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5.1 String Sorts

- *strings in Java*
- *key-indexed counting*
- *LSD radix sort*
- *MSD radix sort*
- *3-way radix quicksort*
- *suffix arrays*

String processing

String. Sequence of characters.

Important fundamental abstraction.

- Genomic sequences.
- Information processing.
- Communication systems (e.g., email).
- Programming systems (e.g., Java programs).
- ...

" The digital information that underlies biochemistry, cell biology, and development can be represented by a simple string of G's, A's, T's and C's. This string is the root data structure of an organism's biology. " — *M. V. Olson*

The char data type

C char data type. Typically an 8-bit integer.

- Supports 7-bit ASCII.
- Can represent at most 256 characters.

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	HT	LF	VT	FF	CR	SO	SI
1	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
2	SP	!	"	#	$	%	&	'	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	DEL

Hexadecimal to ASCII conversion table

A	á	∂	𝔊
U+0041	U+00E1	U+2202	U+1D50A

some Unicode characters

Java char data type. A 16-bit unsigned integer.

- Supports original 16-bit Unicode.
- Supports 21-bit Unicode 3.0 (awkwardly).

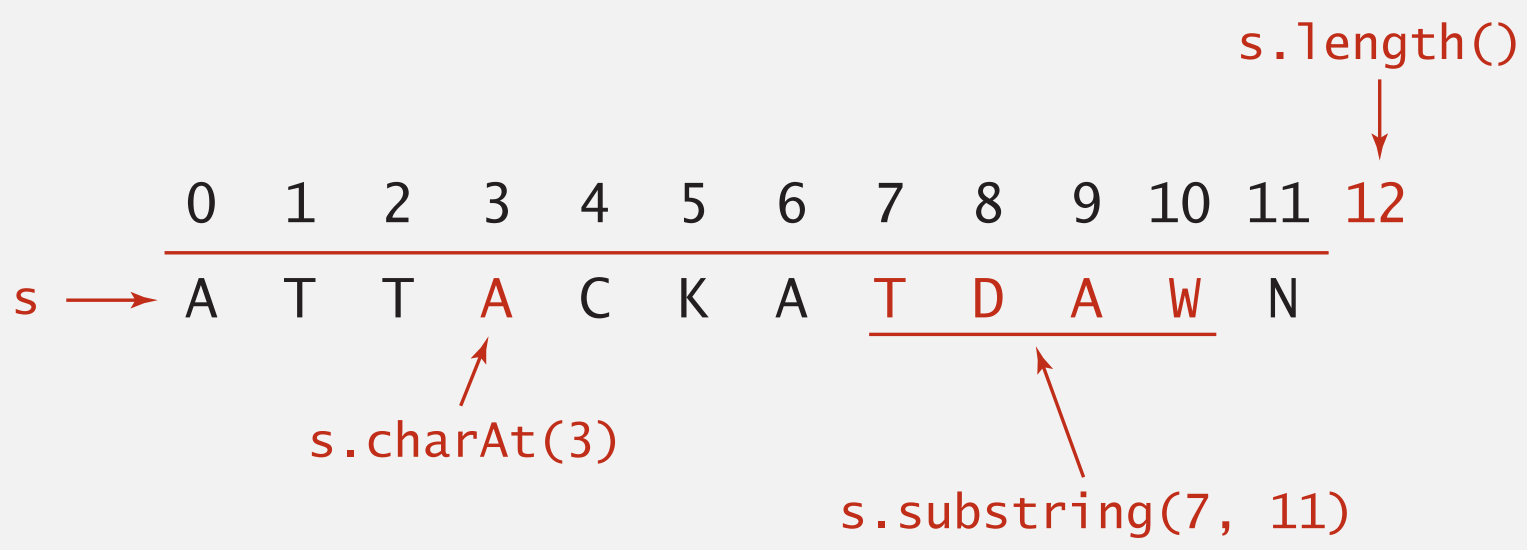
The String data type

String data type in Java. Immutable sequence of characters.

Length. Number of characters.

Indexing. Get the i^{th} character.

Concatenation. Concatenate one string to the end of another.



The String data type: representation

Representation (Java 7). Immutable `char[]` array + cache of hash.

operation	Java	running time
length	`s.length()`	1
indexing	`s.charAt(i)`	1
concatenation	`s + t`	$M + N$
⋮		⋮

String performance trap

Q. How to build a long string, one character at a time?

```
public static String reverse(String s)
{
   String rev = "";
   for (int i = s.length() - 1; i >= 0; i--)
      rev += s.charAt(i);
   return rev;
}
```

quadratic time

A. Use `StringBuilder` data type (mutable `char[]` array).

```
public static String reverse(String s)
{
   StringBuilder rev = new StringBuilder();
   for (int i = s.length() - 1; i >= 0; i--)
      rev.append(s.charAt(i));
   return rev.toString();
}
```

linear time

Comparing two strings

Q. How many character compares to compare two strings of length W ?

p	r	e	f	e	t	c	h
0	1	2	3	4	5	6	7
p	r	e	f	i	x	e	s

Running time. Proportional to length of longest common prefix.

