21

Partial differential equations: separation of variables and other methods

In the previous chapter we demonstrated the methods by which general solutions of some partial differential equations (PDEs) may be obtained in terms of arbitrary functions. In particular, solutions containing the independent variables in definite combinations were sought, thus reducing the effective number of them.

In the present chapter we begin by taking the opposite approach, namely that of trying to keep the independent variables as separate as possible, using the method of separation of variables. We then consider integral transform methods by which one of the independent variables may be eliminated, at least from differential coefficients. Finally, we discuss the use of Green's functions in solving inhomogeneous problems.

21.1 Separation of variables: the general method

Suppose we seek a solution $u(x, y, z, t)$ to some PDE (expressed in Cartesian coordinates). Let us attempt to obtain one that has the product form[§]

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t). \tag{21.1}$$

A solution that has this form is said to be *separable* in x, y, z and t, and seeking solutions of this form is called the method of *separation of variables.*

As simple examples we may observe that, of the functions

$$\text{(i) } xyz^2 \sin bt, \qquad \text{(ii) } xy + zt, \qquad \text{(iii) } (x^2 + y^2)z \cos \omega t,$$

(i) is completely separable, (ii) is inseparable in that no single variable can be separated out from it and written as a multiplicative factor, whilst (iii) is separable in z and t but not in x and y.

[§] It should be noted that the conventional use here of upper-case (capital) letters to denote the functions of the corresponding lower-case variable is intended to enable an easy correspondence between a function and its argument to be made.

When seeking PDE solutions of the form (21.1), we are requiring not that there is no connection at all between the functions X, Y, Z and T (for example, certain parameters may appear in two or more of them), but only that X does not depend upon y, z, t, that Y does not depend on x, z, t, and so on.

For a general PDE it is likely that a separable solution is impossible, but certainly some common and important equations do have useful solutions of this form, and we will illustrate the method of solution by studying the three-dimensional wave equation

$$\nabla^2 u(\mathbf{r}) = \frac{1}{c^2}\frac{\partial^2 u(\mathbf{r})}{\partial t^2}. \tag{21.2}$$

We will work in Cartesian coordinates for the present and assume a solution of the form (21.1); the solutions in alternative coordinate systems, e.g. spherical or cylindrical polars, are considered in section 21.3. Expressed in Cartesian coordinates (21.2) takes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}; \tag{21.3}$$

substituting (21.1) gives

$$\frac{d^2X}{dx^2}YZT + X\frac{d^2Y}{dy^2}ZT + XY\frac{d^2Z}{dz^2}T = \frac{1}{c^2}XYZ\frac{d^2T}{dt^2},$$

which can also be written as

$$X''YZT + XY''ZT + XYZ''T = \frac{1}{c^2}XYZT'', \tag{21.4}$$

where in each case the primes refer to the *ordinary* derivative with respect to the independent variable upon which the function depends. This emphasises the fact that each of the functions X, Y, Z and T has only one independent variable and thus its only derivative is its total derivative. For the same reason, in each term in (21.4) three of the four functions are unaltered by the partial differentiation and behave exactly as constant multipliers.

If we now divide (21.4) throughout by $u = XYZT$ we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2}\frac{T''}{T}. \tag{21.5}$$

This form shows the particular characteristic that is the basis of the method of separation of variables, namely that of the four terms the first is a function of x only, the second of y only, the third of z only and the RHS a function of t only and yet there is an equation connecting them. This can only be so for all x, y, z and t if *each* of the terms does not in fact, despite appearances, depend upon the corresponding independent variable but *is equal to a constant*, the four constants being such that (21.5) is satisfied.

Since there is only one equation to be satisfied and four constants involved, there is considerable freedom in the values they may take. For the purposes of our illustrative example let us make the choice of $-l^2$, $-m^2$, $-n^2$, for the first three constants. The constant associated with $c^{-2}T''/T$ must then have the value $-\mu^2 = -(l^2 + m^2 + n^2)$.

Having recognised that each term of (21.5) is individually equal to a constant (or parameter), we can now replace (21.5) by four separate ordinary differential equations (ODEs):

$$\frac{X''}{X} = -l^2, \qquad \frac{Y''}{Y} = -m^2, \qquad \frac{Z''}{Z} = -n^2, \qquad \frac{1}{c^2}\frac{T''}{T} = -\mu^2. \tag{21.6}$$

The important point to notice is not the simplicity of the equations (21.6) (the corresponding ones for a general PDE are usually far from simple) but that, by the device of assuming a separable solution, a *partial* differential equation (21.3), containing derivatives with respect to the four independent variables all in one equation, has been reduced to four *separate ordinary* differential equations (21.6). The ordinary equations are connected through four constant parameters that satisfy an algebraic relation. These constants are called *separation constants*.

The general solutions of the equations (21.6) can be deduced straightforwardly and are

$$\begin{aligned} X(x) &= A\exp(ilx) + B\exp(-ilx), \\ Y(y) &= C\exp(imy) + D\exp(-imy), \\ Z(z) &= E\exp(inz) + F\exp(-inz), \\ T(t) &= G\exp(ic\mu t) + H\exp(-ic\mu t), \end{aligned} \tag{21.7}$$

where $A, B, \ldots, H$ are constants, which may be determined if boundary condtions are imposed on the solution. Depending on the geometry of the problem and any boundary conditions, it is sometimes more appropriate to write the solutions (21.7) in the alternative form

$$\begin{aligned} X(x) &= A'\cos lx + B'\sin lx, \\ Y(y) &= C'\cos my + D'\sin my, \\ Z(z) &= E'\cos nz + F'\sin nz, \\ T(t) &= G'\cos(c\mu t) + H'\sin(c\mu t), \end{aligned} \tag{21.8}$$

for some different set of constants $A', B', \ldots, H'$. Clearly the choice of how best to represent the solution depends on the problem being considered.

As an example, suppose that we take as particular solutions the four functions

$$X(x) = \exp(ilx), \qquad Y(y) = \exp(imy),$$

$$Z(z) = \exp(inz), \qquad T(t) = \exp(-ic\mu t).$$

This gives a particular solution of the original PDE (21.3)

$$\begin{aligned} u(x,y,z,t) &= \exp(ilx)\exp(imy)\exp(inz)\exp(-ic\mu t) \\ &= \exp[i(lx+my+nz-c\mu t)], \end{aligned}$$

which is a special case of the solution (20.33) obtained in the previous chapter and represents a plane wave of unit amplitude propagating in a direction given by the vector with components l, m, n in a Cartesian coordinate system. In the conventional notation of wave theory, l, m and n are the components of the wave-number vector $\mathbf{k}$, whose magnitude is given by $k = 2\pi/\lambda$, where λ is the wavelength of the wave; $c\mu$ is the angular frequency ω of the wave. This gives the equation in the form

$$\begin{aligned} u(x,y,z,t) &= \exp[i(k_x x + k_y y + k_z z - \omega t)] \\ &= \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \end{aligned}$$

and makes the exponent dimensionless.

The method of separation of variables can be applied to many commonly occurring PDEs encountered in physical applications.

►*Use the method of separation of variables to obtain for the one-dimensional diffusion equation*

$$\kappa\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{21.9}$$

a solution that tends to zero as $t \to \infty$ for all x.

Here we have only two independent variables x and t and we therefore assume a solution of the form

$$u(x,t) = X(x)T(t).$$

Substituting this expression into (21.9) and dividing through by $u = XT$ (and also by κ) we obtain

$$\frac{X''}{X} = \frac{T'}{\kappa T}.$$

Now, arguing exactly as above that the LHS is a function of x only and the RHS is a function of t only, we conclude that each side must equal a constant, which, anticipating the result and noting the imposed boundary condition, we will take as $-\lambda^2$. This gives us two ordinary equations,

$$X'' + \lambda^2 X = 0, \tag{21.10}$$

$$T' + \lambda^2\kappa T = 0, \tag{21.11}$$

which have the solutions

$$\begin{aligned} X(x) &= A\cos\lambda x + B\sin\lambda x, \\ T(t) &= C\exp(-\lambda^2\kappa t). \end{aligned}$$

Combining these to give the assumed solution $u = XT$ yields (absorbing the constant C into A and B)

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\exp(-\lambda^2\kappa t). \tag{21.12}$$

In order to satisfy the boundary condition $u \to 0$ as $t \to \infty$, $\lambda^2\kappa$ must be > 0. Since κ is real and > 0, this implies that λ is a real non-zero number and that the solution is sinusoidal in x and is not a disguised hyperbolic function; this was our reason for choosing the separation constant as $-\lambda^2$. ◄

As a final example we consider Laplace's equation in Cartesian coordinates; this may be treated in a similar manner.

►*Use the method of separation of variables to obtain a solution for the two-dimensional Laplace equation,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{21.13}$$

If we assume a solution of the form $u(x, y) = X(x)Y(y)$ then, following the above method, and taking the separation constant as λ^2, we find

$$X'' = \lambda^2 X, \qquad Y'' = -\lambda^2 Y.$$

Taking λ^2 as > 0, the general solution becomes

$$u(x, y) = (A \cosh \lambda x + B \sinh \lambda x)(C \cos \lambda y + D \sin \lambda y). \tag{21.14}$$

An alternative form, in which the exponentials are written explicitly, may be useful for other geometries or boundary conditions:

$$u(x, y) = [A \exp \lambda x + B \exp(-\lambda x)](C \cos \lambda y + D \sin \lambda y), \tag{21.15}$$

with different constants A and B.

If $\lambda^2 < 0$ then the roles of x and y interchange. The particular combination of sinusoidal and hyperbolic functions and the values of λ allowed will be determined by the geometrical properties of any specific problem, together with any prescribed or necessary boundary conditions. ◄

We note here that a particular case of the solution (21.14) links up with the 'combination' result $u(x, y) = f(x + iy)$ of the previous chapter (equations (20.24) and following), namely that if $A = B$ and $D = iC$ then the solution is the same as $f(p) = AC \exp \lambda p$ with $p = x + iy$.

21.2 Superposition of separated solutions

It will be noticed in the previous two examples that there is considerable freedom in the values of the separation constant λ, the only essential requirement being that λ has the *same* value in both parts of the solution, i.e. the part depending on x and the part depending on y (or t). This is a general feature for solutions in separated form, which, if the original PDE has n independent variables, will contain $n - 1$ separation constants. All that is required in general is that we associate the correct function of one independent variable with the appropriate functions of the others, the correct function being the one with the same values of the separation constants.

If the original PDE is linear (as are the Laplace, Schrödinger, diffusion and wave equations) then mathematically acceptable solutions can be formed by superposing solutions corresponding to different allowed values of the separation constants. To take a two-variable example: if

$$u_{\lambda_1}(x, y) = X_{\lambda_1}(x)Y_{\lambda_1}(y)$$

is a solution of a linear PDE obtained by giving the separation constant the value λ_1, then the superposition

$$u(x, y) = a_1 X_{\lambda_1}(x)Y_{\lambda_1}(y) + a_2 X_{\lambda_2}(x)Y_{\lambda_2}(y) + \cdots = \sum_i a_i X_{\lambda_i}(x)Y_{\lambda_i}(y) \tag{21.16}$$

is also a solution for any constants a_i, provided that the λ_i are the allowed values of the separation constant λ given the imposed boundary conditions. Note that if the boundary conditions allow any of the separation constants to be zero then the form of the general solution is normally different and must be deduced by returning to the separated ordinary differential equations. We will encounter this behaviour in section 21.3.

The value of the superposition approach is that a boundary condition, say that $u(x, y)$ takes a particular form $f(x)$ when $y = 0$, might be met by choosing the constants a_i such that

$$f(x) = \sum_i a_i X_{\lambda_i}(x)Y_{\lambda_i}(0).$$

In general, this will be possible provided that the functions $X_{\lambda_i}(x)$ form a complete set – as do the sinusoidal functions of Fourier series or the spherical harmonics discussed in subsection 18.3.

►*A semi-infinite rectangular metal plate occupies the region $0 \le x \le \infty$ and $0 \le y \le b$ in the xy-plane. The temperature at the far end of the plate and along its two long sides is fixed at $0\,^\circ$C. If the temperature of the plate at $x = 0$ is also fixed and is given by $f(y)$, find the steady-state temperature distribution u(x,y) of the plate. Hence find the temperature distribution if $f(y) = u_0$, where u_0 is a constant.*

The physical situation is illustrated in figure 21.1. With the notation we have used several times before, the two-dimensional heat diffusion equation satisfied by the temperature $u(x, y, t)$ is

$$\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t},$$

with $\kappa = k/(s\rho)$. In this case, however, we are asked to find the steady-state temperature, which corresponds to $\partial u/\partial t = 0$, and so we are led to consider the (two-dimensional) Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We saw that assuming a separable solution of the form $u(x, y) = X(x)Y(y)$ led to solutions such as (21.14) or (21.15), or equivalent forms with x and y interchanged. In the current problem we have to satisfy the boundary conditions $u(x, 0) = 0 = u(x, b)$ and so a solution that is sinusoidal in y seems appropriate. Furthermore, since we require $u(\infty, y) = 0$ it is best to write the x-dependence of the solution explicitly in terms of

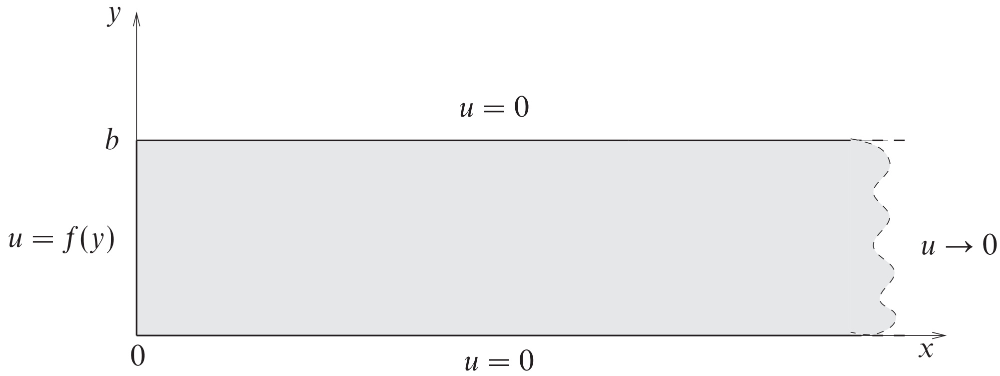

Figure 21.1 A semi-infinite metal plate whose edges are kept at fixed temperatures.

exponentials rather than of hyperbolic functions. We therefore write the separable solution in the form (21.15) as

$$u(x, y) = [A \exp \lambda x + B \exp(-\lambda x)](C \cos \lambda y + D \sin \lambda y).$$

Applying the boundary conditions, we see firstly that $u(\infty, y) = 0$ implies $A = 0$ if we take $\lambda > 0$. Secondly, since $u(x, 0) = 0$ we may set $C = 0$, which, if we absorb the constant D into B, leaves us with

$$u(x, y) = B \exp(-\lambda x) \sin \lambda y.$$

But, using the condition $u(x, b) = 0$, we require $\sin \lambda b = 0$ and so λ must be equal to $n\pi/b$, where n is any positive integer.

Using the principle of superposition (21.16), the general solution satisfying the given boundary conditions can therefore be written

$$u(x, y) = \sum_{n=1}^{\infty} B_n \exp(-n\pi x/b) \sin(n\pi y/b), \qquad (21.17)$$

for some constants B_n. Notice that in the sum in (21.17) we have omitted negative values of n since they would lead to exponential terms that diverge as $x \to \infty$. The $n = 0$ term is also omitted since it is identically zero. Using the remaining boundary condition $u(0, y) = f(y)$ we see that the constants B_n must satisfy

$$f(y) = \sum_{n=1}^{\infty} B_n \sin(n\pi y/b). \qquad (21.18)$$

This is clearly a Fourier sine series expansion of $f(y)$ (see chapter 12). For (21.18) to hold, however, the continuation of $f(y)$ outside the region $0 \le y \le b$ must be an odd periodic function with period $2b$ (see figure 21.2). We also see from figure 21.2 that if the original function $f(y)$ does not equal zero at either of $y = 0$ and $y = b$ then its continuation has a discontinuity at the corresponding point(s); nevertheless, as discussed in chapter 12, the Fourier series will converge to the mid-points of these jumps and hence tend to zero in this case. If, however, the top and bottom edges of the plate were held not at $0\,^\circ$C but at some other non-zero temperature, then, in general, the final solution would possess discontinuities at the corners $x = 0$, $y = 0$ and $x = 0$, $y = b$.

Bearing in mind these technicalities, the coefficients B_n in (21.18) are given by

$$B_n = \frac{2}{b} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy. \qquad (21.19)$$

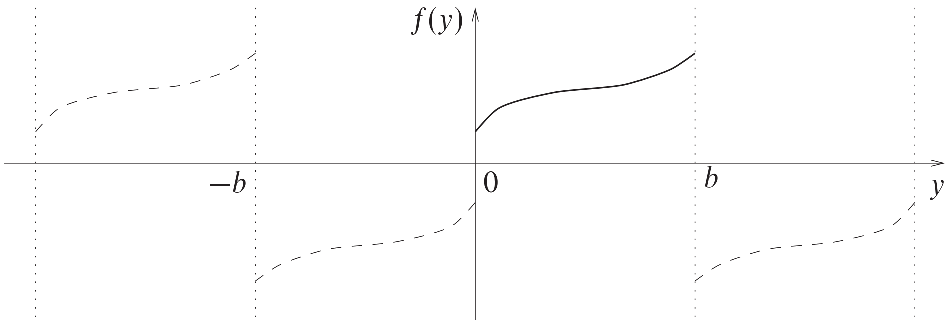

Figure 21.2 The continuation of $f(y)$ for a Fourier sine series.

Therefore, if $f(y) = u_0$ (i.e. the temperature of the side at $x = 0$ is constant along its length), (21.19) becomes

$$\begin{aligned} B_n &= \frac{2}{b}\int_0^b u_0 \sin\left(\frac{n\pi y}{b}\right) dy \\ &= \left[-\frac{2u_0}{b}\frac{b}{n\pi}\cos\left(\frac{n\pi y}{b}\right)\right]_0^b \\ &= -\frac{2u_0}{n\pi}[(-1)^n - 1] = \begin{cases} 4u_0/n\pi & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

Therefore the required solution is

$$u(x, y) = \sum_{n \text{ odd}} \frac{4u_0}{n\pi} \exp\left(-\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right). \blacktriangleleft$$

In the above example the boundary conditions meant that one term in each part of the separable solution could be immediately discarded, making the problem much easier to solve. Sometimes, however, a little ingenuity is required in writing the separable solution in such a way that certain parts can be neglected immediately.

►*Suppose that the semi-infinite rectangular metal plate in the previous example is replaced by one that in the x-direction has finite length a. The temperature of the right-hand edge is fixed at 0 °C and all other boundary conditions remain as before. Find the steady-state temperature in the plate.*

As in the previous example, the boundary conditions $u(x, 0) = 0 = u(x, b)$ suggest a solution that is sinusoidal in y. In this case, however, we require $u = 0$ on $x = a$ (rather than at infinity) and so a solution in which the x-dependence is written in terms of hyperbolic functions, such as (21.14), rather than exponentials is more appropriate. Moreover, since the constants in front of the hyperbolic functions are, at this stage, arbitrary, we may write the separable solution in the most convenient way that ensures that the condition $u(a, y) = 0$ is straightforwardly satisfied. We therefore write

$$u(x, y) = [A\cosh\lambda(a - x) + B\sinh\lambda(a - x)](C\cos\lambda y + D\sin\lambda y).$$

Now the condition $u(a, y) = 0$ is easily satisfied by setting $A = 0$. As before the conditions $u(x, 0) = 0 = u(x, b)$ imply $C = 0$ and $\lambda = n\pi/b$ for integer n. Superposing the solutions for different n we then obtain

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh[n\pi(a - x)/b] \, \sin(n\pi y/b), \tag{21.20}$$

for some constants B_n. We have omitted negative values of n in the sum (21.20) since the relevant terms are already included in those obtained for positive n. Again the $n = 0$ term is identically zero. Using the final boundary condition $u(0, y) = f(y)$ as above we find that the constants B_n must satisfy

$$f(y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi a/b) \sin(n\pi y/b),$$

and, remembering the caveats discussed in the previous example, the B_n are therefore given by

$$B_n = \frac{2}{b \sinh(n\pi a/b)} \int_0^b f(y) \sin(n\pi y/b) \, dy. \tag{21.21}$$

For the case where $f(y) = u_0$, following the working of the previous example gives (21.21) as

$$B_n = \frac{4u_0}{n\pi \sinh(n\pi a/b)} \quad \text{for } n \text{ odd}, \qquad B_n = 0 \quad \text{for } n \text{ even}. \tag{21.22}$$

The required solution is thus

$$u(x, y) = \sum_{n \text{ odd}} \frac{4u_0}{n\pi \sinh(n\pi a/b)} \sinh[n\pi(a - x)/b] \sin\left(n\pi y/b\right).$$

We note that, as required, in the limit $a \to \infty$ this solution tends to the solution of the previous example. ◄

Often the principle of superposition can be used to write the solution to problems with more complicated boundary conditions as the sum of solutions to problems that each satisfy only some part of the boundary condition but when added togther satisfy all the conditions.

