Theoretical and empirical runtime analysis of evolutionary algorithms for the PARTITION problem

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract

A short summary of what is going on here.

Deutsche Zusammenfassung

Kurze Inhaltsangabe auf deutsch.

Contents

1. Introduction

The question of $P = NP$ is still unanswered to this day and solving NP-hard problems for every instance still needs exponential time. To avoid the exponential running time on NP-hard optimisation problems one might use approximation algorithms. Those algorithms do not always return the best possible solution but only a solution with a guaranteed solution quality. For a minimisation problem a (1+0.5)-approximation algorithm will always return a solution that has at most 1.5 the optimal value. An example of an NP-hard optimisation problem is PARTITION. An instance of this problem is a multiset of n positive numbers $\{w_1, \ldots, w_n\}$ which has to be divided in two subsets with sums that are as close as possible. So a solution of PARTITION is a subset of $I \subset \{1, \ldots, n\}$ which splits the multiset into two subsets. The quality of the solution then is $\max\{\sum_{i \in I} w_i, \sum_{i \notin I} w_i\}$. PARTITION is one of the easiest NP-hard problems and has even been dubbed the easiest NP-hard problem [Hay02]. There are multiple algorithms specifically designed for PARTITION. Some of them return approximations and others always return the best solution. The exact algorithms have a runtime exponential in the input size due to the NP-hardness. Problem specific approximation algorithms are mostly deterministic such as the greedy method, the KK-algorithm or the FPTAS for the subsetsum problem which can be used for partition as well. Another class of non-deterministic algorithms are the so-called Evolutionary Algorithms which mimic the behaviour of evolution. Those algorithms start with a random population of solutions which are then changed with random steps. If the solution is good enough it survives and replaces one of the worst individuals in the population. The EA continues to generate new offspring in an endless loop. In practice the algorithm is given stopping conditions such as reaching a number of iterations or a specific solution quality. This is the principle of an anytime algorithm which can be terminated at anytime and output a valid solution. The longer the waiting time the better the solution might get. The main usage of EA lies in problems without a problem specific algorithms as these mostly outperform the EAs. To better understand their behaviour analysing them on well researched problems might still be beneficial to learn more about this class of algorithms. This thesis researches the runtime of basic EAs such as (1+1) EA and variations of RLS. The first part is a theoretical analysis with a main focus of lowering bounds that were previously shown. Additionally there are new results for other algorithm variants and also a lemma on different type of inputs. The remainder of this thesis is an empirical analysis of multiple base algorithms with different parameter setting on different kinds of inputs. Here not only the (1+1) EA with different mutation rate and variants of the RLS with different parameter values are researched but also a heavy tailed mutation operator. Apart from typical distributions such as the uniform, geometric and binomial distribution there are also results for specific problem specific instances such as an input where one values is as large as all other values combined. In the end the empirical results are condensed into a personal suggestion which algorithm should best be chosen to solve the problem, depending on the input but also a general suggestion.

2. Preliminaries

2.1. Notation

1. **RLS:** Randomised Local Search
2. **RSH:** Randomised Search Heuristic referring to all analysed Evolutionary algorithms
3. n**:** The input length of the problem
4. w_i**:** The i-th object of the input. If not mentioned otherwise the weights are sorted in non-increasing order so: $w_1 \geq w_2 \geq \ldots \geq w_{n-1} \geq w_n$
5. W**:** The sum of all objects: $\mathrm{W} = \sum_{i=1}^{n} w_i$
6. **bin:** When solving Partition a set of numbers is divided into two distinct subsets and in this paper both subsets are referred to as bins
7. b_F**:** The fuller bin (the bin with more total weight)
8. b_E**:** The emptier bin (the bin with less total weight)
9. b_{w_i}**:** The bin containing the object w_i
10. *opt***:** The optimal solution for a given partition instance.
11. x**:** A vector $x \in \{0,1\}^n$ describing a solution

2.2. Background

2.2.1. Known algorithms for partition

Multiple methods for generating a solution of PARTITION already exist. Solving the problem with a greedy approach in runtime $\mathcal{O}(n)$ results in an approximation ratio of 3/2 if the elements are not sorted or a ratio of 7/6 if the numbers are sorted. Greedy in this case means putting each element in the currently emptier set while looking at each value exactly once. Another approximation algorithm is the KK-algorithm or also called Largest Differencing Method. With expected time $\mathcal{O}(n \log n)$ it has the same runtime as greedy with sorting and also the same worst case approximation of 7/6. For inputs chosen uniform random from [0,1] the KK has an expected ratio of $1 + \frac{1}{n^{\Theta(\log n)}}$ in comparison to the greedy algorithm which only reaches an approximation ratio of $1 + \mathcal{O}(\frac{1}{n})$. Instead of putting each element in the currently emptier set the currently largest two values are combined to one value by either subtracting or adding them depending on which results in a better solution. Adding them corresponds to putting the elements in the same set whereas subtracting them means putting them in different sets. There is even a fully polynomial time algorithm (FPTAS) for the subsetsum problem [KMPS03] which can be used by setting the required sum to half the sum of all values. FPTAS return a solution of at most $(1+\epsilon)$ the optimum in a time that is polynomial both in n and in $\frac{1}{\epsilon}$. There are lots of other approximation-algorithms as well but also some algorithms that always return the best solution. The Pseudopolynomial time number partitioning algorithm always returns an optimal solution but needs time and space $\mathcal{O}(n\frac{m}{2})$ where m is the largest number in the input. The runtime is only pseudopolynomial because to encode m in the input only $\log_2(m)$ bits are required which causes $m = 2^{\log_2(m)}$ to be exponential in the input size. The Complete Greedy Algorithm (CGA) [Kor98] traverses a binary Tree depth first and searches the complete 2^n search space in a greedy way. It functions the same way as the simple greedy algorithm but instead of only looking at the greedy option it also chooses to evaluate the other option afterwards as well. The algorithm continues the depth first search until it either found a perfect partition or has traversed the whole tree. In the second case it will return the best value found on the way. While the space complexity in only $\mathcal{O}(n)$ the runtime is $\mathcal{O}(2^n)$. Another exact algorithm is the Complete Karmarkar-Karp (CKK) [Kor98]. This algorithm works similar to the greedy approach by traversing the binary tree of all solutions. Instead of greedily selecting the next edge here the algorithm behaves like the KK-algorithm described above. It performs better than the CGA for the same reasons as before but also has the same worst case running time as the GCA.

2.2.2. Evolutionary Algorithm

Evolutionary Algorithms mimic the process of evolution and normally behave mostly the same. A run typically looks like this:

1. Generate initial population at random
2. if stopping condition are met return the currently best solution
3. generate offspring population (e.g. by mutation)
4. evaluate fitness of the offspring
5. select fittest individuals and update population
6. go back to step 2.

For PARTITION a solution $x \in \{0,1\}^n$ where n is the size of the inputs separates all numbers into two different sets with $x_i = 0$ meaning w_i is in set 0 whereas $x_i = 1$ meaning w_i is in set 1. So every possible value of x describes a feasible solution but not necessarily a good one. To evaluate the quality of a solution the EA is given a fitness function. The fitness function in this case is $f(x) = \max\{\sum_{i=1}^{n} w_i \cdot x_i, \sum_{i=1}^{n} w_i \cdot (1 - x_i)\}$. The goal of the algorithm is to return a solution with minimal fitness. A mutation step in the PARTITION problem will change an algorithm-dependent number of bits from 1 to 0 or vice versa. In this case flipping a bit means putting the element in the other set. A simple implementation of an EA is the so called (1+1) EA (Algorithm 2.1). The first 1 in the brackets refers to size of the population and the second to the amount of mutants created in each iteration of the loop. So it always has only one solution and just generates one new solution in each step. The mutation of the current individual is performed by flipping each bit independently with probability $1/n$. The amount of flipped bits is binomial distributed with an expected value of $n \cdot \frac{1}{n} = 1$. By changing the mutation rate $1/n$ to c/n for any constant c the algorithm now flips $n \cdot c/n = c$ bits in expectation.
Another simple EA is the randomised local search or short RLS (Algorithm 2.2). Instead

Algorithm 2.1: (1+1) EA

choose x uniform from $\{0,1\}^n$
while *x not optimal* **do**
  $x' \leftarrow x$
  flip every bit of x' with probability $1/n$
  if $f(x') \leq f(x)$ **then**
    $x \leftarrow x'$

Algorithm 2.2: RLS

choose x uniform from $\{0,1\}^n$
while *x not optimal* **do**
  $x' \leftarrow x$
  flip one uniform random bit of x'
  if $f(x') \leq f(x)$ **then**
    $x \leftarrow x'$

of flipping each bit with probability $1/n$ here one random bit is flipped at uniform random. Apart from that the algorithms are the same.

2.2.3. Literature on the RLS and (1+1) EA for partition

Carsten Witt proved that the RLS and the (1+1) EA find a $(4/3+\epsilon)$ approximation in expected time $\mathcal{O}(n)$ and a $(4/3)$-approximation in expected time $\mathcal{O}(n^2)$ [Wit05]. He then introduced an almost worst case input to prove the bound for the approximation ratio is at least almost tight. The input is defined as followed for any $0 < \epsilon < 1/3$ and even n:
The input contains two numbers of value $1/3 - \epsilon/4$ and $n-2$ elements of value $(1/3 + \epsilon/2)/(n-2)$. The total volume is normalised to 1. When the two large values are in the same bin, the RSHs are tricked into a local optimum, where only w_1 and w_2 are in the first bin and the remaining elements in the other bin. This results in an almost worst case. To leave this $\Omega(n)$ bits must be moved in a step separating the two large values. Such a step will never happen for the RLS and only in expected exponential time for the (1+1) EA. This worst case happens with probability $\Omega(1)$. He also proved the both RSHs return a $(1+\epsilon)$ for $\epsilon \geq 4n$ in expected time $\lceil en\ln(4/\epsilon)\rceil$ for the (1+1) EA and $\lceil en\ln(4/\epsilon)\rceil$ for the RLS with probability at least $2^{-(e\log e+e)\lceil 2/\epsilon\rceil}\ln(4/\epsilon) - \lceil 2/\epsilon$ for the (1+1) EA and at least $2^{-(\log e+1)\lceil 2/\epsilon\rceil}\ln(4/\epsilon) - \lceil 2/\epsilon$ for the RLS. Afterwards he proved both RHSs reach a solution where the difference between the two bins is at most 1 for uniform distributed inputs on [0,1] after expected time $\mathcal{O}(n^2)$ for the (1+1) EA and $\mathcal{O}(n\log n)$. The difference between the two bis is even bounded by $\mathcal{O}(\log n/n)$ after $\mathcal{O}(n^{c+4}\log n)$ steps with probability at least $1-\mathcal{O}(1-1/n^c)$. This leads to an expected difference of $\mathcal{O}(\log n/n)$ after $\mathcal{O}(n^{c+4}\log n)$ steps. He also analysed exponential distributed inputs with parameter 1. With probability $1-\mathcal{O}(1/n^c)$ the difference on those inputs is bounded by $\mathcal{O}(\log n)$ after $\mathcal{O}(n^2\log n)$ steps and even by $\mathcal{O}(\log n/n)$ after $\mathcal{O}(n^{c+4}\log^2 n)$ steps. Additionally he described a polynomial time randomised approximation scheme (PRAS) for the RLS and the (1+1) EA for values of $\epsilon = \Omega(\log\log n/\log n)$.
For MAKESPAN-SCHEDULING a list of processing times has to be distributed on a set of machines while minimising the total time of the fullest machine. With 2 machines this problem is exactly the same as PARTITION. So in a sense MAKESPAN-SCHEDULING is a more general version of PARTITION. This lead to Christian Gunia generalising some

