Theoretical and empirical runtime analysis of evolutionary algorithms for the partition problem

Bachelor Thesis of

Lipp Daniel

At the Department of Informatics and Mathematics
Chair of algorithms for intelligent systems

Reviewers:	?
	?
Advisors:	?
	?

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract

A short summary of what is going on here.

Deutsche Zusammenfassung

Kurze Inhaltsangabe auf deutsch.

Contents

1. Introduction

This chapter should contain

1. A short description of the thesis topic and its background.

2. An overview of related work in this field.

3. Contributions of the thesis.

4. Outline of the thesis.

2. Preliminaries

This chapter should provide the foundations of the thesis.

3. Content Chapters

3.1. Notations

1. **RLS:** Randomised Local Search

2. **RSH:** Randomised Search Heuristic referring to all in this context analysed Evolutionary algorithms

3. **n:** The input length of the problem

4. w_i**:** The i-th object of the input. If not mentioned otherwise the weights are sorted in non increasing order so: $w_1 \geq w_2 \geq \ldots \geq w_{n-1} \geq w_n$

5. **W:** The sum of all objects: $W = \sum_{i=1}^{n} w_i$

6. **bin:** When solving Partition a set of numbers is divided into two distinct subsets and in this paper both subsets are referred to as bins

7. b_F**:** The fuller bin (the bin with more total weight)

8. b_F**:** The emptier bin (the bin with less total weight)

9. b_{w_i}**:** The bin containig the object w_i

10. **opt:** The optimal solution for a given partition instance.

11. **x:** A vector $x \in \{0,1\}^n$ describing a solution

3.2. Improving bounds on the Standard RSHs

Lemma 3.1. *If $w_1 \geq \frac{W}{2}$ then the RLS and the (1+1) EA reach the optimal value in expected time $\Theta(n \log n)$*

Proof. The optimal solution is putting w_1 in one bin and all other elements in the other bin. So the problem is almost identical to OneMax/ZeroMax. A single bit flip of the first bit can only happen, if the emptier bin has a weight of at most $\frac{W-w_1}{2}$. After this flip the weight of the emptier bin is at least $\frac{W-w_1}{2}$ and therefore another single bit flip of w_1 can only happen before a different bit is flipped. After a different bit has been flipped, the RLS wont flip only the first bit again, because it will never result in an improvement. So the run of the RLS can be divided into three phases:

Phase 1: The RLS behaves exactly like OneMax/ZeroMax and flips every bit to the opposite of the first bit (except for the first bit).

Phase 2: The RLS flips only the first bit or bits that do not result in an improvement.

Phase 3: The RLS behaves exactly like ZeroMax/OneMax and flips every bit to the opposite of the first bit (except for the first bit).

The expected length of the first phase is $\mathcal{O}(n)$ because the probability of flipping the first bit is at least $\frac{1}{n} \cdot (1 - \frac{1}{n})^{n-1} \geq \frac{1}{ne}$ and therefore the expected time for such a step is at most $\mathcal{O}(\frac{1}{ne}^{-1}) = \mathcal{O}(ne) = \mathcal{O}(n)$.
The length of the second phase is $\mathcal{O}(n)$ because the solution is either optimal or there is at least one bit that needs to be flipped for an optimal solution. Since the expected length of Phase 1 is $\mathcal{O}(n)$ the solution produced by the RLS won't be optimal in expectation due to the bound of $\Theta(n \log n)$ for OneMax/ZeroMax. This again results in expected time $\mathcal{O}(n)$. The length of the third phase is identical to a run of the RLS on OneMax/ZeroMax where flips of the first bit are ignored as if it was already correctly flipped and therefore the expected time is $\Theta(n \log n)$

So the total expected time is $\mathcal{O}(n) + \mathcal{O}(n) + \Theta(n \log n) = \Theta(n \log n)$