►*Find the steady-state temperature in the (finite) rectangular plate of the previous example, subject to the boundary conditions $u(x, b) = 0$, $u(a, y) = 0$ and $u(0, y) = f(y)$ as before, but now, in addition, $u(x, 0) = g(x)$.*

Figure 21.3(c) shows the imposed boundary conditions for the metal plate. Although we could find a solution to this problem using the methods presented above, we can arrive at the answer almost immediately by using the principle of superposition and the result of the previous example.

Let us suppose the required solution $u(x, y)$ is made up of two parts:

$$u(x, y) = v(x, y) + w(x, y),$$

where $v(x, y)$ is the solution satisfying the boundary conditions shown in figure 21.3(a),

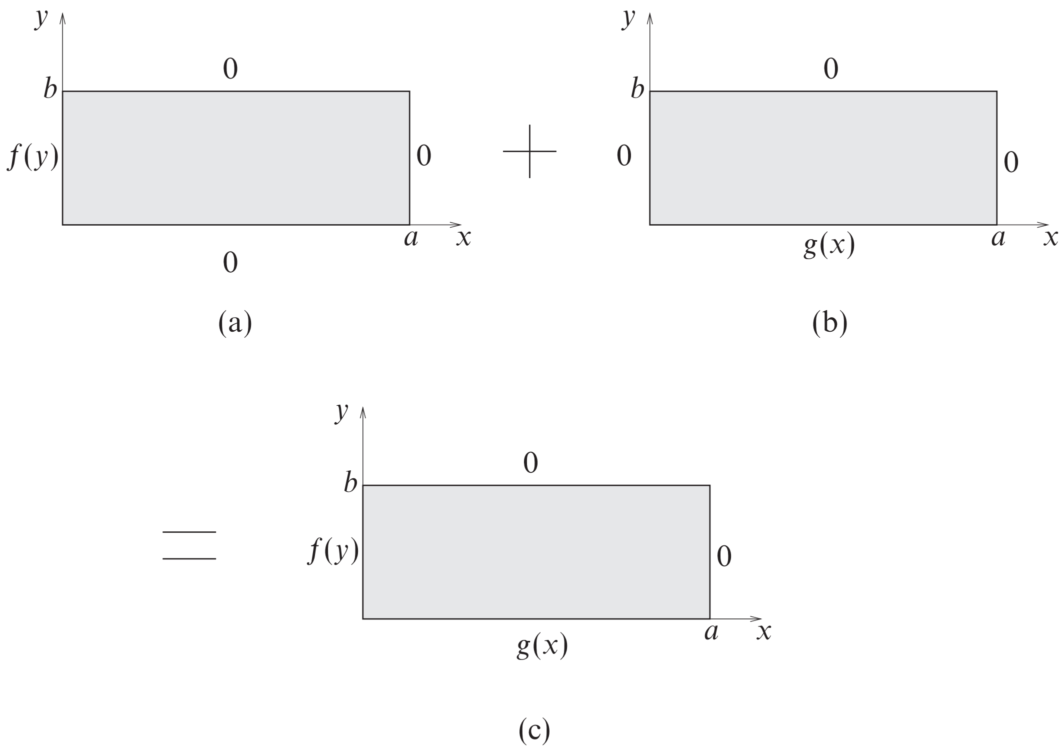

Figure 21.3 Superposition of boundary conditions for a metal plate.

whilst $w(x, y)$ is the solution satisfying the boundary conditions in figure 21.3(b). It is clear that $v(x, y)$ is simply given by the solution to the previous example,

$$v(x, y) = \sum_{n \text{ odd}} B_n \sinh\left[\frac{n\pi(a - x)}{b}\right] \sin\left(\frac{n\pi y}{b}\right),$$

where B_n is given by (21.21). Moreover, by symmetry, $w(x, y)$ must be of the same form as $v(x, y)$ but with x and a interchanged with y and b, respectively, and with $f(y)$ in (21.21) replaced by $g(x)$. Therefore the required solution can be written down immediately without further calculation as

$$u(x, y) = \sum_{n \text{ odd}} B_n \sinh\left[\frac{n\pi(a - x)}{b}\right] \sin\left(\frac{n\pi y}{b}\right) + \sum_{n \text{ odd}} C_n \sinh\left[\frac{n\pi(b - y)}{a}\right] \sin\left(\frac{n\pi x}{a}\right),$$

the B_n being given by (21.21) and the C_n by

$$C_n = \frac{2}{a \sinh(n\pi b/a)} \int_0^a g(x) \sin(n\pi x/a)\, dx.$$

Clearly, this method may be extended to cases in which three or four sides of the plate have non-zero boundary conditions. ◀

As a final example of the usefulness of the principle of superposition we now consider a problem that illustrates how to deal with inhomogeneous boundary conditions by a suitable change of variables.

►*A bar of length L is initially at a temperature of* $0\,^\circ$C. *One end of the bar* ($x = 0$) *is held at* $0\,^\circ$C *and the other is supplied with heat at a constant rate per unit area of H. Find the temperature distribution within the bar after a time t.*

With our usual notation, the heat diffusion equation satisfied by the temperature $u(x,t)$ is

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

with $\kappa = k/(s\rho)$, where k is the thermal conductivity of the bar, s is its specific heat capacity and ρ is its density.

The boundary conditions can be written as

$$u(x,0) = 0, \qquad u(0,t) = 0, \qquad \frac{\partial u(L,t)}{\partial x} = \frac{H}{k},$$

the last of which is inhomogeneous. In general, inhomogeneous boundary conditions can cause difficulties and it is usual to attempt a transformation of the problem into an equivalent homogeneous one. To this end, let us assume that the solution to our problem takes the form

$$u(x,t) = v(x,t) + w(x),$$

where the function $w(x)$ is to be suitably determined. In terms of v and w the problem becomes

$$\begin{aligned}
\kappa\left(\frac{\partial^2 v}{\partial x^2} + \frac{d^2 w}{dx^2}\right) &= \frac{\partial v}{\partial t},\\
v(x,0) + w(x) &= 0,\\
v(0,t) + w(0) &= 0,\\
\frac{\partial v(L,t)}{\partial x} + \frac{dw(L)}{dx} &= \frac{H}{k}.
\end{aligned}$$

There are several ways of choosing $w(x)$ so as to make the new problem straightforward. Using some physical insight, however, it is clear that ultimately (at $t = \infty$), when all transients have died away, the end $x = L$ will attain a temperature u_0 such that $ku_0/L = H$ and there will be a constant temperature gradient $u(x,\infty) = u_0 x/L$. We therefore choose

$$w(x) = \frac{Hx}{k}.$$

Since the second derivative of $w(x)$ is zero, v satisfies the diffusion equation and the boundary conditions on v are now given by

$$v(x,0) = -\frac{Hx}{k}, \qquad v(0,t) = 0, \qquad \frac{\partial v(L,t)}{\partial x} = 0,$$

which are homogeneous in x.

From (21.12) a separated solution for the one-dimensional diffusion equation is

$$v(x,t) = (A\cos\lambda x + B\sin\lambda x)\exp(-\lambda^2\kappa t),$$

corresponding to a separation constant $-\lambda^2$. If we restrict λ to be real then all these solutions are transient ones decaying to zero as $t \to \infty$. These are just what is required to add to $w(x)$ to give the correct solution as $t \to \infty$. In order to satisfy $v(0,t) = 0$, however, we require $A = 0$. Furthermore, since

$$\frac{\partial v}{\partial x} = B\exp(-\lambda^2\kappa t)\lambda\cos\lambda x,$$

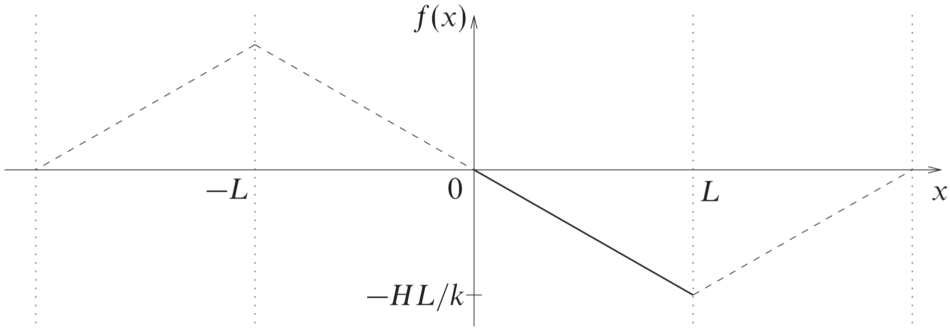

Figure 21.4 The appropriate continuation for a Fourier series containing only sine terms.

in order to satisfy $\partial v(L,t)/\partial x = 0$ we require $\cos \lambda L = 0$, and so λ is restricted to the values

$$\lambda = \frac{n\pi}{2L},$$

where n is an odd non-negative integer, i.e. $n = 1, 3, 5, \ldots$.

Thus, to satisfy the boundary condition $v(x,0) = -Hx/k$, we must have

$$\sum_{n \text{ odd}} B_n \sin\left(\frac{n\pi x}{2L}\right) = -\frac{Hx}{k},$$

in the range $x = 0$ to $x = L$. In this case we must be more careful about the continuation of the function $-Hx/k$, for which the Fourier sine series is required. We want a series that is odd in x (sine terms only) and continuous as $x = 0$ and $x = L$ (no discontinuities, since the series must converge at the end-points). This leads to a continuation of the function as shown in figure 21.4, with a period of $L' = 4L$. Following the discussion of section 12.3, since this continuation is odd about $x = 0$ and even about $x = L'/4 = L$ it can indeed be expressed as a Fourier sine series containing only odd-numbered terms.

The corresponding Fourier series coefficients are found to be

$$B_n = \frac{-8HL}{k\pi^2} \frac{(-1)^{(n-1)/2}}{n^2} \qquad \text{for } n \text{ odd},$$

and thus the final formula for $u(x,t)$ is

$$u(x,t) = \frac{Hx}{k} - \frac{8HL}{k\pi^2} \sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi x}{2L}\right) \exp\left(-\frac{kn^2\pi^2 t}{4L^2 s\rho}\right),$$

giving the temperature for all positions $0 \leq x \leq L$ and for all times $t \geq 0$. ◀

We note that in all the above examples the boundary conditions restricted the separation constant(s) to an infinite number of *discrete* values, usually integers. If, however, the boundary conditions allow the separation constant(s) λ to take a *continuum* of values then the summation in (21.16) is replaced by an integral over λ. This is discussed further in connection with integral transform methods in section 21.4.

21.3 Separation of variables in polar coordinates

So far we have considered the solution of PDEs only in Cartesian coordinates, but many systems in two and three dimensions are more naturally expressed in some form of polar coordinates, in which full advantage can be taken of any inherent symmetries. For example, the potential associated with an isolated point charge has a very simple expression, $q/(4\pi\epsilon_0 r)$, when polar coordinates are used, but involves all three coordinates and square roots when Cartesians are employed. For these reasons we now turn to the separation of variables in plane polar, cylindrical polar and spherical polar coordinates.

Most of the PDEs we have considered so far have involved the operator ∇^2, e.g. the wave equation, the diffusion equation, Schrödinger's equation and Poisson's equation (and of course Laplace's equation). It is therefore appropriate that we recall the expressions for ∇^2 when expressed in polar coordinate systems. From chapter 10, in plane polars, cylindrical polars and spherical polars, respectively, we have

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}, \tag{21.23}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}, \tag{21.24}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{21.25}$$

Of course the first of these may be obtained from the second by taking z to be identically zero.

21.3.1 Laplace's equation in polar coordinates

The simplest of the equations containing ∇^2 is Laplace's equation,

$$\nabla^2 u(\mathbf{r}) = 0. \tag{21.26}$$

Since it contains most of the essential features of the other more complicated equations, we will consider its solution first.

Laplace's equation in plane polars

Suppose that we need to find a solution of (21.26) that has a prescribed behaviour on the circle $\rho = a$ (e.g. if we are finding the shape taken up by a circular drumskin when its rim is slightly deformed from being planar). Then we may seek solutions of (21.26) that are separable in ρ and ϕ (measured from some arbitrary radius as $\phi = 0$) and hope to accommodate the boundary condition by examining the solution for $\rho = a$.

Thus, writing $u(\rho, \phi) = P(\rho)\Phi(\phi)$ and using the expression (21.23), Laplace's equation (21.26) becomes

$$\frac{\Phi}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial P}{\partial \rho}\right) + \frac{P}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

Now, employing the same device as previously, that of dividing through by $u = P\Phi$ and multiplying through by ρ^2, results in the separated equation

$$\frac{\rho}{P}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial P}{\partial \rho}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

Following our earlier argument, since the first term on the RHS is a function of ρ only, whilst the second term depends only on ϕ, we obtain the two *ordinary* equations

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) = n^2, \tag{21.27}$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2, \tag{21.28}$$

where we have taken the separation constant to have the form n^2 for later convenience; for the present, n is a general (complex) number.

Let us first consider the case in which $n \neq 0$. The second equation, (21.28), then has the general solution

$$\Phi(\phi) = A\exp(in\phi) + B\exp(-in\phi). \tag{21.29}$$

Equation (21.27), on the other hand, is the homogeneous equation

$$\rho^2 P'' + \rho P' - n^2 P = 0,$$

which must be solved either by trying a power solution in ρ or by making the substitution $\rho = \exp t$ as described in subsection 15.2.1 and so reducing it to an equation with constant coefficients. Carrying out this procedure we find

$$P(\rho) = C\rho^n + D\rho^{-n}. \tag{21.30}$$

Returning to the solution (21.29) of the azimuthal equation (21.28), we can see that if Φ, and hence u, is to be single-valued and so not change when ϕ increases by 2π then n must be an integer. Mathematically, other values of n are permissible, but for the description of real physical situations it is clear that this limitation must be imposed. Having thus restricted the possible values of n in one part of the solution, the same limitations must be carried over into the radial part, (21.30). Thus we may write a particular solution of the two-dimensional Laplace equation as

$$u(\rho, \phi) = (A\cos n\phi + B\sin n\phi)(C\rho^n + D\rho^{-n}),$$

where A, B, C, D are arbitrary constants and n is any integer.

We have not yet, however, considered the solution when $n = 0$. In this case, the solutions of the separated ordinary equations (21.28) and (21.27), respectively, are easily shown to be

$$\Phi(\phi) = A\phi + B,$$
$$P(\rho) = C \ln \rho + D.$$

But, in order that $u = P\Phi$ is single-valued, we require $A = 0$, and so the solution for $n = 0$ is simply (absorbing B into C and D)

$$u(\rho, \phi) = C \ln \rho + D.$$

Superposing the solutions for the different allowed values of n, we can write the general solution to Laplace's equation in plane polars as

$$u(\rho, \phi) = (C_0 \ln \rho + D_0) + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi)(C_n \rho^n + D_n \rho^{-n}), \tag{21.31}$$

where n can take only integer values. Negative values of n have been omitted from the sum since they are already included in the terms obtained for positive n. We note that, since $\ln \rho$ is singular at $\rho = 0$, whenever we solve Laplace's equation in a region containing the origin, C_0 must be identically zero.

►*A circular drumskin has a supporting rim at $\rho = a$. If the rim is twisted so that it is displaced vertically by a small amount $\epsilon(\sin \phi + 2 \sin 2\phi)$, where ϕ is the azimuthal angle with respect to a given radius, find the resulting displacement $u(\rho, \phi)$ over the entire drumskin.*

The transverse displacement of a circular drumskin is usually described by the two-dimensional wave equation. In this case, however, there is no time dependence and so $u(\rho, \phi)$ solves the two-dimensional Laplace equation, subject to the imposed boundary condition.

Referring to (21.31), since we wish to find a solution that is finite everywhere inside $\rho = a$, we require $C_0 = 0$ and $D_n = 0$ for all $n > 0$. Now the boundary condition at the rim requires

$$u(a, \phi) = D_0 + \sum_{n=1}^{\infty} C_n a^n (A_n \cos n\phi + B_n \sin n\phi) = \epsilon(\sin \phi + 2 \sin 2\phi).$$

Firstly we see that we require $D_0 = 0$ and $A_n = 0$ for all n. Furthermore, we must have $C_1 B_1 a = \epsilon$, $C_2 B_2 a^2 = 2\epsilon$ and $B_n = 0$ for $n > 2$. Hence the appropriate shape for the drumskin (valid over the whole skin, not just the rim) is

$$u(\rho, \phi) = \frac{\epsilon\rho}{a} \sin \phi + \frac{2\epsilon\rho^2}{a^2} \sin 2\phi = \frac{\epsilon\rho}{a} \left(\sin \phi + \frac{2\rho}{a} \sin 2\phi \right). \blacktriangleleft$$

Laplace's equation in cylindrical polars

Passing to three dimensions, we now consider the solution of Laplace's equation in cylindrical polar coordinates,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial z^2}=0. \tag{21.32}$$

We note here that, even when considering a cylindrical physical system, if there is no dependence of the physical variables on z (i.e. along the length of the cylinder) then the problem may be treated using two-dimensional plane polars, as discussed above.

For the more general case, however, we proceed as previously by trying a solution of the form

$$u(\rho,\phi,z)=P(\rho)\Phi(\phi)Z(z),$$

which, on substitution into (21.32) and division through by $u = P\Phi Z$, gives

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z}\frac{d^2 Z}{dz^2}=0.$$

The last term depends only on z, and the first and second (taken together) depend only on ρ and ϕ. Taking the separation constant to be k^2, we find

$$\frac{1}{Z}\frac{d^2 Z}{dz^2}=k^2,$$

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2}+k^2=0.$$

The first of these equations has the straightforward solution

$$Z(z)=E\exp(-kz)+F\exp kz.$$

Multiplying the second equation through by ρ^2, we obtain

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+k^2\rho^2=0,$$

in which the second term depends only on Φ and the other terms depend only on ρ. Taking the second separation constant to be m^2, we find

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2, \tag{21.33}$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+(k^2\rho^2-m^2)P=0. \tag{21.34}$$

The equation in the azimuthal angle ϕ has the very familiar solution

$$\Phi(\phi)=C\cos m\phi+D\sin m\phi.$$

As in the two-dimensional case, single-valuedness of u requires that m is an integer. However, in the particular case $m = 0$ the solution is

$$\Phi(\phi) = C\phi + D.$$

This form is appropriate to a solution with axial symmetry ($C = 0$) or one that is multivalued, but manageably so, such as the magnetic scalar potential associated with a current I (in which case $C = I/(2\pi)$ and D is arbitrary).

Finally, the ρ-equation (21.34) may be transformed into Bessel's equation of order m by writing $\mu = k\rho$. This has the solution

$$P(\rho) = AJ_m(k\rho) + BY_m(k\rho).$$

The properties of these functions were investigated in chapter 16 and will not be pursued here. We merely note that $Y_m(k\rho)$ is singular at $\rho = 0$, and so, when seeking solutions to Laplace's equation in cylindrical coordinates within some region containing the $\rho = 0$ axis, we require $B = 0$.

The complete separated-variable solution in cylindrical polars of Laplace's equation $\nabla^2 u = 0$ is thus given by

$$u(\rho, \phi, z) = [AJ_m(k\rho) + BY_m(k\rho)][C\cos m\phi + D\sin m\phi][E\exp(-kz) + F\exp kz]. \tag{21.35}$$

Of course we may use the principle of superposition to build up more general solutions by adding together solutions of the form (21.35) for all allowed values of the separation constants k and m.

►*A semi-infinite solid cylinder of radius a has its curved surface held at $0\,°$C and its base held at a temperature T_0. Find the steady-state temperature distribution in the cylinder.*

The physical situation is shown in figure 21.5. The steady-state temperature distribution $u(\rho, \phi, z)$ must satisfy Laplace's equation subject to the imposed boundary conditions. Let us take the cylinder to have its base in the $z = 0$ plane and to extend along the positive z-axis. From (21.35), in order that u is finite everywhere in the cylinder we immediately require $B = 0$ and $F = 0$. Furthermore, since the boundary conditions, and hence the temperature distribution, are axially symmetric, we require $m = 0$, and so the general solution must be a superposition of solutions of the form $J_0(k\rho)\exp(-kz)$ for all allowed values of the separation constant k.

The boundary condition $u(a, \phi, z) = 0$ restricts the allowed values of k, since we must have $J_0(ka) = 0$. The zeros of Bessel functions are given in most books of mathematical tables, and we find that, to two decimal places,

$$J_0(x) = 0 \qquad \text{for } x = 2.40,\ 5.52,\ 8.65,\ \ldots.$$

Writing the allowed values of k as k_n for $n = 1, 2, 3, \ldots$ (so, for example, $k_1 = 2.40/a$), the required solution takes the form

$$u(\rho, \phi, z) = \sum_{n=1}^{\infty} A_n J_0(k_n\rho)\exp(-k_n z).$$

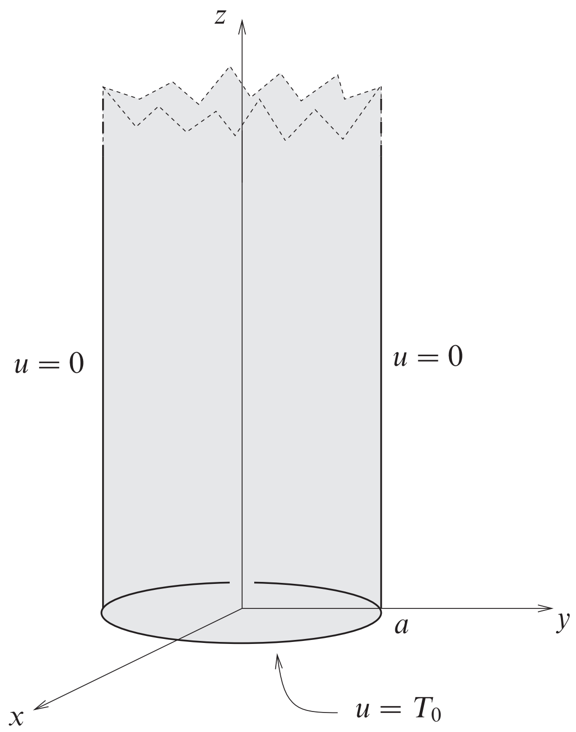

Figure 21.5 A uniform metal cylinder whose curved surface is kept at 0 °C and whose base is held at a temperature T_0.

