^2}\cos\left(\frac{\pi r x}{2}\right) \qquad \text{for } 0 < x \leq 2. \blacktriangleleft$$

We note that in the above example we could have extended the range so as to make the function odd. In other words we could have set $f(x) = -f(-x)$ and then made $f(x)$ periodic in such a way that $f(x + 4) = f(x)$. In this case the resulting Fourier series would be a series of just sine terms. However, although this will faithfully represent the function inside the required range, it does not converge to the correct values of $f(x) = \pm 4$ at $x = \pm 2$; it converges, instead, to zero, the average of the values at the two ends of the range.

12.6 Integration and differentiation

It is sometimes possible to find the Fourier series of a function by integration or differentiation of another Fourier series. If the Fourier series of $f(x)$ is integrated term by term then the resulting Fourier series converges to the integral of $f(x)$. Clearly, when integrating in such a way there is a constant of integration that must be found. If $f(x)$ is a continuous function of x for all x and $f(x)$ is also periodic then the Fourier series that results from differentiating term by term converges to $f'(x)$, provided that $f'(x)$ itself satisfies the Dirichlet conditions. These properties of Fourier series may be useful in calculating complicated Fourier series, since simple Fourier series may easily be evaluated (or found from standard tables) and often the more complicated series can then be built up by integration and/or differentiation.

►*Find the Fourier series of* $f(x) = x^3$ *for* $0 < x \leq 2$.

In the example discussed in the previous section we found the Fourier series for $f(x) = x^2$ in the required range. So, if we *integrate* this term by term, we obtain

$$\frac{x^3}{3} = \frac{4}{3}x + 32\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi^3 r^3}\sin\left(\frac{\pi r x}{2}\right) + c,$$

where c is, so far, an arbitrary constant. We have not yet found the Fourier series for x^3 because the term $\frac{4}{3}x$ appears in the expansion. However, by now *differentiating* the same initial expression for x^2 we obtain

$$2x = -8\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi r}\sin\left(\frac{\pi r x}{2}\right).$$

We can now write the full Fourier expansion of x^3 as

$$x^3 = -16\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi r}\sin\left(\frac{\pi r x}{2}\right) + 96\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi^3 r^3}\sin\left(\frac{\pi r x}{2}\right) + c.$$

Finally, we can find the constant, c, by considering $f(0)$. At $x = 0$, our Fourier expansion gives $x^3 = c$ since all the sine terms are zero, and hence $c = 0$. ◄

12.7 Complex Fourier series

As a Fourier series expansion in general contains both sine and cosine parts, it may be written more compactly using a complex exponential expansion. This simplification makes use of the property that $\exp(irx) = \cos rx + i\sin rx$. The complex Fourier series expansion is written

$$f(x) = \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi i r x}{L}\right), = \sum_r c_r\, y_r(x) \qquad (12.9)$$

where the Fourier coefficients are given by

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\exp\left(-\frac{2\pi irx}{L}\right)dx. = \frac{\langle y_r | f\rangle}{\langle y_r | y_r\rangle} \tag{12.10}$$

This relation can be derived, in a similar manner to that of section 12.2, by multiplying (12.9) by $\exp(-2\pi ipx/L)$ before integrating and using the orthogonality relation

$$\langle y_p | y_r\rangle = \int_{x_0}^{x_0+L} \exp\left(-\frac{2\pi ipx}{L}\right)\exp\left(\frac{2\pi irx}{L}\right)dx = \begin{cases} L & \text{for } r = p, \\ 0 & \text{for } r \neq p. \end{cases}$$

The complex Fourier coefficients in (12.9) have the following relations to the real Fourier coefficients:

$$\begin{aligned} c_r &= \tfrac{1}{2}(a_r - ib_r), \\ c_{-r} &= \tfrac{1}{2}(a_r + ib_r). \end{aligned} \tag{12.11}$$

Note that if $f(x)$ is real then $c_{-r} = c_r^*$, where the asterisk represents complex conjugation.