- Proportional to W in the worst case.
- But, often sublinear in W.

Alphabets

Digital key. Sequence of digits over fixed alphabet.

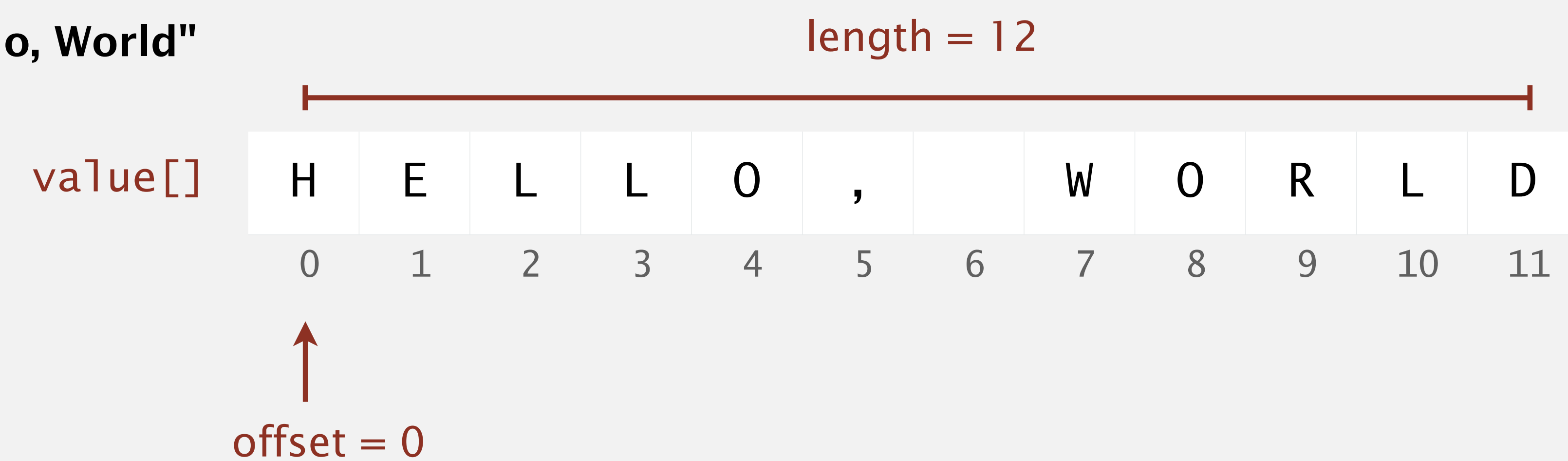
Radix. Number of digits R in alphabet.

name	R()	lgR()	characters
BINARY	2	1	01
OCTAL	8	3	01234567
DECIMAL	10	4	0123456789
HEXADECIMAL	16	4	0123456789ABCDEF
DNA	4	2	ACTG
LOWERCASE	26	5	abcdefghijklmnopqrstuvwxyz
UPPERCASE	26	5	ABCDEFGHIJKLMNOPQRSTUVWXYZ
PROTEIN	20	5	ACDEFGHIKLMNPQRSTVWY
BASE64	64	6	ABCDEFGHIJKLMNOPQRSTUVWXYZabcdef ghijklmnopqrstuvwxyz0123456789+/
ASCII	128	7	*ASCII characters*
EXTENDED_ASCII	256	8	*extended ASCII characters*
UNICODE16	65536	16	*Unicode characters*

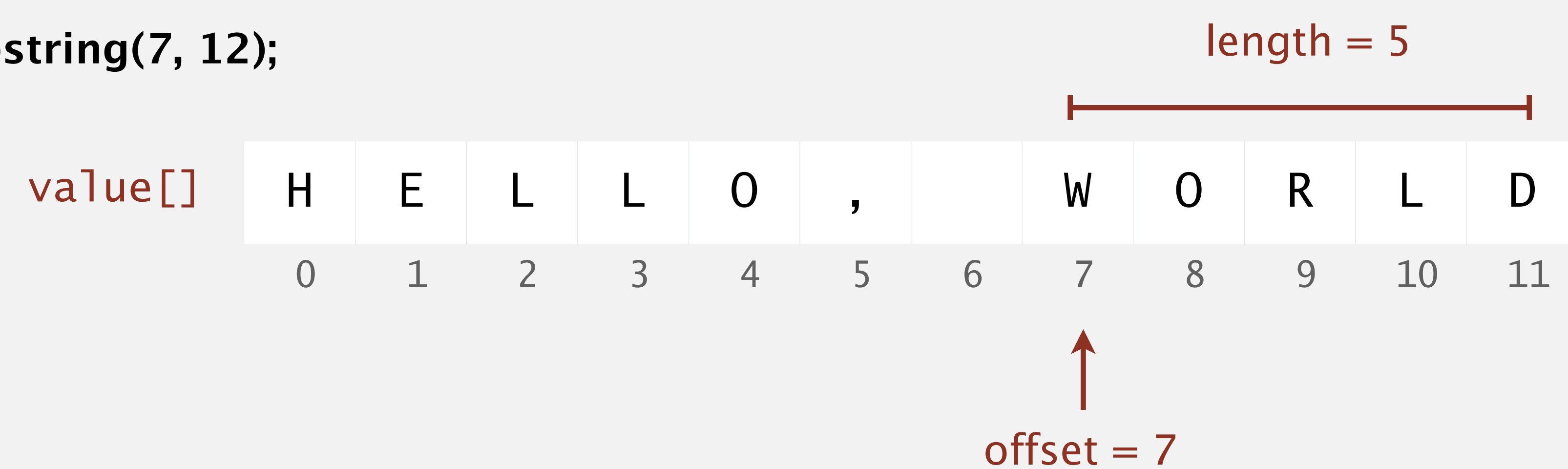
The String data type: Java 7u5 implementation

```
public final class String implements Comparable<String>
{
   private char[] value;   // characters
   private int offset;     // index of first char in array
   private int length;     // length of string
   private int hash;       // cache of hashCode()
   …
```