By imposing the remaining boundary condition $u(\rho, \phi, 0) = T_0$, the coefficients A_n can be found in a similar way to Fourier coefficients but this time by exploiting the orthogonality of the Bessel functions, as discussed in chapter 16. From this boundary condition we require

$$u(\rho, \phi, 0) = \sum_{n=1}^{\infty} A_n J_0(k_n \rho) = T_0.$$

If we multiply this expression by $\rho J_0(k_r \rho)$ and integrate from $\rho = 0$ to $\rho = a$, and use the orthogonality of the Bessel functions $J_0(k_n \rho)$, then the coefficients are given by (18.91) as

$$A_n = \frac{2T_0}{a^2 J_1^2(k_n a)} \int_0^a J_0(k_n \rho) \rho \, d\rho. \tag{21.36}$$

The integral on the RHS can be evaluated using the recurrence relation (18.92) of chapter 16,

$$\frac{d}{dz}[z J_1(z)] = z J_0(z),$$

which on setting $z = k_n \rho$ yields

$$\frac{1}{k_n} \frac{d}{d\rho}[k_n \rho J_1(k_n \rho)] = k_n \rho J_0(k_n \rho).$$

Therefore the integral in (21.36) is given by

$$\int_0^a J_0(k_n \rho) \rho \, d\rho = \left[\frac{1}{k_n} \rho J_1(k_n \rho) \right]_0^a = \frac{1}{k_n} a J_1(k_n a),$$

and the coefficients A_n may be expressed as

$$A_n = \frac{2T_0}{a^2 J_1^2(k_n a)} \left[\frac{a J_1(k_n a)}{k_n} \right] = \frac{2T_0}{k_n a J_1(k_n a)}.$$

The steady-state temperature in the cylinder is then given by

$$u(\rho, \phi, z) = \sum_{n=1}^{\infty} \frac{2T_0}{k_n a J_1(k_n a)} J_0(k_n \rho) \exp(-k_n z). \blacktriangleleft$$

We note that if, in the above example, the base of the cylinder were not kept at a uniform temperature T_0, but instead had some fixed temperature distribution $T(\rho, \phi)$, then the solution of the problem would become more complicated. In such a case, the required temperature distribution $u(\rho, \phi, z)$ is in general *not* axially symmetric, and so the separation constant m is not restricted to be zero but may take any integer value. The solution will then take the form

$$u(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{nm}\rho)(C_{nm} \cos m\phi + D_{nm} \sin m\phi) \exp(-k_{nm} z),$$

where the separation constants k_{nm} are such that $J_m(k_{nm} a) = 0$, i.e. $k_{nm} a$ is the nth zero of the mth-order Bessel function. At the base of the cylinder we would then require

$$u(\rho, \phi, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{nm}\rho)(C_{nm} \cos m\phi + D_{nm} \sin m\phi) = T(\rho, \phi). \tag{21.37}$$

The coefficients C_{nm} could be found by multiplying (21.37) by $J_q(k_{rq}\rho) \cos q\phi$, integrating with respect to ρ and ϕ over the base of the cylinder and exploiting the orthogonality of the Bessel functions and of the trigonometric functions. The D_{nm} could be found in a similar way by multiplying (21.37) by $J_q(k_{rq}\rho) \sin q\phi$.

Laplace's equation in spherical polars

We now come to an equation that is very widely applicable in physical science, namely $\nabla^2 u = 0$ in spherical polar coordinates:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} = 0. \tag{21.38}$$

Our method of procedure will be as before; we try a solution of the form

$$u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi).$$

Substituting this in (21.38), dividing through by $u = R\Theta\Phi$ and multiplying by r^2, we obtain

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi \sin^2\theta} \frac{d^2\Phi}{d\phi^2} = 0. \tag{21.39}$$

The first term depends only on r and the second and third terms (taken together) depend only on θ and ϕ. Thus (21.39) is equivalent to the two equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda, \tag{21.40}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = -\lambda. \tag{21.41}$$

Equation (21.40) is a homogeneous equation,

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda R = 0,$$

which can be reduced, by the substitution $r = \exp t$ (and writing $R(r) = S(t)$), to

$$\frac{d^2S}{dt^2} + \frac{dS}{dt} - \lambda S = 0.$$

This has the straightforward solution

$$S(t) = A\exp\lambda_1 t + B\exp\lambda_2 t,$$

and so the solution to the radial equation is

$$R(r) = Ar^{\lambda_1} + Br^{\lambda_2},$$

where $\lambda_1 + \lambda_2 = -1$ and $\lambda_1\lambda_2 = -\lambda$. We can thus take λ_1 and λ_2 as given by ℓ and $-(\ell+1)$; λ then has the form $\ell(\ell+1)$. (It should be noted that at this stage nothing has been either assumed or proved about whether ℓ is an integer.)

Hence we have obtained some information about the first factor in the separated-variable solution, which will now have the form

$$u(r,\theta,\phi) = \left[Ar^{\ell} + Br^{-(\ell+1)}\right]\Theta(\theta)\Phi(\phi), \tag{21.42}$$

where Θ and Φ must satisfy (21.41) with $\lambda = \ell(\ell+1)$.

The next step is to take (21.41) further. Multiplying through by $\sin^2\theta$ and substituting for λ, it too takes a separated form:

$$\left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta\right] + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \tag{21.43}$$

Taking the separation constant as m^2, the equation in the azimuthal angle ϕ has the same solution as in cylindrical polars, namely

$$\Phi(\phi) = C\cos m\phi + D\sin m\phi.$$

As before, single-valuedness of u requires that m is an integer; for $m = 0$ we again have $\Phi(\phi) = C\phi + D$.

Having settled the form of $\Phi(\phi)$, we are left only with the equation satisfied by $\Theta(\theta)$, which is

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\ell(\ell+1)\sin^2\theta=m^2. \tag{21.44}$$

A change of independent variable from θ to $\mu = \cos\theta$ will reduce this to a form for which solutions are known, and of which some study has been made in chapter 16. Putting

$$\mu=\cos\theta, \qquad \frac{d\mu}{d\theta}=-\sin\theta, \qquad \frac{d}{d\theta}=-(1-\mu^2)^{1/2}\frac{d}{d\mu},$$

the equation for $M(\mu)\equiv\Theta(\theta)$ reads

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-\mu^2}\right]M=0. \tag{21.45}$$

This equation is the *associated Legendre equation*, which was mentioned in subsection 18.2 in the context of Sturm–Liouville equations.

We recall that for the case $m = 0$, (21.45) reduces to Legendre's equation, which was studied at length in chapter 16, and has the solution

$$M(\mu)=EP_\ell(\mu)+FQ_\ell(\mu). \tag{21.46}$$

We have not solved (21.45) explicitly for general m, but the solutions were given in subsection 18.2 and are the associated Legendre functions $P_\ell^m(\mu)$ and $Q_\ell^m(\mu)$, where

$$P_\ell^m(\mu)=(1-\mu^2)^{|m|/2}\frac{d^{|m|}}{d\mu^{|m|}}P_\ell(\mu), \tag{21.47}$$

and similarly for $Q_\ell^m(\mu)$. We then have

$$M(\mu)=EP_\ell^m(\mu)+FQ_\ell^m(\mu); \tag{21.48}$$

here m must be an integer, $0\le|m|\le\ell$. We note that if we require solutions to Laplace's equation that are finite when $\mu=\cos\theta=\pm1$ (i.e. on the polar axis where $\theta=0,\pi$), then we must have $F=0$ in (21.46) and (21.48) since $Q_\ell^m(\mu)$ diverges at $\mu=\pm1$.

It will be remembered that one of the important conditions for obtaining finite polynomial solutions of Legendre's equation is that ℓ is an integer ≥ 0. This condition therefore applies also to the solutions (21.46) and (21.48) and is reflected back into the radial part of the general solution given in (21.42).

Now that the solutions of each of the three ordinary differential equations governing R, Θ and Φ have been obtained, we may assemble a complete separated-variable solution of Laplace's equation in spherical polars. It is

$$u(r, \theta, \phi) = (Ar^{\ell} + Br^{-(\ell+1)})(C \cos m\phi + D \sin m\phi)[EP_{\ell}^{m}(\cos\theta) + FQ_{\ell}^{m}(\cos\theta)], \tag{21.49}$$

where the three bracketted factors are connected only through the *integer* parameters ℓ and m, $0 \leq |m| \leq \ell$. As before, a general solution may be obtained by superposing solutions of this form for the allowed values of the separation constants ℓ and m. As mentioned above, if the solution is required to be finite on the polar axis then $F = 0$ for all ℓ and m.

►*An uncharged conducting sphere of radius a is placed at the origin in an initially uniform electrostatic field E. Show that it behaves as an electric dipole.*

The uniform field, taken in the direction of the polar axis, has an electrostatic potential

$$u = -Ez = -Er\cos\theta,$$

where u is arbitrarily taken as zero at $z = 0$. This satisfies Laplace's equation $\nabla^2 u = 0$, as must the potential v when the sphere is present; for large r the asymptotic form of v must still be $-Er\cos\theta$.

Since the problem is clearly axially symmetric, we have immediately that $m = 0$, and since we require v to be finite on the polar axis we must have $F = 0$ in (21.49). Therefore the solution must be of the form

$$v(r, \theta, \phi) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-(\ell+1)}) P_{\ell}(\cos\theta).$$

Now the $\cos\theta$-dependence of v for large r indicates that the (θ, ϕ)-dependence of $v(r, \theta, \phi)$ is given by $P_1^0(\cos\theta) = \cos\theta$. Thus the r-dependence of v must also correspond to an $\ell = 1$ solution, and the most general such solution (outside the sphere, i.e. for $r \geq a$) is

$$v(r, \theta, \phi) = (A_1 r + B_1 r^{-2}) P_1(\cos\theta).$$

The asymptotic form of v for large r immediately gives $A_1 = -E$ and so yields the solution

$$v(r, \theta, \phi) = \left(-Er + \frac{B_1}{r^2}\right)\cos\theta.$$

Since the sphere is conducting, it is an equipotential region and so v must not depend on θ for $r = a$. This can only be the case if $B_1/a^2 = Ea$, thus fixing B_1. The final solution is therefore

$$v(r, \theta, \phi) = -Er\left(1 - \frac{a^3}{r^3}\right)\cos\theta.$$

Since a dipole of moment p gives rise to a potential $p/(4\pi\epsilon_0 r^2)$, this result shows that the sphere behaves as a dipole of moment $4\pi\epsilon_0 a^3 E$, because of the charge distribution induced on its surface; see figure 21.6. ◄

Often the boundary conditions are not so easily met, and it is necessary to use the mutual orthogonality of the associated Legendre functions (and the trigonometric functions) to obtain the coefficients in the general solution.

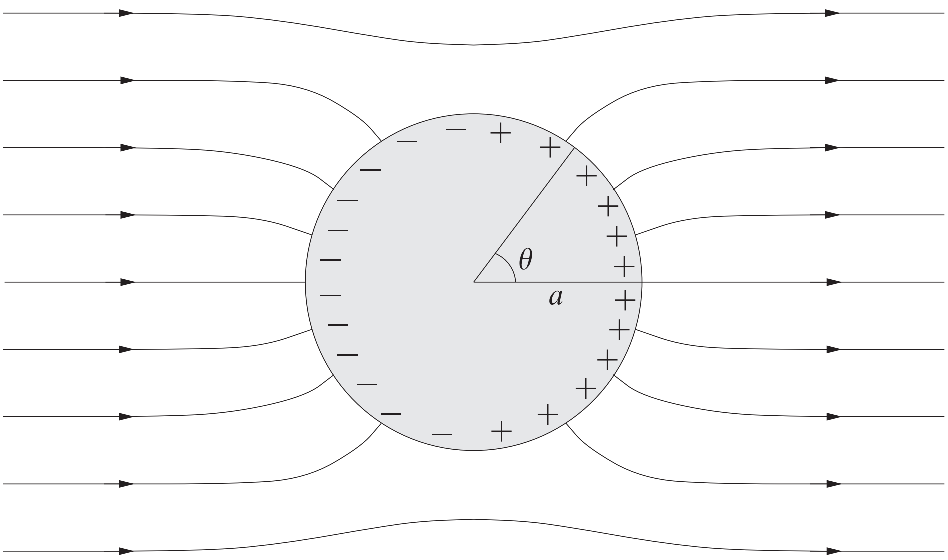

Figure 21.6 Induced charge and field lines associated with a conducting sphere placed in an initially uniform electrostatic field.

►*A hollow split conducting sphere of radius a is placed at the origin. If one half of its surface is charged to a potential v_0 and the other half is kept at zero potential, find the potential v inside and outside the sphere.*

Let us choose the top hemisphere to be charged to v_0 and the bottom hemisphere to be at zero potential, with the plane in which the two hemispheres meet perpendicular to the polar axis; this is shown in figure 21.7. The boundary condition then becomes

$$v(a,\theta,\phi) = \begin{cases} v_0 & \text{for } 0 < \theta < \pi/2 \quad (0 < \cos\theta < 1), \\ 0 & \text{for } \pi/2 < \theta < \pi \quad (-1 < \cos\theta < 0). \end{cases} \tag{21.50}$$

The problem is clearly axially symmetric and so we may set $m = 0$. Also, we require the solution to be finite on the polar axis and so it cannot contain $Q_\ell(\cos\theta)$. Therefore the general form of the solution to (21.38) is

$$v(r,\theta,\phi) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta). \tag{21.51}$$

Inside the sphere (for $r < a$) we require the solution to be finite at the origin and so $B_\ell = 0$ for all ℓ in (21.51). Imposing the boundary condition at $r = a$ we must then have

$$v(a,\theta,\phi) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta),$$

where $v(a,\theta,\phi)$ is also given by (21.50). Exploiting the mutual orthogonality of the Legendre polynomials, the coefficients in the Legendre polynomial expansion are given by (18.14) as (writing $\mu = \cos\theta$)

$$\begin{aligned} A_\ell a^\ell &= \frac{2\ell+1}{2} \int_{-1}^{1} v(a,\theta,\phi) P_\ell(\mu)\,d\mu \\ &= \frac{2\ell+1}{2} v_0 \int_0^1 P_\ell(\mu)\,d\mu, \end{aligned}$$

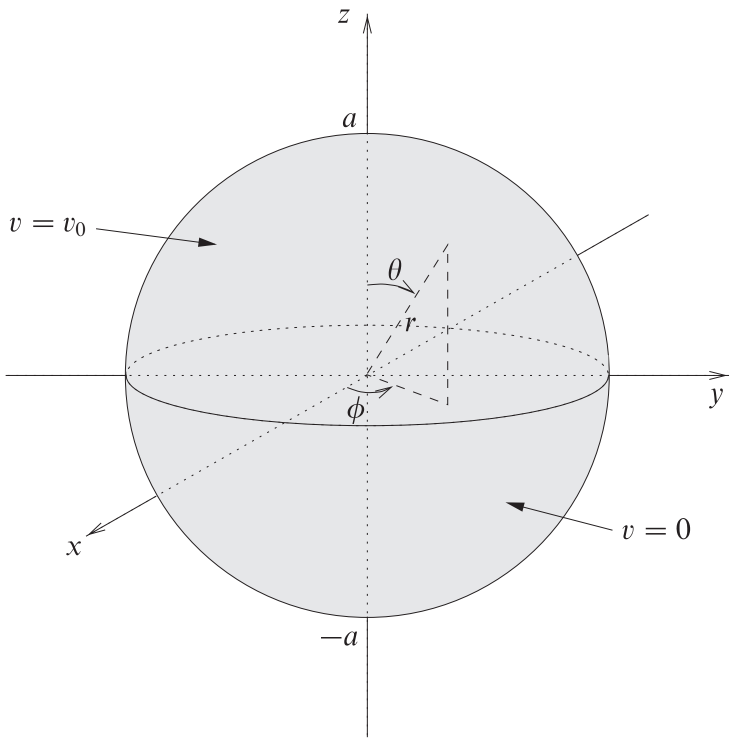

Figure 21.7 A hollow split conducting sphere with its top half charged to a potential v_0 and its bottom half at zero potential.

where in the last line we have used (21.50). The integrals of the Legendre polynomials are easily evaluated (see exercise 17.3) and we find

$$A_0 = \frac{v_0}{2}, \qquad A_1 = \frac{3v_0}{4a}, \qquad A_2 = 0, \qquad A_3 = -\frac{7v_0}{16a^3}, \qquad \cdots,$$

so that the required solution inside the sphere is

$$v(r, \theta, \phi) = \frac{v_0}{2}\left[1 + \frac{3r}{2a}P_1(\cos\theta) - \frac{7r^3}{8a^3}P_3(\cos\theta) + \cdots\right].$$

Outside the sphere (for $r > a$) we require the solution to be bounded as r tends to infinity and so in (21.51) we must have $A_\ell = 0$ for all ℓ. In this case, by imposing the boundary condition at $r = a$ we require

$$v(a, \theta, \phi) = \sum_{\ell=0}^{\infty} B_\ell a^{-(\ell+1)} P_\ell(\cos\theta),$$

where $v(a, \theta, \phi)$ is given by (21.50). Following the above argument the coefficients in the expansion are given by

$$B_\ell a^{-(\ell+1)} = \frac{2\ell + 1}{2} v_0 \int_0^1 P_\ell(\mu)\, d\mu,$$

so that the required solution outside the sphere is

$$v(r, \theta, \phi) = \frac{v_0 a}{2r}\left[1 + \frac{3a}{2r}P_1(\cos\theta) - \frac{7a^3}{8r^3}P_3(\cos\theta) + \cdots\right]. \blacktriangleleft$$

In the above example, on the equator of the sphere (i.e. at $r = a$ and $\theta = \pi/2$) the potential is given by

$$v(a, \pi/2, \phi) = v_0/2,$$

i.e. mid-way between the potentials of the top and bottom hemispheres. This is so because a Legendre polynomial expansion of a function behaves in the same way as a Fourier series expansion, in that it converges to the average of the two values at any discontinuities present in the original function.

If the potential on the surface of the sphere had been given as a function of θ *and* ϕ, then we would have had to consider a double series summed over ℓ and m (for $-\ell \leq m \leq \ell$), since, in general, the solution would not have been axially symmetric.

Finally, we note in general that, when obtaining solutions of Laplace's equation in spherical polar coordinates, one finds that, for solutions that are finite on the polar axis, the angular part of the solution is given by

$$\Theta(\theta)\Phi(\phi) = P_\ell^m(\cos\theta)(C\cos m\phi + D\sin m\phi),$$

where ℓ and m are integers with $-\ell \leq m \leq \ell$. This general form is sufficiently common that particular functions of θ and ϕ called *spherical harmonics* are defined and tabulated (see section 18.3).

21.3.2 Other equations in polar coordinates

The development of the solutions of $\nabla^2 u = 0$ carried out in the previous subsection can be employed to solve other equations in which the ∇^2 operator appears. Since we have discussed the general method in some depth already, only an outline of the solutions will be given here.

Let us first consider the wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{21.52}$$

and look for a separated solution of the form $u = F(\mathbf{r})T(t)$, so that initially we are separating only the spatial and time dependences. Substituting this form into (21.52) and taking the separation constant as k^2 we obtain

$$\nabla^2 F + k^2 F = 0, \qquad \frac{d^2 T}{dt^2} + k^2 c^2 T = 0. \tag{21.53}$$

The second equation has the simple solution

$$T(t) = A\exp(i\omega t) + B\exp(-i\omega t), \tag{21.54}$$

where $\omega = kc$; this may also be expressed in terms of sines and cosines, of course. The first equation in (21.53) is referred to as *Helmholtz's equation*; we discuss it below.

We may treat the diffusion equation

$$\kappa \nabla^2 u = \frac{\partial u}{\partial t}$$

in a similar way. Separating the spatial and time dependences by assuming a solution of the form $u = F(\mathbf{r})T(t)$, and taking the separation constant as k^2, we find

$$\nabla^2 F + k^2 F = 0, \qquad \frac{dT}{dt} + k^2 \kappa T = 0.$$

Just as in the case of the wave equation, the spatial part of the solution satisfies Helmholtz's equation. It only remains to consider the time dependence, which has the simple solution

$$T(t) = A \exp(-k^2 \kappa t).$$

Helmholtz's equation is clearly of central importance in the solutions of the wave and diffusion equations. It can be solved in polar coordinates in much the same way as Laplace's equation, and indeed reduces to Laplace's equation when $k = 0$. Therefore, we will merely sketch the method of its solution in each of the three polar coordinate systems.

