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Fixed Point Iteration Method

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Abstract Our aim is to discuss the problem of finding approximate solutions of the equation. Sometimes we can find the exact roots of the equation 1, for example, when (fx) is a quadrantic on cubic polynomial then, in general, is convenient finding approximate solutions using same numerical methods. Here, we will discuss a method called fixed point iteration method and particular case of this method called Newton's Method.

Keywords: *Numerical , Iteration , Alegorithm , Varient*

1. INTRODUCTION

In this section we consider methods for determining the solution to an equation expressed, for some functions g .in the form

$$g(x) = x \quad (2)$$

A solution to such an equation is said to be a fixed point of the function g. Let's we found a fixed point for any given g. Then every root finding problem could also be solved for example. The root finding problem $f(x) = 0$ has solutions that correspond precisely to the fixed points of $g(x) = x$ when $g(x) = x - f(x)$. The first task, then, is to decide when a function will have a fixed point and how the fixed points can be determined. (In numerical analysis, "determined" generally means approximated to a sufficient degree of accuracy.)

EXAMPLE 1.

(a) The function $g(x) = x, 0 \le x \le 1$ has a fixed point at each x in $[0,1]$.

(b) The function $g(x) = x - \sin \pi$ has exactly two fixed points in $[0,1]$. x = 0 and x = 1. (see figure 1.1)

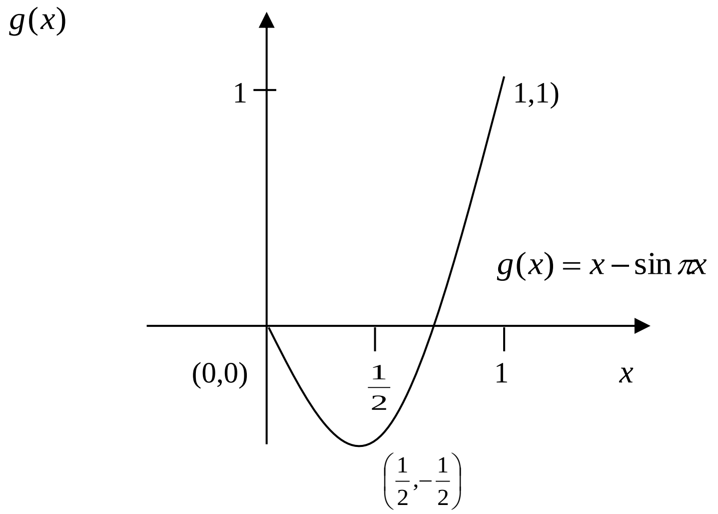

Figure 1.1.

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

Theorem 1.1.

If $g \in [a,b]$ and $g(x) \in [a,b]$. then g has a fixed point in $[a,b]$. Further, suppose $g'(x)$ exists on $[a,b]$ and then a positive constant k < 1 exists with

(1.1) $$|g'(x)| \le k < 1 \qquad \text{for all } x \in (a,b).$$

Then g has a unique fixed point p in $[a,b]$. (see figure 1.1)

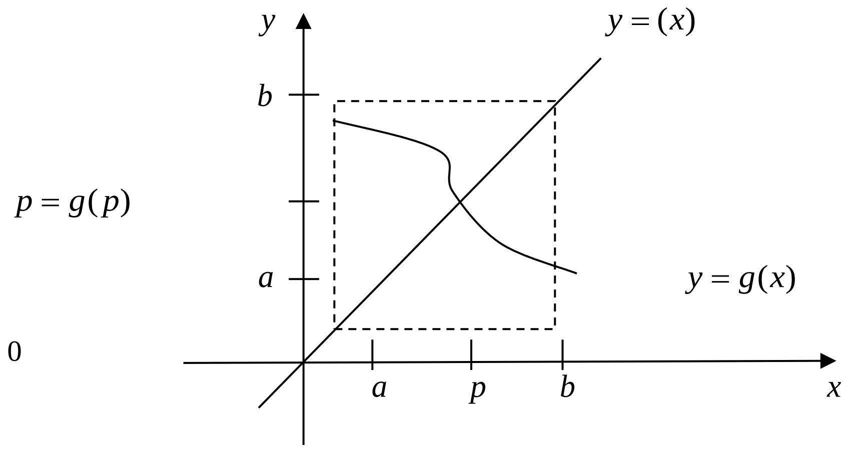

Figure 1.1.