results previously shown by C. Witt to MAKESPAN-SCHEDULING on k machines [Gun05]. Solutions for MAKESPAN-SCHEDULING are $x \in \{0, \dots, k-1\}^n$ and therefore during a mutation x_i is set to a uniform random value from $\{0, \dots, k-1\} \backslash \{x_i\}$ instead of $1 - x_i$. The adapted RSHs reach an approximation ratio of $(2k/k+1)$ in expected time $\mathcal{O}(Wn^{2k-2}/w_n)$. On an instance where every weight is the same the expected optimisation time is bounded by $\mathcal{O}(n \log n)$. He also adapted the almost worse case to the problem and proved the RLS does not find a solution better than $(2k/k+1) - \epsilon$ in finite time for any $\epsilon > 0$. The second statement for PARTITION on the uniform distributed input on [0,1] are the exact same for MAKESPAN-SCHEDULING on k machines as well.

3. Improving bounds on the RLS and (1+1) EA

3.1. Improving bounds on the RLS and the (1+1) EA

Lemma 3.1. *Let $w_1 \geq W/2$, then for any $\gamma > 1$ and $0 < \delta < 1$, the (1+1) EA with mutation rate c/n with constant $0 < c < \sqrt{n}$ (RLS_k^S) reaches an f-value at most $w_1 + \delta(W - w_1)$ in at most $\lceil \frac{e^c}{c\cdot(1-o(1))} n \ln(\gamma/\delta) \rceil$ ($\lceil kn \ln(\gamma/\delta) \rceil$) steps with probability at least $1-\gamma^{-1}$. Moreover, the expected number of steps is at most $2\lceil \frac{e^c}{c\cdot(1-o(1))} n \ln(2/\delta) \rceil$ ($2\lceil kn \ln(2/\delta) \rceil$).*

Proof. This Lemma is very similar to C. Witt's Lemma 2 from section '2. Definitions and Proof Methods' in [Wit05]. The proof is almost the same. He first defines a potential function $p(x) = f(x) - l$. While $p(x) > 0$ all steps moving only a small object to the emptier bin are accepted. The expected p-decrease is at least $p_0 \cdot q$ where q is a lower bound on the probability of the algorithm to flip one specific bit. This leads to a next p value of $(1-q)p_0$. Since all steps of both algorithms are independent this argumentation remains valid even if the p value is only an expected value. With $q = 1/yn$ for a constant $y > 0$ the expected p value after $t = yn \ln(\gamma/\delta)$ steps is at most

$$p_t \leq p_0(1 - 1/yn)^t = p_0(1 - 1/yn)^{yn \ln(\gamma/\delta)} \leq p_0 \cdot e^{-\frac{1}{y}\cdot yn \ln(\gamma/\delta)} = p_0(\gamma/\delta)^{-1} = p_0(\delta/\gamma)$$

Applying Markov's inequality to the non-negative p value implies $p_t \leq p_0\delta$ with probability $1 - 1/\gamma$. Repeating independent phases of length $\lceil yn \ln(2/\delta) \rceil$ the expected number of phases is at most 2. Up until here the proof is the same.
Instead of choosing $W/2$ as the general upper bound for p_0 as in the original lemma here the lower value $W - w_1 \leq W/2$ is chosen because it is more tight for the special case $w_1 \geq W/2$ with $l = w_1$. The probability of the RLS_k^S to flip one specific bit is $\frac{1}{k} \cdot \frac{1}{n}$ and for (1+1) EA with mutation rate c/n at least

$$\frac{c}{n}(1 - \frac{c}{n})^{n-1} \geq \frac{c}{n}(1 - \frac{c}{n})^n \geq \frac{c}{n} e^{-c}(1 - \frac{c^2}{n}) = \frac{c}{e^c n}(1 - o(1))$$

The inequality $(1 + x/n)^n \geq e^x(1 - x^2/n)$ requires $n \geq 1, |c| \leq n$ which both hold. Setting $y = \frac{e^c n}{c\cdot(1-o(1))}$ for the (1+1) EA and $y = k$ for the RLS_k^S concludes the result. □

Lemma 3.2. *For instances with $w_1 > W/2$ the probability of flipping w_1 when $b_E = c\cdot\frac{W-w_1}{2}$ for $1 < c < 2$ holds, is at most $\frac{2y(y-1)^2}{n(n-1)(c-1)}$ in a step where any algorithm flips $2 \leq y \leq n/2$ bits.*

Proof. For a successful flip of w_1 after $b_E \geq \frac{W-w_1}{2}$ holds a total volume of at least $z \geq 2 \cdot (b_E - \frac{W-w_1}{2})$ must be shifted from b_E to b_F. Otherwise the step is rejected because

$$b'_F = b_E + w_1 - z > b_E + w_1 - 2 \cdot (b_E - \frac{W - w_1}{2}) = b_E + w_1 - 2b_E + W - w_1 = W - b_E = b_F$$

which results in an increase of the fitness ($b_F = f(x), b'_F = f(x')$). Let I be the set of indices of all elements moved from b_E to b_F and $w_{\max} = \max\{w_i | i \in I\}$. $|I| \leq y - 1$ holds because at least w_1 is moved from b_F to b_E. The sum of all element is at most $w_{\max} \cdot |I| \leq (y-1)w_{\max}$. If $w_{\max} < 2 \cdot (b_E - \frac{W-w_1}{2})/(y-1)$ then $(y-1)w_{\max} < 2 \cdot (b_E - \frac{W-w_1}{2})$ and the step is rejected. Thus at least one of the objects moved from b_E to b_F must have a volume of at least $2 \cdot (b_E - \frac{W-w_1}{2})/(y-1)$. At most $d \leq \frac{b_E}{w_{\max}}$ of these objects can be in b_E if they made up the complete volume of b_E. Simplifying this inequality leads to at most

$$d \leq \frac{b_E}{w_{\max}} \leq \frac{W - w_1}{w_{\max}} \leq \frac{W - w_1}{2(c\frac{W-w_1}{2} - \frac{W-w_1}{2})/(y-1)} = \frac{(W - w_1)(y-1)}{(W - w_1)(c-1)} = \frac{(y-1)}{(c-1)}$$

objects having at least a volume of $w_{\max}$. For a successful flip w_1 and at least one of these d objects must switch bins and the probability for such a step flipping y bits is therefore at most

$$\mathbb{P}(y \text{ bits are flipped}) \cdot \mathbb{P}(\text{the correct } y \text{ bits are flipped} | y \text{ bits are flipped})$$

$$\leq 1 \cdot \frac{\binom{1}{1} \cdot \binom{\lceil d \rceil}{1} \cdot \binom{n-2}{y-2}}{\binom{n}{y}} = \frac{\lceil d \rceil \frac{(n-2)!}{(n-2-y+2)! \cdot (y-2)!}}{\frac{n!}{(n-y)! \cdot y!}} = \frac{\lceil d \rceil \cdot (n-2)! \cdot (n-y)! \cdot y!}{n! \cdot (n-y)! \cdot (y-2)!} = \frac{\lceil d \rceil y(y-1)}{n(n-1)}$$

$$\leq \frac{y(y-1)}{n(n-1)} \cdot (\frac{y-1}{c-1} + 1) = \frac{y(y-1)}{n(n-1)} \cdot (\frac{y-1+c-1}{c-1}) \leq \frac{2y(y-1)^2}{n(n-1)(c-1)}$$

□

Theorem 3.3. *If $w_1 \geq \frac{W}{2}$ then the RLS and the (1+1) EA with mutation rate k/n with constant $0 < k < \sqrt{n}$ reach the optimal solution in expected time $\Theta(n \log n)$*

Proof. The optimal solution is putting w_1 in one bin and all other elements in the other bin. So the problem is almost identical to OneMax/ZeroMax. A single bit flip of the first bit can only happen, if the emptier bin has a weight of at most $\frac{W-w_1}{2}$. After this flip the weight of the emptier bin is at least $\frac{W-w_1}{2}$ and therefore another single bit flip of w_1 can onl The run of the RLS can be divided into two phases:

Phase 1: The RLS reaches a search point with $b_E > \frac{W-w_1}{2}$.

Phase 2: The RLS reaches an optimal solution $\Rightarrow$ w_1 is in one bin and all other elements are in the other bin.

The expected length of the first phase is at most $2n$ because the probability of flipping the first bit is $\frac{1}{n}$ and the expected time for such a step then is at most n. After such a step $b_E \geq \frac{W-w_1}{2}$ holds. If the solution is already optimal $b_E = W - w_1 > \frac{W-w_1}{2}$, otherwise there is at least one bit that can be flipped. This bit will be flipped in expected time at most n for the same reason as for w_1. The total length of first phase is at most $2n$. In the second phase the RLS can no longer flip w_1 as it does not result in an improvement ever again. Therefore the RLS behaves exactly as on OneMax/ZeroMax depending on the value of the first bit and reaches an optimal solution in $\Theta(n \log n)$ resulting in a total runtime of $\Theta(n \log n)$ (Theorem 3 in [Wit14]).
As long as w_1 does not flip the (1+1) EA has to minimize a linear function of $n-1$ bits which takes $(1 + o(1))\frac{e^k}{k} n \ln n$ time (Corollary 4.2 in [Wit13]). The only steps that could

hinder the algorithm from optimising the linear function in $\Theta(n \log n)$ would be a flip of the first bit. Such steps invert the optimal solution which could decrease the progress of minimising the linear function. If such a step has an expected time of $\omega(n \log n)$ the linear function is likely to be optimised in expectation before such a step happens.