Helmholtz's equation in plane polars

In two-dimensional plane polar coordinates, Helmholtz's equation takes the form

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial F}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \phi^2} + k^2 F = 0.$$

If we try a separated solution of the form $F(\mathbf{r}) = P(\rho)\Phi(\phi)$, and take the separation constant as m^2, we find

$$\frac{d^2\Phi}{d\phi^2} + m^2 \phi = 0,$$
$$\frac{d^2 P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right) P = 0.$$

As for Laplace's equation, the angular part has the familiar solution (if $m \neq 0$)

$$\Phi(\phi) = A \cos m\phi + B \sin m\phi,$$

or an equivalent form in terms of complex exponentials. The radial equation differs from that found in the solution of Laplace's equation, but by making the substitution $\mu = k\rho$ it is easily transformed into Bessel's equation of order m (discussed in chapter 16), and has the solution

$$P(\rho) = CJ_m(k\rho) + DY_m(k\rho),$$

where Y_m is a Bessel function of the second kind, which is infinite at the origin and is not to be confused with a spherical harmonic (these are written with a superscript as well as a subscript).

Putting the two parts of the solution together we have

$$F(\rho, \phi) = [A \cos m\phi + B \sin m\phi][C J_m(k\rho) + D Y_m(k\rho)]. \qquad (21.55)$$

Clearly, for solutions of Helmholtz's equation that are required to be finite at the origin, we must set $D = 0$.

►*Find the four lowest frequency modes of oscillation of a circular drumskin of radius a whose circumference is held fixed in a plane.*

The transverse displacement $u(\mathbf{r}, t)$ of the drumskin satisfies the two-dimensional wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

with $c^2 = T/\sigma$, where T is the tension of the drumskin and σ is its mass per unit area. From (21.54) and (21.55) a separated solution of this equation, in plane polar coordinates, that is finite at the origin is

$$u(\rho, \phi, t) = J_m(k\rho)(A \cos m\phi + B \sin m\phi) \exp(\pm i\omega t),$$

where $\omega = kc$. Since we require the solution to be single-valued we must have m as an integer. Furthermore, if the drumskin is clamped at its outer edge $\rho = a$ then we also require $u(a, \phi, t) = 0$. Thus we need

$$J_m(ka) = 0,$$

which in turn restricts the allowed values of k. The zeros of Bessel functions can be obtained from most books of tables; the first few are

$$\begin{aligned} J_0(x) &= 0 \quad \text{for } x \approx 2.40,\ 5.52,\ 8.65, \ldots, \\ J_1(x) &= 0 \quad \text{for } x \approx 3.83,\ 7.02,\ 10.17, \ldots, \\ J_2(x) &= 0 \quad \text{for } x \approx 5.14,\ 8.42,\ 11.62 \ldots. \end{aligned}$$

The smallest value of x for which any of the Bessel functions is zero is $x \approx 2.40$, which occurs for $J_0(x)$. Thus the lowest-frequency mode has $k = 2.40/a$ and angular frequency $\omega = 2.40c/a$. Since $m = 0$ for this mode, the shape of the drumskin is

$$u \propto J_0\left(2.40\frac{\rho}{a}\right);$$

this is illustrated in figure 21.8.

Continuing in the same way, the next three modes are given by

$$\begin{aligned} \omega &= 3.83\frac{c}{a}, \qquad u \propto J_1\left(3.83\frac{\rho}{a}\right)\cos\phi, \qquad J_1\left(3.83\frac{\rho}{a}\right)\sin\phi; \\ \omega &= 5.14\frac{c}{a}, \qquad u \propto J_2\left(5.14\frac{\rho}{a}\right)\cos 2\phi, \qquad J_2\left(5.14\frac{\rho}{a}\right)\sin 2\phi; \\ \omega &= 5.52\frac{c}{a}, \qquad u \propto J_0\left(5.52\frac{\rho}{a}\right). \end{aligned}$$

These modes are also shown in figure 21.8. We note that the second and third frequencies have *two* corresponding modes of oscillation; these frequencies are therefore two-fold degenerate. ◄

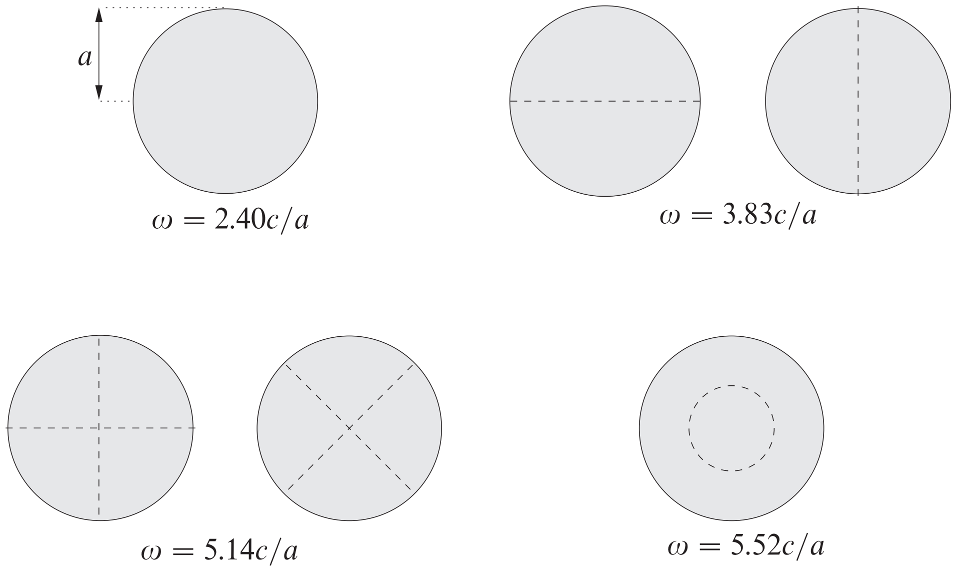

Figure 21.8 The modes of oscillation with the four lowest frequencies for a circular drumskin of radius a. The dashed lines indicate the nodes, where the displacement of the drumskin is always zero.

Helmholtz's equation in cylindrical polars

Generalising the above method to three-dimensional cylindrical polars is straightforward, and following a similar procedure to that used for Laplace's equation we find the separated solution of Helmholtz's equation takes the form

$$\begin{aligned} F(\rho, \phi, z) = &\left[AJ_m\left(\sqrt{k^2-\alpha^2}\,\rho\right) + BY_m\left(\sqrt{k^2-\alpha^2}\,\rho\right)\right] \\ &\times (C\cos m\phi + D\sin m\phi)[E\exp(i\alpha z) + F\exp(-i\alpha z)], \end{aligned}$$

where α and m are separation constants. We note that the angular part of the solution is the same as for Laplace's equation in cylindrical polars.

Helmholtz's equation in spherical polars

In spherical polars, we find again that the angular parts of the solution $\Theta(\theta)\Phi(\phi)$ are identical to those of Laplace's equation in this coordinate system, i.e. they are the spherical harmonics $Y_\ell^m(\theta, \phi)$, and so we shall not discuss them further.

The radial equation in this case is given by

$$r^2R'' + 2rR' + [k^2r^2 - \ell(\ell+1)]R = 0, \tag{21.56}$$

which has an additional term k^2r^2R compared with the radial equation for the Laplace solution. The equation (21.56) looks very much like Bessel's equation. In fact, by writing $R(r) = r^{-1/2}S(r)$ and making the change of variable $\mu = kr$, it can be reduced to Bessel's equation of order $\ell + \frac{1}{2}$, which has as its solutions $S(\mu) = J_{\ell+1/2}(\mu)$ and $Y_{\ell+1/2}(\mu)$ (see section 18.6). The separated solution to

Helmholtz's equation in spherical polars is thus

$$F(r,\theta,\phi) = r^{-1/2}[AJ_{\ell+1/2}(kr) + BY_{\ell+1/2}(kr)](C\cos m\phi + D\sin m\phi)$$
$$\times[EP_\ell^m(\cos\theta) + FQ_\ell^m(\cos\theta)]. \tag{21.57}$$

For solutions that are finite at the origin we require $B = 0$, and for solutions that are finite on the polar axis we require $F = 0$. It is worth mentioning that the solutions proportional to $r^{-1/2}J_{\ell+1/2}(kr)$ and $r^{-1/2}Y_{\ell+1/2}(kr)$, when suitably normalised, are called *spherical Bessel functions* of the first and second kind, respectively, and are denoted by $j_\ell(kr)$ and $n_\ell(\mu)$ (see section 18.6).

As mentioned at the beginning of this subsection, the separated solution of the wave equation in spherical polars is the product of a time-dependent part (21.54) and a spatial part (21.57). It will be noticed that, although this solution corresponds to a solution of definite frequency $\omega = kc$, the zeros of the radial function $j_\ell(kr)$ are not equally spaced in r, except for the case $\ell = 0$ involving $j_0(kr)$, and so there is no precise wavelength associated with the solution.

To conclude this subsection, let us mention briefly the Schrödinger equation for the electron in a hydrogen atom, the nucleus of which is taken at the origin and is assumed massive compared with the electron. Under these circumstances the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2 u - \frac{e^2}{4\pi\epsilon_0}\frac{u}{r} = i\hbar\frac{\partial u}{\partial t}.$$

For a 'stationary-state' solution, for which the energy is a constant E and the time-dependent factor T in u is given by $T(t) = A\exp(-iEt/\hbar)$, the above equation is similar to, but not quite the same as, the Helmholtz equation.§ However, as with the wave equation, the angular parts of the solution are identical to those for Laplace's equation and are expressed in terms of spherical harmonics.

The important point to note is that for *any* equation involving ∇^2, provided θ and ϕ do not appear in the equation other than as part of ∇^2, a separated-variable solution in spherical polars will always lead to spherical harmonic solutions. This is the case for the Schrödinger equation describing an atomic electron in a central potential $V(r)$.

21.3.3 Solution by expansion

It is sometimes possible to use the uniqueness theorem discussed in the previous chapter, together with the results of the last few subsections, in which Laplace's equation (and other equations) were considered in polar coordinates, to obtain solutions of such equations appropriate to particular physical situations.

§ For the solution by series of the r-equation in this case the reader may consult, for example, L. Schiff, *Quantum Mechanics* (New York: McGraw-Hill, 1955), p. 82.

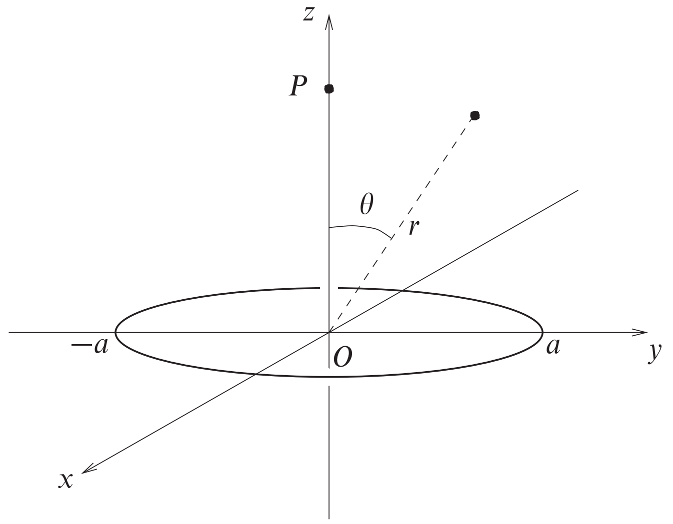

Figure 21.9 The polar axis Oz is taken as normal to the plane of the ring of matter and passing through its centre.

We will illustrate the method for Laplace's equation in spherical polars and first assume that the required solution of $\nabla^2 u = 0$ can be written as a superposition in the normal way:

$$u(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (Ar^{\ell} + Br^{-(\ell+1)})P_{\ell}^{m}(\cos\theta)(C\cos m\phi + D\sin m\phi). \tag{21.58}$$

Here, all the constants A, B, C, D may depend upon ℓ and m, and we have assumed that the required solution is finite on the polar axis. As usual, boundary conditions of a physical nature will then fix or eliminate some of the constants; for example, u finite at the origin implies all $B = 0$, or axial symmetry implies that only $m = 0$ terms are present.

The essence of the method is then to find the remaining constants by determining u at values of r, θ, ϕ for which it can be evaluated *by other means*, e.g. by direct calculation on an axis of symmetry. Once the remaining constants have been fixed by these special considerations to have particular values, the uniqueness theorem can be invoked to establish that they must have these values in general.

►*Calculate the gravitational potential at a general point in space due to a uniform ring of matter of radius a and total mass M.*

Everywhere except on the ring the potential $u(\mathbf{r})$ satisfies the Laplace equation, and so if we use polar coordinates with the normal to the ring as polar axis, as in figure 21.9, a solution of the form (21.58) can be assumed.

We expect the potential $u(r, \theta, \phi)$ to tend to zero as $r \to \infty$, and also to be finite at $r = 0$. At first sight this might seem to imply that all A and B, and hence u, must be identically zero, an unacceptable result. In fact, what it means is that different expressions must apply to different regions of space. On the ring itself we no longer have $\nabla^2 u = 0$ and so it is not surprising that the form of the expression for u changes there. Let us therefore take two separate regions.

In the region $r > a$

(i) we must have $u \to 0$ as $r \to \infty$, implying that all $A = 0$, *and*
(ii) the system is axially symmetric and so only $m = 0$ terms appear.

With these restrictions we can write as a trial form

$$u(r, \theta, \phi) = \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell^0(\cos\theta). \tag{21.59}$$

The constants B_ℓ are still to be determined; this we do by calculating *directly* the potential where this can be done simply – in this case, on the polar axis.

Considering a point P on the polar axis at a distance z ($> a$) from the plane of the ring (taken as $\theta = \pi/2$), all parts of the ring are at a distance $(z^2 + a^2)^{1/2}$ from it. The potential at P is thus straightforwardly

$$u(z, 0, \phi) = -\frac{GM}{(z^2 + a^2)^{1/2}}, \tag{21.60}$$

where G is the gravitational constant. This must be the same as (21.59) for the particular values $r = z$, $\theta = 0$, and ϕ undefined. Since $P_\ell^0(\cos\theta) = P_\ell(\cos\theta)$ with $P_\ell(1) = 1$, putting $r = z$ in (21.59) gives

$$u(z, 0, \phi) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{z^{\ell+1}}. \tag{21.61}$$

However, expanding (21.60) for $z > a$ (as it applies to this region of space) we obtain

$$u(z, 0, \phi) = -\frac{GM}{z}\left[1 - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{3}{8}\left(\frac{a}{z}\right)^4 - \cdots\right],$$

which on comparison with (21.61) gives[§]

$$\begin{aligned} B_0 &= -GM, \\ B_{2\ell} &= -\frac{GMa^{2\ell}(-1)^\ell(2\ell-1)!!}{2^\ell \ell!} \qquad \text{for } \ell \geq 1, \\ B_{2\ell+1} &= 0. \end{aligned} \tag{21.62}$$

We now conclude the argument by saying that if a solution for a general point (r, θ, ϕ) exists at all, which of course we very much expect on physical grounds, then it must be (21.59) with the B_ℓ given by (21.62). This is so because thus defined it is a function with no arbitrary constants and which satisfies all the boundary conditions, and the uniqueness theorem states that there is only one such function. The expression for the potential in the region $r > a$ is therefore

$$u(r, \theta, \phi) = -\frac{GM}{r}\left[1 + \sum_{\ell=1}^{\infty} \frac{(-1)^\ell(2\ell-1)!!}{2^\ell \ell!}\left(\frac{a}{r}\right)^{2\ell} P_{2\ell}(\cos\theta)\right].$$

The expression for $r < a$ can be found in a similar way. The finiteness of u at $r = 0$ and the axial symmetry give

$$u(r, \theta, \phi) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell^0(\cos\theta).$$

[§] $(2\ell - 1)!! = 1 \times 3 \times \cdots \times (2\ell - 1)$.

Comparing this expression for $r = z$, $\theta = 0$ with the $z < a$ expansion of (21.60), which is valid for any z, establishes $A_{2\ell+1} = 0$, $A_0 = -GM/a$ and

$$A_{2\ell} = -\frac{GM}{a^{2\ell+1}} \frac{(-1)^\ell (2\ell - 1)!!}{2^\ell \ell!},$$

so that the final expression valid, and convergent, for $r < a$ is thus

$$u(r, \theta, \phi) = -\frac{GM}{a} \left[1 + \sum_{\ell=1}^{\infty} \frac{(-1)^\ell (2\ell - 1)!!}{2^\ell \ell!} \left(\frac{r}{a} \right)^{2\ell} P_{2\ell}(\cos\theta) \right].$$

It is easy to check that the solution obtained has the expected physical value for large r and for $r = 0$ and is continuous at $r = a$. ◀

21.3.4 Separation of variables for inhomogeneous equations

So far our discussion of the method of separation of variables has been limited to the solution of homogeneous equations such as the Laplace equation and the wave equation. The solutions of inhomogeneous PDEs are usually obtained using the Green's function methods to be discussed below in section 21.5. However, as a final illustration of the usefulness of the separation of variables, we now consider its application to the solution of inhomogeneous equations.

Because of the added complexity in dealing with inhomogeneous equations, we shall restrict our discussion to the solution of Poisson's equation,

$$\nabla^2 u = \rho(\mathbf{r}), \tag{21.63}$$

in spherical polar coordinates, although the general method can accommodate other coordinate systems and equations. In physical problems the RHS of (21.63) usually contains some multiplicative constant(s). If u is the electrostatic potential in some region of space in which ρ is the density of electric charge then $\nabla^2 u = -\rho(\mathbf{r})/\epsilon_0$. Alternatively, u might represent the gravitational potential in some region where the matter density is given by ρ, so that $\nabla^2 u = 4\pi G \rho(\mathbf{r})$.

We will simplify our discussion by assuming that the required solution u is finite on the polar axis and also that the system possesses axial symmetry about that axis – in which case ρ does not depend on the azimuthal angle ϕ. The key to the method is then to assume a separated form for both the solution u *and* the density term ρ.

From the discussion of Laplace's equation, for systems with axial symmetry only $m = 0$ terms appear, and so the angular part of the solution can be expressed in terms of Legendre polynomials $P_\ell(\cos\theta)$. Since these functions form an orthogonal set let us expand both u and ρ in terms of them:

$$u = \sum_{\ell=0}^{\infty} R_\ell(r) P_\ell(\cos\theta), \tag{21.64}$$

$$\rho = \sum_{\ell=0}^{\infty} F_\ell(r) P_\ell(\cos\theta), \tag{21.65}$$

where the coefficients $R_\ell(r)$ and $F_\ell(r)$ in the Legendre polynomial expansions are functions of r. Since in any particular problem ρ is given, we can find the coefficients $F_\ell(r)$ in the expansion in the usual way (see subsection 18.1.2). It then only remains to find the coefficients $R_\ell(r)$ in the expansion of the solution u.

Writing ∇^2 in spherical polars and substituting (21.64) and (21.65) into (21.63) we obtain

$$\sum_{\ell=0}^{\infty}\left[\frac{P_\ell(\cos\theta)}{r^2}\frac{d}{dr}\left(r^2\frac{dR_\ell}{dr}\right)+\frac{R_\ell}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_\ell(\cos\theta)}{d\theta}\right)\right]=\sum_{\ell=0}^{\infty}F_\ell(r)P_\ell(\cos\theta). \tag{21.66}$$

However, if, in equation (21.44) of our discussion of the angular part of the solution to Laplace's equation, we set $m=0$ we conclude that

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_\ell(\cos\theta)}{d\theta}\right)=-\ell(\ell+1)P_\ell(\cos\theta).$$

Substituting this into (21.66), we find that the LHS is greatly simplified and we obtain

$$\sum_{\ell=0}^{\infty}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_\ell}{dr}\right)-\frac{\ell(\ell+1)R_\ell}{r^2}\right]P_\ell(\cos\theta)=\sum_{\ell=0}^{\infty}F_\ell(r)P_\ell(\cos\theta).$$

This relation is most easily satisfied by equating terms on both sides for each value of ℓ separately, so that for $\ell=0,1,2,\ldots$ we have

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_\ell}{dr}\right)-\frac{\ell(\ell+1)R_\ell}{r^2}=F_\ell(r). \tag{21.67}$$

This is an ODE in which $F_\ell(r)$ is given, and it can therefore be solved for $R_\ell(r)$. The solution to Poisson's equation, u, is then obtained by making the superposition (21.64).

►*In a certain system, the electric charge density ρ is distributed as follows:*

$$\rho=\begin{cases}Ar\cos\theta & \textit{for } 0\le r<a,\\ 0 & \textit{for } r\ge a.\end{cases}$$

Find the electrostatic potential inside and outside the charge distribution, given that both the potential and its radial derivative are continuous everywhere.

The electrostatic potential u satisfies

$$\nabla^2 u=\begin{cases}-(A/\epsilon_0)r\cos\theta & \text{for } 0\le r<a,\\ 0 & \text{for } r\ge a.\end{cases}$$

For $r<a$ the RHS can be written $-(A/\epsilon_0)rP_1(\cos\theta)$, and the coefficients in (21.65) are simply $F_1(r)=-(Ar/\epsilon_0)$ and $F_\ell(r)=0$ for $\ell\neq 1$. Therefore we need only calculate $R_1(r)$, which satisfies (21.67) for $\ell=1$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_1}{dr}\right)-\frac{2R_1}{r^2}=-\frac{Ar}{\epsilon_0}.$$

This can be rearranged to give

$$r^2R_1'' + 2rR_1' - 2R_1 = -\frac{Ar^3}{\epsilon_0},$$

where the prime denotes differentiation with respect to r. The LHS is homogeneous and the equation can be reduced by the substitution $r = \exp t$, and writing $R_1(r) = S(t)$, to

$$\ddot{S} + \dot{S} - 2S = -\frac{A}{\epsilon_0}\exp 3t, \tag{21.68}$$

where the dots indicate differentiation with respect to t.

