# FIXED POINT ITERATION METHOD

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ABSTRACT

We discuss the problem of finding approximate solutions of the equation

$$f(x) = 0 \qquad (1)$$

In some cases it is possible to find the exact roots of the equation (1) for example when $f(x)$ is $a$ quadratic on cubic polynomial otherwise, in general, is interested in finding approximate solutions using some numerical methods. Here, we will discuss $a$ method called fixed point iteration method and $a$ particular case of this method called Newton's method.

INTRODUCTION

In this section we consider methods for determining the solutıon to an equation expressed, for some functions $g$ .in the form

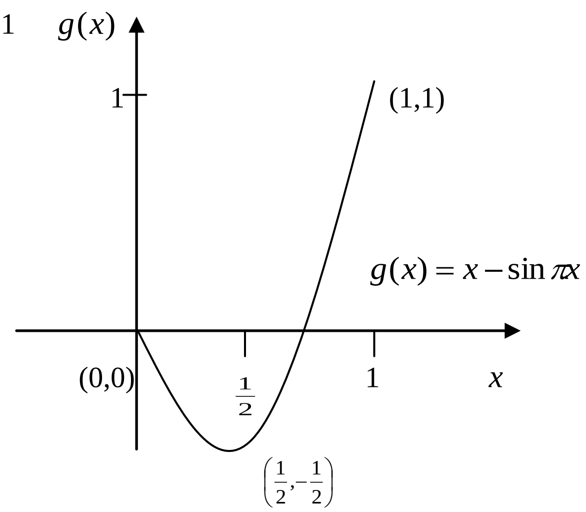
$$g(x) = x \qquad (2)$$

A solutıon to such an equation is said to be $a$ fixed point of the function $g$ .Let's we found a fixed point for any given g. Then every root finding problem could also be solved for example. The root finding problem $f(x) = 0$ has solutions that correspond precisely to the fixed points of $g(x) = x$ when $g(x) = x - f(x)$. The first task, then, is to decide when a function will have a fixed point and how the fixed points can be determined. (In numerical analysis, "determined" generally means approximated to a sufficient degree of accuracy.)

EXAMPLE 1

(a) The function $g\ (x) = x$, $0 \le x \le 1$ has a fixed point at each $x$ in $[0,1]$.
(b) The function $g\ (x) = x - \sin\pi$ has exactly two fixed points in $[0,1]$. $x$ = 0 and $x$ = 1.(see figure 1.1)

Figure 1.1



The following theorem gives sufficient conditions for the existence and uniqueness of $a$ fixed point.

Theorem 1.1

If $g \in [a,b]$ and $g(x) \in [a,b]$.then $g$ has a fixed point in $[a,b]$.Further, suppose $g'(x)$ exists on $[a,b]$ and then a positive constant $k < 1$ exists with

(1.1) $\qquad |g'(x)| \le k < 1 \qquad$ for all $x \in (a,b)$.

Then g has $a$ unique fixed point p in $[a,b]$. (see figure 1.1)

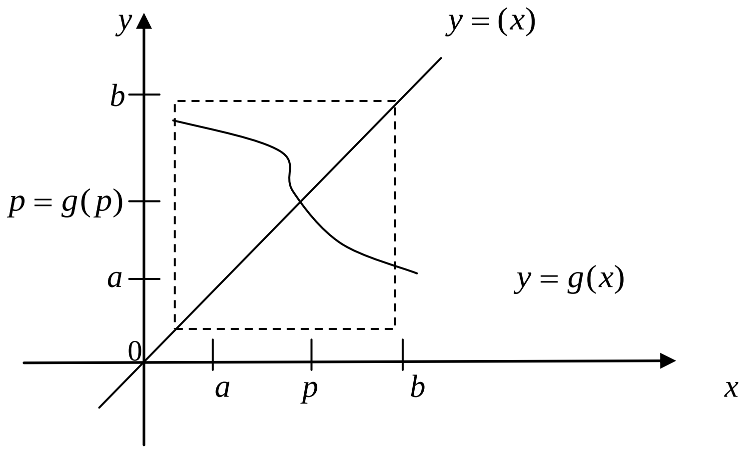



Fıgure 1.1

Proof: if $g(a)= a$ or $g(b)=b$, the existence of $a$ fixed point is obvious. Suppose not; then it must be true that $g(a)> a$ and $g(b)<b$. Decline $h(x)=g(x)-x$. Then his continuous on $[a,b]$ and