►*Find a complex Fourier series for $f(x) = x$ in the range $-2 < x < 2$.*

Using (12.10), for $r \neq 0$,

$$\begin{aligned} c_r &= \frac{1}{4}\int_{-2}^{2} x\exp\left(-\frac{\pi irx}{2}\right)dx \\ &= \left[-\frac{x}{2\pi ir}\exp\left(-\frac{\pi irx}{2}\right)\right]_{-2}^{2} + \int_{-2}^{2}\frac{1}{2\pi ir}\exp\left(-\frac{\pi irx}{2}\right)dx \\ &= -\frac{1}{\pi ir}[\exp(-\pi ir) + \exp(\pi ir)] + \left[\frac{1}{r^2\pi^2}\exp\left(-\frac{\pi irx}{2}\right)\right]_{-2}^{2} \\ &= \frac{2i}{\pi r}\cos\pi r - \frac{2i}{r^2\pi^2}\sin\pi r = \frac{2i}{\pi r}(-1)^r. \end{aligned} \tag{12.12}$$

For $r = 0$, we find $c_0 = 0$ and hence

$$x = \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} \frac{2i(-1)^r}{r\pi}\exp\left(\frac{\pi irx}{2}\right).$$

We note that the Fourier series derived for x in section 12.6 gives $a_r = 0$ for all r and

$$b_r = -\frac{4(-1)^r}{\pi r},$$

and so, using (12.11), we confirm that c_r and c_{-r} have the forms derived above. It is also apparent that the relationship $c_r^* = c_{-r}$ holds, as we expect since $f(x)$ is real. ◄

12.8 Parseval's theorem

Parseval's theorem gives a useful way of relating the Fourier coefficients to the function that they describe. Essentially a conservation law, it states that

$$\frac{1}{L}\langle f|f\rangle = \frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 dx = \sum_{r=-\infty}^{\infty} |c_r|^2$$
$$= \left(\tfrac{1}{2}a_0\right)^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2). \qquad (12.13)$$

In a more memorable form, this says that the sum of the moduli squared of the complex Fourier coefficients is equal to the average value of $|f(x)|^2$ over one period. Parseval's theorem can be proved straightforwardly by writing $f(x)$ as a Fourier series and evaluating the required integral, but the algebra is messy. Therefore, we shall use an alternative method, for which the algebra is simple and which in fact leads to a more general form of the theorem.

Let us consider two functions $f(x)$ and $g(x)$, which are (or can be made) periodic with period L and which have Fourier series (expressed in complex form)

$$f(x) = \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi i r x}{L}\right),$$
$$g(x) = \sum_{r=-\infty}^{\infty} \gamma_r \exp\left(\frac{2\pi i r x}{L}\right),$$

where c_r and γ_r are the complex Fourier coefficients of $f(x)$ and $g(x)$ respectively. Thus

$$f(x)g^*(x) = \sum_{r=-\infty}^{\infty} c_r g^*(x) \exp\left(\frac{2\pi i r x}{L}\right).$$

Integrating this equation with respect to x over the interval $(x_0, x_0 + L)$ and dividing by L, we find

$$\underbrace{\frac{1}{L}\int_{x_0}^{x_0+L} f(x)g^*(x)\,dx} = \sum_{r=-\infty}^{\infty} c_r \frac{1}{L}\int_{x_0}^{x_0+L} g^*(x)\exp\left(\frac{2\pi i r x}{L}\right) dx$$
$$= \sum_{r=-\infty}^{\infty} c_r \left[\frac{1}{L}\int_{x_0}^{x_0+L} g(x)\exp\left(\frac{-2\pi i r x}{L}\right) dx\right]^*$$
$$\frac{1}{L}\langle g|f\rangle = \quad = \sum_{r=-\infty}^{\infty} c_r\gamma_r^*,$$

where the last equality uses (12.10). Finally, if we let $g(x) = f(x)$ then we obtain Parseval's theorem (12.13). This result can be proved in a similar manner using the sine and cosine form of the Fourier series, but the algebra is slightly more complicated.