This is an inhomogeneous second-order ODE with constant coefficients and can be straightforwardly solved by the methods of subsection 15.2.1 to give

$$S(t) = c_1 \exp t + c_2 \exp(-2t) - \frac{A}{10\epsilon_0}\exp 3t.$$

Recalling that $r = \exp t$ we find

$$R_1(r) = c_1 r + c_2 r^{-2} - \frac{A}{10\epsilon_0}r^3.$$

Since we are interested in the region $r < a$ we must have $c_2 = 0$ for the solution to remain finite. Thus inside the charge distribution the electrostatic potential has the form

$$u_1(r,\theta,\phi) = \left(c_1 r - \frac{A}{10\epsilon_0}r^3\right) P_1(\cos\theta). \tag{21.69}$$

Outside the charge distribution (for $r \geq a$), however, the electrostatic potential obeys Laplace's equation, $\nabla^2 u = 0$, and so given the symmetry of the problem and the requirement that $u \to \infty$ as $r \to \infty$ the solution must take the form

$$u_2(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta). \tag{21.70}$$

We can now use the boundary conditions at $r = a$ to fix the constants in (21.69) and (21.70). The requirement of continuity of the potential and its radial derivative at $r = a$ imply that

$$u_1(a,\theta,\phi) = u_2(a,\theta,\phi),$$
$$\frac{\partial u_1}{\partial r}(a,\theta,\phi) = \frac{\partial u_2}{\partial r}(a,\theta,\phi).$$

Clearly $B_\ell = 0$ for $\ell \neq 1$; carrying out the necessary differentiations and setting $r = a$ in (21.69) and (21.70) we obtain the simultaneous equations

$$c_1 a - \frac{A}{10\epsilon_0}a^3 = \frac{B_1}{a^2},$$
$$c_1 - \frac{3A}{10\epsilon_0}a^2 = -\frac{2B_1}{a^3},$$

which may be solved to give $c_1 = Aa^2/(6\epsilon_0)$ and $B_1 = Aa^5/(15\epsilon_0)$. Since $P_1(\cos\theta) = \cos\theta$, the electrostatic potentials inside and outside the charge distribution are given, respectively, by

$$u_1(r,\theta,\phi) = \frac{A}{\epsilon_0}\left(\frac{a^2 r}{6} - \frac{r^3}{10}\right)\cos\theta, \qquad u_2(r,\theta,\phi) = \frac{Aa^5}{15\epsilon_0}\frac{\cos\theta}{r^2}. \blacktriangleleft$$

21.4 Integral transform methods

In the method of separation of variables our aim was to keep the independent variables in a PDE as separate as possible. We now discuss the use of integral transforms in solving PDEs, a method by which one of the independent variables can be eliminated from the differential coefficients. It will be assumed that the reader is familiar with Laplace and Fourier transforms and their properties, as discussed in chapter 13.

The method consists simply of transforming the PDE into one containing derivatives with respect to a smaller number of variables. Thus, if the original equation has just two independent variables, it may be possible to reduce the PDE into a soluble ODE. The solution obtained can then (where possible) be transformed back to give the solution of the original PDE. As we shall see, boundary conditions can usually be incorporated in a natural way.

Which sort of transform to use, and the choice of the variable(s) with respect to which the transform is to be taken, is a matter of experience; we illustrate this in the example below. In practice, transforms can be taken with respect to each variable in turn, and the transformation that affords the greatest simplification can be pursued further.

►*A semi-infinite tube of constant cross-section contains initially pure water. At time $t = 0$, one end of the tube is put into contact with a salt solution and maintained at a concentration u_0. Find the total amount of salt that has diffused into the tube after time t, if the diffusion constant is κ.*

The concentration $u(x, t)$ at time t and distance x from the end of the tube satisfies the diffusion equation

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{21.71}$$

which has to be solved subject to the boundary conditions $u(0, t) = u_0$ for all t and $u(x, 0) = 0$ for all $x > 0$.

Since we are interested only in $t > 0$, the use of the Laplace transform is suggested. Furthermore, it will be recalled from chapter 13 that one of the major virtues of Laplace transformations is the possibility they afford of replacing derivatives of functions by simple multiplication by a scalar. If the derivative with respect to time were so removed, equation (21.71) would contain only differentiation with respect to a single variable. Let us therefore take the Laplace transform of (21.71) with respect to t:

$$\int_0^\infty \kappa \frac{\partial^2 u}{\partial x^2} \exp(-st)\, dt = \int_0^\infty \frac{\partial u}{\partial t} \exp(-st)\, dt.$$

On the LHS the (double) differentiation is with respect to x, whereas the integration is with respect to the independent variable t. Therefore the derivative can be taken outside the integral. Denoting the Laplace transform of $u(x, t)$ by $\bar{u}(x, s)$ and using result (13.57) to rewrite the transform of the derivative on the RHS (or by integrating directly by parts), we obtain

$$\kappa \frac{\partial^2 \bar{u}}{\partial x^2} = s\bar{u}(x, s) - u(x, 0).$$

But from the boundary condition $u(x, 0) = 0$ the last term on the RHS vanishes, and the solution is immediate:

$$\bar{u}(x,s) = A\exp\left(\sqrt{\frac{s}{\kappa}}\,x\right) + B\exp\left(-\sqrt{\frac{s}{\kappa}}\,x\right),$$

where the constants A and B may depend on s.

We require $u(x,t) \to 0$ as $x \to \infty$ and so we must also have $\bar{u}(\infty,s) = 0$; consequently we require that $A = 0$. The value of B is determined by the need for $u(0,t) = u_0$ and hence that

$$\bar{u}(0,s) = \int_0^\infty u_0 \exp(-st)\,dt = \frac{u_0}{s}.$$

We thus conclude that the appropriate expression for the Laplace transform of $u(x,t)$ is

$$\bar{u}(x,s) = \frac{u_0}{s}\exp\left(-\sqrt{\frac{s}{\kappa}}\,x\right). \tag{21.72}$$

To obtain $u(x,t)$ from this result requires the inversion of this transform – a task that is generally difficult and requires a contour integration. This is discussed in chapter 24, but for completeness we note that the solution is

$$u(x,t) = u_0\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right)\right],$$

where $\operatorname{erf}(x)$ is the error function discussed in the Appendix. (The more complete sets of mathematical tables list this inverse Laplace transform.)

In the present problem, however, an alternative method is available. Let $w(t)$ be the amount of salt that has diffused into the tube in time t; then

$$w(t) = \int_0^\infty u(x,t)\,dx,$$

and its transform is given by

$$\begin{aligned}\bar{w}(s) &= \int_0^\infty dt\;\exp(-st)\int_0^\infty u(x,t)\,dx \\ &= \int_0^\infty dx\int_0^\infty u(x,t)\exp(-st)\,dt \\ &= \int_0^\infty \bar{u}(x,s)\,dx.\end{aligned}$$

Substituting for $\bar{u}(x,s)$ from (21.72) into the last integral and integrating, we obtain

$$\bar{w}(s) = u_0\kappa^{1/2}s^{-3/2}.$$

This expression is much simpler to invert, and referring to the table of standard Laplace transforms (table 13.1) we find

$$w(t) = 2(\kappa/\pi)^{1/2}u_0 t^{1/2},$$

which is thus the required expression for the amount of diffused salt at time t. ◄

The above example shows that in some circumstances the use of a Laplace transformation can greatly simplify the solution of a PDE. However, it will have been observed that (as with ODEs) the easy elimination of some derivatives is usually paid for by the introduction of a difficult inverse transformation. This problem, although still present, is less severe for Fourier transformations.

►*An infinite metal bar has an initial temperature distribution $f(x)$ along its length. Find the temperature distribution at a later time t.*

We are interested in values of x from $-\infty$ to ∞, which suggests Fourier transformation with respect to x. Assuming that the solution obeys the boundary conditions $u(x,t) \to 0$ and $\partial u/\partial x \to 0$ as $|x| \to \infty$, we may Fourier-transform the one-dimensional diffusion equation (21.71) to obtain

$$\frac{\kappa}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2} \exp(-ikx)\, dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x,t) \exp(-ikx)\, dx,$$

where on the RHS we have taken the partial derivative with respect to t outside the integral. Denoting the Fourier transform of $u(x,t)$ by $\widetilde{u}(k,t)$, and using equation (13.28) to rewrite the Fourier transform of the second derivative on the LHS, we then have

$$-\kappa k^2 \widetilde{u}(k,t) = \frac{\partial \widetilde{u}(k,t)}{\partial t}.$$

This first-order equation has the simple solution

$$\widetilde{u}(k,t) = \widetilde{u}(k,0) \exp(-\kappa k^2 t),$$

where the initial conditions give

$$\begin{aligned}\widetilde{u}(k,0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) \exp(-ikx)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx = \widetilde{f}(k).\end{aligned}$$

Thus we may write the Fourier transform of the solution as

$$\widetilde{u}(k,t) = \widetilde{f}(k) \exp(-\kappa k^2 t) = \sqrt{2\pi}\, \widetilde{f}(k) \widetilde{G}(k,t), \tag{21.73}$$

where we have defined the function $\widetilde{G}(k,t) = (\sqrt{2\pi})^{-1} \exp(-\kappa k^2 t)$. Since $\widetilde{u}(k,t)$ can be written as the product of two Fourier transforms, we can use the convolution theorem, subsection 13.1.7, to write the solution as

$$u(x,t) = \int_{-\infty}^{\infty} G(x-x',t) f(x')\, dx',$$

where $G(x,t)$ is the Green's function for this problem (see subsection 15.2.5). This function is the inverse Fourier transform of $\widetilde{G}(k,t)$ and is thus given by

$$\begin{aligned}G(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\kappa k^2 t) \exp(ikx)\, dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\kappa t \left(k^2 - \frac{ix}{\kappa t} k\right)\right] dk.\end{aligned}$$

Completing the square in the integrand we find

$$\begin{aligned}G(x,t) &= \frac{1}{2\pi} \exp\left(-\frac{x^2}{4\kappa t}\right) \int_{-\infty}^{\infty} \exp\left[-\kappa t \left(k - \frac{ix}{2\kappa t}\right)^2\right] dk \\ &= \frac{1}{2\pi} \exp\left(-\frac{x^2}{4\kappa t}\right) \int_{-\infty}^{\infty} \exp\left(-\kappa t k'^2\right) dk' \\ &= \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right),\end{aligned}$$

where in the second line we have made the substitution $k' = k - ix/(2\kappa t)$, and in the last

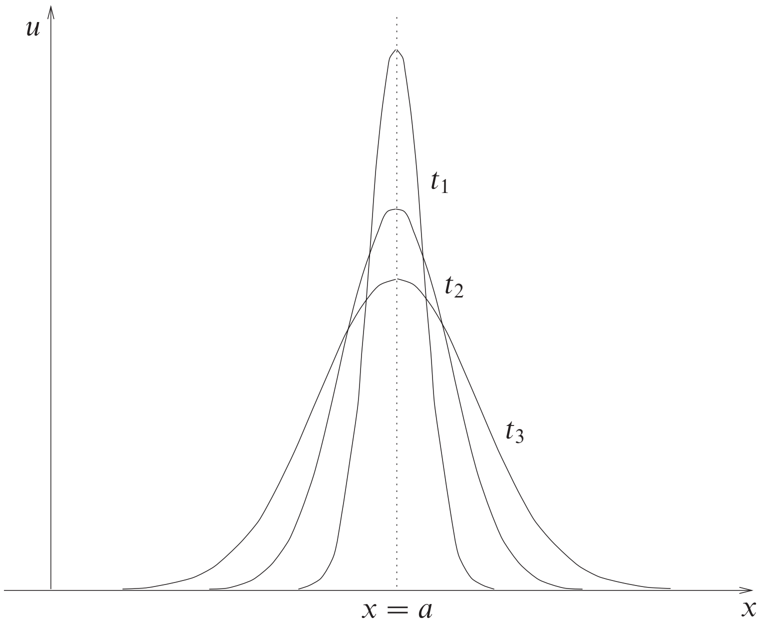

Figure 21.10 Diffusion of heat from a point source in a metal bar: the curves show the temperature u at position x for various times $t_1 < t_2 < t_3$. The area under the curves remains constant, since the total heat energy is conserved.

line we have used the standard result for the integral of a Gaussian, given in subsection 6.4.2. (Strictly speaking the change of variable from k to k' shifts the path of integration off the real axis, since k' is complex for real k, and so results in a complex integral, as will be discussed in chapter 24. Nevertheless, in this case the path of integration can be shifted back to the real axis without affecting the value of the integral.)

Thus the temperature in the bar at a later time t is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-x')^2}{4\kappa t}\right] f(x')\, dx', \qquad (21.74)$$

which may be evaluated (numerically if necessary) when the form of $f(x)$ is given. ◄

As we might expect from our discussion of Green's functions in chapter 15, we see from (21.74) that, if the initial temperature distribution is $f(x) = \delta(x-a)$, i.e. a 'point' source at $x = a$, then the temperature distribution at later times is simply given by

$$u(x,t) = G(x-a,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{(x-a)^2}{4\kappa t}\right].$$

The temperature at several later times is illustrated in figure 21.10, which shows that the heat diffuses out from its initial position; the width of the Gaussian increases as $\sqrt{t}$, a dependence on time which is characteristic of diffusion processes.

The reader may have noticed that in both examples using integral transforms the solutions have been obtained in closed form – albeit in one case in the form of an integral. This differs from the infinite series solutions usually obtained via the separation of variables. It should be noted that this behaviour is a result of the infinite range in x, rather than of the transform method itself. In fact the method of separation of variables would yield the same solutions, since in the infinite-range case the separation constant is not restricted to take on an infinite set of discrete values but may have any real value, with the result that the sum over λ becomes an integral, as mentioned at the end of section 21.2.

►*An infinite metal bar has an initial temperature distribution $f(x)$ along its length. Find the temperature distribution at a later time t using the method of separation of variables.*

This is the same problem as in the previous example, but we now seek a solution by separating variables. From (21.12) a separated solution for the one-dimensional diffusion equation is given by

$$u(x,t) = [A\exp(i\lambda x) + B\exp(-i\lambda x)]\exp(-\kappa\lambda^2 t),$$

where $-\lambda^2$ is the separation constant. Since the bar is infinite we do not require the solution to take a given form at any finite value of x (for instance at $x = 0$) and so there is no restriction on λ other than its being real. Therefore instead of the superposition of such solutions in the form of a sum over allowed values of λ we have an integral over all λ,

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\lambda)\exp(-\kappa\lambda^2 t)\exp(i\lambda x)\,d\lambda, \qquad (21.75)$$

where in taking λ from $-\infty$ to ∞ we need include only one of the complex exponentials; we have taken a factor $1/\sqrt{2\pi}$ out of $A(\lambda)$ for convenience. We can see from (21.75) that the expression for $u(x,t)$ has the form of an inverse Fourier transform (where λ is the transform variable). Therefore, Fourier-transforming both sides and using the Fourier inversion theorem, we find

$$\widetilde{u}(\lambda,t) = A(\lambda)\exp(-\kappa\lambda^2 t).$$

Now, the initial boundary condition requires

$$u(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\lambda)\exp(i\lambda x)\,d\lambda = f(x),$$

from which, using the Fourier inversion theorem once more, we see that $A(\lambda) = \widetilde{f}(\lambda)$. Therefore we have

$$\widetilde{u}(\lambda,t) = \widetilde{f}(\lambda)\exp(-\kappa\lambda^2 t),$$

which is identical to (21.73) in the previous example (but with k replaced by λ), and hence leads to the same result. ◄

21.5 Inhomogeneous problems – Green's functions

In chapters 15 and 17 we encountered Green's functions and found them a useful tool for solving inhomogeneous linear ODEs. We now discuss their usefulness in solving inhomogeneous linear PDEs.

For the sake of brevity we shall again denote a linear PDE by

$$\mathcal{L}u(\mathbf{r}) = \rho(\mathbf{r}), \qquad (21.76)$$

where $\mathcal{L}$ is a linear partial differential operator. For example, in Laplace's equation we have $\mathcal{L} = \nabla^2$, whereas for Helmholtz's equation $\mathcal{L} = \nabla^2 + k^2$. Note that we have not specified the dimensionality of the problem, and (21.76) may, for example, represent Poisson's equation in two or three (or more) dimensions. The reader will also notice that for the sake of simplicity we have not included any time dependence in (21.76). Nevertheless, the following discussion can be generalised to include it.

As we discussed in subsection 20.3.2, a problem is inhomogeneous if the fact that $u(\mathbf{r})$ is a solution does *not* imply that any constant multiple $\lambda u(\mathbf{r})$ is also a solution. This inhomogeneity may derive from either the PDE itself or from the boundary conditions imposed on the solution.

In our discussion of Green's function solutions of inhomogeneous ODEs (see subsection 15.2.5) we dealt with inhomogeneous boundary conditions by making a suitable change of variable such that in the new variable the boundary conditions were homogeneous. In an analogous way, as illustrated in the final example of section 21.2, it is usually possible to make a change of variables in PDEs to transform between inhomogeneity of the boundary conditions and inhomogeneity of the equation. Therefore let us assume for the moment that the boundary conditions imposed on the solution $u(\mathbf{r})$ of (21.76) are homogeneous. This most commonly means that if we seek a solution to (21.76) in some region V then on the surface S that bounds V the solution obeys the conditions $u(\mathbf{r}) = 0$ or $\partial u/\partial n = 0$, where $\partial u/\partial n$ is the normal derivative of u at the surface S.

We shall discuss the extension of the Green's function method to the direct solution of problems with inhomogeneous boundary conditions in subsection 21.5.2, but we first highlight how the Green's function approach to solving ODEs can be simply extended to PDEs for homogeneous boundary conditions.

21.5.1 Similarities to Green's functions for ODEs

As in the discussion of ODEs in chapter 15, we may consider the Green's function for a system described by a PDE as the response of the system to a 'unit impulse' or 'point source'. Thus if we seek a solution to (21.76) that satisfies some homogeneous boundary conditions on $u(\mathbf{r})$ then the Green's function $G(\mathbf{r}, \mathbf{r}_0)$ for the problem is a solution of

$$\mathcal{L}G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0), \tag{21.77}$$

where $\mathbf{r}_0$ lies in V. The Green's function $G(\mathbf{r}, \mathbf{r}_0)$ must also satisfy the imposed (homogeneous) boundary conditions.

It is understood that in (21.77) the $\mathcal{L}$ operator expresses differentiation with respect to $\mathbf{r}$ as opposed to $\mathbf{r}_0$. Also, $\delta(\mathbf{r} - \mathbf{r}_0)$ is the Dirac delta function (see chapter 13) of dimension appropriate to the problem; it may be thought of as representing a unit-strength point source at $\mathbf{r} = \mathbf{r}_0$.

Following an analogous argument to that given in subsection 15.2.5 for ODEs, if the boundary conditions on $u(\mathbf{r})$ are homogeneous then a solution to (21.76) that satisfies the imposed boundary conditions is given by

$$u(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r}_0)\, dV(\mathbf{r}_0), \tag{21.78}$$

where the integral on $\mathbf{r}_0$ is over some appropriate 'volume'. In two or more dimensions, however, the task of finding directly a solution to (21.77) that satisfies the imposed boundary conditions on S can be a difficult one, and we return to this in the next subsection.

An alternative approach is to follow a similar argument to that presented in chapter 17 for ODEs and so to construct the Green's function for (21.76) as a superposition of eigenfunctions of the operator $\mathcal{L}$, provided $\mathcal{L}$ is Hermitian. By analogy with an ordinary differential operator, a partial differential operator is Hermitian if it satisfies

$$\int_V v^*(\mathbf{r})\mathcal{L}w(\mathbf{r})\, dV = \left[\int_V w^*(\mathbf{r})\mathcal{L}v(\mathbf{r})\, dV\right]^*,$$

where the asterisk denotes complex conjugation and v and w are arbitrary functions obeying the imposed (homogeneous) boundary condition on the solution of $\mathcal{L}u(\mathbf{r}) = 0$.

The eigenfunctions $u_n(\mathbf{r})$, $n = 0, 1, 2, \ldots$, of $\mathcal{L}$ satisfy

$$\mathcal{L}u_n(\mathbf{r}) = \lambda_n u_n(\mathbf{r}),$$

where λ_n are the corresponding eigenvalues, which are all real for an Hermitian operator $\mathcal{L}$. Furthermore, each eigenfunction must obey any imposed (homogeneous) boundary conditions. Using an argument analogous to that given in chapter 17, the Green's function for the problem is given by

$$G(\mathbf{r}, \mathbf{r}_0) = \sum_{n=0}^{\infty} \frac{u_n(\mathbf{r})u_n^*(\mathbf{r}_0)}{\lambda_n}. \tag{21.79}$$

From (21.79) we see immediately that the Green's function (irrespective of how it is found) enjoys the property

$$G(\mathbf{r}, \mathbf{r}_0) = G^*(\mathbf{r}_0, \mathbf{r}).$$

Thus, if the Green's function is real then it is symmetric in its two arguments.