The probability of the (1+1) EA to flip more than $k + \sqrt{6k\ln(n)} = k + k\sqrt{6\ln(n)/k}$ is limited by Chernoff bounds:

$$\begin{aligned}
&\mathbb{P}((1+1)\text{ EA flips more than } k + k\sqrt{6\ln(n)/k} \text{ bits})\\
&\leq \mathbb{P}(X \geq (1 + \sqrt{6\ln(n)/k}) \cdot k) \leq e^{-k\cdot\sqrt{6\ln(n)/k}^2/3} = e^{-k\cdot\frac{6\ln n}{3k}} = n^{-2}
\end{aligned}$$

So the expected time for such a step is at least $n^2 = \omega(n\ln(n))$. Now let's look at steps that flip at most $k + \sqrt{6k\ln(n)}$ bits in a single step. Such a step only successfully flips w_1 if both w_1 is flipped and enough total volume is shifted from b_E to b_F. Due to Lemma 3.1 with $\delta = \frac{1}{n}$ (for $n > 1$) the solution is at most $w_1 + \delta(W - w_1)$ after expected time

$$\begin{aligned}
2\lceil\frac{e^k}{k(1-o(1))}n\ln(2/\delta)\rceil &= 2\lceil\frac{e^k}{k(1-o(1))}n\ln(2/\frac{1}{n})\rceil = 2\lceil\frac{e^k}{k(1-o(1))}n\ln(2n)\rceil\\
&= 2\lceil\frac{e^k}{k(1-o(1))}n(\ln(n)+\ln(2))\rceil \leq \frac{2e^k}{k(1-o(1))}n\ln(n) + 4
\end{aligned}$$

The value of b_E is then at least $W - (w_1 + \delta(W - w_1)) = (1-\delta)(W - w_1) = (1 - \frac{1}{n})(W - w_1)$.

Lemma 3.2 states that the probability of a step flipping with w_1 together with $y - 1$ other bits is at most $\frac{2y(y-1)^2}{n(n-1)(c-1)}$. Applying the bound $y \leq k + \sqrt{6k\ln(n)}$ and the value $c = 2(1 - \frac{1}{n})$ this simplifies to

$$\frac{2y(y-1)^2}{n(n-1)(c-1)} \leq \frac{2(k+\sqrt{6k\ln(n)})^3}{n(n-1)(1-\frac{2}{n})} = \frac{2(k+\sqrt{6k\ln(n)})^3}{n(n-1)\frac{n-2}{n}} = \frac{2(k+\sqrt{6k\ln(n)})^3}{(n-1)(n-2)}$$

The probability of one of these steps to happen for any value of y is given by

$$\begin{aligned}
&\sum_{y=2}^{k+\sqrt{6k\ln(n)}} \mathbb{P}(y \text{ bits are flipped}) \cdot \mathbb{P}(\text{the correct } y \text{ bits are flipped} | y \text{ bits are flipped})\\
&\leq (k + \sqrt{6k\ln(n)}) \cdot \frac{2(k+\sqrt{6k\ln(n)})^3}{(n-2)(n-1)} = \frac{2(k+\sqrt{6k\ln(n)})^4}{(n-2)(n-1)}\\
&= \frac{2(o(n^{1/8}))^4}{(n-2)(n-1)} = \frac{o(n^{0.5})}{\mathcal{O}(n^2)} = \mathcal{O}(n^{-1.5})
\end{aligned}$$

The expected time for any step successfully flipping w_1 is then $\Omega(n^{1.5}) = \omega(n\ln n)$. Let T be the time until the linear function is optimised and p the probability of successfully flipping w_1. Then the probability that w_1 flips after expected time $\frac{2e^k}{k(1-o(1))}n\ln(n) + 4$ before the linear function is optimised is at most

$$p \cdot E(T) \leq \frac{1}{\mathcal{O}(n^{1.5})} \cdot (1+o(1))\frac{e^k}{k}n\ln n = \frac{(1+o(1))\frac{e^k}{k}\ln n}{\mathcal{O}(n^{0.5})} = \frac{o(n^{0.1})}{\mathcal{O}(n^{0.5})} = o(\frac{1}{n^{0.4}}) = o(1)$$

The probability of w_1 not being flipped after expected time $\frac{2e^k}{k(1-o(1))}n\ln(n)+4$ is $1-o(\frac{1}{n^{0.4}}) = 1-o(1)$. If such a step happens the fitness does not decrease and the bound on the probability of flipping w_1 still holds. The algorithm will still find the solution in expected time at most $(1+o(1))\frac{e^k}{k}n\ln n$. Since even after a flip all condition are still true the expected time of optimising the linear function after expected time $\frac{2e^k}{k(1-o(1))} n \ln(n) + 4$ is given by $\frac{1}{1-o(1)} \cdot (1+o(1)) \frac{e^k}{k} n \ln n = \Theta(n \log n)$

The total runtime for the (1+1) EA is $\frac{2e^k}{k(1-o(1))} n \ln(n) + 4 + \frac{1+o(1)}{1-o(1)} \cdot \frac{e^k}{k} n \ln n = \Theta(n \log n)$. ☐

Lemma 3.4. *If $b_F \leq \frac{2}{3} \cdot W$ the approximation ratio is at most $\frac{4}{3}$*

Proof. $\frac{b_F}{opt} \leq \frac{(2/3) \cdot W}{opt} \leq \frac{(2/3) \cdot W}{(1/2) \cdot W} = \frac{4}{3}$, since $opt \geq \frac{W}{2}$ ☐

Corollary 3.5. *If $w_1 \geq \frac{W}{3}$ and w_1 is in the emptier bin, then the approximation ratio is at most $\frac{4}{3}$*

Proof. w_1 is in the emptier bin, so $b_F \leq W - w_1 \leq W - \frac{W}{3} = \frac{2W}{3}$ and with Lemma 3.4 the assumption follows. ☐

Lemma 3.6. *Any object of weight v can be moved from b_F to b_E if and only if $b_F - b_E \geq v$*

Proof. $'' \Leftarrow''$:
$b_F - b_E \geq v \Leftrightarrow b_F \geq b_E + v$, so after moving an object with weight v from b_F to b_E, the new weight of b_E is at most the weight of b_F before moving the object, thus the RSH accepts the step.
$'' \Rightarrow''$:
$b_F - b_E < v \Leftrightarrow b_F < b_E + v$, so moving an object of weight v results in ${b_F}' = b_E + v > b_F$ which results in the step being rejected. ☐

Corollary 3.7. *The RLS is stuck in a local optimum if $b_F - b_E \leq w_n$ holds and $b_F > opt$.*

Proof. A single bit flip of weight v can only happen if $b_F - b_E \geq v$ (Corollary 3.7). If $b_F - b_E < w_n$ there is no weight which satisfies the condition and therefore no single bit flip is possible. If $b_F - b_E = w_n$ then only objects with weight w_n can be flipped, but those do not change the fitness ($b'_F = b_E + w_n = b_F - w_n + w_n = b_F$). Since the RLS can only move one bit at a time and only if it results in an improvement, the RLS is stuck. ☐

Corollary 3.8. *Every object $\leq \frac{W}{3}$ can be moved from b_F to b_E if $b_F \geq \frac{2W}{3}$*

Proof. $b_F \geq \frac{2W}{3} \Rightarrow b_E \leq W - \frac{2W}{3} \leq \frac{W}{3} \Rightarrow b_F - b_E \geq \frac{2W}{3} - \frac{W}{3} = \frac{W}{3}$ and with Lemma 3.6 the assumption follows. ☐

Lemma 3.9. *In expected time $\mathcal{O}(n \log n)$ the weight of the fuller bin can be decreased to $\leq \frac{2W}{3}$ by the RLS and the (1+1) EA if every object besides the biggest in the fuller bin is at most $\frac{W}{3}$ and $w_1 \leq \frac{W}{2}$.*

Proof. In expected time $\mathcal{O}(n \log n)$ the RSH can move every object $\leq \frac{W}{3}$ to the emptier bin as long as $b_F \geq \frac{2W}{3}$ due to Corollary 3.8 and Theorem 3.3. So in expected time $\mathcal{O}(n \log n)$ the solution can be shifted to w_1 being in one bin and all other objects in the other bin. The RSH will only stop moving the elements if the condition $b_F \geq \frac{2W}{3}$ is no longer satisfied (Corollary 3.8). If $w_1 \geq \frac{W}{3}$ and every object was moved to the bin without w_1, then $b_F = \max\{W - w_1, w_1\} = W - w_1 \leq \frac{2W}{3}$, because $w_1 \leq \frac{W}{2}$. So either the RSH moves all

objects to the emptier bin or stops moving objects because $b_F < \frac{2W}{3}$ both resulting in $b_F \leq \frac{2W}{3}$. If w_1 is not in the fuller bin, then the result follows by Corollary 3.5.
Now assume $w_1 < \frac{W}{3}$. In this case the RLS will move one object per step to the emptier bin. Each object has weight $< \frac{W}{3}$ and therefore one step cannot decrease the weight of the fuller bin from $> \frac{2W}{3}$ to $\leq \frac{W}{3}$. If all objects except one where moved to one bin, the other bin would have a weight of at least $W - w_1 > \frac{2W}{3}$. Therefore the RLS will find a solution with $b_F < \frac{2W}{3}$ before moving all elements from the first to the second bin.
The proof for the (1+1) EA is mostly the same. The main difference is the (1+1) EA being able to flip more than one bit in a single step. Such a step could make the emptier bin the fuller bin or increase the number of bits that must be shifted to the emptier bin. But with the results of Theorem 3.3 the proof works exactly the same as for the RLS. The case $w_1 \geq \frac{W}{3}$ does not change only the bin containing w_1 might change. Apart from that there is no difference for the (1+1) EA. The case $w_1 < \frac{W}{3}$ is also rather similar. The (1+1) EA will move elements from the fuller bin to the emptier bin until $b_F < \frac{2W}{3}$ holds. The (1+1) EA can make the emptier bin the fuller bin by moving multiple objects in one step, but this does not hinder it from reaching $b_F < \frac{2W}{3}$. After such a step it will continue moving elements until the condition holds. □

Lemma 3.10. *The RLS and the (1+1) EA reach an approximation ratio of at most $\frac{4}{3}$ in expected time $\mathcal{O}(n \log n)$ if $w_1 < W/2$*