Parseval's theorem is sometimes used to sum series. However, if one is presented with a series to sum, it is not usually possible to decide which Fourier series should be used to evaluate it. Rather, useful summations are nearly always found serendipitously. The following example shows the evaluation of a sum by a Fourier series method.

►*Using Parseval's theorem and the Fourier series for $f(x) = x^2$ found in section 12.5, calculate the sum $\sum_{r=1}^{\infty} r^{-4}$.*

Firstly we find the average value of $[f(x)]^2$ over the interval $-2 < x \leq 2$:

$$\frac{1}{4}\int_{-2}^{2} x^4\,dx = \frac{16}{5}.$$

Now we evaluate the right-hand side of (12.13):

$$\left(\tfrac{1}{2}a_0\right)^2 + \tfrac{1}{2}\sum_{1}^{\infty} a_r^2 + \tfrac{1}{2}\sum_{1}^{\infty} b_n^2 = \left(\tfrac{4}{3}\right)^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}\frac{16^2}{\pi^4 r^4}.$$

Equating the two expression we find

$$\sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{\pi^4}{90}. \blacktriangleleft$$

12.9 Exercises

12.1 Prove the orthogonality relations stated in section 12.1.

12.2 Derive the Fourier coefficients b_r in a similar manner to the derivation of the a_r in section 12.2.

12.3 Which of the following functions of x could be represented by a Fourier series over the range indicated?

(a) $\tanh^{-1}(x)$, $\quad -\infty < x < \infty$;
(b) $\tan x$, $\quad -\infty < x < \infty$;
(c) $|\sin x|^{-1/2}$, $\quad -\infty < x < \infty$;
(d) $\cos^{-1}(\sin 2x)$, $\quad -\infty < x < \infty$;
(e) $x\sin(1/x)$, $\quad -\pi^{-1} < x \leq \pi^{-1}$, cyclically repeated.

12.4 By moving the origin of t to the centre of an interval in which $f(t) = +1$, i.e. by changing to a new independent variable $t' = t - \frac{1}{4}T$, express the square-wave function in the example in section 12.2 as a cosine series. Calculate the Fourier coefficients involved (a) directly and (b) by changing the variable in result (12.8).

12.5 Find the Fourier series of the function $f(x) = x$ in the range $-\pi < x \leq \pi$. Hence show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

12.6 For the function

$$f(x) = 1 - x, \qquad 0 \leq x \leq 1,$$

find (a) the Fourier sine series and (b) the Fourier cosine series. Which would be better for numerical evaluation? Relate your answer to the relevant periodic continuations.

12.7 For the continued functions used in exercise 12.6 and the derived corresponding series, consider (i) their derivatives and (ii) their integrals. Do they give meaningful equations? You will probably find it helpful to sketch all the functions involved.

12.8 The function $y(x) = x \sin x$ for $0 \le x \le \pi$ is to be represented by a Fourier series of period 2π that is either even or odd. By sketching the function and considering its derivative, determine which series will have the more rapid convergence. Find the full expression for the better of these two series, showing that the convergence $\sim n^{-3}$ and that alternate terms are missing.

12.9 Find the Fourier coefficients in the expansion of $f(x) = \exp x$ over the range $-1 < x < 1$. What value will the expansion have when $x = 2$?

12.10 By integrating term by term the Fourier series found in the previous question and using the Fourier series for $f(x) = x$ found in section 12.6, show that $\int \exp x \, dx = \exp x + c$. Why is it not possible to show that $d(\exp x)/dx = \exp x$ by differentiating the Fourier series of $f(x) = \exp x$ in a similar manner?