Once the Green's function has been obtained, the solution to (21.76) is again given by (21.78). For PDEs this approach can become very cumbersome, however, and so we shall not pursue it further here.

21.5.2 General boundary-value problems

As mentioned above, often inhomogeneous boundary conditions can be dealt with by making an appropriate change of variables, such that the boundary conditions in the new variables are homogeneous although the equation itself is generally inhomogeneous. In this section, however, we extend the use of Green's functions to problems with inhomogeneous boundary conditions (and equations). This provides a more consistent and intuitive approach to the solution of such *boundary-value problems.*

For definiteness we shall consider Poisson's equation

$$\nabla^2 u(\mathbf{r}) = \rho(\mathbf{r}), \tag{21.80}$$

but the material of this section may be extended to other linear PDEs of the form (21.76). Clearly, Poisson's equation reduces to Laplace's equation for $\rho(\mathbf{r}) = 0$ and so our discussion is equally applicable to this case.

We wish to solve (21.80) in some region V bounded by a surface S, which may consist of several disconnected parts. As stated above, we shall allow the possibility that the boundary conditions on the solution $u(\mathbf{r})$ may be inhomogeneous on S, although as we shall see this method reduces to those discussed above in the special case that the boundary conditions are in fact homogeneous.

The two common types of inhomogeneous boundary condition for Poisson's equation are (as discussed in subsection 20.6.2):

(i) Dirichlet conditions, in which $u(\mathbf{r})$ is specified on S, and
(ii) Neumann conditions, in which $\partial u/\partial n$ is specified on S.

In general, specifying *both* Dirichlet *and* Neumann conditions on S overdetermines the problem and leads to there being no solution.

The specification of the surface S requires some further comment, since S may have several disconnected parts. If we wish to solve Poisson's equation inside some closed surface S then the situation is straightforward and is shown in figure 21.11(a). If, however, we wish to solve Poisson's equation in the gap between two closed surfaces (for example in the gap between two concentric conducting cylinders) then the volume V is bounded by a surface S that has two disconnected parts S_1 and S_2, as shown in figure 21.11(b); the direction of the normal to the surface is always taken as pointing *out* of the volume V. A similar situation arises when we wish to solve Poisson's equation *outside* some closed surface S_1. In this case the volume V is infinite but is treated formally by taking the surface S_2 as a large sphere of radius R and letting R tend to infinity.

In order to solve (21.80) subject to either Dirichlet or Neumann boundary conditions on S, we will remind ourselves of Green's second theorem, equation (11.20), which states that, for two scalar functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ defined in some volume V bounded by a surface S,

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \hat{\mathbf{n}}\, dS, \tag{21.81}$$

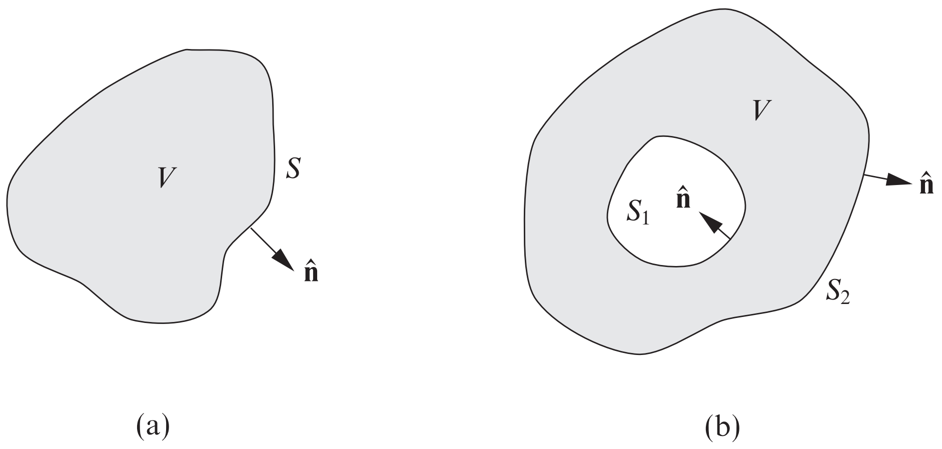

Figure 21.11 Surfaces used for solving Poisson's equation in different regions V.

where on the RHS it is common to write, for example, $\nabla\psi \cdot \hat{\mathbf{n}}\, dS$ as $(\partial\psi/\partial n)\, dS$. The expression $\partial\psi/\partial n$ stands for $\nabla\psi \cdot \hat{\mathbf{n}}$, the rate of change of ψ in the direction of the unit outward normal $\hat{\mathbf{n}}$ to the surface S.

The Green's function for Poisson's equation (21.80) must satisfy

$$\nabla^2 G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0), \tag{21.82}$$

where $\mathbf{r}_0$ lies in V. (As mentioned above, we may think of $G(\mathbf{r}, \mathbf{r}_0)$ as the solution to Poisson's equation for a unit-strength point source located at $\mathbf{r} = \mathbf{r}_0$.) Let us for the moment impose no boundary conditions on $G(\mathbf{r}, \mathbf{r}_0)$.

If we now let $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r}, \mathbf{r}_0)$ in Green's theorem (21.81) then we obtain

$$\int_V \left[u(\mathbf{r})\nabla^2 G(\mathbf{r}, \mathbf{r}_0) - G(\mathbf{r}, \mathbf{r}_0)\, \nabla^2 u(\mathbf{r})\right]\, dV(\mathbf{r})$$
$$= \int_S \left[u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right]\, dS(\mathbf{r}),$$

where we have made explicit that the volume and surface integrals are with respect to $\mathbf{r}$. Using (21.80) and (21.82) the LHS can be simplified to give

$$\int_V [u(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0) - G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})]\, dV(\mathbf{r})$$
$$= \int_S \left[u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right]\, dS(\mathbf{r}). \tag{21.83}$$

Since $\mathbf{r}_0$ lies within the volume V,

$$\int_V u(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0)\, dV(\mathbf{r}) = u(\mathbf{r}_0),$$

and thus on rearranging (21.83) the solution to Poisson's equation (21.80) can be written as

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \int_S \left[u(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0) \frac{\partial u(\mathbf{r})}{\partial n} \right] dS(\mathbf{r}). \tag{21.84}$$

Clearly, we can interchange the roles of $\mathbf{r}$ and $\mathbf{r}_0$ in (21.84) if we wish. (Remember also that, for a real Green's function, $G(\mathbf{r}, \mathbf{r}_0) = G(\mathbf{r}_0, \mathbf{r})$.)

Equation (21.84) is *central* to the extension of the Green's function method to problems with inhomogeneous boundary conditions, and we next discuss its application to both Dirichlet and Neumann boundary-value problems. But, before doing so, we also note that if the boundary condition on S is in fact homogeneous, so that $u(\mathbf{r}) = 0$ or $\partial u(\mathbf{r})/\partial n = 0$ on S, then demanding that the Green's function $G(\mathbf{r}, \mathbf{r}_0)$ also obeys the same boundary condition causes the surface integral in (21.84) to vanish, and we are left with the familiar form of solution given in (21.78). The extension of (21.84) to a PDE other than Poisson's equation is discussed in exercise 21.28.

21.5.3 Dirichlet problems

In a Dirichlet problem we require the solution $u(\mathbf{r})$ of Poisson's equation (21.80) to take specific values on some surface S that bounds V, i.e. we require that $u(\mathbf{r}) = f(\mathbf{r})$ on S where f is a given function.

If we seek a Green's function $G(\mathbf{r}, \mathbf{r}_0)$ for this problem it must clearly satisfy (21.82), but we are free to choose the boundary conditions satisfied by $G(\mathbf{r}, \mathbf{r}_0)$ in such a way as to make the solution (21.84) as simple as possible. From (21.84), we see that by choosing

$$G(\mathbf{r}, \mathbf{r}_0) = 0 \qquad \text{for } \mathbf{r} \text{ on } S \tag{21.85}$$

the second term in the surface integral vanishes. Since $u(\mathbf{r}) = f(\mathbf{r})$ on S, (21.84) then becomes

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \int_S f(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n}\, dS(\mathbf{r}). \tag{21.86}$$

Thus we wish to find the *Dirichlet Green's function* that

(i) satisfies (21.82) and hence is singular at $\mathbf{r} = \mathbf{r}_0$, and
(ii) obeys the boundary condition $G(\mathbf{r}, \mathbf{r}_0) = 0$ for $\mathbf{r}$ on S.

In general, it is difficult to obtain this function directly, and so it is useful to separate these two requirements. We therefore look for a solution of the form

$$G(\mathbf{r}, \mathbf{r}_0) = F(\mathbf{r}, \mathbf{r}_0) + H(\mathbf{r}, \mathbf{r}_0),$$

where $F(\mathbf{r}, \mathbf{r}_0)$ satisfies (21.82) and has the required singular character at $\mathbf{r} = \mathbf{r}_0$ but does not necessarily obey the boundary condition on S, whilst $H(\mathbf{r}, \mathbf{r}_0)$ satisfies the corresponding homogeneous equation (i.e. Laplace's equation) inside V but is adjusted in such a way that the sum $G(\mathbf{r}, \mathbf{r}_0)$ equals zero on S. The Green's function $G(\mathbf{r}, \mathbf{r}_0)$ is still a solution of (21.82) since

$$\nabla^2 G(\mathbf{r}, \mathbf{r}_0) = \nabla^2 F(\mathbf{r}, \mathbf{r}_0) + \nabla^2 H(\mathbf{r}, \mathbf{r}_0) = \nabla^2 F(\mathbf{r}, \mathbf{r}_0) + 0 = \delta(\mathbf{r} - \mathbf{r}_0).$$

The function $F(\mathbf{r}, \mathbf{r}_0)$ is called the *fundamental solution* and will clearly take different forms depending on the dimensionality of the problem. Let us first consider the fundamental solution to (21.82) in three dimensions.

►*Find the fundamental solution to Poisson's equation in three dimensions that tends to zero as* $|\mathbf{r}| \to \infty$.

We wish to solve

$$\nabla^2 F(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0) \tag{21.87}$$

in three dimensions, subject to the boundary condition $F(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$. Since the problem is spherically symmetric about $\mathbf{r}_0$, let us consider a large sphere S of radius R centred on $\mathbf{r}_0$, and integrate (21.87) over the enclosed volume V. We then obtain

$$\int_V \nabla^2 F(\mathbf{r}, \mathbf{r}_0)\, dV = \int_V \delta(\mathbf{r} - \mathbf{r}_0)\, dV = 1, \tag{21.88}$$

since V encloses the point $\mathbf{r}_0$. However, using the divergence theorem,

$$\int_V \nabla^2 F(\mathbf{r}, \mathbf{r}_0)\, dV = \int_S \nabla F(\mathbf{r}, \mathbf{r}_0) \cdot \hat{\mathbf{n}}\, dS, \tag{21.89}$$

where $\hat{\mathbf{n}}$ is the unit normal to the large sphere S at any point.

Since the problem is spherically symmetric about $\mathbf{r}_0$, we expect that

$$F(\mathbf{r}, \mathbf{r}_0) = F(|\mathbf{r} - \mathbf{r}_0|) = F(r),$$

i.e. that F has the same value everywhere on S. Thus, evaluating the surface integral in (21.89) and equating it to unity from (21.88), we have[§]

$$4\pi r^2 \frac{dF}{dr}\bigg|_{r=R} = 1.$$

Integrating this expression we obtain

$$F(r) = -\frac{1}{4\pi r} + \text{constant},$$

but, since we require $F(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$, the constant must be zero. The fundamental solution in three dimensions is consequently given by

$$F(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|}. \tag{21.90}$$

This is clearly also the full Green's function for Poisson's equation subject to the boundary condition $u(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$. ◄

Using (21.90) we can write down the solution of Poisson's equation to find,

§ A vertical bar to the right of an expression is a common alternative to enclosing the expression in square brackets; as usual, the subscript shows the value of the variable at which the expression is to be evaluated.

for example, the electrostatic potential $u(\mathbf{r})$ due to some distribution of electric charge $\rho(\mathbf{r})$. The electrostatic potential satisfies

$$\nabla^2 u(\mathbf{r}) = -\frac{\rho}{\epsilon_0},$$

where $u(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$. Since the boundary condition on the surface at infinity is homogeneous the surface integral in (21.86) vanishes, and using (21.90) we recover the familiar solution

$$u(\mathbf{r}_0) = \int \frac{\rho(\mathbf{r})}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_0|}\, dV(\mathbf{r}), \qquad (21.91)$$

where the volume integral is over all space.

We can develop an analogous theory in two dimensions. As before the fundamental solution satisfies

$$\nabla^2 F(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0), \qquad (21.92)$$

where $\delta(\mathbf{r}-\mathbf{r}_0)$ is now the two-dimensional delta function. Following an analogous method to that used in the previous example, we find the fundamental solution in two dimensions to be given by

$$F(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_0| + \text{constant}. \qquad (21.93)$$

From the form of the solution we see that in two dimensions we cannot apply the condition $F(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$, and in this case the constant does not necessarily vanish.

We now return to the task of constructing the full Dirichlet Green's function. To do so we wish to add to the fundamental solution a solution of the homogeneous equation (in this case Laplace's equation) such that $G(\mathbf{r}, \mathbf{r}_0) = 0$ on S, as required by (21.86) and its attendant conditions. The appropriate Green's function is constructed by adding to the fundamental solution 'copies' of itself that represent 'image' sources at different locations *outside* V. Hence this approach is called the *method of images*.

In summary, if we wish to solve Poisson's equation in some region V subject to Dirichlet boundary conditions on its surface S then the procedure and argument are as follows.

(i) To the single source $\delta(\mathbf{r} - \mathbf{r}_0)$ inside V add image sources *outside* V

$$\sum_{n=1}^{N} q_n \delta(\mathbf{r} - \mathbf{r}_n) \quad \text{with } \mathbf{r}_n \text{ outside } V,$$

where the positions $\mathbf{r}_n$ and the strengths q_n of the image sources are to be determined as described in step (iii) below.

(ii) Since all the image sources lie outside V, the fundamental solution corresponding to each source satisfies Laplace's equation *inside* V. Thus we may add the fundamental solutions $F(\mathbf{r}, \mathbf{r}_n)$ corresponding to each image source to that corresponding to the single source inside V, obtaining the Green's function

$$G(\mathbf{r}, \mathbf{r}_0) = F(\mathbf{r}, \mathbf{r}_0) + \sum_{n=1}^{N} q_n F(\mathbf{r}, \mathbf{r}_n).$$

(iii) Now adjust the positions $\mathbf{r}_n$ and strengths q_n of the image sources so that the required boundary conditions are satisfied on S. For a Dirichlet Green's function we require $G(\mathbf{r}, \mathbf{r}_0) = 0$ for $\mathbf{r}$ on S.

(iv) The solution to Poisson's equation subject to the Dirichlet boundary condition $u(\mathbf{r}) = f(\mathbf{r})$ on S is then given by (21.86).

In general it is very difficult to find the correct positions and strengths for the images, i.e. to make them such that the boundary conditions on S are satisfied. Nevertheless, it is possible to do so for certain problems that have simple geometry. In particular, for problems in which the boundary S consists of straight lines (in two dimensions) or planes (in three dimensions), positions of the image points can be deduced simply by imagining the boundary lines or planes to be mirrors in which the single source in V (at $\mathbf{r}_0$) is reflected.

►*Solve Laplace's equation $\nabla^2 u = 0$ in three dimensions in the half-space $z > 0$, given that $u(\mathbf{r}) = f(\mathbf{r})$ on the plane $z = 0$.*

The surface S bounding V consists of the xy-plane and the surface at infinity. Therefore, the Dirichlet Green's function for this problem must satisfy $G(\mathbf{r}, \mathbf{r}_0) = 0$ on $z = 0$ and $G(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$. Thus it is clear in this case that we require one image source at a position $\mathbf{r}_1$ that is the reflection of $\mathbf{r}_0$ in the plane $z = 0$, as shown in figure 21.12 (so that $\mathbf{r}_1$ lies in $z < 0$, outside the region in which we wish to obtain a solution). It is also clear that the strength of this image should be -1.

Therefore by adding the fundamental solutions corresponding to the original source and its image we obtain the Green's function

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} + \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_1|}, \tag{21.94}$$

where $\mathbf{r}_1$ is the reflection of $\mathbf{r}_0$ in the plane $z = 0$, i.e. if $\mathbf{r}_0 = (x_0, y_0, z_0)$ then $\mathbf{r}_1 = (x_0, y_0, -z_0)$. Clearly $G(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$ as required. Also $G(\mathbf{r}, \mathbf{r}_0) = 0$ on $z = 0$, and so (21.94) is the desired Dirichlet Green's function.

The solution to Laplace's equation is then given by (21.86) with $\rho(\mathbf{r}) = 0$,

$$u(\mathbf{r}_0) = \int_S f(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} \, dS(\mathbf{r}). \tag{21.95}$$

Clearly the surface at infinity makes no contribution to this integral. The outward-pointing unit vector normal to the xy-plane is simply $\hat{\mathbf{n}} = -\mathbf{k}$ (where $\mathbf{k}$ is the unit vector in the z-direction), and so

$$\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} = -\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial z} = -\mathbf{k} \cdot \nabla G(\mathbf{r}, \mathbf{r}_0).$$

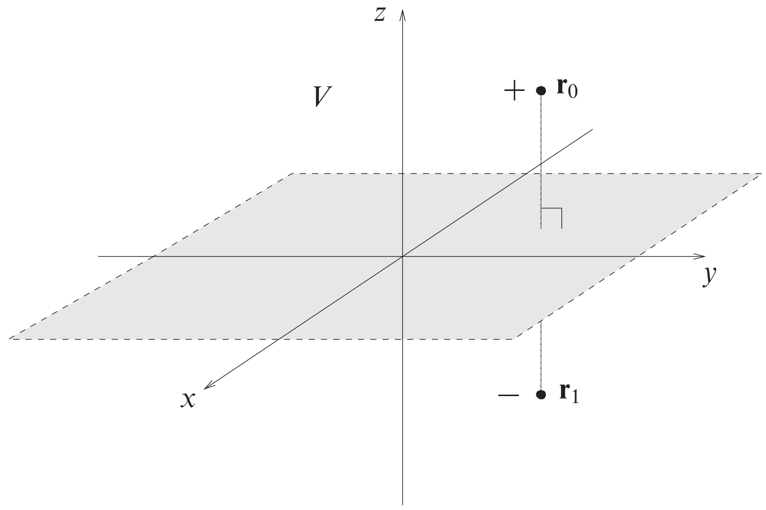

Figure 21.12 The arrangement of images for solving Laplace's equation in the half-space $z > 0$.

We may evaluate this normal derivative by writing the Green's function (21.94) explicitly in terms of x, y and z (and x_0, y_0 and z_0) and calculating the partial derivative with respect to z directly. It is usually quicker, however, to use the fact that[§]

$$\nabla|\mathbf{r} - \mathbf{r}_0| = \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}; \tag{21.96}$$

thus

$$\nabla G(\mathbf{r}, \mathbf{r}_0) = \frac{\mathbf{r} - \mathbf{r}_0}{4\pi|\mathbf{r} - \mathbf{r}_0|^3} - \frac{\mathbf{r} - \mathbf{r}_1}{4\pi|\mathbf{r} - \mathbf{r}_1|^3}.$$

Since $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{r}_1 = (x_0, y_0, -z_0)$ the normal derivative is given by

$$\begin{aligned} -\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial z} &= -\mathbf{k} \cdot \nabla G(\mathbf{r}, \mathbf{r}_0) \\ &= -\frac{z - z_0}{4\pi|\mathbf{r} - \mathbf{r}_0|^3} + \frac{z + z_0}{4\pi|\mathbf{r} - \mathbf{r}_1|^3}. \end{aligned}$$

Therefore on the surface $z = 0$, and writing out the dependence on x, y and z explicitly, we have

$$-\left.\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial z}\right|_{z=0} = \frac{2z_0}{4\pi[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}}.$$

Inserting this expression into (21.95) we obtain the solution

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x, y)}{[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}}\, dx\, dy. \blacktriangleleft$$

An analogous procedure may be applied in two-dimensional problems. For

[§] Since $|\mathbf{r} - \mathbf{r}_0|^2 = (\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r} - \mathbf{r}_0)$ we have $\nabla|\mathbf{r} - \mathbf{r}_0|^2 = 2(\mathbf{r} - \mathbf{r}_0)$, from which we obtain

$$\nabla(|\mathbf{r} - \mathbf{r}_0|^2)^{1/2} = \frac{1}{2} \frac{2(\mathbf{r} - \mathbf{r}_0)}{(|\mathbf{r} - \mathbf{r}_0|^2)^{1/2}} = \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}.$$

Note that this result holds in two *and* three dimensions.

example, in solving Poisson's equation in two dimensions in the half-space $x > 0$ we again require just one image charge, of strength $q_1 = -1$, at a position $\mathbf{r}_1$ that is the reflection of $\mathbf{r}_0$ in the line $x = 0$. Since we require $G(\mathbf{r}, \mathbf{r}_0) = 0$ when $\mathbf{r}$ lies on $x = 0$, the constant in (21.93) must equal zero, and so the Dirichlet Green's function is

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi}\left(\ln|\mathbf{r} - \mathbf{r}_0| - \ln|\mathbf{r} - \mathbf{r}_1|\right).$$

Clearly $G(\mathbf{r}, \mathbf{r}_0)$ tends to zero as $|\mathbf{r}| \to \infty$. If, however, we wish to solve the two-dimensional Poisson equation in the quarter space $x > 0$, $y > 0$, then more image points are required.