Proof: if $g(a) = a$ or $g(b) = b$, the existence of a fixed point is obvious. Suppose not; then it must be true that $g(a) > a$ and $g(b) < b$. Decline $h(x) = g(x) - x$. Then his continuous on $[a,b]$ and

$$h(a) = g(a) - a > 0, h(b) = g(b) - b < 0$$

The intermediate value theorem implies that there exists $p \in (a,b)$ for which $h(p) = 0$ thus, $g(p) - p = 0$ and p is a fixed point of g.

Suppose in addition that inequality (1.1) holds and that p and q are both fixed points in $[a,b]$ with $p \neq q$. by the mean value theorem a number ξ exists between p and q. And hence in $[a,b]$ with.

$$|p-q| = |g(p) - g(q)| = |g'(f)|\ |p-q| \leq k|p-q| < |p-q|$$

Which is a contradiction this contradiction must come from the only supposition $p \neq q$ .hence $p = q$ and the fixed point in $[a,b]$ is unique

EXAMPLE 2.

(a) Let $g(x) = (x^2\text{-}1)/3$ on [-1, 1] using the extreme value theorem, it is easy to show that the absolute minimum or g occurs at $x = 0$ and $g(0) = -\frac{1}{3}$. Similarly. The absolute maximum of g occurs at $x = \pm 1$ and has the value g(± 1)= 0.moreover. g is continuous and

$$|g'(x)| = \left|\frac{2x}{3}\right| \leq \frac{2}{3} \qquad \text{for all } x \in [-1,1].$$

So g satisfies the hypotheses of theorem 1.1 and has a unique fixed in [-1, 1].

In this example the unique fixed point p in the interval [-1, 1] can be determined exactly. If

$$P = g(p) = \frac{p^2 - 1}{3}, \text{ then } p^2 - 3_p - 1 = 0$$

Which by the quadratic Formula implies that?

$$p = \frac{3 - \sqrt{13}}{2}.$$

1

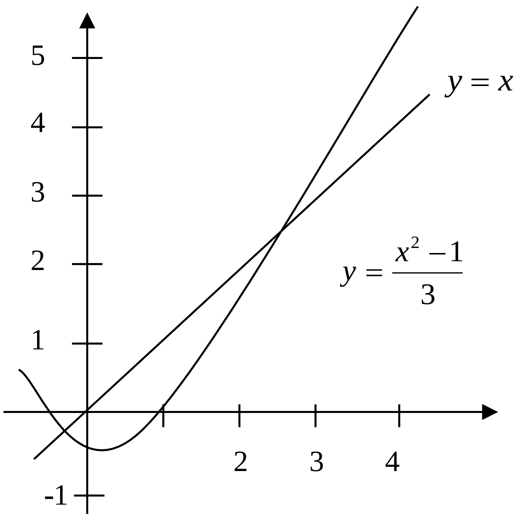

Figure 1.2.

That g also has a unique fixed point p = (3 + $\sqrt{(13)}$ / 2 for interval [3,4] forever $g(4) = 5$ and $g'(4) = \frac{1}{3} > 1: so$ g does not satisfy their hypotheses of theorem 1.1 this shows that the hypotheses of theorem 1.1 sufficient guarantee a unique fixed point, but are not necessary. (see figure 1.2).