String s = "Hello, World"



String t = s.substring(7, 12);



The String data type: Java 7u6 implementation

```
public final class String implements Comparable<String>
{
   private char[] value;   // characters
   private int hash;       // cache of hashCode()
   …
```

String s = "Hello, World"

value[]	H	E	L	L	O	,		W	O	R	L	D
	0	1	2	3	4	5	6	7	8	9	10	11

String t = s.substring(7, 12);

value[]	W	O	R	L	D
	0	1	2	3	4

A Reddit exchange

```
I'm the author of the substring() change. As has
been suggested in the analysis here there were two
motivations for the change
   • Reduce the size of String instances. Strings
     are typically 20-40% of common apps footprint.
   • Avoid memory leakage caused by retained
     substrings holding the entire character array.
```

bondolo

```
Changing this function, in a bugfix release no
less, was totally irresponsible. It broke backwards
compatibility for numerous applications with errors
that didn't even produce a message, just freezing
and timeouts...  All pain, no gain. Your work was
not just vain, it was thoroughly destructive, even
beyond its immediate effect.
```

cypherpunks

http://www.reddit.com/r/programming/comments/1qw73v/til_oracle_changed_the_internal_string

The String data type: performance

String data type (in Java). Sequence of characters (immutable).

Java 7u5. Immutable `char[]` array, offset, length, hash cache.

Java 7u6. Immutable `char[]` array, hash cache.

operation	Java 7u5	Java 7u6
length	1	1
indexing	1	1
substring extraction	1	N
concatenation	$M + N$	$M + N$
immutable?	✔	✔
memory	$64 + 2N$	$56 + 2N$

MSD string sort vs. quicksort for strings

Disadvantages of MSD string sort.

- Extra space for `aux[]`.
- Extra space for `count[]`.
- Inner loop has a lot of instructions.
- Accesses memory "randomly" (cache inefficient).

Disadvantage of quicksort.

- Linearithmic number of string compares (not linear).
- Has to rescan many characters in keys with long prefix matches.

doesn't rescan characters

tight inner loop, cache friendly

Goal. Combine advantages of MSD and quicksort.

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5.1 String Sorts

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- *key-indexed counting*
- *LSD radix sort*
- *MSD radix sort*
- *3-way radix quicksort*
- *suffix arrays*

Key-indexed counting demo

Goal. Sort an array a[] of N integers between 0 and $R - 1$.

R = 6

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

use a for 0
b for 1
c for 2
d for 3
e for 4
f for 5

Key-indexed counting demo

Goal. Sort an array a[] of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

count frequencies

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

offset by 1
[stay tuned]

r	count[r]
a	0
b	2
c	3
d	1
e	2
f	1
-	3

Key-indexed counting demo

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

compute cumulates

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	0
b	2
c	5
d	6
e	8
f	9
-	12

6 keys < d, 8 keys < e
so d's go in a[6] and a[7]

Key-indexed counting demo

Goal. Sort an array `a[]` of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

move items

i	a[i]
0	d
1	a
2	c
3	f
4	f
5	b
6	d
7	b
8	f
9	b
10	e
11	a

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting demo

Goal. Sort an array a[] of N integers between 0 and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

copy back

i	a[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

r	count[r]
a	2
b	5
c	6
d	8
e	9
f	12
-	12

i	aux[i]
0	a
1	a
2	b
3	b
4	b
5	c
6	d
7	d
8	e
9	f
10	f
11	f

Key-indexed counting: analysis

Proposition. Key-indexed takes time proportional to $N + R$.

Proposition. Key-indexed counting uses extra space proportional to $N + R$.