►*A line charge in the z-direction of charge density λ is placed at some position $\mathbf{r}_0$ in the quarter-space $x > 0$, $y > 0$. Calculate the force per unit length on the line charge due to the presence of thin earthed plates along $x = 0$ and $y = 0$.*

Here we wish to solve Poisson's equation,

$$\nabla^2 u = -\frac{\lambda}{\epsilon_0}\delta(\mathbf{r} - \mathbf{r}_0),$$

in the quarter space $x > 0$, $y > 0$. It is clear that we require three image line charges with positions and strengths as shown in figure 21.13 (all of which lie outside the region in which we seek a solution). The boundary condition that the electrostatic potential u is zero on $x = 0$ and $y = 0$ (shown as the 'curve' C in figure 21.13) is then automatically satisfied, and so this system of image charges is directly equivalent to the original situation of a single line charge in the presence of the earthed plates along $x = 0$ and $y = 0$. Thus the electrostatic potential is simply equal to the Dirichlet Green's function

$$u(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}_0) = -\frac{\lambda}{2\pi\epsilon_0}\left(\ln|\mathbf{r} - \mathbf{r}_0| - \ln|\mathbf{r} - \mathbf{r}_1| + \ln|\mathbf{r} - \mathbf{r}_2| - \ln|\mathbf{r} - \mathbf{r}_3|\right),$$

which equals zero on C and on the 'surface' at infinity.

The force on the line charge at $\mathbf{r}_0$, therefore, is simply that due to the three line charges at $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. The elecrostatic potential due to a line charge at $\mathbf{r}_i$, $i = 1$, 2 or 3, is given by the fundamental solution

$$u_i(\mathbf{r}) = \mp\frac{\lambda}{2\pi\epsilon_0}\ln|\mathbf{r} - \mathbf{r}_i| + c,$$

the upper or lower sign being taken according to whether the line charge is positive or negative, respectively. Therefore the force per unit length on the line charge at $\mathbf{r}_0$, due to the one at $\mathbf{r}_i$, is given by

$$-\lambda\nabla u_i(\mathbf{r})\bigg|_{\mathbf{r}=\mathbf{r}_0} = \pm\frac{\lambda^2}{2\pi\epsilon_0}\frac{\mathbf{r}_0 - \mathbf{r}_i}{|\mathbf{r}_0 - \mathbf{r}_i|^2}.$$

Adding the contributions from the three image charges shown in figure 21.13, the total force experienced by the line charge at $\mathbf{r}_0$ is given by

$$\mathbf{F} = \frac{\lambda^2}{2\pi\epsilon_0}\left(-\frac{\mathbf{r}_0 - \mathbf{r}_1}{|\mathbf{r}_0 - \mathbf{r}_1|^2} + \frac{\mathbf{r}_0 - \mathbf{r}_2}{|\mathbf{r}_0 - \mathbf{r}_2|^2} - \frac{\mathbf{r}_0 - \mathbf{r}_3}{|\mathbf{r}_0 - \mathbf{r}_3|^2}\right),$$

where, from the figure, $\mathbf{r}_0 - \mathbf{r}_1 = 2y_0\mathbf{j}$, $\mathbf{r}_0 - \mathbf{r}_2 = 2x_0\mathbf{i} + 2y_0\mathbf{j}$ and $\mathbf{r}_0 - \mathbf{r}_3 = 2x_0\mathbf{i}$. Thus, in

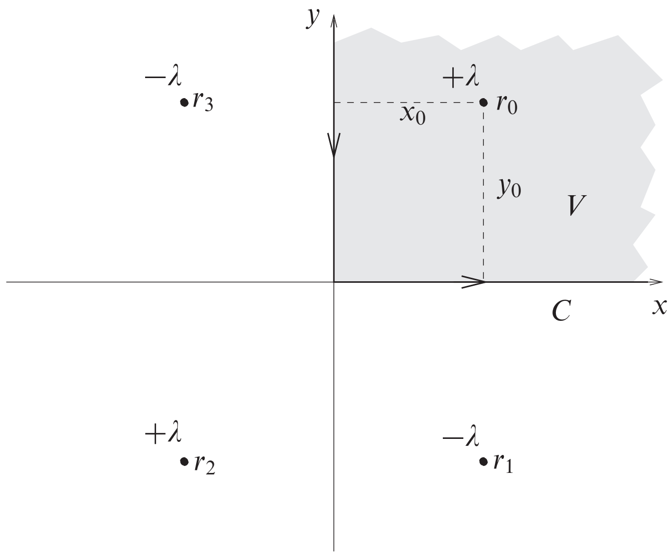

Figure 21.13 The arrangement of images for finding the force on a line charge situated in the (two-dimensional) quarter-space $x > 0$, $y > 0$, when the planes $x = 0$ and $y = 0$ are earthed.

terms of x_0 and y_0, the total force on the line charge due to the charge induced on the plates is given by

$$\mathbf{F} = \frac{\lambda^2}{2\pi\epsilon_0}\left(-\frac{1}{2y_0}\mathbf{j} + \frac{2x_0\,\mathbf{i} + 2y_0\,\mathbf{j}}{4x_0^2 + 4y_0^2} - \frac{1}{2x_0}\mathbf{i}\right)$$
$$= -\frac{\lambda^2}{4\pi\epsilon_0(x_0^2 + y_0^2)}\left(\frac{y_0^2}{x_0}\mathbf{i} + \frac{x_0^2}{y_0}\mathbf{j}\right). \blacktriangleleft$$

Further generalisations are possible. For instance, solving Poisson's equation in the two-dimensional strip $-\infty < x < \infty$, $0 < y < b$ requires an infinite series of image points.

So far we have considered problems in which the boundary S consists of straight lines (in two dimensions) or planes (in three dimensions), in which simple reflections of the source at $\mathbf{r}_0$ in these boundaries fix the positions of the image points. For more complicated (curved) boundaries this is no longer possible, and finding the appropriate position(s) and strength(s) of the image source(s) requires further work.

►*Use the method of images to find the Dirichlet Green's function for solving Poisson's equation outside a sphere of radius a centred at the origin.*

We need to find a solution of Poisson's equation valid outside the sphere of radius a. Since an image point $\mathbf{r}_1$ cannot lie in this region, it must be located within the sphere. The Green's function for this problem is therefore

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} - \frac{q}{4\pi|\mathbf{r} - \mathbf{r}_1|},$$

where $|\mathbf{r}_0| > a$, $|\mathbf{r}_1| < a$ and q is the strength of the image which we have yet to determine. Clearly, $G(\mathbf{r}, \mathbf{r}_0) \to 0$ on the surface at infinity.

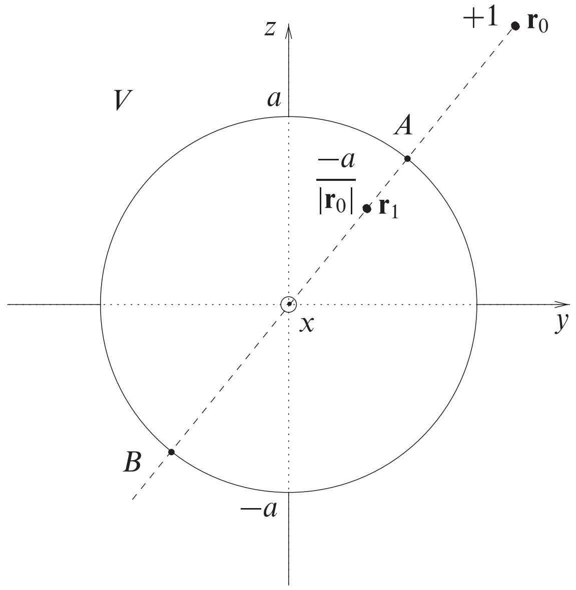

Figure 21.14 The arrangement of images for solving Poisson's equation outside a sphere of radius a centred at the origin. For a charge $+1$ at $\mathbf{r}_0$, the image point $\mathbf{r}_1$ is given by $(a/|\mathbf{r}_0|)^2\mathbf{r}_0$ and the strength of the image charge is $-a/|\mathbf{r}_0|$.

By symmetry we expect the image point $\mathbf{r}_1$ to lie on the same radial line as the original source, $\mathbf{r}_0$, as shown in figure 21.14, and so $\mathbf{r}_1 = k\mathbf{r}_0$ where $k < 1$. However, for a Dirichlet Green's function we require $G(\mathbf{r}-\mathbf{r}_0) = 0$ on $|\mathbf{r}| = a$, and the form of the Green's function suggests that we need

$$|\mathbf{r}-\mathbf{r}_0| \propto |\mathbf{r}-\mathbf{r}_1| \qquad \text{for all } |\mathbf{r}| = a. \tag{21.97}$$

Referring to figure 21.14, if this relationship is to hold over the whole surface of the sphere, then it must certainly hold for the points A and B. We thus require

$$\frac{|\mathbf{r}_0|-a}{a-|\mathbf{r}_1|} = \frac{|\mathbf{r}_0|+a}{a+|\mathbf{r}_1|},$$

which reduces to $|\mathbf{r}_1| = a^2/|\mathbf{r}_0|$. Therefore the image point must be located at the position

$$\mathbf{r}_1 = \frac{a^2}{|\mathbf{r}_0|^2}\mathbf{r}_0.$$

It may now be checked that, for this location of the image point, (21.97) is satisfied over the whole sphere. Using the geometrical result

$$\begin{aligned}|\mathbf{r}-\mathbf{r}_1|^2 &= |\mathbf{r}|^2 - \frac{2a^2}{|\mathbf{r}_0|^2}\mathbf{r}\cdot\mathbf{r}_0 + \frac{a^4}{|\mathbf{r}_0|^2}\\ &= \frac{a^2}{|\mathbf{r}_0|^2}\left(|\mathbf{r}_0|^2 - 2\mathbf{r}\cdot\mathbf{r}_0 + a^2\right) \qquad \text{for } |\mathbf{r}| = a,\end{aligned} \tag{21.98}$$

we see that, on the surface of the sphere,

$$|\mathbf{r}-\mathbf{r}_1| = \frac{a}{|\mathbf{r}_0|}|\mathbf{r}-\mathbf{r}_0| \qquad \text{for } |\mathbf{r}| = a. \tag{21.99}$$

Therefore, in order that $G = 0$ at $|\mathbf{r}| = a$, the strength of the image charge must be $-a/|\mathbf{r}_0|$. Consequently, the Dirichlet Green's function for the exterior of the sphere is

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} + \frac{a/|\mathbf{r}_0|}{4\pi\,|\mathbf{r} - (a^2/|\mathbf{r}_0|^2)\mathbf{r}_0|}.$$

For a less formal treatment of the same problem see exercise 21.22. ◄

If we seek solutions to Poisson's equation in the *interior* of a sphere then the above analysis still holds, but $\mathbf{r}$ and $\mathbf{r}_0$ are now inside the sphere and the image $\mathbf{r}_1$ lies outside it.

For two-dimensional Dirichlet problems outside the circle $|\mathbf{r}| = a$, we are led by arguments similar to those employed previously to use the same image point as in the three-dimensional case, namely

$$\mathbf{r}_1 = \frac{a^2}{|\mathbf{r}_0|^2}\mathbf{r}_0. \tag{21.100}$$

As illustrated below, however, it is usually necessary to take the image strength as -1 in two-dimensional problems.

►*Solve Laplace's equation in the two-dimensional region $|\mathbf{r}| \le a$, subject to the boundary condition $u = f(\phi)$ on $|\mathbf{r}| = a$.*

In this case we wish to find the Dirichlet Green's function in the interior of a disc of radius a, so the image charge must lie outside the disc. Taking the strength of the image to be -1, we have

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi}\ln|\mathbf{r} - \mathbf{r}_0| - \frac{1}{2\pi}\ln|\mathbf{r} - \mathbf{r}_1| + c,$$

where $\mathbf{r}_1 = (a^2/|\mathbf{r}_0|^2)\mathbf{r}_0$ lies outside the disc, and c is a constant that includes the strength of the image charge and does not necessarily equal zero.

Since we require $G(\mathbf{r}, \mathbf{r}_0) = 0$ when $|\mathbf{r}| = a$, the value of the constant c is determined, and the Dirichlet Green's function for this problem is given by

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi}\left(\ln|\mathbf{r} - \mathbf{r}_0| - \ln\left|\mathbf{r} - \frac{a^2}{|\mathbf{r}_0|^2}\mathbf{r}_0\right| - \ln\frac{|\mathbf{r}_0|}{a}\right). \tag{21.101}$$

Using plane polar coordinates, the solution to the boundary-value problem can be written as a line integral around the circle $\rho = a$:

$$\begin{aligned} u(\mathbf{r}_0) &= \int_C f(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n}\,dl \\ &= \int_0^{2\pi} f(\mathbf{r})\left.\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \rho}\right|_{\rho=a} a\,d\phi. \end{aligned} \tag{21.102}$$

The normal derivative of the Green's function (21.101) is given by

$$\begin{aligned} \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \rho} &= \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\nabla G(\mathbf{r}, \mathbf{r}_0) \\ &= \frac{\mathbf{r}}{2\pi|\mathbf{r}|}\cdot\left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^2} - \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^2}\right). \end{aligned} \tag{21.103}$$

Using the fact that $\mathbf{r}_1 = (a^2/|\mathbf{r}_0|^2)\mathbf{r}_0$ and the geometrical result (21.99), we find that

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial \rho}\right|_{\rho=a} = \frac{a^2 - |\mathbf{r}_0|^2}{2\pi a|\mathbf{r}-\mathbf{r}_0|^2}.$$

In plane polar coordinates, $\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j}$ and $\mathbf{r}_0 = \rho_0\cos\phi_0\,\mathbf{i} + \rho_0\sin\phi_0\,\mathbf{j}$, and so

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial \rho}\right|_{\rho=a} = \left(\frac{1}{2\pi a}\right)\frac{a^2-\rho_0^2}{a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)}.$$

On substituting into (21.102), we obtain

$$u(\rho_0,\phi_0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-\rho_0^2)f(\phi)\,d\phi}{a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)}, \tag{21.104}$$

which is the solution to the problem. ◄

21.5.4 Neumann problems

In a Neumann problem we require the normal derivative of the solution of Poisson's equation to take on specific values on some surface S that bounds V, i.e. we require $\partial u(\mathbf{r})/\partial n = f(\mathbf{r})$ on S, where f is a given function. As we shall see, much of our discussion of Dirichlet problems can be immediately taken over into the solution of Neumann problems.

As we proved in section 20.7 of the previous chapter, specifying Neumann boundary conditions determines the relevant solution of Poisson's equation to within an (unimportant) additive constant. Unlike Dirichlet conditions, Neumann conditions impose a self-consistency requirement. In order for a solution u to exist, it is necessary that the following consistency condition holds:

$$\int_S f\,dS = \int_S \nabla u\cdot\hat{\mathbf{n}}\,dS = \int_V \nabla^2 u\,dV = \int_V \rho\,dV, \tag{21.105}$$

where we have used the divergence theorem to convert the surface integral into a volume integral. As a physical example, the integral of the normal component of an electric field over a surface bounding a given volume cannot be chosen arbitrarily when the charge inside the volume has already been specified (Gauss's theorem).

Let us again consider (21.84), which is central to our discussion of Green's functions in inhomogeneous problems. It reads

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r},\mathbf{r}_0)\rho(\mathbf{r})\,dV(\mathbf{r}) + \int_S\left[u(\mathbf{r})\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} - G(\mathbf{r},\mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right]dS(\mathbf{r}).$$

As always, the Green's function must obey

$$\nabla^2 G(\mathbf{r},\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0),$$

where $\mathbf{r}_0$ lies in V. In the solution of Dirichlet problems in the previous subsection, we chose the Green's function to obey the boundary condition $G(\mathbf{r},\mathbf{r}_0) = 0$ on S

and, in a similar way, we might wish to choose $\partial G(\mathbf{r},\mathbf{r}_0)/\partial n = 0$ in the solution of Neumann problems. However, in general this is *not* permitted since the Green's function must obey the consistency condition

$$\int_S \frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n}\, dS = \int_S \nabla G(\mathbf{r},\mathbf{r}_0)\cdot \hat{\mathbf{n}}\, dS = \int_V \nabla^2 G(\mathbf{r},\mathbf{r}_0)\, dV = 1.$$

The simplest permitted boundary condition is therefore

$$\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} = \frac{1}{A} \qquad \text{for } \mathbf{r} \text{ on } S,$$

where A is the area of the surface S; this defines a *Neumann Green's function.*

If we require $\partial u(\mathbf{r})/\partial n = f(\mathbf{r})$ on S, the solution to Poisson's equation is given by

$$\begin{aligned} u(\mathbf{r}_0) &= \int_V G(\mathbf{r},\mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \frac{1}{A}\int_S u(\mathbf{r})\, dS(\mathbf{r}) - \int_S G(\mathbf{r},\mathbf{r}_0) f(\mathbf{r})\, dS(\mathbf{r}) \\ &= \int_V G(\mathbf{r},\mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \langle u(\mathbf{r})\rangle_S - \int_S G(\mathbf{r},\mathbf{r}_0) f(\mathbf{r})\, dS(\mathbf{r}), \end{aligned} \tag{21.106}$$

where $\langle u(\mathbf{r})\rangle_S$ is the average of u over the surface S and is a freely specifiable constant. For Neumann problems in which the volume V is bounded by a surface S at infinity, we do not need the $\langle u(\mathbf{r})\rangle_S$ term. For example, if we wish to solve a Neumann problem outside the unit sphere centred at the origin then $r > a$ is the region V throughout which we require the solution; this region may be considered as being bounded by two disconnected surfaces, the surface of the sphere and a surface at infinity. By requiring that $u(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, the term $\langle u(\mathbf{r})\rangle_S$ becomes zero.

As mentioned above, much of our discussion of Dirichlet problems can be taken over into the solution of Neumann problems. In particular, we may use the method of images to find the appropriate Neumann Green's function.

►*Solve Laplace's equation in the two-dimensional region $|\mathbf{r}| \le a$ subject to the boundary condition $\partial u/\partial n = f(\phi)$ on $|\mathbf{r}| = a$, with $\int_0^{2\pi} f(\phi)\, d\phi = 0$ as required by the consistency condition (21.105).*

Let us assume, as in Dirichlet problems with this geometry, that a single image charge is placed outside the circle at

$$\mathbf{r}_1 = \frac{a^2}{|\mathbf{r}_0|^2}\mathbf{r}_0,$$

where $\mathbf{r}_0$ is the position of the source inside the circle (see equation (21.100)). Then, from (21.99), we have the useful geometrical result

$$|\mathbf{r}-\mathbf{r}_1| = \frac{a}{|\mathbf{r}_0|}|\mathbf{r}-\mathbf{r}_0| \qquad \text{for } |\mathbf{r}| = a. \tag{21.107}$$

Leaving the strength q of the image as a parameter, the Green's function has the form

$$G(\mathbf{r},\mathbf{r}_0) = \frac{1}{2\pi}\left(\ln|\mathbf{r}-\mathbf{r}_0| + q\ln|\mathbf{r}-\mathbf{r}_1| + c\right). \tag{21.108}$$

Using plane polar coordinates, the radial (i.e. normal) derivative of this function is given by

$$\begin{aligned}\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial \rho} &= \frac{\mathbf{r}}{|\mathbf{r}|}\cdot \nabla G(\mathbf{r},\mathbf{r}_0)\\ &= \frac{\mathbf{r}}{2\pi|\mathbf{r}|}\cdot\left[\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^2}+\frac{q(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^2}\right].\end{aligned}$$

Using (21.107), on the perimeter of the circle $\rho = a$ the radial derivative takes the form

$$\begin{aligned}\left.\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial \rho}\right|_{\rho=a} &= \frac{1}{2\pi|\mathbf{r}|}\left[\frac{|\mathbf{r}|^2-\mathbf{r}\cdot\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^2} + \frac{q|\mathbf{r}|^2-q(a^2/|\mathbf{r}_0|^2)\mathbf{r}\cdot\mathbf{r}_0}{(a^2/|\mathbf{r}_0|^2)|\mathbf{r}-\mathbf{r}_0|^2}\right]\\ &= \frac{1}{2\pi a}\frac{1}{|\mathbf{r}-\mathbf{r}_0|^2}\left[|\mathbf{r}|^2+q|\mathbf{r}_0|^2-(1+q)\mathbf{r}\cdot\mathbf{r}_0\right],\end{aligned}$$

where we have set $|\mathbf{r}|^2 = a^2$ in the second term on the RHS, but not in the first. If we take $q = 1$, the radial derivative simplifies to

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial \rho}\right|_{\rho=a} = \frac{1}{2\pi a},$$

or $1/L$, where L is the circumference, and so (21.108) with $q = 1$ is the required Neumann Green's function.