Proof. If $w_1 + w_2 > \frac{2W}{3}$ after time $\mathcal{O}(n)$ w_1 and w_2 are separated and will remain separated afterwards (3. Average case analysis, Theorem 1 in [Wit05]). From then on the following holds. If w_1 is in the emptier bin, then the result follows directly by Corollary 3.5. Otherwise all elements in the fuller bin except w_1 have a weight of at most $\frac{1}{3}$ and therefore the result follows by Lemma 3.9 and Lemma 3.4. If $w_1 + w_2 \leq \frac{2W}{3}$ the result follows directly by Lemma 3.9 and Lemma 3.4. □

Corollary 3.11. *The RLS and the (1+1) EA reach an approximation ratio of at most $\frac{4}{3}$ for every input in expected time $\mathcal{O}(n \log n)$*

Proof. This follows directly from Theorem 3.3 and Lemma 3.10. □

3.2. Binomial distributed input

Lemma 3.12. *A binomial distributed input ~$B(m, p)$ has a perfect partition ($b_F - b_E = 0$ for even W and $b_F - b_E = 1$ for uneven W) with high probability if the input size n is large enough.*

Proof. Sketch:

- The initial distribution is likely rather close to the optimum
- The difference between the bins is probably not more than 10 expected values
- the large values

Consider a random separation of all values into two sets with equal size if n is even or one set with one value more than the other if n is odd. The sum X of one set is a sum of $\frac{n}{2} \cdot m$ independent Bernoulli trials with probability p. With Chernoff Bounds the following inequality follows:

$$\mathbb{P}(X \geq (\frac{n}{2} + \sqrt{\frac{n}{2}}) \cdot m \cdot p) = \mathbb{P}(X \geq (1 + 2\sqrt{\frac{2}{n}}) \cdot \frac{nmp}{2}) \leq e^{-\frac{mnp}{2} \cdot 2\sqrt{\frac{2}{n}}^2/3} = e^{-\frac{2mp}{3}}$$

For $mp \geq 1.5$ the probability is less than $\frac{1}{e}$. Otherwise the input is rather trivial, since the numbers will be concentrated around $mp \leq 1.5$ and most values will be below 10.
After moving $\mathcal{O}(\sqrt{\frac{n}{2}}/2)$ objects to the emptier set, the difference between the two sets is at most half the expected value mp of a single value. ... □

Lemma 3.13. *With high probability the RLS does not find a perfect partition for an input with distribution* ~$B(m,p)$ *if n is large enough,* $mp \geq 50$ *and* $m - mp \geq 50$*.*

Proof. The RLS will always move one object per step from fuller to the emptier bin. As long as $b_F - b_E > w_i$ holds moving any object of weight at most w_i from b_F to b_E results in a decrease of the fitness ($b'_F = b_E + w_i < b_F - w_i + w_i = b_F$). If the RLS does not decrease the difference $b_F - b_E$ to at most w_n in any step the solution is never optimal and the RLS therefore must be stuck. So now assume the RLS eventually reaches a search point with $b_F - b_E \leq w_n$. Consider the step which decreases the difference from more than w_n to at most w_n. If this step does not decrease the difference to 0 for even n or 1 for odd n the RLS is stuck due to Corollary 3.7. For a step to decrease $b_F - b_E$ from more than w_n to 0 the RLS must move an object of weight y which is given by $b'_F - b'_E = (b_F - y) - (b_E + y) = 0 \Leftrightarrow b_F - b_E - 2y = 0 \Leftrightarrow y = (b_F - b_E)/2$. Such an element can only exist for even n because only for even n either both bins have an even sum or neither of them. Odd inputs will always have exactly one bin with an even sum and one with an odd sum. For odd n a difference of 1 suffices for a perfect partition and therefore the value of y must be either $\lceil (b_F - b_E)/2 \rceil$ or $\lfloor (b_F - b_E)/2 \rfloor$. For any binomial distribution the value which occurs the most in expectation is the expected value if $mp \in \mathbb{N}$ or $\lceil mp \rceil$ and $\lfloor mp \rfloor$ otherwise. Even if the behaviour of the algorithm forces values smaller or bigger than mp in the emptier bin, the number of bigger/smaller values in the fuller bin will still be less in expectation than the amount of mp in both bins. Let's assume that objects with exactly this volume must be selected and that there are two options because $mp \notin \mathbb{N}$. This will give an upper bound on the probability of flipping the right object in the crucial step. The probability of a number to be $\lfloor mp \rfloor$ is $p^{\lfloor mp \rfloor}(1-p)^{m - \lfloor mp \rfloor} \leq 0.5^{50} = \frac{1}{1024^5} < 10^{-15}$ because both $mp \geq 50$ and $m - mp \geq 50$ and $p \leq 0.5$ or $1 - p \leq 0.5$. For $\lceil mp \rceil$ the same bound applies with the same arguments. The probability of either of these numbers to be drawn from the binomial distribution is at most $2 \cdot 0.5^{50} = 0.5^{49} \leq 10^{-15}$. This leads to $n \cdot 10^{-15}$ values with the right value in expectation. All bits are flipped uniformly at random and therefore the probability of flipping a good bit is at most $\frac{n \cdot 10^{-15}}{|b_F|} \leq \frac{n \cdot 10^{-15}}{n} = 10^{-15}$.

This means the RLS is very unlikely to find a perfect partition. Given the solution has a perfect partition the RLS will get stuck in a local optimum.

Note: The restriction of $mp \geq 50$ and $m - mp \geq 50$ was not chosen to make the proof work but is also necessary. If the algorithm produces many small values and especially elements close to 1. When there are many small values these can be used to fill the small gaps which makes it easy for the RLS to find a perfect partition. On the other hand if p is almost one, then almost every element will be same for smaller values of m. If every element is the same then the RLS must only find a search point with equal amounts of 0s and 1s. □

4. Higher mutation rates and heavy tailed mutations

In this chapter different ways to change the mutation rate for EAs are discussed so that they flip more bits in expectation. For OneMax and all other linear functions the mutation rate of $p_m = 1/n$ was proven to be optimal[Wit13]. This is not the case for every fitness function. Jump_k has a optimal mutation rate of k/n and a small constant factor deviation from k/n results in an increase of the runtime exponential in $\Omega(k)$[DLMN17].

4.1. Algorithms

For the (1+1) EA changing the expected amount of flipped bits per step can be done easily. By changing the mutation rate $1/n$ to c/n for any constant c the algorithm now flips $n \cdot c/n = c$ bits in expectation.

For the RLS it is not that simple, as the RLS chooses a random bit and flips it. Instead of flipping c bits in every step there should be the possibility to flip different amounts of bits in every step. The standard RLS chooses a random neighbour with Theorem 3.3 one. So the variant of the RLS could simply choose neighbours that have a Theorem 3.3 larger than one. The selection should still be uniform random to keep the idea of the RLS intact. One possible way is to choose a random neighbour with Theorem 3.3 $\leq k$. The amount of neighbours with Theorem 3.3 y is given by $\binom{n}{y}$. For $k = 4$, this results in n neighbours with Theorem 3.3 1, $n(n-1)/2$ neighbours with Theorem 3.3 2, $n(n-1)(n-2)/6$ and

Algorithm 4.1: (1+1) EA WITH STATIC MUTATION RATE

```
choose x uniform from {0,1}^n
while x not optimal do
    x' ← x
    flip every bit of x' with probability c/n
    if f(x') ≤ f(x) then
        x ← x'
```

Algorithm 4.2: RLS_k^B

choose x uniform from $\{0,1\}^n$
while x *not optimal* **do**
  $x' \leftarrow$ uniform random neighbour of x with Theorem 3.3 $\leq k$
  if $f(x') \leq f(x)$ **then**
    $x \leftarrow x'$

Algorithm 4.3: RLS_k^S

choose x uniform from $\{0,1\}^n$
while x *not optimal* **do**
  $y \leftarrow$ uniform random value $\in \{1,\dots,k\}$
  $x' \leftarrow$ uniform random neighbour of x with Hamming Distance y
  if $f(x') \leq f(x)$ **then**
    $x \leftarrow x'$

$n(n-1)(n-2)(n-3)/24$. The probability to choose a random neighbour with Theorem 3.3 $y \leq k$ for $k = \mathcal{O}(1)$ is given by

$$P(\mathrm{RLS}_k^B{}_k \text{ flips } y \text{ bits}) = \frac{\binom{n}{y}}{\sum_{i=1}^{k}\binom{n}{i}} = \frac{\Theta(n^y)}{\sum_{i=1}^{k}\Theta(n^i)} = \frac{\Theta(n^y)}{\Theta(n^k)} = \Theta(n^{y-k}) = \Theta(\frac{1}{n^{k-y}})$$

This variant of the RLS is likely to choose a neighbour with Theorem 3.3 k as the number of neighbours with hamming distance k rises with k for $k \leq n/2$. The probability of flipping only one bit is $\mathcal{O}(\frac{1}{n^{k-1}})$. For some inputs flipping only one bit might be more optimal which is rather unlikely for this variant of the RLS which will be called RLS_k^B from now on.

An alternative way of changing the RLS is to first choose $y \in \{1,\dots,k\}$ uniform random and then choose a neighbour with Theorem 3.3 y uniform random. Here the probability of flipping $y \leq k$ bits is given by $1/k$, so the algorithm is much more likely to choose to flip only one bit. This variant of the RLS will be referred to as RLS_k^S.

Both variants of the RLS change at most k bits in each step and therefore only a constant amount of bits. For the (1+1) EA the algorithm will also flip mostly $\mathcal{O}(c)$ bits which is also constant. So neither of the new variants is likely to change up to $\mathcal{O}(n)$ bits. Quinzan *et al.* therefore introduced another mutation operator in [FGQW18] called $pmut_\beta$. This operator chooses k from a powerlaw distribution D_n^β with exponent β and maximum value n and then k uniform random bits are flipped. This algorithm will mostly flip a small number of bits but occasionally up to n bits. Distributions like this are called heavy tailed mutations because their tail is not bounded exponentially.

4.2. Runtime Analysis of higher mutation rates

The first analysed input is again the input with $w_1 \geq W/2$. This input is very close to a weighted OneMax and therefore more or less the OneMax equivalent for Partition. The first analysed algorithm will be the RLS variants.