Stable? ✔

a[0]	Anderson	2	Harris	1	aux[0]
a[1]	Brown	3	Martin	1	aux[1]
a[2]	Davis	3	Moore	1	aux[2]
a[3]	Garcia	4	Anderson	2	aux[3]
a[4]	Harris	1	Martinez	2	aux[4]
a[5]	Jackson	3	Miller	2	aux[5]
a[6]	Johnson	4	Robinson	2	aux[6]
a[7]	Jones	3	White	2	aux[7]
a[8]	Martin	1	Brown	3	aux[8]
a[9]	Martinez	2	Davis	3	aux[9]
a[10]	Miller	2	Jackson	3	aux[10]
a[11]	Moore	1	Jones	3	aux[11]
a[12]	Robinson	2	Taylor	3	aux[12]
a[13]	Smith	4	Williams	3	aux[13]
a[14]	Taylor	3	Garcia	4	aux[14]
a[15]	Thomas	4	Johnson	4	aux[15]
a[16]	Thompson	4	Smith	4	aux[16]
a[17]	White	2	Thomas	4	aux[17]
a[18]	Williams	3	Thompson	4	aux[18]
a[19]	Wilson	4	Wilson	4	aux[19]

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5.1 String Sorts

- strings in Java
- key-indexed counting
- **LSD radix sort**
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

Least-significant-digit-first string sort

LSD string (radix) sort.

- Consider characters from right to left.
- Stably sort using d^{th} character as the key (using key-indexed counting).

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

sort key (d = 2)

0	d	a	b
1	c	a	b
2	e	b	b
3	a	d	d
4	f	a	d
5	b	a	d
6	d	a	d
7	f	e	d
8	b	e	d
9	f	e	e
10	b	e	e
11	a	c	e

sort is stable
(arrows do not cross)

sort key (d = 1)

0	d	a	b
1	c	a	b
2	f	a	d
3	b	a	d
4	d	a	d
5	e	b	b
6	a	c	e
7	a	d	d
8	f	e	d
9	b	e	d
10	f	e	e
11	b	e	e

sort key (d = 0)

0	a	c	e
1	a	d	d
2	b	a	d
3	b	e	d
4	b	e	e
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	d
11	f	e	e

LSD string sort: correctness proof

Proposition. LSD sorts fixed-length strings in ascending order.

Pf. [by induction on i]

After pass i, strings are sorted by last i characters.

- If two strings differ on sort key, key-indexed sort puts them in proper relative order.
- If two strings agree on sort key, stability keeps them in proper relative order.

0	d	a	b
1	c	a	b
2	f	a	d
3	b	a	d
4	d	a	d
5	e	b	b
6	a	c	e
7	a	d	d
8	f	e	d
9	b	e	d
10	f	e	e
11	b	e	e

sorted from previous passes (by induction)

sort key

0	a	c	e
1	a	d	d
2	b	a	d
3	b	e	d
4	b	e	e
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	d
11	f	e	e

Proposition. LSD sort is stable.

Pf. Key-indexed counting is stable.

LSD string sort: Java implementation

```
public class LSD
{
   public static void sort(String[] a, int W)
   {
      int R = 256;
      int N = a.length;
      String[] aux = new String[N];

      for (int d = W-1; d >= 0; d--)
      {
         int[] count = new int[R+1];
         for (int i = 0; i < N; i++)
            count[a[i].charAt(d) + 1]++;
         for (int r = 0; r < R; r++)
            count[r+1] += count[r];
         for (int i = 0; i < N; i++)
            aux[count[a[i].charAt(d)]++] = a[i];
         for (int i = 0; i < N; i++)
            a[i] = aux[i];
      }
   }
}
```

fixed-length W strings

radix R

do key-indexed counting for each digit from right to left

key-indexed counting

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	½ N^2	¼ N^2	1	✔	`compareTo()`
mergesort	$N \lg N$	$N \lg N$	N	✔	`compareTo()`
quicksort	$1.39\, N \lg N$ *	$1.39\, N \lg N$	$c \lg N$		`compareTo()`
heapsort	$2\, N \lg N$	$2\, N \lg N$	1		`compareTo()`
LSD sort †	$2\, W (N + R)$	$2\, W (N + R)$	$N + R$	✔	`charAt()`

* probabilistic
† fixed-length W keys

Q. What if strings are not all of same length?

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5.1 String Sorts

Reverse LSD

- Consider characters from left to right.
- Stably sort using d^{th} character as the key (using key-indexed counting).

0	d	a	b
1	a	d	d
2	c	a	b
3	f	a	d
4	f	e	e
5	b	a	d
6	d	a	d
7	b	e	e
8	f	e	d
9	b	e	d
10	e	b	b
11	a	c	e

sort key (d = 0)

0	a	d	d
1	a	c	e
2	b	a	d
3	b	e	e
4	b	e	d
5	c	a	b
6	d	a	b
7	d	a	d
8	e	b	b
9	f	a	d
10	f	e	e
11	f	e	d

sort key (d = 1)

0	b	a	d
1	c	a	b
2	d	a	b
3	d	a	d
4	f	a	d
5	e	b	b
6	a	c	e
7	a	d	d
8	b	e	e
9	b	e	d
10	f	e	e
11	f	e	d

sort key (d = 2)