$$h(a)=g(a)-a>0, h(b)=g(b)-b<0$$

The ıntermediate value theorem implies that there exists $p \in (a,b)$ for which $h(p)=0$ thus, $g(p)-p=0$ and p is a fixed point of g.

Suppose in addition that inequality (1.1) holds and that $p$ and $q$ are both fixed points in $[a,b]$ with $p \neq q$. by the mean value theorem a number $\xi$ exists between $p$ and $q$. And hence in $[a,b]$ with.

$$|p-q| = |g(p)-g(q)| = |g'(f)|\ |p-q| \le k|p-q| < |p-q|$$

Which is a contradiction this contradiction must come from the only supposition $p \neq q$ .hence $p = q$ and the fixed point in $[a,b]$ is unique

EXAMPLE 2

($a$) Let $g(x)=(x^2-1)/3$ on [ -1,1] using the extreme value theorem ,it is easy to show that the absolute minimum or $g$ occurs at $x = 0$ and $g(0)=-\frac{1}{3}$. Similarly. The absolute maximum of g occurs at $x = \pm 1$ and has the value g$(\pm 1)$= 0.moreover. $g$ is continuous and

$$|g'(x)| = \left|\frac{2x}{3}\right| \le \frac{2}{3} \quad \text{for all } x \in [-1,1].$$

So $g$ satisfies the hypotheses of theorem 1.1 and has $a$ unique fixed in [-1,1].

In this example the unique fixed point $p$ in the interval [-1 , 1] can be determined exactly. If

$$P = g(p) = \frac{p^2 - 1}{3}, \text{ then } \; p^2 - 3_p - 1 = 0$$

Which by the quadratic Formula implies that?

$$p = \frac{3 - \sqrt{13}}{2}.$$

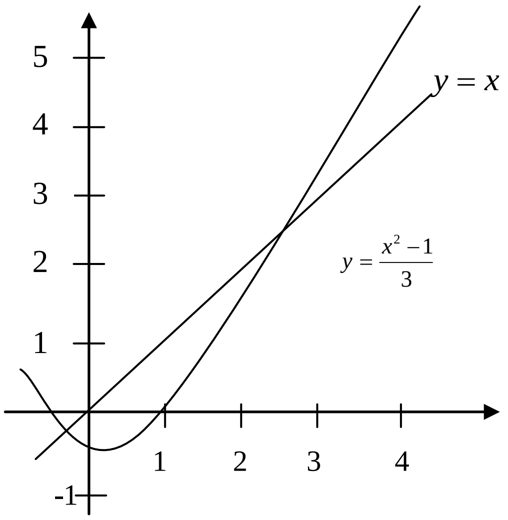



Figure 1.2

That g also has a unique fixed point $p$ = ( 3 + $\sqrt{(13)}$ / 2 for interval [3,4] forever $g(4) = 5$ and $g'(4) = \frac{1}{3} > 1 : so$ $g$ does not satisfy their hypotheses of theorem 1.1 this shows that the hypotheses of theorem 1.1 sufficient guarantee $a$ unique fixed point, but are not necessary.(see figure 1.2).