$G(x) =$ 3^{-x}. since $g'(x) = -3^{-x}\ln 3 < 0 = on[.0.1]$, the function this decreasing [0,1] hence g (1) = $\frac{1}{3} \leq g(x) \leq 1 = g(0)\, for\, 0 \leq x \leq 1$. this for $x \in [0,1]$ $g(x)$ $\in [0,1]$ therefore, g has a fixed point in [0,1] since

g' (0) = - in 3 = -1.098612289

$f(x) \not\leq 1$ on [0, 1] theorem 1.1 cannot be used determinant unequation forever g is decreasing so it is clear that the fixed point must the unique (see figure 1.3)

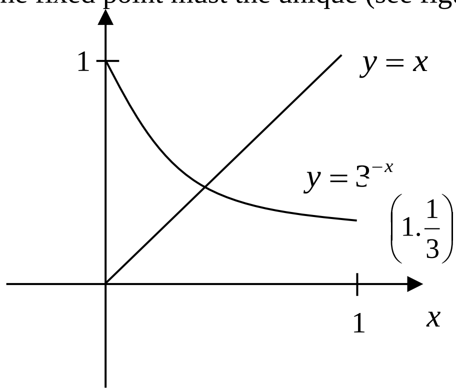

Figure 1.3.

Approximate point of a function g,we choose an initial information p and sequence $\{p_n\}^1{}_n = 0$ by letting $p_n = q(p_{n-1})$ h n ≥ 1 if the for p and g is continuous then by

Theorem 1.2

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g(\lim_{n\to\infty} p_{n-1}) = g(p)$$

and a solution to $x = g(x)$ is obtained this technique is called fixed – point or functional iteration the procedure is detailed in algorithm 1.2 and described in figure 1.4

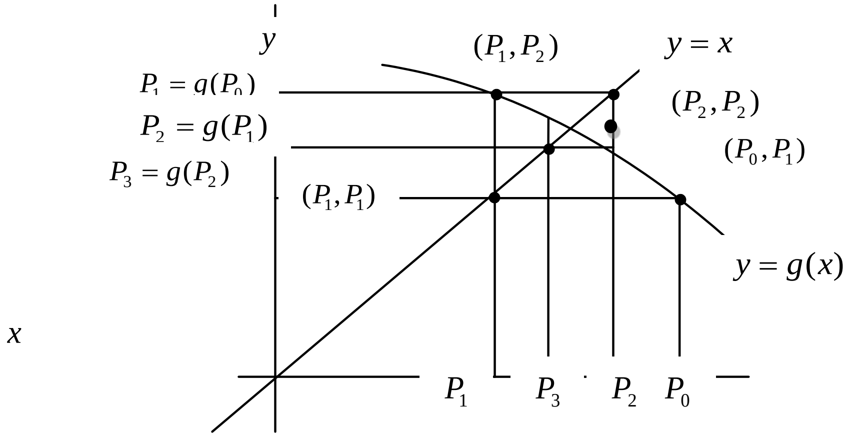

Figure 1.4

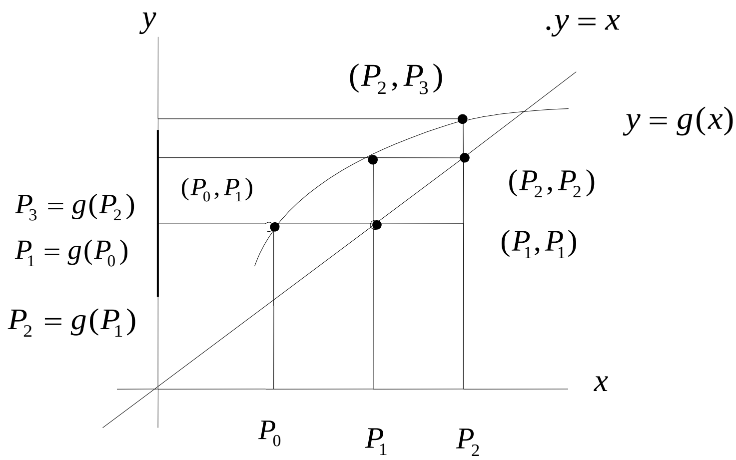

Figure 1.5

FIXED – POINT ALGORITHM 1

To find a solution to $p = g(p)$ given an initial approximation p_0 : INPUT initial approximation p_0; tolerance TOL; maximum number of iterations no: OUTPUT approximate solution p or message failure.