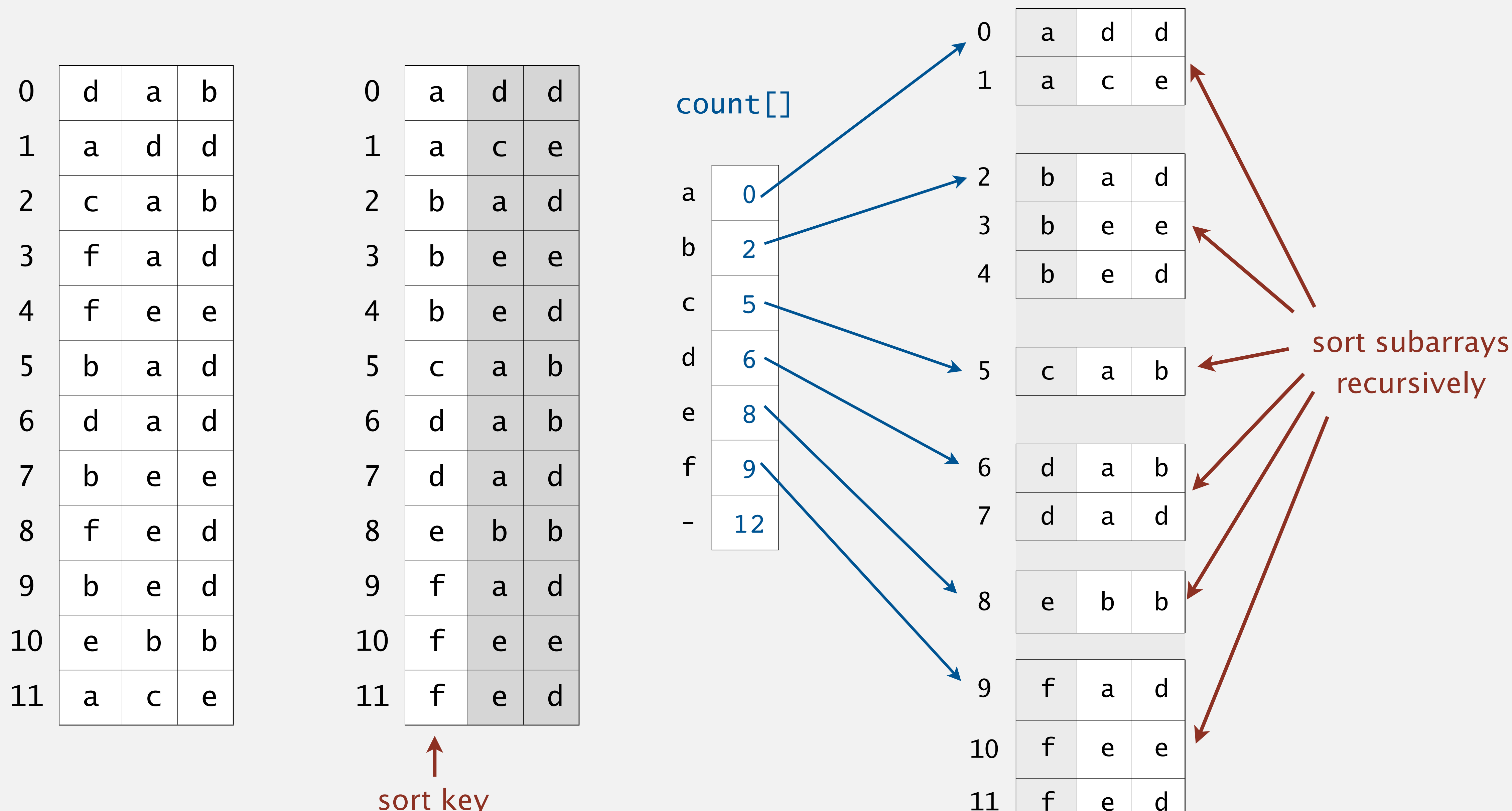
0	c	a	b
1	d	a	b
2	e	b	b
3	b	a	d
4	d	a	d
5	f	a	d
6	a	d	d
7	b	e	d
8	f	e	d
9	a	c	e
10	b	e	e
11	f	e	e

not sorted!

Most-significant-digit-first string sort

MSD string (radix) sort.

- Partition array into R pieces according to first character (use key-indexed counting).
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).



MSD string sort: example

Trace of recursive calls for MSD string sort (no cutoff for small subarrays, subarrays of size 0 and 1 omitted)

Variable-length strings

Treat strings as if they had an extra char at end (smaller than any char).

why smaller?

0	s	e	a	-1						
1	s	e	a	s	h	e	l	l	s	-1
2	s	e	l	l	s	-1				
3	s	h	e	-1						
4	s	h	e	-1						
5	s	h	e	l	l	s	-1			
6	s	h	o	r	e	-1				
7	s	u	r	e	l	y	-1			

she before shells

```
private static int charAt(String s, int d)
{
   if (d < s.length()) return s.charAt(d);
   else return -1;
}
```

C strings. Have extra char '\0' at end $\Rightarrow$ no extra work needed.