12.11 Consider the function $f(x) = \exp(-x^2)$ in the range $0 \le x \le 1$. Show how it should be continued to give as its Fourier series a series (the actual form is not wanted) (a) with only cosine terms, (b) with only sine terms, (c) with period 1 and (d) with period 2.

Would there be any difference between the values of the last two series at (i) $x = 0$, (ii) $x = 1$?

12.12 Find, without calculation, which terms will be present in the Fourier series for the periodic functions $f(t)$, of period T, that are given in the range $-T/2$ to $T/2$ by:

(a) $f(t) = 2$ for $0 \le |t| < T/4$, $f = 1$ for $T/4 \le |t| < T/2$;
(b) $f(t) = \exp[-(t - T/4)^2]$;
(c) $f(t) = -1$ for $-T/2 \le t < -3T/8$ and $3T/8 \le t < T/2, f(t) = 1$ for $-T/8 \le t < T/8$; the graph of f is completed by two straight lines in the remaining ranges so as to form a continuous function.

12.13 Consider the representation as a Fourier series of the displacement of a string lying in the interval $0 \le x \le L$ and fixed at its ends, when it is pulled aside by y_0 at the point $x = L/4$. Sketch the continuations for the region outside the interval that will

(a) produce a series of period L,
(b) produce a series that is antisymmetric about $x = 0$, and
(c) produce a series that will contain only cosine terms.
(d) What are (i) the periods of the series in (b) and (c) and (ii) the value of the 'a_0-term' in (c)?
(e) Show that a typical term of the series obtained in (b) is

$$\frac{32 y_0}{3n^2\pi^2} \sin \frac{n\pi}{4} \sin \frac{n\pi x}{L}.$$

12.14 Show that the Fourier series for the function $y(x) = |x|$ in the range $-\pi \le x < \pi$ is

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}.$$

By integrating this equation term by term from 0 to x, find the function $g(x)$ whose Fourier series is

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3}.$$

Deduce the value of the sum S of the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots .$$

12.15 Using the result of exercise 12.14, determine, as far as possible by inspection, the forms of the functions of which the following are the Fourier series:

(a)

$$\cos\theta + \frac{1}{9}\cos 3\theta + \frac{1}{25}\cos 5\theta + \cdots ;$$

(b)

$$\sin\theta + \frac{1}{27}\sin 3\theta + \frac{1}{125}\sin 5\theta + \cdots ;$$

(c)

$$\frac{L^2}{3} - \frac{4L^2}{\pi^2}\left[\cos\frac{\pi x}{L} - \frac{1}{4}\cos\frac{2\pi x}{L} + \frac{1}{9}\cos\frac{3\pi x}{L} - \cdots\right].$$

(You may find it helpful to first set $x = 0$ in the quoted result and so obtain values for $S_o = \sum(2m+1)^{-2}$ and other sums derivable from it.)

12.16 By finding a cosine Fourier series of period 2 for the function $f(t)$ that takes the form $f(t) = \cosh(t-1)$ in the range $0 \le t \le 1$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 + 1} = \frac{1}{e^2 - 1}.$$

Deduce values for the sums $\sum(n^2\pi^2+1)^{-1}$ over odd n and even n separately.

12.17 Find the (real) Fourier series of period 2 for $f(x) = \cosh x$ and $g(x) = x^2$ in the range $-1 \le x \le 1$. By integrating the series for $f(x)$ twice, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2(n^2\pi^2+1)} = \frac{1}{2}\left(\frac{1}{\sinh 1} - \frac{5}{6}\right).$$

12.18 Express the function $f(x) = x^2$ as a Fourier sine series in the range $0 < x \le 2$ and show that it converges to zero at $x = \pm 2$.

12.19 Demonstrate explicitly for the square-wave function discussed in section 12.2 that Parseval's theorem (12.13) is valid. You will need to use the relationship

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

Show that a filter that transmits frequencies only up to $8\pi/T$ will still transmit more than 90% of the power in such a square-wave voltage signal.