Corollary 4.1. *The expected time until the RLS_k^S and RLS_k^B flips $w_1 \geq W/2$ after they reached a solution with $b_E = c \cdot \frac{W-w_1}{2}$ with $1 < c < 2$ is at least $\frac{n(n-1)(c-1)}{2k^2(k-1)^2}$. For constant values of c the expected time is $= \Omega(n^2)$.*

Proof. Using Lemma 3.2, the upper bound of $y \leq k$ and the fact that both algorithms flip at most k bits leads to the first part

$$(k \cdot \frac{2y(y-1)^2}{n(n-1)(c-1)})^{-1} = \frac{n(n-1)(c-1)}{2ky(y-1)^2} \leq \frac{n(n-1)(c-1)}{2k^2(k-1)^2}$$

k is constant and if c is constant too, this leads to expected time $= \Omega(n^2)$ for a flip of the first bit if $b_E = c \cdot \frac{W-w_1}{2}$ with $1 < c < 2$ which concludes the second statement. □

Lemma 4.2. *The RLS$_k^S$ reaches the optimal solution on an input with $w_1 \geq W/2$ in expected time $\mathcal{O}(n \log n)$*

Proof. This proof is similar to proof for the (1+1) EA in Theorem 3.3 and is divided in the same two parts. The first part is proving the RLS$_k^S$ does not flip the first bit in expected time $\mathcal{O}(n \log n)$ after some time T has passed. After time T the algorithm then minimises the linear function in expected time $\mathcal{O}(n \log n)$ before the first bit is flipped and the progress of the linear function might be reset. After expected time $(2\lceil kn \ln(2/0.4)\rceil) = 2\lceil kn \ln(5)\rceil \leq 2\lceil 1.61 \cdot kn\rceil \leq 4kn$ the RLS$_k^S$ reaches a solution of $f(x) \leq w_1 + 0.4(W - w_1)$ (Lemma 3.1). This means $b_E \geq 0.6(W - w_1) = 1.2 \cdot \frac{W-w_1}{2}$. With Lemma 4.1 the expected time of at least $\frac{n(n-1)(c-1)}{k^2(k-1)^2} \frac{n(n-1)}{5k^2(k-1)^2} = \Omega(n^2)$ for a successful flip of w_1 follows. This concludes the first part
The RLS$_k^S$ can be seen as an unbiased unary black box algorithm with a sequence $\mathcal{D} = (p_1, \ldots, p_n)$ where $p_i = 1/k$ for $1 \leq i \leq k$ and $p_i = 0$ otherwise. The mean of this sequence is $\mathcal{X} = \sum_{i=1}^{n} \mathrm{I\!P}(X = i)i = \sum_{i=1}^{k} \frac{i}{k} = \frac{1}{k} \cdot \sum_{i=1}^{k} i = \frac{k+1}{2}$. For any constant value k both the probability $p_1 = \frac{1}{k} = \Theta(1)$ and mean $\mathcal{X} = \frac{k+1}{2} = O(1)$ meet the conditions of Theorem 1 in [DJL23] and therefore the RLS$_k^S$ optimises the linear function in expected time $(1+o(1))\frac{1}{p_1} n \ln n = (1+o(1))kn \ln n$. Applying the same arguments as in Theorem 3.3 this results in a probability of not flipping w_1 after expected time $4kn$ of at least

$$1 - \frac{(1+o(1))kn \ln n \cdot 5k^2(k-1)^2}{n(n-1)} = 1 - \frac{(1+o(1))5k^5 \ln n}{n-1} = 1 - \frac{o(k^5 n^{0.5})}{n-1} = 1 - \frac{1}{o(\sqrt{n})}$$

The expected time of minimising the linear function then is

$$\frac{1}{1 - \frac{1}{o(\sqrt{n})}} \cdot (1+o(1))kn \ln n = \frac{1}{1-o(1)} \cdot (1+o(1))kn \ln n = \Theta(n \log n)$$

The total expected runtime therefore is at most $4kn + \frac{1+o(1)}{1-o(1)} \cdot kn \ln n = \Theta(n \log n)$.

□

5. Experimental Results

In the following chapter the different variants of the RLS and the (1+1) EA are now analysed empirically for the best algorithm depending on the input. Additionally for most lemmas from the previous chapters there are also tests if they actually hold in practice.

5.1. Code

The complete java code used for all empirical studies is available on GitHub.

5.1.1. The Algorithms

All different variants of the RLS function more or the less the same. They start with an initial random value and then optimise this one value in the loop. The loop can be summarised like this:

1. generate a number k of bits to be flipped (algorithm specific)
2. flip k random bits
3. evaluate fitness of the mutated individual
4. replace old value with new value if new value is better
5. repeat if not optimal

The (1+1) EA variants behave differently at first glance as they flip bit each bit independently with probability c/n. This can be seen as n independent Bernoulli trials with probability c/n. The amount of bits that are flipped is therefore binomial distributed and the algorithm can be implemented exactly as the versions of the RLS. The same holds for the *pmut* operator which generates a number k from a powerlaw distribution and then flips k bits. This leads to only one implementation of a partition solving algorithm which is not only given the input array of numbers but also a generator for the amount of bits to be flipped in each step. The random values for the amount of bits to be flipped are generated according to this table:

Algorithm	Returned value
RLS	1
RLS_k^B	$y \in \{1, \ldots, k\}$ with probability $\frac{\binom{n}{y}}{\sum_{i=1}^{k} \binom{n}{i}}$
RLS_k^S	uniform random value $y \in \{1, \ldots, k\}$
(1+1) EA	binomial distributed value from $\sim B(n, c/n)$
pmut	powerlaw distributed value from $\sim D_n^\beta$

Algorithm 5.1: GenericPartitionSolver

choose x uniform random from $\{0,1\}^n$
while x *not optimal* **do**
    $x' \leftarrow x$
    $k \leftarrow$ kGenerator.generate()
    flip k uniform random bits of x'
    if $f(x') \leq f(x)$ **then**
        $x \leftarrow x'$

5.1.2. Random number generation

Java only provides a random number generator for uniform distributed values for any integer interval or random double values $\in [0, 1)$. For this project this does not suffice as for an efficient way of implementing the (1+1) EA or simply for generating a binomial distributed input another random number generator is needed. One of the needed distributions is a binomial distribution. The simplest way to generate a number ~$B(m, p)$ would be to run a loop m times and add 1 to the generated number if a uniform random value $\in [0, 1)$ is less than p. This works perfectly fine and generates numbers according to the distribution. With low values for p this approach is rather inefficient and especially for values of $p = 1/m$. The expected value in this case is 1 but generating a random number takes time $\mathcal{O}(m)$. Another more efficient way was implemented by StackOverflow user pjs on stackoverflow inspired by Devroyes method introduced in [Dev06]. This method has an expected running time of $\mathcal{O}(mp)$ which is equal to the expected value of the distribution. For the case of $p = 1/m$ this runs in expected constant time in comparison to $\mathcal{O}(m)$ for the naive way. This number generation was also used for the implementation of the (1+1) EA instead of running a loop in every step. Algorithm 5.2 uses the second waiting time method which uses the fact that X~$B(m, p)$ if X is the smallest integer so that $\sum_{i=1}^{X+1} \frac{E_i}{n-i+1} < -\log(1-p)$ for E_i iid exponential random variables (Lemma 4.5 section X.4 [Dev06]).

Algorithm 5.2: Binomial random number generator

$q \leftarrow \ln(1.0 - p)$
$x \leftarrow 0$
$sum \leftarrow 0$
while *true* **do**
    $sum \leftarrow sum + \ln(\text{random}())/(n - x)$
    `// random() generates a random value` $\in [0, 1)$
    if $sum < q$ **then**
        return x
    $x \leftarrow x + 1$

The next generator needed is for geometric distributed values. This generator is only necessary for the generation of geometric distributed inputs but not for the algorithms. The easiest way to generate geometric distributed values is the naive way: generating a uniform random value p' until $p' < p$ holds. The expected running time of this algorithm is equal to the expected value of the distribution $1/p$. So this method is comparably effective to the approach used for binomial random number generation.

The last generator needed is for powerlaw distributed values. This generator is in contrast to the geometric number generator needed for both the algorithm with the $pmut_\beta$ mutation

Algorithm 5.3: Geometric random number generator

```
1 sum ← 0
  // random() generates a random value ∈ [0,1)
2 while random() ≥ q do
3     sum ← sum + 1
4 return sum
```

operator and for generating inputs. This implementation is also from stackoverflow. The user gnovice provided the following formula on this page on stackoverflow:

$$x = [(b^{n+1} - a^{n+1}) * y + a^{n+1}]^{1/(n+1)}$$

a is the lower bound, b the upper bound, n the parameter of the distribution and y the number generated uniform random $\in [0, 1)$. The idea behind the formula and the formula itself is explained in a mathworld page. For a powerlaw distribution $P(x) = Cx^n$ for $x \in [a, b]$ normalisation gives

$$\int_a^b P(x)dx = C\frac{[x^{n+1}]_a^b}{n+1} = 1 \Leftrightarrow C = \frac{n+1}{b^{n+1} - a^{n+1}}$$

Let Y be a uniformly random distributed variate on [0,1]. Then

$$D(x) = \int_a^x P(x')dx' = C\int_a^x x'^n dx' = \frac{C}{n+1}(x^{n+1} - a^{n+1}) = \frac{(x^{n+1} - a^{n+1})}{b^{n+1} - a^{n+1}} \equiv y$$

and the variate is given by

$$X = [(b^{n+1} - a^{n+1}) * y + a^{n+1}]^{1/(n+1)}$$

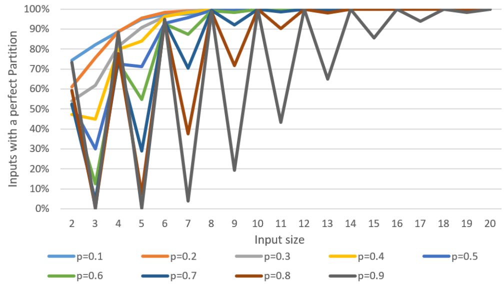
5.2. Do inputs have perfect partitions?

5.2.1. Binomial Inputs

Lemma 3.12 is only valid for larger n. In practice the bound is much smaller depending on the expected value of a single number. Another factor deciding how likely an input is to have a perfect partition is whether n is even or odd. To determine the influence of all factors multiple experiments were conducted. The goal of the first experiment was to determine the influence of the array size to the input having a perfect partition and the fact if n is even or odd. So for every possible combination of $p \in \{0.1, 0.2, \dots, 0.8, 0.9\}, m \in \{10, 100, 1000, 10^4, 10^5\}$ and $n \in \{2, 3, 4, \dots, 19, 20\}$ 1000 randomly generated inputs of size n were tested for a perfect partition. Due to the small values for n it was possible to brute force the results in a short amount of time. The results are visualised in figure 5.1 to figure 5.9.