MSD string sort: Java implementation

```
public static void sort(String[] a)
{
   aux = new String[a.length];
   sort(a, aux, 0, a.length - 1, 0);
}

private static void sort(String[] a, String[] aux, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int[] count = new int[R+2];
   for (int i = lo; i <= hi; i++)
      count[charAt(a[i], d) + 2]++;
   for (int r = 0; r < R+1; r++)
      count[r+1] += count[r];
   for (int i = lo; i <= hi; i++)
      aux[count[charAt(a[i], d) + 1]++] = a[i];
   for (int i = lo; i <= hi; i++)
      a[i] = aux[i - lo];

   for (int r = 0; r < R; r++)
      sort(a, aux, lo + count[r], lo + count[r+1] - 1, d+1);
}
```

recycles aux[] array
but not count[] array

key-indexed counting

sort R subarrays recursively

Cutoff to insertion sort

Solution. Cutoff to insertion sort for small subarrays.

- Insertion sort, but start at d^{th} character.

```
private static void sort(String[] a, int lo, int hi, int d)
{
   for (int i = lo; i <= hi; i++)
      for (int j = i; j > lo && less(a[j], a[j-1], d); j--)
         exch(a, j, j-1);
}
```

- Implement `less()` so that it compares starting at d^{th} character.

```
private static boolean less(String v, String w, int d)
{
   for (int i = d; i < Math.min(v.length(), w.length()); i++)
   {
      if (v.charAt(i) < w.charAt(i)) return true;
      if (v.charAt(i) > w.charAt(i)) return false;
   }
   return v.length() < w.length();
}
```

MSD string sort: performance

Number of characters examined.

- MSD examines just enough characters to sort the keys.
- Number of characters examined depends on keys.
- Can be sublinear in input size!

compareTo() based sorts can also be sublinear!

Random (sublinear)	Non-random with duplicates (nearly linear)	Worst case (linear)
1EIO402	are	1DNB377
1HYL490	by	1DNB377
1ROZ572	sea	1DNB377
2HXE734	seashells	1DNB377
2IYE230	seashells	1DNB377
2XOR846	sells	1DNB377
3CDB573	sells	1DNB377
3CVP720	she	1DNB377
3IGJ319	she	1DNB377
3KNA382	shells	1DNB377
3TAV879	shore	1DNB377
4CQP781	surely	1DNB377
4QGI284	the	1DNB377
4YHV229	the	1DNB377

Characters examined by MSD string sort

Summary of the performance of sorting algorithms

Frequency of operations.

algorithm	guarantee	random	extra space	stable?	operations on keys
insertion sort	½ N^2	¼ N^2	1	✔	`compareTo()`
mergesort	$N \lg N$	$N \lg N$	N	✔	`compareTo()`
quicksort	$1.39\, N \lg N$ *	$1.39\, N \lg N$	$c \lg N$		`compareTo()`
heapsort	$2\, N \lg N$	$2\, N \lg N$	1		`compareTo()`
LSD sort †	$2\, W (N + R)$	$2\, W (N + R)$	$N + R$	✔	`charAt()`
MSD sort ‡	$2\, W (N + R)$	$N \log_R N$	$N + D\, R$	✔	`charAt()`

D = function-call stack depth
(length of longest prefix match)

* probabilistic
† fixed-length W keys
‡ average-length W keys

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5.3 Substring Search

- *introduction*
- *brute force*
- *Knuth-Morris-Pratt*
- *Boyer-Moore*
- *Rabin-Karp*

Substring search

Goal. Find pattern of length M in a text of length N.

typically N >> M

pattern → N E E D L E

text → I N A H A Y S T A C K N E E D L E I N A

match

Substring search applications

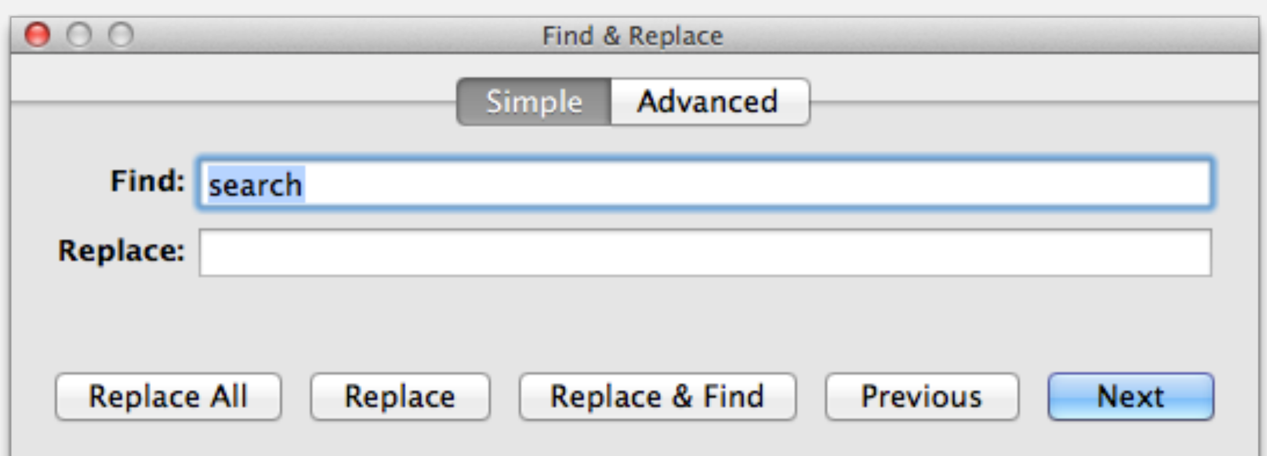
Goal. Find pattern of length M in a text of length N.