Since $\rho(\mathbf{r}) = 0$, the solution to our boundary-value problem is now given by (21.106) as

$$u(\mathbf{r}_0) = \langle u(\mathbf{r})\rangle_C - \int_C G(\mathbf{r},\mathbf{r}_0) f(\mathbf{r})\, dl(\mathbf{r}),$$

where the integral is around the circumference of the circle C. In plane polar coordinates $\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j}$ and $\mathbf{r}_0 = \rho_0\cos\phi_0\,\mathbf{i} + \rho_0\sin\phi_0\,\mathbf{j}$, and again using (21.107) we find that on C the Green's function is given by

$$\begin{aligned}G(\mathbf{r},\mathbf{r}_0)|_{\rho=a} &= \frac{1}{2\pi}\left[\ln|\mathbf{r}-\mathbf{r}_0| + \ln\left(\frac{a}{|\mathbf{r}_0|}|\mathbf{r}-\mathbf{r}_0|\right)+c\right]\\ &= \frac{1}{2\pi}\left(\ln|\mathbf{r}-\mathbf{r}_0|^2 + \ln\frac{a}{|\mathbf{r}_0|}+c\right)\\ &= \frac{1}{2\pi}\left\{\ln\left[a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)\right]+\ln\frac{a}{\rho_0}+c\right\}. \qquad (21.109)\end{aligned}$$

Since $dl = a\,d\phi$ on C, the solution to the problem is given by

$$u(\rho_0,\phi_0) = \langle u\rangle_C - \frac{a}{2\pi}\int_0^{2\pi} f(\phi)\ln[a^2+\rho_0^2-2a\rho_0\cos(\phi-\phi_0)]\,d\phi.$$

The contributions of the final two terms terms in the Green's function (21.109) vanish because $\int_0^{2\pi} f(\phi)\,d\phi = 0$. The average value of u around the circumference, $\langle u\rangle_C$, is a freely specifiable constant as we would expect for a Neumann problem. This result should be compared with the result (21.104) for the corresponding Dirichlet problem, but it should be remembered that in the one case $f(\phi)$ is a potential, and in the other the gradient of a potential. ◀

21.6 Exercises

21.1 Solve the following first-order partial differential equations by separating the variables:

$$\text{(a)}\ \frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0; \qquad \text{(b)}\ x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0.$$

21.2 A cube, made of material whose conductivity is k, has as its six faces the planes $x = \pm a$, $y = \pm a$ and $z = \pm a$, and contains no internal heat sources. Verify that the temperature distribution

$$u(x, y, z, t) = A \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp \left(-\frac{2\kappa \pi^2 t}{a^2} \right)$$

obeys the appropriate diffusion equation. Across which faces is there heat flow? What is the direction and rate of heat flow at the point $(3a/4, a/4, a)$ at time $t = a^2/(\kappa\pi^2)$?

21.3 The wave equation describing the transverse vibrations of a stretched membrane under tension T and having a uniform surface density ρ is

$$T \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \rho \frac{\partial^2 u}{\partial t^2}.$$

Find a separable solution appropriate to a membrane stretched on a frame of length a and width b, showing that the natural angular frequencies of such a membrane are given by

$$\omega^2 = \frac{\pi^2 T}{\rho} \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right),$$

where n and m are any positive integers.

21.4 Schrödinger's equation for a non-relativistic particle in a constant potential region can be taken as

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = i\hbar \frac{\partial u}{\partial t}.$$

(a) Find a solution, separable in the four independent variables, that can be written in the form of a plane wave,

$$\psi(x, y, z, t) = A \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)].$$

Using the relationships associated with de Broglie ($\mathbf{p} = \hbar \mathbf{k}$) and Einstein ($E = \hbar\omega$), show that the separation constants must be such that

$$p_x^2 + p_y^2 + p_z^2 = 2mE.$$

(b) Obtain a different separable solution describing a particle confined to a box of side a (ψ must vanish at the walls of the box). Show that the energy of the particle can only take the quantised values

$$E = \frac{\hbar^2 \pi^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2),$$

where n_x, n_y and n_z are integers.

21.5 Denoting the three terms of ∇^2 in spherical polars by ∇_r^2, ∇_θ^2, ∇_ϕ^2 in an obvious way, evaluate $\nabla_r^2 u$, etc. for the two functions given below and verify that, in each case, although the individual terms are not necessarily zero their sum $\nabla^2 u$ is zero. Identify the corresponding values of ℓ and m.

(a) $u(r, \theta, \phi) = \left(Ar^2 + \dfrac{B}{r^3} \right) \dfrac{3\cos^2\theta - 1}{2}.$

(b) $u(r, \theta, \phi) = \left(Ar + \dfrac{B}{r^2} \right) \sin\theta \exp i\phi.$

21.6 Prove that the expression given in equation (21.47) for the associated Legendre function $P_\ell^m(\mu)$ satisfies the appropriate equation, (21.45), as follows.

(a) Evaluate $dP_\ell^m(\mu)/d\mu$ and $d^2P_\ell^m(\mu)/d\mu^2$, using the forms given in (21.47), and substitute them into (21.45).
(b) Differentiate Legendre's equation m times using Leibnitz' theorem.
(c) Show that the equations obtained in (a) and (b) are multiples of each other, and hence that the validity of (b) implies that of (a).

21.7 Continue the analysis of exercise 10.20, concerned with the flow of a very viscous fluid past a sphere, to find the full expression for the stream function $\psi(r,\theta)$. At the surface of the sphere $r=a$, the velocity field $\mathbf{u}=\mathbf{0}$, whilst far from the sphere $\psi \simeq (Ur^2\sin^2\theta)/2$.

Show that $f(r)$ can be expressed as a superposition of powers of r, and determine which powers give acceptable solutions. Hence show that

$$\psi(r,\theta) = \frac{U}{4}\left(2r^2 - 3ar + \frac{a^3}{r}\right)\sin^2\theta.$$

21.8 The motion of a very viscous fluid in the two-dimensional (wedge) region $-\alpha < \phi < \alpha$ can be described, in (ρ,ϕ) coordinates, by the (biharmonic) equation

$$\nabla^2\nabla^2\psi \equiv \nabla^4\psi = 0,$$

together with the boundary conditions $\partial\psi/\partial\phi = 0$ at $\phi = \pm\alpha$, which represent the fact that there is no radial fluid velocity close to either of the bounding walls because of the viscosity, and $\partial\psi/\partial\rho = \pm\rho$ at $\phi = \pm\alpha$, which impose the condition that azimuthal flow increases linearly with r along any radial line. Assuming a solution in separated-variable form, show that the full expression for ψ is

$$\psi(\rho,\phi) = \frac{\rho^2}{2}\,\frac{\sin 2\phi - 2\phi\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$$

21.9 A circular disc of radius a is heated in such a way that its perimeter $\rho = a$ has a steady temperature distribution $A + B\cos^2\phi$, where ρ and ϕ are plane polar coordinates and A and B are constants. Find the temperature $T(\rho,\phi)$ everywhere in the region $\rho < a$.

21.10 Consider possible solutions of Laplace's equation inside a circular domain as follows.

(a) Find the solution in plane polar coordinates ρ, ϕ, that takes the value $+1$ for $0 < \phi < \pi$ and the value -1 for $-\pi < \phi < 0$, when $\rho = a$.
(b) For a point (x,y) on or inside the circle $x^2+y^2 = a^2$, identify the angles α and β defined by

$$\alpha = \tan^{-1}\frac{y}{a+x} \qquad \text{and} \qquad \beta = \tan^{-1}\frac{y}{a-x}.$$

Show that $u(x,y) = (2/\pi)(\alpha+\beta)$ is a solution of Laplace's equation that satisfies the boundary conditions given in (a).
(c) Deduce a Fourier series expansion for the function

$$\tan^{-1}\frac{\sin\phi}{1+\cos\phi} + \tan^{-1}\frac{\sin\phi}{1-\cos\phi}.$$

21.11 The free transverse vibrations of a thick rod satisfy the equation

$$a^4\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0.$$

Obtain a solution in separated-variable form and, for a rod clamped at one end,

$x = 0$, and free at the other, $x = L$, show that the angular frequency of vibration ω satisfies

$$\cosh\left(\frac{\omega^{1/2}L}{a}\right) = -\sec\left(\frac{\omega^{1/2}L}{a}\right).$$

[At a clamped end both u and $\partial u/\partial x$ vanish, whilst at a free end, where there is no bending moment, $\partial^2 u/\partial x^2$ and $\partial^3 u/\partial x^3$ are both zero.]

21.12 A membrane is stretched between two concentric rings of radii a and b ($b > a$). If the smaller ring is transversely distorted from the planar configuration by an amount $c|\phi|$, $-\pi \leq \phi \leq \pi$, show that the membrane then has a shape given by

$$u(\rho,\phi) = \frac{c\pi}{2}\frac{\ln(b/\rho)}{\ln(b/a)} - \frac{4c}{\pi}\sum_{m\ \text{odd}}\frac{a^m}{m^2(b^{2m}-a^{2m})}\left(\frac{b^{2m}}{\rho^m}-\rho^m\right)\cos m\phi.$$

21.13 A string of length L, fixed at its two ends, is plucked at its mid-point by an amount A and then released. Prove that the subsequent displacement is given by

$$u(x,t) = \sum_{n=0}^{\infty}\frac{8A\,(-1)^n}{\pi^2(2n+1)^2}\sin\left[\frac{(2n+1)\pi x}{L}\right]\cos\left[\frac{(2n+1)\pi ct}{L}\right],$$

where, in the usual notation, $c^2 = T/\rho$.

Find the total kinetic energy of the string when it passes through its unplucked position, by calculating it in each mode (each n) and summing, using the result

$$\sum_{0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Confirm that the total energy is equal to the work done in plucking the string initially.

21.14 Prove that the potential for $\rho < a$ associated with a vertical split cylinder of radius a, the two halves of which ($\cos\phi > 0$ and $\cos\phi < 0$) are maintained at equal and opposite potentials $\pm V$, is given by

$$u(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{\rho}{a}\right)^{2n+1}\cos(2n+1)\phi.$$

21.15 A conducting spherical shell of radius a is cut round its equator and the two halves connected to voltages of $+V$ and $-V$. Show that an expression for the potential at the point (r,θ,ϕ) anywhere inside the two hemispheres is

$$u(r,\theta,\phi) = V\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!(4n+3)}{2^{2n+1}n!(n+1)!}\left(\frac{r}{a}\right)^{2n+1}P_{2n+1}(\cos\theta).$$

[This is the spherical polar analogue of the previous question.]

21.16 A slice of biological material of thickness L is placed into a solution of a radioactive isotope of constant concentration C_0 at time $t = 0$. For a later time t find the concentration of radioactive ions at a depth x inside one of its surfaces if the diffusion constant is κ.

21.17 Two identical copper bars are each of length a. Initially, one is at $0\,^\circ$C and the other is at $100\,^\circ$C; they are then joined together end to end and thermally isolated. Obtain in the form of a Fourier series an expression $u(x,t)$ for the temperature at any point a distance x from the join at a later time t. Bear in mind the heat flow conditions at the free ends of the bars.

Taking $a = 0.5$ m estimate the time it takes for one of the free ends to attain a temperature of $55\,^\circ$C. The thermal conductivity of copper is $3.8 \times 10^2\,\text{J m}^{-1}\,\text{K}^{-1}\,\text{s}^{-1}$, and its specific heat capacity is $3.4 \times 10^6\,\text{J m}^{-3}\,\text{K}^{-1}$.

21.18 A sphere of radius a and thermal conductivity k_1 is surrounded by an infinite medium of conductivity k_2 in which far away the temperature tends to T_∞. A distribution of heat sources $q(\theta)$ embedded in the sphere's surface establish steady temperature fields $T_1(r,\theta)$ inside the sphere and $T_2(r,\theta)$ outside it. It can be shown, by considering the heat flow through a small volume that includes part of the sphere's surface, that

$$k_1\frac{\partial T_1}{\partial r} - k_2\frac{\partial T_2}{\partial r} = q(\theta) \quad \text{on} \quad r = a.$$

Given that

$$q(\theta) = \frac{1}{a}\sum_{n=0}^{\infty} q_n P_n(\cos\theta),$$

find complete expressions for $T_1(r,\theta)$ and $T_2(r,\theta)$. What is the temperature at the centre of the sphere?

21.19 Using result (21.74) from the worked example in the text, find the general expression for the temperature $u(x,t)$ in the bar, given that the temperature distribution at time $t = 0$ is $u(x,0) = \exp(-x^2/a^2)$.

21.20 Working in *spherical* polar coordinates $\mathbf{r} = (r,\theta,\phi)$, but for a system that has azimuthal symmetry around the polar axis, consider the following gravitational problem.

(a) Show that the gravitational potential due to a uniform disc of radius a and mass M, centred at the origin, is given for $r < a$ by

$$\frac{2GM}{a}\left[1 - \frac{r}{a}P_1(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{1}{8}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \cdots\right],$$

and for $r > a$ by

$$\frac{GM}{r}\left[1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) - \cdots\right],$$

where the polar axis is normal to the plane of the disc.

(b) Reconcile the presence of a term $P_1(\cos\theta)$, which is odd under $\theta \to \pi - \theta$, with the symmetry with respect to the plane of the disc of the physical system.

(c) Deduce that the gravitational field near an infinite sheet of matter of constant density ρ per unit area is $2\pi G\rho$.

21.21 In the region $-\infty < x,\, y < \infty$ and $-t \le z \le t$, a charge-density wave $\rho(\mathbf{r}) = A\cos qx$, in the x-direction, is represented by

$$\rho(\mathbf{r}) = \frac{e^{iqx}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{\rho}(\alpha)e^{i\alpha z}\,d\alpha.$$

The resulting potential is represented by

$$V(\mathbf{r}) = \frac{e^{iqx}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{V}(\alpha)e^{i\alpha z}\,d\alpha.$$

Determine the relationship between $\tilde{V}(\alpha)$ and $\tilde{\rho}(\alpha)$, and hence show that the potential at the point $(0,0,0)$ is

$$\frac{A}{\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\sin kt}{k(k^2+q^2)}\,dk.$$

21.22 Point charges q and $-qa/b$ (with $a < b$) are placed, respectively, at a point P, a distance b from the origin O, and a point Q between O and P, a distance a^2/b from O. Show, by considering similar triangles QOS and SOP, where S is any point on the surface of the sphere centred at O and of radius a, that the net potential anywhere on the sphere due to the two charges is zero.

Use this result (backed up by the uniqueness theorem) to find the force with which a point charge q placed a distance b from the centre of a spherical conductor of radius a ($< b$) is attracted to the sphere (i) if the sphere is earthed, and (ii) if the sphere is uncharged and insulated.

21.23 Find the Green's function $G(\mathbf{r}, \mathbf{r}_0)$ in the half-space $z > 0$ for the solution of $\nabla^2\Phi = 0$ with Φ specified in cylindrical polar coordinates (ρ, ϕ, z) on the plane $z = 0$ by

$$\Phi(\rho, \phi, z) = \begin{cases} 1 & \text{for } \rho \le 1, \\ 1/\rho & \text{for } \rho > 1. \end{cases}$$

Determine the variation of $\Phi(0, 0, z)$ along the z-axis.

21.24 Electrostatic charge is distributed in a sphere of radius R centred on the origin. Determine the form of the resultant potential $\phi(\mathbf{r})$ at distances much greater than R, as follows.

(a) Express in the form of an integral over all space the solution of

$$\nabla^2\phi = -\frac{\rho(\mathbf{r})}{\epsilon_0}.$$

(b) Show that, for $r \gg r'$,

$$|\mathbf{r} - \mathbf{r}'| = r - \frac{\mathbf{r} \cdot \mathbf{r}'}{r} + \mathrm{O}\left(\frac{1}{r}\right).$$

(c) Use results (a) and (b) to show that $\phi(\mathbf{r})$ has the form

$$\phi(\mathbf{r}) = \frac{M}{r} + \frac{\mathbf{d} \cdot \mathbf{r}}{r^3} + \mathrm{O}\left(\frac{1}{r^3}\right).$$

Find expressions for M and $\mathbf{d}$, and identify them physically.

21.25 Find, in the form of an infinite series, the Green's function of the ∇^2 operator for the Dirichlet problem in the region $-\infty < x < \infty$, $-\infty < y < \infty$, $-c \le z \le c$.

21.26 Find the Green's function for the three-dimensional Neumann problem

$$\nabla^2\phi = 0 \quad \text{for } z > 0 \qquad \text{and} \qquad \frac{\partial\phi}{\partial z} = f(x, y) \quad \text{on } z = 0.$$

Determine $\phi(x, y, z)$ if

$$f(x, y) = \begin{cases} \delta(y) & \text{for } |x| < a, \\ 0 & \text{for } |x| \ge a. \end{cases}$$

21.27 Determine the Green's function for the Klein–Gordon equation in a half-space as follows.

(a) By applying the divergence theorem to the volume integral

$$\int_V \left[\phi(\nabla^2 - m^2)\psi - \psi(\nabla^2 - m^2)\phi\right] dV,$$

obtain a Green's function expression, as the sum of a volume integral and a surface integral, for the function $\phi(\mathbf{r}')$ that satisfies

$$\nabla^2\phi - m^2\phi = \rho$$

in V and takes the specified form $\phi = f$ on S, the boundary of V. The Green's function, $G(\mathbf{r}, \mathbf{r}')$, to be used satisfies

$$\nabla^2 G - m^2 G = \delta(\mathbf{r} - \mathbf{r}')$$

and vanishes when $\mathbf{r}$ is on S.

(b) When V is all space, $G(\mathbf{r}, \mathbf{r}')$ can be written as $G(t) = g(t)/t$, where $t = |\mathbf{r} - \mathbf{r}'|$ and $g(t)$ is bounded as $t \to \infty$. Find the form of $G(t)$.

(c) Find $\phi(\mathbf{r})$ in the half-space $x > 0$ if $\rho(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_1)$ and $\phi = 0$ both on $x = 0$ and as $r \to \infty$.

21.28 Consider the PDE $\mathcal{L}u(\mathbf{r}) = \rho(\mathbf{r})$, for which the differential operator $\mathcal{L}$ is given by

$$\mathcal{L} = \nabla \cdot [p(\mathbf{r})\nabla] + q(\mathbf{r}),$$

where $p(\mathbf{r})$ and $q(\mathbf{r})$ are functions of position. By proving the generalised form of Green's theorem,

$$\int_V (\phi\mathcal{L}\psi - \psi\mathcal{L}\phi)\, dV = \oint_S p(\phi\nabla\psi - \psi\nabla\phi) \cdot \hat{\mathbf{n}}\, dS,$$

show that the solution of the PDE is given by

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \oint_S p(\mathbf{r}) \left[u(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0) \frac{\partial u(\mathbf{r})}{\partial n} \right] dS(\mathbf{r}),$$

where $G(\mathbf{r}, \mathbf{r}_0)$ is the Green's function satisfying $\mathcal{L}G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$.

21.7 Hints and answers

21.1 (a) $C \exp[\lambda(x^2 + 2y)]$; (b) $C(x^2 y)^\lambda$.

21.3 $u(x, y, t) = \sin(n\pi x/a) \sin(m\pi y/b)(A \sin \omega t + B \cos \omega t)$.

21.5 (a) $6u/r^2$, $-6u/r^2$, 0, $\ell = 2$ (or -3), $m = 0$;
(b) $2u/r^2$, $(\cot^2\theta - 1)u/r^2$; $-u/(r^2 \sin^2\theta)$, $\ell = 1$ (or -2), $m = \pm 1$.

21.7 Solutions of the form r^ℓ give ℓ as $-1, 1, 2, 4$. Because of the asymptotic form of ψ, an r^4 term cannot be present. The coefficients of the three remaining terms are determined by the two boundary conditions $\mathbf{u} = \mathbf{0}$ on the sphere and the form of ψ for large r.

21.9 Express $\cos^2\phi$ in terms of $\cos 2\phi$; $T(\rho, \phi) = A + B/2 + (B\rho^2/2a^2)\cos 2\phi$.

21.11 $(A \cos mx + B \sin mx + C \cosh mx + D \sinh mx)\cos(\omega t + \epsilon)$, with $m^4 a^4 = \omega^2$.

21.13 $E_n = 16\rho A^2 c^2/[(2n+1)^2 \pi^2 L]$; $E = 2\rho c^2 A^2/L = \int_0^A [2Tv/(\frac{1}{2}L)]\, dv$.

21.15 Note that the boundary value function is a square wave that is *symmetric* in ϕ.

21.17 Since there is no heat flow at $x = \pm a$, use a series of period $4a$, $u(x, 0) = 100$ for $0 < x \leq 2a$, $u(x, 0) = 0$ for $-2a \leq x < 0$.

$$u(x, t) = 50 + \frac{200}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[\frac{(2n+1)\pi x}{2a}\right] \exp\left[-\frac{k(2n+1)^2\pi^2 t}{4a^2 s}\right].$$

Taking only the $n = 0$ term gives $t \approx 2300$ s.

21.19 $u(x, t) = [a/(a^2 + 4\kappa t)^{1/2}] \exp[-x^2/(a^2 + 4\kappa t)]$.

21.21 Fourier-transform Poisson's equation to show that $\tilde{\rho}(\alpha) = \epsilon_0(\alpha^2 + q^2)\tilde{V}(\alpha)$.

21.23 Follow the worked example that includes result (21.95). For part of the explicit integration, substitute $\rho = z \tan\alpha$.

$$\Phi(0, 0, z) = \frac{z(1 + z^2)^{1/2} - z^2 + (1 + z^2)^{1/2} - 1}{z(1 + z^2)^{1/2}}.$$