On the x-axis is the size of the input and on the y-axis the percentage of inputs that had a perfect partition. The different graphs in each figure resemble the different values of p used for generating the inputs. The graph for $p = 0.1$ resembles the percentage of inputs that had a perfect partition with values generated from the distribution ~$B(m, 0.1)$ with m being dependent on the figure. For figure 5.1 m has the value 10.
Figure 5.1 is a bit overloaded with information and the zigzag makes it hard to gain any benefit from the graphs. That's why for $m \in \{10, 100, 1000\}$ there is one figure for the even input sizes of n and one for the odd. Figures which show only results for either even or odd values of n have dotted graphs, because the values in between the points do not

Figure 5.1.: Percentage of Binomial inputs with perfect partitions for $m = 10$



exist. The dotted lines are only in the figure for a better visualisation of the trend and not meant for interpretation apart from the marked values. For $n \geq 10,000$ all values for the odd input sizes are 0 %, so there is no point in showing the data in a separate figure.

It is easy to see that for small inputs sizes it is relevant if n is even or odd for higher expected values as all curves in figure 5.9 oscillate between 0% and 100% for $n \geq 14$. For odd inputs the probability of a perfect partition decreases much more drastically with m as for even inputs because the expected value of a single number increases with m. If all values are much higher the small differences between the values can no longer even out the fact of one set having more elements than the other. The oscillation therefore increases with increasing m. For $n = 20$ all 1000 inputs had a perfect partition for every combination of p and m but for $n = 19$ only combinations where $mp \leq 300$ holds had at least one input with a perfect partition. For expected values of up to 10^5 it seems to be almost granted that an input of length 20 has a perfect partition if it is binomial distributed. Even for only 12 binomial generated values more than 50% of the inputs had a perfect partition (see figure 5.9). Another visible effect is the decreasing percentage with rising p. This may be a direct result of the value chosen for p but can also be an indirect result as the value for p changes the expected value for a constant m. The expected value may have an influence on the number of perfect partitions because it influences the highest value of the input. For uniform distributed inputs Borgs *et. al.* showed that the coefficient of number of bits needed to encode the max value/n has a huge impact on the number of perfect partitions [BCP01]. For a coefficient < 1 the probability of a perfect partition tends to 1 and for a coefficient > 1 it tends to 0. This was only proven for the uniform distributed input, but it might also hold for a binomial distributed input. This leads to the second experiment.

Figure 5.2.: Percentage of Binomial inputs with perfect partitions for $m = 10$ for even n

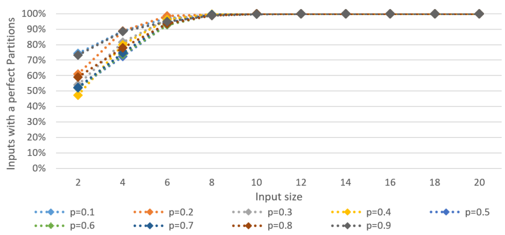
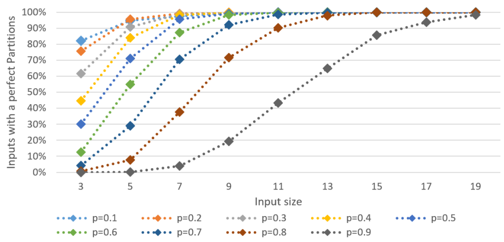


Figure 5.3.: Percentage of Binomial inputs with perfect partitions for $m = 10$ for odd n



In the second experiment the inputs were generated a bit differently. Here the goal was to keep the expected value fixed for any combination of p and n and set the value of m to e/p for all $e \in \{10, 20, 30, 40, 50, 100, 200, 500, 1000, 2000, 5000, 10000, 50000\}$ so that $E(X) = mp = e/p \cdot p = e$. With this setup the influence of the expected value is almost

Figure 5.4.: Percentage of Binomial inputs with perfect partitions for $m = 100$ for even n

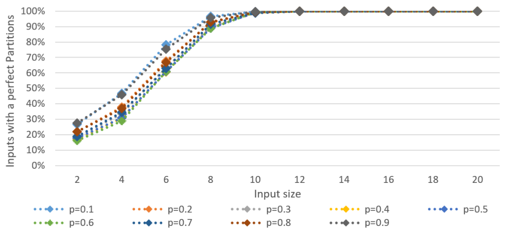


Figure 5.5.: Percentage of Binomial inputs with perfect partitions for $m = 100$ for odd n

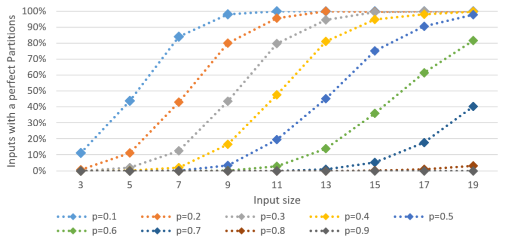


Figure 5.6.: Percentage of Binomial inputs with perfect partitions for $m = 1000$ for even n

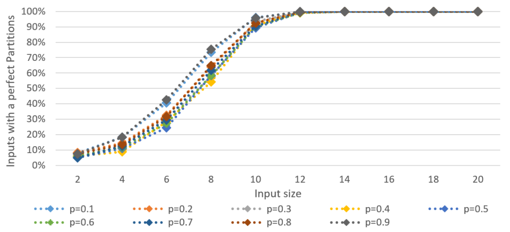
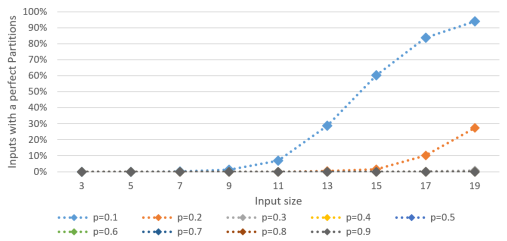


Figure 5.7.: Percentage of Binomial inputs with perfect partitions for $m = 1000$ for odd n



isolated from the other parameters. The probability is still linked to p as p also influences the variance $mp(1-p)$.
Figure 5.10 to figure 5.13 again show the percentage of perfect inputs with different settings of m, p, n. The x-axis is the expected value mp of a single number of the input. The different graphs show the percentage for different input sizes. It seems as if the value of p has a much smaller influence than the expected value. For a fixed expected value and a fixed input size a higher value for p seems to only slightly increase the percentage of inputs with a perfect partition. The expected value influences the percentage significantly more. For $p = 0.1, n = 14$ the value decreases from 100% at $E(X) = 10$ to below 20% at $E(X) = 50000$ (figure 5.10). For $p = 0.9$ the percentage only drops below 50% but still decreases by a factor of 2 (figure 5.13).

The last experiment showed that for $n = 20$, 1000/1000 inputs had a perfect partition. This raised the question of how the amount of perfect partition changes with changing values for m, p, n. Figure 5.14 to figure 5.17 show the amount of perfect partitions a binomial distribution $\sim B(m, p)$ has. For these figures 10,000 random binomial inputs with the given values for m and p were generated. Each input was then tested for the number of perfect partitions it has. The used method was again brute force to ensure correctness which was only possible due to the small input sizes. After all runs the average values were combined in the given figures. The value of p is dependent on the picture and each value of $m \in \{10, 100, 1000, 10000\}$ has its own graph within the figure. The x-axis is the size of the input and the y-axis the number of perfect partitions the input has. Notice that all graphs have a y-axis with a logarithmic scale. Since the graphs are all linear the actual values rise exponentially. The number of perfect partitions is mostly multiplied by a factor between 3 and 4 when the input size increases by 2.

The higher the value of m the closer the curves of p and $1-p$ get. For $m = 1000$ and $m = 10000$ the values are almost the same for every input size. For $p = 0.1, m = 10$ an input with expected values of 1 seems to much more likely to have a perfect partition than an input with expected value $10 \cdot 0.9 = 9$. With growing m this has less impact.

Figure 5.8.: Percentage of Binomial inputs with perfect partitions for $m = 10,000$

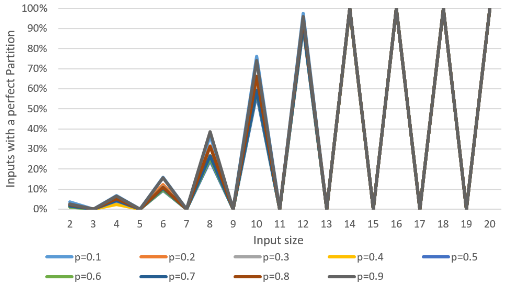


Figure 5.9.: Percentage of Binomial inputs with perfect partitions for $m = 100,000$

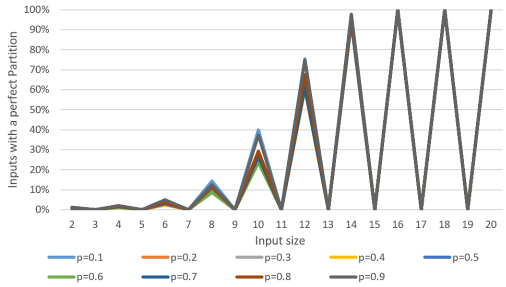


Figure 5.10.: Percentage of Binomial inputs with perfect partitions for p = 0.1

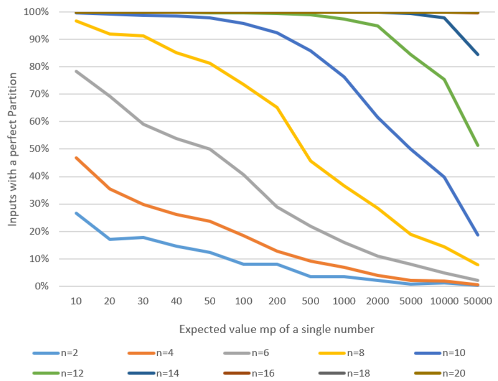
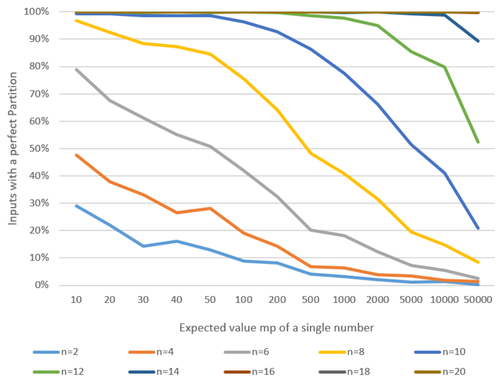


Figure 5.11.: Percentage of Binomial inputs with perfect partitions for p = 0.2



So the binomial input should be easy to solve due to the exponential number of perfect partitions. It might be harder for the smaller values of n as there are only a few perfect partitions. Due to the small number of total possibilities it should still be easy to solve for the small values of n as long as the RSH is not stuck in a local optimum. The number of iterations might be high in terms of the big-O notation but should still be small in the absolute value.