$G(x) = 3^{-x}$ .since $g'(x) = -3^{-x} \ln 3 < 0 = on[.0.1]$ ,the function this decreasing [0,1] hence $g$ (1) = $\frac{1}{3} \le g(x) \le 1 = g(0)\, for 0 \le x \le 1.$ this for $x \in [0,1]$ $g(x)$ $\in [0,1]$ therefore, $g$ has $a$ fixed point in [0,1] since

$$g'(0) = - \text{ın } 3 = -1.098612289$$

$f(x) \not\le 1$ on [0,1] theorem 1.1 cannot be used determinant unequation forever $g$ is decreasing so it is clear that the fixed point must the unique (see figure 1.3)

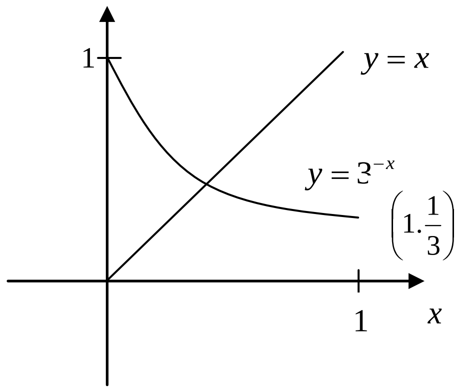



Figure 1.3

Approximate point of $a$ function $g$ ,we choose an initial information $p$ and sequence $\{p_n\}^1{}_n = 0$ by letting $p_n = q\left(p_{n-1}\right)$ $h$ $n \geq 1$ ıf the for $p$ and $g$ is continuous then by

Theorem 1.2

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g(\lim_{n\to\infty} p_{n-1}) = g(p)$$

and $a$ solution to $x = g(x)$ is obtained this technique is called fixed – point or functional iteration the procedure is detailed in algorithm 1.2 and described in figure 1.4

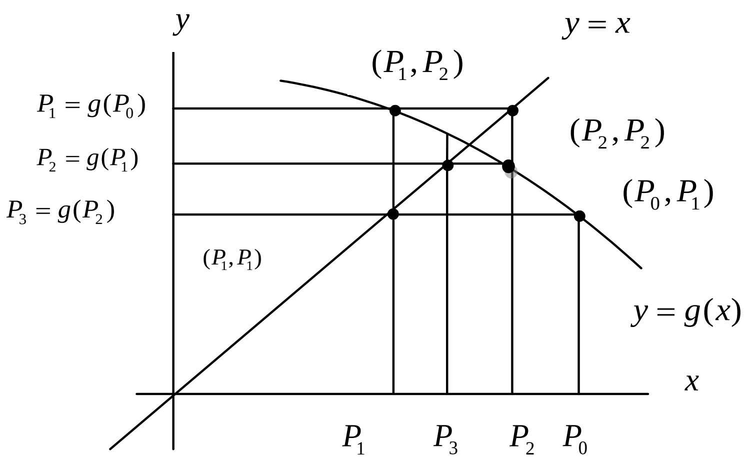



Figure 1.4

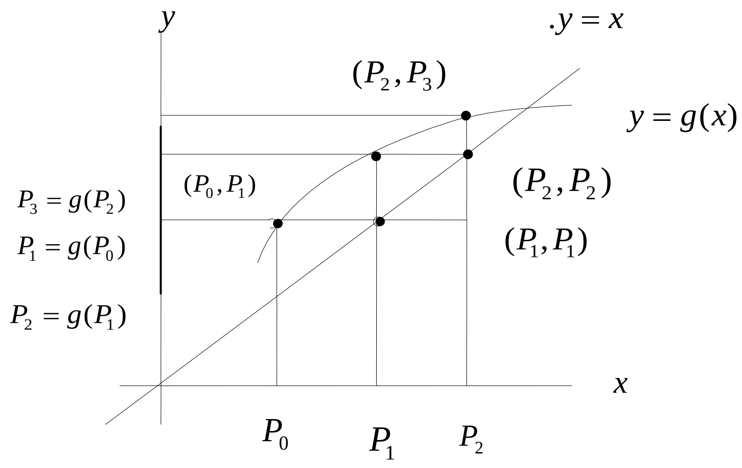



Figure 1.5

FIXED – POINT ALGORITHM 1

To find $a$ solution to $p = g(p)$ given an initial approximation $p_0$ : INPUT initial approximation $p_0$; tolerance TOL; maximum number of iterations no: OUTPUT approximate solution $p$ or message failure.