12.20 Show that the Fourier series for $|\sin\theta|$ in the range $-\pi \le \theta \le \pi$ is given by

$$|\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2 - 1}.$$

By setting $\theta = 0$ and $\theta = \pi/2$, deduce values for

$$\sum_{m=1}^{\infty} \frac{1}{4m^2 - 1} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{1}{16m^2 - 1}.$$

12.21 Find the complex Fourier series for the periodic function of period 2π defined in the range $-\pi \le x \le \pi$ by $y(x) = \cosh x$. By setting $x = 0$ prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{1}{2}\left(\frac{\pi}{\sinh \pi} - 1\right).$$

12.22 The repeating output from an electronic oscillator takes the form of a sine wave $f(t) = \sin t$ for $0 \le t \le \pi/2$; it then drops instantaneously to zero and starts again. The output is to be represented by a complex Fourier series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{4nti}.$$

Sketch the function and find an expression for c_n. Verify that $c_{-n} = c_n^*$. Demonstrate that setting $t = 0$ and $t = \pi/2$ produces differing values for the sum

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}.$$

Determine the correct value and check it using the result of exercise 12.20.

12.23 Apply Parseval's theorem to the series found in the previous exercise and so derive a value for the sum of the series

$$\frac{17}{(15)^2} + \frac{65}{(63)^2} + \frac{145}{(143)^2} + \cdots + \frac{16n^2+1}{(16n^2-1)^2} + \cdots.$$

12.24 A string, anchored at $x = \pm L/2$, has a fundamental vibration frequency of $2L/c$, where c is the speed of transverse waves on the string. It is pulled aside at its centre point by a distance y_0 and released at time $t = 0$. Its subsequent motion can be described by the series

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$

Find a general expression for a_n and show that only the odd harmonics of the fundamental frequency are present in the sound generated by the released string. By applying Parseval's theorem, find the sum S of the series $\sum_0^{\infty} (2m+1)^{-4}$.

12.25 Show that Parseval's theorem for two real functions whose Fourier expansions have cosine and sine coefficients a_n, b_n and α_n, β_n takes the form

$$\frac{1}{L}\int_0^L f(x)g^*(x)\,dx = \frac{1}{4}a_0\alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n).$$

(a) Demonstrate that for $g(x) = \sin mx$ or $\cos mx$ this reduces to the definition of the Fourier coefficients.
(b) Explicitly verify the above result for the case in which $f(x) = x$ and $g(x)$ is the square-wave function, both in the interval $-1 \le x \le 1$.

12.26 An odd function $f(x)$ of period 2π is to be approximated by a Fourier sine series having only m terms. The error in this approximation is measured by the square deviation

$$E_m = \int_{-\pi}^{\pi}\left[f(x) - \sum_{n=1}^{m} b_n \sin nx\right]^2 dx.$$

By differentiating E_m with respect to the coefficients b_n, find the values of b_n that minimise E_m.

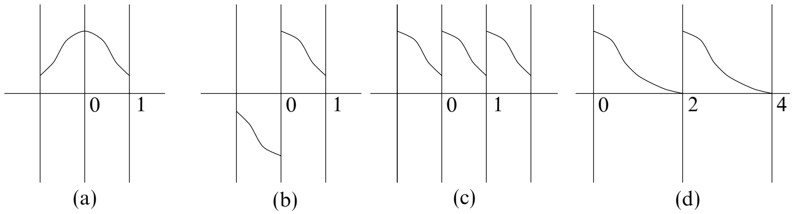

Figure 12.6 Continuations of $\exp(-x^2)$ in $0 \le x \le 1$ to give: (a) cosine terms only; (b) sine terms only; (c) period 1; (d) period 2.