typically N >> M

pattern ⟶ N E E D L E

text ⟶ I N A H A Y S T A C K N E E D L E I N A

match



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5.3 Substring Search

Brute-force substring search

Check for pattern starting at each text position.

i	j	i+j	0	1	2	3	4	5	6	7	8	9	10
		txt →	A	B	A	C	A	D	A	B	R	A	C
0	2	2	A	B	R	A ← pat							
1	0	1		A	B	R	A						
2	1	3			A	B	R	A					
3	0	3				A	B	R	A				
4	1	5					A	B	R	A			
5	0	5						A	B	R	A		
6	4	10							A	B	R	A	

entries in red are mismatches

entries in gray are for reference only

entries in black match the text

return `i` *when* `j` *is* `M`

match

Brute-force substring search: Java implementation

Check for pattern starting at each text position.

i	j	i+j	0	1	2	3	4	5	6	7	8	9	10
			A	B	A	C	A	D	A	B	R	A	C
4	3	7					A	D	A	C	R		
5	0	5						A	D	A	C	R	

```
public static int search(String pat, String txt)
{
   int M = pat.length();
   int N = txt.length();
   for (int i = 0; i <= N - M; i++)
   {
      int j;
      for (j = 0; j < M; j++)
         if (txt.charAt(i+j) != pat.charAt(j))
            break;
      if (j == M) return i;    ← index in text where pattern starts
   }
   return N;    ← not found
}
```

Brute-force substring search: worst case

Brute-force algorithm can be slow if text and pattern are repetitive.

i	j	i+j	0	1	2	3	4	5	6	7	8	9	
		txt →	A	A	A	A	A	A	A	A	A	B	
0	4	4	A	A	A	A	B						← pat
1	4	5		A	A	A	A	B					
2	4	6			A	A	A	A	B				
3	4	7				A	A	A	A	B			
4	4	8					A	A	A	A	B		
5	5	10						A	A	A	A	B	

match

Worst case. $\sim M N$ char compares.

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5.3 Substring Search

- introduction
- brute force
- Knuth-Morris-Pratt
- Boyer-Moore
- Rabin-Karp

Michael Rabin
Dick Karp

Rabin-Karp fingerprint search

Basic idea = modular hashing.

- Compute a hash of `pat[0..M-1]`.
- For each `i`, compute a hash of `txt[i..M+i-1]`.
- If pattern hash = text substring hash, check for a match.

pat.charAt(i)

i	0	1	2	3	4	
	2	6	5	3	5	% 997 = 613

txt.charAt(i)

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3
0	3	1	4	1	5	% 997 = 508										
1		1	4	1	5	9	% 997 = 201									
2			4	1	5	9	2	% 997 = 715								
3				1	5	9	2	6	% 997 = 971							
4					5	9	2	6	5	% 997 = 442						
5						9	2	6	5	3	% 997 = 929					
6 ← *return* i = 6							2	6	5	3	5	% 997 = 613 *match*				

modular hashing with R = 10 and hash(s) = s (mod 997)

Modular arithmetic

Math trick. To keep numbers small, take intermediate results modulo Q.

Ex.

$$
\begin{aligned}
(10000 + 535) * 1000 &\quad (\text{mod } 997) \\
= (30 + 535) * 3 &\quad (\text{mod } 997) \\
= 1695 &\quad (\text{mod } 997) \\
= 698 &\quad (\text{mod } 997)
\end{aligned}
$$

$$(a + b) \bmod Q = ((a \bmod Q) + (b \bmod Q)) \bmod Q$$

$$(a * b) \bmod Q = ((a \bmod Q) * (b \bmod Q)) \bmod Q$$

two useful modular arithmetic identities

Efficiently computing the hash function

Modular hash function. Using the notation t_i for `txt.charAt(i)`, we wish to compute

$$x_i = t_i R^{M-1} + t_{i+1} R^{M-2} + \ldots + t_{i+M-1} R^0 \pmod{Q}$$

Intuition. M-digit, base-R integer, modulo Q.

Horner's method. Linear-time method to evaluate degree-M polynomial.

pat.charAt()

i	0	1	2	3	4	
	2	6	5	3	5	
0	2					% 997 = 2
1	2	6				% 997 = (2*10 + 6) % 997 = 26
2	2	6	5			% 997 = (26*10 + 5) % 997 = 265
3	2	6	5	3		% 997 = (265*10 + 3) % 997 = 659
4	2	6	5	3	5	% 997 = (659*10 + 5) % 997 = 613

(R = 10, Q = 997)

```
// Compute hash for M-digit key
private long hash(String key, int M)
{
   long h = 0;
   for (int j = 0; j < M; j++)
      h = (h * R + key.charAt(j)) % Q;
   return h;
}
```

26535 = 2*10000 + 6*1000 + 5*100 + 3*10 + 5

= ((((2) *10 + 6) * 10 + 5) * 10 + 3) * 10 + 5

Efficiently computing the hash function

Challenge. How to efficiently compute x_{i+1} given that we know x_i.