5.3. Binomial distributed values

In the following subsections the performance of the different algorithms is tested for different kinds of inputs. The exact distributions of the input are explained separately in each subsection. The procedure for each comparison is always the same. A random input is generated according to the distribution and then solved by every algorithm. All algorithms had the same two stopping conditions. The first was reaching a perfect partition and the second was taking more than $10 \cdot n \ln(n)$ steps or $100 \cdot n \ln(n)$ in some cases. For the lower values of n the step limit of 100,000 was used instead. For $n = 20$ giving the algorithm only 600 steps is rather small. In some cases the smaller inputs are even more difficult to solve. Most modern computer should be able to handle 100,000 iterations in a short amount of time anyway. So the minimum step limit of 100,000 seemed reasonable. If either of these conditions was met, the algorithm returned its current best solution. This step is repeated 1000 times. The results are presented in a table containing multiple statistics for each algorithm over all 1000 runs. The data is explained in the table below.

Figure 5.12.: Percentage of Binomial inputs with perfect partitions for p = 0.5

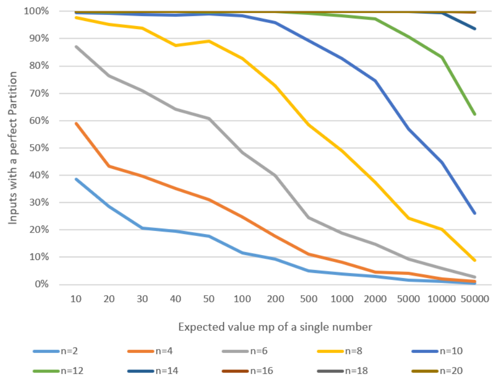


Figure 5.13.: Percentage of Binomial inputs with perfect partitions for p = 0.9

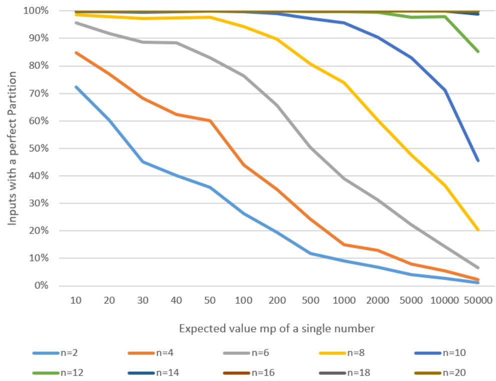


Figure 5.14.: Amount of perfect partitions for $p = 0.1$

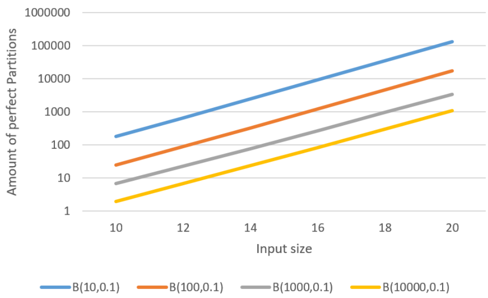
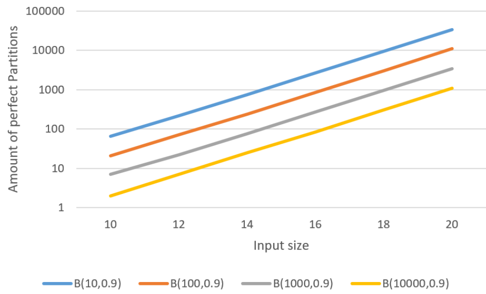


Figure 5.15.: Amount of perfect partitions for $p = 0.9$



column name	meaning
algo type	type of algorithm (RLS, RLS_k^B, RLS_k^S, (1+1)EA or pmut)
algo param	parameter of the algorithm or '-' if it is the standard variant
avg mut/change	average #bits flipped for iterations leading to an improvement
avg mut/step	average #bits flipped for any iteration
total avg count	average #iterations for all runs
avg eval count	average #iterations of runs returning an optimal solution
max eval count	maximum #iterations of runs returning an optimal solution
min eval count	minimum #iterations of runs returning an optimal solution
fails	number of runs that did not find an optimal solution
fail ratio	ratio of unsuccessful runs to all runs
avg fail dif	average value of $b_F - f(opt)$ for non-optimal solutions

Firstly the different variants of the RLS are compared with values of $k \in \{2, 3, 4\}$, then the performance of the (1+1) EA with static mutation rate c/n with $c \in \{1, 2, 3, 5, 10, 50, 100\}$ and lastly the performance of the $pmut_\beta$ mutation operator with the parameter $\beta \in \{1.25, 1.5, \ldots, 2.75, 3.0, 3.25\}$. Additionally the best variants of each algorithm are compared in another 1000 runs. Afterwards there is also a comparison for multiple input sizes of the best algorithms because the best algorithm is often dependent on the size of the input. Normally there are three tables for each input. The first states how often the algorithms did not find an optimal solution for the different input sizes ('fails' in top left cell). The second gives their average performance for the successful runs ('avg' in top left cell) and the last the performance for all runs ('total avg' in top left cell). The last two tables differ in the unsuccessful runs. Often the algorithm is stuck in a local optima it won't leave

Figure 5.16.: Amount of perfect partitions for $p = 0.2$

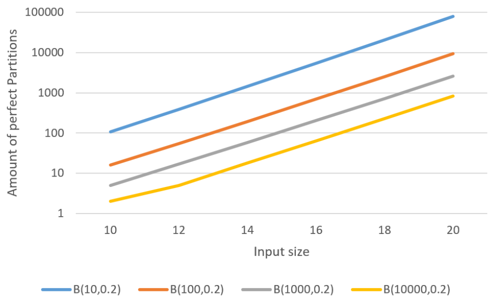
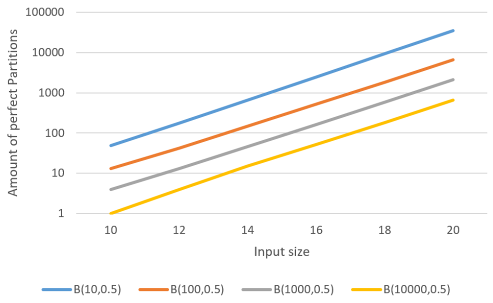


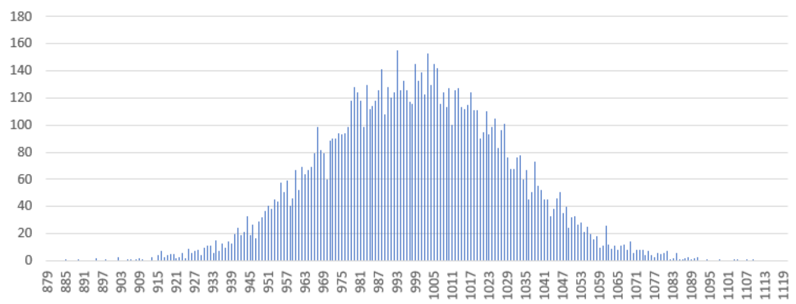
Figure 5.17.: Amount of perfect partitions for $p = 0.5$



in reasonable time or never for variants of the RLS. In these cases the step limit is the deciding factor on how big the penality for this run is. So neither of the two average values alone is enough to give a complete insight on the performance. Sometimes a variant of the RLS is much faster than the other algorithms for a specific input but is also the only algorithm to get stuck in a local optimum. This creates the possibility to start the RLS variant with a low step limit and switch to the (1+1) EA if the RLS variant does not return an optimal solution. Giving both tables for the different average values might help with this decision.

The first analysed inputs are inputs following a binomial distribution $\sim B(m,p)$ as those inputs have been researched in the previous subsection. The results showed that the expected value of a single number is the main driver for the amount of perfect partitions the input has. The results also suggested the inputs tend to have more perfect partitions if the expected value is lower. The more perfect partitions an input has relative to the number of all possible partitions, the more likely the different RSHs are to find one of those. Therefore researching inputs with higher expected values seems more interesting but generating higher values takes more time with a random number generator that needs $\mathcal{O}(mp)$ time. To keep the time for generating one set of numbers reasonable the values chosen for all tests are $m = 10000, p = 0.1, n = 10000$ with the expected value for a single number being $mp = 1000$. Figure 5.18 shows a random binomial distributed input of length $n = 10000$. For this input type almost every time all elements were sharply concentrated around the expected value with all values being at 1000 ± 200 (for figure 5.18 even closer at 1000 ± 121). So after reaching a difference between the two bins of below $(1000 - 200)/2 = 400$ the algorithm can no longer achieve an improvement by flipping a single bit.

Figure 5.18.: Distribution of a random binomial input



5.3.1. RLS Comparison

algo type	RLS_b^B	RLS_b^B	RLS_s^S	RLS_s^S	RLS_s^S	RLS_b^B	RLS
algo param	b=2	b=4	s=2	s=4	s=3	b=3	-
avg mut/change	2.000	4.000	1.603	2.553	2.000	2.728	1.000
avg mut/step	2.000	4.000	1.500	2.500	1.999	3.000	1.000
total avg count	318	434	499	579	681	518,428	920,109
avg eval count	318	434	499	579	681	395,440	50
max eval count	1,648	3,243	3,094	3,737	4,717	917,134	50
min eval count	20	28	11	17	16	0	50
fail ratio	0.000	0.000	0.000	0.000	0.000	0.234	0.999
avg fail dif	-	-	-	-	-	1	254

The RLS_2^B seems to perform the best as it mostly switches two elements which works great for binomial distributed inputs. The same algorithm with $k = 4$ performs a bit worse but still good as switching 4 elements can be beneficial as well. The variant of RLS_3^B on the other hand does not reach the optimal solution in 23.4% of the inputs with an average difference of 1. It also needs 1000 times more iterations to find an optimum on average compared to the best algorithm RLS_2^B. The RLS_k^S variants behave mostly the same with $k = 2$ being the best, followed by $k = 4$ and $k = 3$. In this case the variant of $k = 3$ is by far not as bad as for the RLS_3^B because the probability of flipping 2 bits is 1/3 as compared to $\mathcal{O}(n^{-1})$ for the RLS_3^B. The RLS_k^S seem all to be a good option for binomial inputs with values of $k \in \{2, 3, 4\}$. The standard RLS on the other hand performs by far the worst as it only moves one element per step. It only managed to reach the optimal solution once for 1000 different inputs. The number of iterations for this input was only 50 so the RLS likely had a good initialisation with a few lucky steps leading directly to the optimum. For all other cases the average difference between the bins was 254 which is close to the median of the values from 0 to $(1000 - 100)/2 = 450$. This is likely due to the RLS being unable to improve the solution once the current solution has a difference below half of the lowest value (Corollary 3.7).