Step 1 set i = 1.
Step 2 white i $\le$ $N_0$
Step 3 set $p = g\ (p_0)$. ( compare p.)
Step 4 ıf $|p - p_0| < TOL$ then
OUTPUT (P), (Procedure completed successfully.)
STOP.
Step 5 set i = i + 1.
Step 6 set $p_0$ = p. (Update $p_0$)
Step 7 OUTPUT (Method failed after $N_0$ iterations $N_0 = N_0$;
(Procedure completed unsuccessfully.)
STOP.

To illustrate the technique of functional iteration consider the following example.

Example:

a) Let us take the problem given where $g(x) = \frac{1}{7}(x^3 + 2)$. Then $g : [0,1] \to [0,1]$ and $|g'(x)| < \frac{3}{7}$ for all $x \in [0,1]$. Home by the previous theorem sequence $P_n$ defined by the process $P_{n+1} = \frac{1}{7}(P_n^{\ 3} + 2)$ converges to a root of $x^3 - 7x + 2 = 0$

b) Consider $f:[0,2]\to R$ defined by $f(x)=(1+x)^{\frac{1}{5}}$. Observe that $f$ maps [0,2] onto itself. Moreover $\left|f'(x)\right|\le\frac{1}{5}<1$ for $x\in[0,2]$. By the previous theorem the sequence ($P_n$) defined by $P_{n+1}=(1+P_n)^{1/5}$ converges to a root of $x^2-x-1=0$ in the interval [0,2]
In practice, it is often difficult to check the condition $f([a,b]\le[a,b])$ given in the previous theorem. We now present a variant of theorem.

Theorem 1.2 (Fixed point theorem) let $g\in[a,b]$ and suppose that $g(x)\in[a,b]$ for all $x$ in $[a,b]$.further,
Suppose $g'$ exists on $[a,b]$ with

$$|g'(x)|\le k<1 \qquad \text{for all } x\in(a,b)$$

If $p_0$ is any number in $[a,b]$ then the sequence defined by

$$p_n=g(p_n-1) \qquad n\ge1.$$

Converges to the unique fixed point $p$ in $[a,b]$

Proof by theorem 1.1 a unıque fixed point exist in $[a,b]$ since $g$ maps $[a,b]$ into itself the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n\ge0$ and $p_n\in[a,b]$ for all n. Using inequality and the mean value theorem.

$$|p_n-p|=|g(p_n-1)-g(p)|=|g'(\xi)||p_{n-1}-p|\le k|p_{n-1}-p|.$$

Where $\xi\in(a,b)$ applying inequality (1.3) inductively gives:

$$|p_n-p|\le k|p_{n-1}-p|\le k^2|p_{n-2}-p|\le.......\le k^n|p_0-p|.$$

Since $k<1$,

$$\lim_{n\to\infty}|p_n-p|\le\lim_{n\to\infty}k^n|p_0-p|=0$$

and $\{p_n\}_{n=0}^{\infty}$ converges to $p$.
Corollary 1.3 If $g$ satisfies the hypotheses of theorem 1.2 $a$ bound for the error involve in using $p_n$ to apporoximate p is given by.

$$|p_n-p|\le k^n\max\{p_0-a,b-p_0\} \qquad \text{for all} \quad n\ge1.$$

Proof from inequality,

$$|p_n-p|\le k^n|p_0-p|\le k^n\max\{p_0-a,b-p_0\},$$

Since $p\in[a,b]$.
Corollary 1.4 If g satisfies the hypotheses of theorem 1.2,then

$$|p_n-p|\le\frac{k^n}{1-k}|p_0-p_1| \quad \text{for all } n\ge1$$

Proof for $n\ge1$ the procedure used in the proof of theorem 1.2 implies that

$$|p_{n+1}-p_n|=|g(p_n)-g(p_{n-1})|\le k|p_n-p_{n-1}|\le....\le k_n|p_1-p_0|$$

Thus, for $m > n \geq 1$

$$\begin{aligned}|p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - \ldots + p_{n+1} - p_n| \\ &\leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \ldots. + |p_{n+1} - p_n| \\ &\leq k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + \ldots + k^n|p_1 - p_0| \\ &= k^n\left(1 + k + k^2 + \ldots. + k^{m-n-}1\right)|p_1 - p_0|\end{aligned}$$