$$x_i \;=\; t_i\, R^{\,M-1} + t_{i+1}\, R^{\,M-2} + \ldots + t_{i+M-1}\, R^0$$

$$x_{i+1} = t_{i+1}\, R^{\,M-1} + t_{i+2}\, R^{\,M-2} + \ldots + t_{i+M}\, R^0$$

Key property. Can update "rolling" hash function in constant time!

$$x_{i+1} \;=\; (\underbrace{x_i}_{\text{current value}} \;-\; \underbrace{t_i\, R^{\,M-1}}_{\text{subtract leading digit}})\; \underbrace{R}_{\text{multiply by radix}} \;+\; \underbrace{t_{i+M}}_{\text{add new trailing digit}}$$

(can precompute R^{M-1})

i	...	2	3	4	5	6	7	...	
current value	1	4	1	5	9	2	6	5	*text*
new value		4	1	5	9	2	6	5	*text*

	4	1	5	9	2	*current value*
-	4	0	0	0	0	
		1	5	9	2	*subtract leading digit*
			*	1	0	*multiply by radix*
	1	5	9	2	0	
				+	6	*add new trailing digit*
	1	5	9	2	6	*new value*

Rabin-Karp substring search example

First R entries: Use Horner's rule.

Remaining entries: Use rolling hash (and % to avoid overflow).

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
	3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	
0	3																`% 997 = 3`
1	3	1															`% 997 = (3*10 + 1) % 997 = 31`
2	3	1	4														`% 997 = (31*10 + 4) % 997 = 314`
3	3	1	4	1													`% 997 = (314*10 + 1) % 997 = 150`
4	3	1	4	1	5												`% 997 = (150*10 + 5) % 997 = 508`
5		1	4	1	5	9											`% 997 = ((508 + 3*(997 - 30))*10 + 9) % 997 = 201`
6			4	1	5	9	2										`% 997 = ((201 + 1*(997 - 30))*10 + 2) % 997 = 715`
7				1	5	9	2	6									`% 997 = ((715 + 4*(997 - 30))*10 + 6) % 997 = 971`
8					5	9	2	6	5								`% 997 = ((971 + 1*(997 - 30))*10 + 5) % 997 = 442`
9						9	2	6	5	3							`% 997 = ((442 + 5*(997 - 30))*10 + 3) % 997 = 929`
10 ← *return* `i-M+1 = 6`							2	6	5	3	5						`% 997 = ((929 + 9*(997 - 30))*10 + 5) % 997 = 613`

Q

RM

R

Horner's rule

rolling hash

match

-30 (mod 997) = 997 – 30

10000 (mod 997) = 30

Rabin-Karp: Java implementation

```
public class RabinKarp
{
   private long patHash;    // pattern hash value
   private int M;           // pattern length
   private long Q;          // modulus
   private int R;           // radix
   private long RM1;        // R^(M-1) % Q

   public RabinKarp(String pat) {
      M = pat.length();
      R = 256;
      Q = longRandomPrime();

      RM1 = 1;
      for (int i = 1; i <= M-1; i++)
         RM1 = (R * RM1) % Q;
      patHash = hash(pat, M);
   }

   private long hash(String key, int M)
   {  /* as before */  }

   public int search(String txt)
   {  /* see next slide */  }
}
```

a large prime
(but avoid overflow)

precompute R^{M-1} (mod Q)

Rabin-Karp: Java implementation (continued)

Monte Carlo version. Return match if hash match.

```
public int search(String txt)
{
    int N = txt.length();
    int txtHash = hash(txt, M);
    if (patHash == txtHash) return 0;
    for (int i = M; i < N; i++)
    {
        txtHash = (txtHash + Q - RM*txt.charAt(i-M) % Q) % Q;
        txtHash = (txtHash*R + txt.charAt(i)) % Q;
        if (patHash == txtHash) return i - M + 1;
    }
    return N;
}
```

check for hash collision
using rolling hash function

Las Vegas version. Check for substring match if hash match; continue search if false collision.

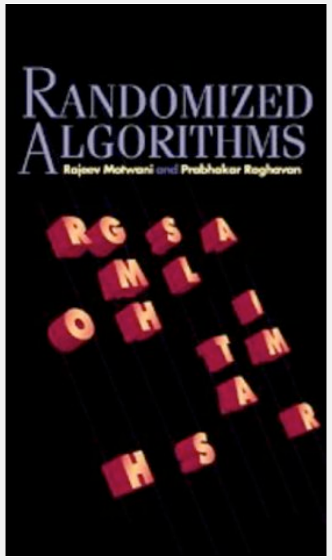
Rabin-Karp analysis

Theory. If Q is a sufficiently large random prime (about $M N^2$), then the probability of a false collision is about $1 / N$.

Practice. Choose Q to be a large prime (but not so large to cause overflow). Under reasonable assumptions, probability of a collision is about $1 / Q$.

Monte Carlo version.

- Always runs in linear time.
- Extremely likely to return correct answer (but not always!).



Las Vegas version.

- Always returns correct answer.
- Extremely likely to run in linear time (but worst case is $M N$).