Sketch the graph of the function $f(x)$, where

$$f(x) = \begin{cases} -x(\pi + x) & \text{for } -\pi \le x < 0, \\ x(x - \pi) & \text{for } 0 \le x < \pi. \end{cases}$$

If $f(x)$ is to be approximated by the first three terms of a Fourier sine series, what values should the coefficients have so as to minimise E_3? What is the resulting value of E_3?

12.10 Hints and answers

12.1 Note that the only integral of a sinusoid around a complete cycle of length L that is not zero is the integral of $\cos(2\pi nx/L)$ when $n = 0$.

12.3 Only (c). In terms of the Dirichlet conditions (section 12.1), the others fail as follows: (a) (i); (b) (ii); (d) (ii); (e) (iii).

12.5 $f(x) = 2\sum_1^\infty (-1)^{n+1} n^{-1} \sin nx$; set $x = \pi/2$.

12.7 (i) Series (a) from exercise 12.6 does not converge and cannot represent the function $y(x) = -1$. Series (b) reproduces the square-wave function of equation (12.8).
(ii) Series (a) gives the series for $y(x) = -x - \frac{1}{2}x^2 - \frac{1}{2}$ in the range $-1 \le x \le 0$ and for $y(x) = x - \frac{1}{2}x^2 - \frac{1}{2}$ in the range $0 \le x \le 1$. Series (b) gives the series for $y(x) = x + \frac{1}{2}x^2 + \frac{1}{2}$ in the range $-1 \le x \le 0$ and for $y(x) = x - \frac{1}{2}x^2 + \frac{1}{2}$ in the range $0 \le x \le 1$.

12.9 $f(x) = (\sinh 1)\left\{1 + 2\sum_1^\infty (-1)^n (1 + n^2\pi^2)^{-1} [\cos(n\pi x) - n\pi \sin(n\pi x)]\right\}$.
The series will converge to the same value as it does at $x = 0$, i.e. $f(0) = 1$.

12.11 See figure 12.6. (c) (i) $(1 + e^{-1})/2$, (ii) $(1 + e^{-1})/2$; (d) (i) $(1 + e^{-4})/2$, (ii) e^{-1}.

12.13 (d) (i) The periods are both $2L$; (ii) $y_0/2$.

12.15 $S_o = \pi^2/8$. If $S_e = \sum(2m)^{-2}$ then $S_e = \frac{1}{4}(S_e + S_o)$, yielding $S_o - S_e = \pi^2/12$ and $S_e + S_o = \pi^2/6$.
(a) $(\pi/4)(\pi/2 - |\theta|)$; (b) $(\pi\theta/4)(\pi/2 - |\theta|/2)$ from integrating (a). (c) Even function; average value $L^2/3$; $y(0) = 0$; $y(L) = L^2$; probably $y(x) = x^2$. Compare with the worked example in section 12.5.

12.17 $\cosh x = (\sinh 1)[1 + 2\sum_{n=1}^\infty (-1)^n (\cos n\pi x)/(n^2\pi^2 + 1)]$ and after integrating twice this form must be recovered. Use $x^2 = \frac{1}{3} + 4\sum(-1)^n (\cos n\pi x)/(n^2\pi^2)]$ to eliminate the quadratic term arising from the constants of integration; there is no linear term.

12.19 $C_{\pm(2m+1)} = \mp 2i/[(2m+1)\pi]$; $\sum |C_n|^2 = (4/\pi^2) \times 2 \times (\pi^2/8)$; the values $n = \pm 1$, ± 3 contribute $> 90\%$ of the total.

12.21 $c_n = [(-1)^n \sinh \pi]/[\pi(1 + n^2)]$. Having set $x = 0$, separate out the $n = 0$ term and note that $(-1)^n = (-1)^{-n}$.

12.23 $(\pi^2 - 8)/16$.

12.25 (b) All a_n and α_n are zero; $b_n = 2(-1)^{n+1}/(n\pi)$ and $\beta_n = 4/(n\pi)$. You will need the result quoted in exercise 12.19.