Step 1 set i = 1.

Step 2 white i $\leq$ N_0

Step 3 set $p = g\,(p_0)$. (compare p.)

Step 4 if $|p - p_0| < TOL$ then

OUTPUT (P), (Procedure completed successfully)
STOP.
Step 5 set i = i + 1.
Step 6 set p_0 = p. (Update p_0)
Step 7 OUTPUT (Method failed after N_0 iterations $N_0 = N_0$;
(Procedure completed unsuccessfully.)
STOP.
To illustrate the technique of functional iteration consider the following example.

EXAMPLE 3.

a) Let us take the problem given where $g(x) = \frac{1}{7}(x^3 + 2)$. Then $g:[0,1] \to [0,1]$ and $|g'(x)| < \frac{3}{7}$ for all $x \in [0,1]$. Home by the previous theorem sequence P_n defined by the process $P_{n+1} = \frac{1}{7}(P_n^{\ 3} + 2)$ converges to a root of $x^3 - 7x + 2 = 0$

b) Consider $f:[0,2] \to R$ defined by $f(x) = (1+x)^{\frac{1}{5}}$. Observe that f maps [0, 2] onto itself. Moreover $|f'(x)| \le \frac{1}{5} < 1$ for $x \in [0,2]$. By the previous theorem the sequence (P_n) defined by $P_{n+1} = (1+P_n)^{1/5}$ converges to a root of $x^2 - x - 1 = 0$ in the interval [0,2]

In practice, it is often difficult to check the condition $f([a,b] \le [a,b])$ given in the previous theorem. We now present a variant of theorem.

Theorem 1.2. (Fixed point theorem) let $g \in [a,b]$ and suppose that $g(x) \in [a,b]$ for all x in $[a,b]$. further,
Suppose g' exists on $[a,b]$ with

$|g'(x)| \le k < 1$ for all $x \in (a,b)$

If p_0 is any number in $[a,b]$ then the sequence defined by

$p_n = g(p_n - 1)$ $n \ge 1.$

Converges to the unique fixed point p in $[a,b]$

Proof by theorem 1.1 a unique fixed point exist in $[a,b]$ since g maps $[a,b]$ into itself the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \ge 0$ and $p_n \in [a,b]$ for all n. Using inequality and the mean value theorem.

$$|p_n - p| = |g(p_n - 1) - g(p)| = |g'(\xi)||p_{n-1} - p| \le k|p_{n-1} - p|.$$

Where $\xi \in (a,b)$ applying inequality (1.3) inductively gives:

$$|p_n - p| \le k|p_{n-1} - p| \le k^2|p_{n-2} - p| \le .... \le k^n|p_0 - p|.$$

Since $k < 1$,

$$\lim_{n \to \infty} |p_n - p| \le \lim_{n \to \infty} k^n |p_0 - p| = 0$$

and $\{p_n\}_{n=0}^{\infty}$ converges to p.

Corollary 1.3 If g satisfies the hypotheses of theorem 1.2 a bound for the error involve in using p_n to apporoximate p is given by.

$$|p_n - p| \le k^n \max\{p_0 - a, b - p_0\} \qquad \text{for all } n \ge 1.$$

Proof from inequality,

$$|p_n - p| \le k^n |p_0 - p| \le k^n \max\{p_0 - a, b - p_0\},$$

Since $p \in [a,b]$.