By theorem 1.2, $\lim_{m \to \infty} p_m = p$ so

$$|p - p_n| = \lim_{m \to \infty}|p_m - p_n| \leq k^n|p_1 - p_0|\sum_{p=0}^{\infty} k^p = \frac{k^n}{1-k}|p_1 - p_0|$$

Both corollaries relate the rate of convergence to the bound k on the first derivate ıt is clear that the rate of convergence depends on the factor $k^n(1-k)$ and that the smaller $k$ can be made the faster the convergence the convergence may be very slow if $k$ is close to 1.In the following example the fixed-point methods in example 3 are reconsidered in light of the results described in theorem 1.2.

EXAMPLE 4

(a) When $g_1(x) = x - x^3 - 4x^2 + 10, g_1'(x) = 1 \quad 3x^2 - 8x$. Then is no interval $[a,b]$ containing $p$ for which $|g_1'(x)| < 1$ though theorem (1.2) does not guarantee that the method must fail for this choice of $g$, there is no reason to expect convergence.

(b) With $g_2(x) = [(10/x) - 4x]^{1/2}$, we can see that $p_2$ does not map [1,5] into [1,2] and the sequence $\{p_n\}_{n=0}^{\infty}$ is not defined with $p$ =1.5 moreover there is no interval containing such that

$$|g_2'(x)| < 1, \qquad \text{since } |g_2'(p)| \approx 3.4$$

(c) for the function $g_3(x) = \frac{1}{2}\left(10 - x^3\right)^{1/2}$

$$g_3(x) = -\frac{3}{4}x^2\left(10 - x^3\right)^{-1/2} < 0 \qquad \text{on } [1,2],$$

So $g$ is strictly decreasing on [1,2] however, $|g_3'(2)| \approx 2.12$, so inequality (1.2) does not hold on [1,2].A closer examination of the sequence $\{p_0\}_{n=0}^{\infty}$ starting with $p_0 = 1.5$ will show $g_3'(x) < 0$ and $g$ is strictly decreasing but additionally,

$$1 < 1.28 \approx g_3(1.5) \leq g_3(x) \leq g_3(1) = 1.5$$

For all $x \in [1,1.5]$ this shows that $g_3$ maps the interval [1,1.5] into itself. Since it is also true that $|g_3'(x)| \leq |g_3'(1.5)| \approx 0.66$ on this interval, theorem 1.2 configures the convergence which we were already aware

(c) for $g_4(x)=\left(\frac{10}{4+x}\right)^{1/2}$,

$$\left|g_4'(x)\right|=\left|\frac{-5}{\sqrt{10}(4+x)^{3/2}}\right|<\frac{5}{\sqrt{10}(5)^{3/2}}<0.15 \qquad \text{for all } [1.2]$$

The bound on the magnitude $g_4'(x)$ is much smaller than the bound on the magnitude of $g_3'(x)$ which explains the more rapid convergence using $g_4$ the other part of example 3 can be handled in a similar manner.

**Remark:** If $g$ is invertible then $P$ is a fixed point of $g$ if and only if $q$ is a fixed point of $g^{-1}$, in view of this fact, sometimes we can apply the fixed point iteration method for $g^{-1}$ instead of $g$ .For understanding, consider $g(x)=3x-21$ then $\left|g'(x)\right|=3$ for all $x$. So the fixed point iteration method may not work. However, $g^{-1}(x);=\frac{1}{3}x+7$ and in this case $\left|(g^{-1})'(x)\right|=\frac{1}{3}$ for all $x$.

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