The (1+1) EA can do multiple bit flips in a single step so the first bit can be flipped multiple times if the combined moved weight $y \leq b_F - b_E$. **TODO insert proof for (1+1) EA**. □

Lemma 3.2. *The RLS and the (1+1) EA reach an approximation ratio of at most 4/3 in expected time $\mathcal{O}(n \log n)$ if $w_1 < W/2$*

Proof. Without loss of generality the weights are normalized to a total weight W = 1.
Helpful statements

(1) If $b_F \leq \frac{2}{3}$ the approximation ratio is at most $\frac{b_F}{opt} = \frac{2/3}{opt} \leq \frac{2/3}{1/2} = \frac{4}{3}$

(2) If $w_1 \geq \frac{1}{3}$ and w_1 is in the emptier bin, then $b_F \leq 1 - w_1 \leq 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow$ approximation $\leq \frac{4}{3}$ (1)

(3) $b_F - b_E \geq v \Leftrightarrow b_F \geq b_E + v$ and therefore any object of weight at most v can be moved from b_F to b_E if $b_F - b_E \geq v$

(4) $b_F \geq \frac{2}{3} \Rightarrow b_E \leq \frac{1}{3} \Rightarrow b_F - b_E \geq \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \Rightarrow$ Every object $\leq \frac{1}{3}$ can be moved from b_F to b_E as long as $b_F \geq \frac{2}{3}$ (3)

(5) In Time $\mathcal{O}(n \log n)$ the weight of the fuller bin can be decreased to $\leq \frac{2}{3}$ if every object besides the biggest in the fuller bin is at most $\frac{1}{3}$ and $w_1 \leq \frac{W}{2}$.

Proof of (5):

In time $\mathcal{O}(n \log n)$ the RSH can move every object $\leq \frac{1}{3}$ to the emptier bin as long as $b_F \geq \frac{2}{3}$ (4). So in Time $\mathcal{O}(n \log n)$ the solution can be shifted to w_1 being in one bin and all other objects in the other bin. The RSH will only stop moving the elements if the condition $b_F \geq \frac{2}{3}$ is no longer satisfied (4). If $w_1 \geq \frac{1}{3}$ and every object was moved to the bin without w_1, then $b_F = \max\{1 - w_1, w_1\} = 1 - w_1 \leq \frac{2}{3}$. So either the RSH moves all objects to the emptier bin or stops moving objects because $b_F < \frac{2}{3}$ both resulting in $b_F \leq \frac{2}{3}$.

Now assume $w_1 < \frac{1}{3}$. In this case the RLS will move one object per step to the emptier bin. If all objects except the biggest where moved the other bin, the other bin would have a weight of at least $1 - w_1 > \frac{2}{3}$. Each object has weight $< \frac{1}{3}$ and therefore one step can not decrease the weight of the fuller bin from $> \frac{2}{3}$ to $\leq \frac{1}{3}$. Therefore the RLS will find a solution with $b_F < \frac{2}{3}$ before moving all elements from the first to the second bin. ***TODO: insert proof for (1+1) EA***

Proof of the Lemma:

If $w_1 + w_2 > \frac{2}{3}$ after time $\mathcal{O}(n)$ w_1 and w_2 are separated (Proof by by C.Witt) and will remain separated afterwards. From then on the following holds. If w_1 is in the emptier bin, the result follows by (2). Otherwise all elements in the fuller bin except w_1 have a weight smaller than $\frac{1}{3}$ and therefore the result follows by (5) and (1). If $w_1 + w_2 \leq \frac{2}{3}$ the result follows directly by (5) and (1). $\qquad \square$

Corollary 3.3. *The RLS and the (1+1) EA reach an approximation ratio of at most $\frac{4}{3}$ in expected time $\mathcal{O}(n \log n)$*

Proof. This follows directly from Lemma 3.1 and Lemma 3.2 $\qquad \square$

4. Conclusion

Summary and outlook.

Bibliography

Appendix

A. Appendix Section 1

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Figure A.1.: A figure