Corollary 1.4 If g satisfies the hypotheses of theorem 1.2, then

$$|p_n - p| \le \frac{k^n}{1-k}|p_0 - p_1| \text{ for all } n \ge 1$$

Proof for $n \ge 1$ the procedure used in the proof of theorem 1.2 implies that

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \le k|p_n - p_{n-1}| \le .... \le k_n|p_1 - p_0|$$

Thus, for $m > n \ge 1$

$$|p_m - p_n| = |p_m - p_{m-1} + p_{m-1} - ... + p_{n+1} - p_n|$$

$$\le |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + ... + |p_{n+1} - p_n|$$

$$\le k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + ... + k^n|p_1 - p_0|$$

$$= k^n (1 + k + k^2 + ... + k^{m-n-} 1)|p_1 - p_0|$$

By theorem 1.2, lim. $p_m = p$ so

$m \to \infty$

$$|p - p_n| = \lim |p_m - p_n| \le k^n |p_1 - p_0| \sum_{p=0}^{\infty} k^p = \frac{k^n}{1-k}|p_1 - p_0|$$

$m \to \infty$

Both corollaries relate the rate of convergence to the bound k on the first derivate it is clear that the rate of convergence depends on the factor $k^n(1-k)$ and that the smaller k can be made the faster the convergence the convergence may be very slow if k is close to 1.In the following example the fixed-point methods in example 3 are reconsidered in light of the results described in theorem 1.2.

EXAMPLE 4.

(a) When $g_1(x) = x - x^3 - 4x^2 + 10, g_1'(x) = 1\ \ 3x^2 - 8x$. Then is no interval $[a,b]$ containing p for which $|g_1'(x)| < 1$ though theorem (1.2) does not guarantee that the method must fail for this choice of g, there is no reason to expect convergence.

(b) With $g_2(x) = [(10/x) - 4x]^{1/2}$, we can see that p_2 does not map [1,5] into [1,2] and the sequence $\{p_n\}_{n=0}^{\infty}$ is not defined with p =1.5 moreover there is no interval containing such that $|g_2'(x)| < 1,$ since $|g_2'(p)| \approx 3.4$

(c) for the function $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

$$g_3(x) = -\frac{3}{4}x^2\left(10 - x^3\right)^{-1/2} < 0 \quad \text{on } [1,2],$$

So g is strictly decreasing on [1,2] however, $|g_3'(2)| \approx 2.12,$ so inequality (1.2) does not hold on [1,2].A closer examination of the sequence $\{p_0\}_{n=0}^{\infty}$ starting with $p_0 = 1.5$ will show $g_3'(x) < 0$ and g is strictly decreasing but additionally,

$1 < 1.28 \approx g_3(1.5) \leq g_3(x) \leq g_3(1) = 1.5$

For all $x \in [1,1.5]$ this shows that g_3 maps the interval [1,1.5] into itself. Since it is also true that $|g_3'(x)| \leq |g_3'(1.5)| \approx 0.66$ on this interval, theorem 1.2 configures the convergence which we were already aware

(c) for $g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$,

$$|g_4'(x)| = \left|\frac{-5}{\sqrt{10}(4+x)^{3/2}}\right| < \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15 \quad \text{for all } [1.2]$$

The bound on the magnitude $g_4'(x)$ is much smaller than the bound on the magnitude of $g_3'(x)$ which explains the more rapid convergence using g_4 the other part of example 3 can be handled in a similar manner.

REMARK: If g is invertible then P is a fixed point of g if and only if q is a fixed point of g^{-1}, in view of this fact, sometimes we can apply the fixed point iteration method for g^{-1} instead of g .For understanding, consider $g(x) = 3x - 21$ then $|g'(x)| = 3$ for all x. So the fixed point iteration method may not work. However, $g^{-1}(x); = \frac{1}{3}x + 7$ and in this case $|(g^{-1})'(x)| = \frac{1}{3}$ for all x.

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