5.3.2. (1+1) EA Comparison

For the (1+1) EA the best static mutation rate seems to be $3/n$. The probability of flipping 2 or 4 bits as n goes to infinity for mutation rate $1/n$ approaches $13/24e \approx 0.199$, for $2/n$ approaches $8/3e^2 \approx 0.361$, for $3/n$ approaches $63/8e^3 \approx 0.392$, for $4/n$ approaches $56/3e^4 \approx 0.342$ and for $5/n$ approaches $77/2e^5 \approx 0.259$. So the highest probability has $c = 3$, followed by $c = 4$ and $c = 2$ then $c = 5$ and lastly $c = 1$. For higher values of c the probability decreases further as the expected number of flipped bits is c for mutation rate c/n.

algo type	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM
algo param	$3/n$	$4/n$	$2/n$	$5/n$	-	$10/n$	$50/n$	$100/n$
avg mut/change	3.101	3.968	2.343	4.859	1.698	9.732	49.544	99.494
avg mut/step	2.999	4.003	2.002	4.999	1.001	9.998	49.998	99.997
avg eval count	646	701	706	857	1,123	1,508	8,175	15,485
max eval count	5,346	5,692	3,415	5,572	7,001	12,112	52,831	145,269
min eval count	23	4	30	9	23	14	27	69
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The static mutation rate $3/n$ seems to perform the best with both $4/n$ and $2/n$ being a close second place. The next best values are $5/n$ and the standard $1/n$ both having a clear difference between each other and the better parameters. From then on the number of iterations rises monotonically with rising mutation rate. The higher mutation rates perform significantly worse but the still find a solution within the limit as opposed to the standard RLS.

5.3.3. pmut Comparison

algo type	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut
algo param	2.25	2.00	1.75	2.50	2.75	3.00	3.25	1.50	1.25
avg mut/change	3.822	6.266	14.371	2.804	2.347	1.995	1.843	38.318	92.365
avg mut/step	4.344	8.504	22.176	2.878	2.272	1.933	1.732	70.476	224.535
avg eval count	652	668	675	688	697	718	758	785	1,050
max eval count	4,340	4,506	5,616	5,098	9,140	5,081	6,189	6,542	7,837
min eval count	14	4	9	12	27	10	11	21	7
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

For the $pmut_\beta$ mutation operator the choice of β seems to be much more insignificant than for the RLS or (1+1) EA. Here all values perform comparably good with only the value of $\beta = 1.25$ having a clear performance difference compared to next best value. All values of β reach an optimal solution in every case. The worst variant of the $pmut_\beta$ operator still performs much better than the worst value for the (1+1) EA and even better than the worst RLS variant. There is no clear winner but because $\beta = 2.25$ had the best performance in this experiment, it was used for the comparison of the best variants.

5.3.4. Comparison of the best variants

algo type	RLS_b^B	EA-SM	pmut
algo param	b=2	$3/n$	2.25
avg mut/change	2.000	3.092	3.965
avg mut/step	2.000	2.999	4.339
avg eval count	302	677	691
max eval count	1,610	6,404	5,205
min eval count	9	33	17
fail ratio	0.000	0.000	0.000

For this setting of $m = 10000, p = 0.1, n = 10000$ the RLS_2^B performs better than the (1+1) EA and $pmut_\beta$ mutation for all values of c/n and β by a factor of at least 2. This is likely from the fact that this version of the RLS flips almost only two bits which seems to be close to optimal for this kind of input. There are many values close to the expected value which can be switched to make small adjustments to the fitness value. The (1+1) EA with $p_m = 3/n$ and $pmut_\beta$ algorithm with $\beta = 2.25$ perform almost the same. To further investigate which input performs best on all binomial distributed inputs now a comparison with different input lengths follows. The parameters of the distribution were not changed. The following table shows the number of runs in which the algorithms did not find an optimal solution within 50,000 steps. The time limit was set 50,000 because the algorithms normally reach the optimal solution within a few thousand steps. If the solutions is not found after 50,000 steps, the algorithm is most likely stuck in a local optimum which could only be left by flipping more bits than possible for the algorithm.

fails in 1000 runs	20	50	100	500	1000	5000	10000
RLS_2^S	231	22	3	0	0	0	0
RLS_4^B	0	1	2	0	0	0	0
RLS_2^S	243	4	0	0	0	0	0
RLS_4^S	1	0	0	0	0	0	0
(1+1) EA ($3/n$)	0	0	0	0	0	0	0
pmut(2.5)	0	0	0	0	0	0	0

The (1+1) EA and $pmut_\beta$ always reach an optimal solution but the RLS does not. The RLS variants that can only flip two bits per step perform significantly worse for small inputs. They are probably more likely to get stuck in a local optimum where a step flipping 4 bits or more would be necessary. So the RLS_2^B does perform better for larger inputs but is much more likely to get stuck in a local optima. The next table contains the average number of iterations the algorithm needed to find an optimal solution for all runs where the algorithms managed to find an optimal solution. Here it still looks like the RLS_2^B finds the solution with the lowest amount of steps, because the cases where the algorithm is stuck in a local optima are not contained in this table.

avg	20	50	100	500	1000	5000	10000
RLS_2^S	164	257	337	256	257	293	310
RLS_4^B	349	355	661	412	412	436	425
RLS_2^S	267	435	408	428	442	452	496
RLS_4^S	601	491	492	491	534	562	560
(1+1) EA ($3/n$)	716	570	569	580	614	625	650
pmut(2.5)	1596	576	600	637	657	700	685

The next table contains the overall average amount of iterations for every run. So runs where no optimal result was found add 50,000 to the sum of all iterations.

total avg	20	50	100	500	1000	5000	10000
RLS_2^S	11676	1351	486	256	257	293	310
RLS_4^B	349	405	760	412	412	436	425
RLS_2^S	12352	633	408	428	442	452	496
RLS_4^S	650	491	492	491	534	562	560
(1+1) EA ($3/n$)	716	570	569	580	614	625	650
pmut(2.5)	1596	576	600	637	657	700	685

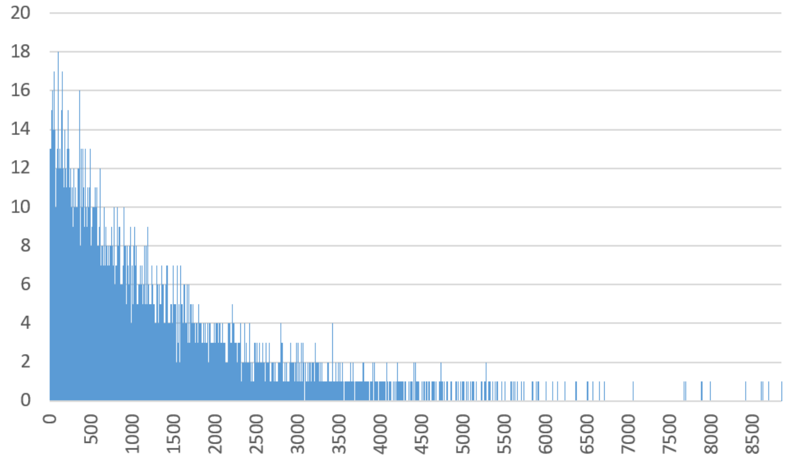
Here the RLS_2^B is only the best algorithm for values of $n \geq 500$. Below this bound choosing the (1+1) EA with static mutation rate $3/n$ is a safer choice as the (1+1) EA reaches an optimal solution for every input (in this experiment).

5.4. Geometric distributed values

For the geometric distribution the chosen default value is $p = 0.001$. This results in an expected value of 1000 which is the same as for the binomial distribution in the last subsection. This should make the results more comparable. The maximum value is theoretically not limited but for the implementation in Java the maximum value was set to the maximum value of a long value $= 2^{63} - 1 = 9,223,372,036,854,775,807$. Without this maximum value the value might overflow and instead be negative with high absolute value. Figure 5.19 shows a random geometric distributed input. The span of all values is way higher than for the binomial distribution, although they have same expected value. Here the values are not in the interval $[800, 1200]$ but rather between 0 and the 9000. The theoretical limitation of the values being at most $2^{63} - 1$ seems to not have an influence on the results.

The geometric distribution does not only have low values close or equal to 1 but also has mostly values that are very small. This should lead to 1-bit flips being effective as the small values can remove the small differences. Because there are so many small values moving only one bit might be better than switching two elements.

Figure 5.19.: Distribution of a random geometric input



5.4.1. RLS Comparison

algo type	RLS$_s^S$	RLS$_s^S$	RLS$_s^S$	RLS$_b^B$	RLS$_b^B$	RLS$_b^B$	RLS
algo param	s=2	s=3	s=4	b=2	b=3	b=4	-
avg mut/change	1.477	1.959	2.431	2.000	3.000	4.000	1.000
avg mut/step	1.500	2.001	2.501	2.000	3.000	4.000	1.000
total avg count	2,592	2,945	3,259	3,497	4,463	5,345	6,650
avg eval count	2,592	2,945	3,259	3,497	4,463	5,345	2,055
max eval count	19,845	23,932	28,532	23,824	30,881	41,600	25,889
min eval count	8	22	19	18	43	19	23
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.005
avg fail dif	-	-	-	-	-	-	1

For these inputs the variants of the RLS perform differently to the binomial input. The only similarity is the RLS being the worst as the RLS is the only algorithm that did not find an optimal solution for every input. If the RLS did find an optimal solution in those 5 cases it would instead be the best RLS variant. The other algorithms are ranked by the probability of flipping only one bit. This means at first the three RLS$_k^S$ variants from 2 to 3 to 4 and then the same for the RLS$_k^B$ variants. So it does seem like moving mostly one element at once is better for the geometric input in comparison to two elements for the binomial distribution. In the 5 cases where the RLS did not find an optimal solution it was most likely stuck in a local optimum.

5.4.2. (1+1) EA Comparison

algo type	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM
algo param	$2/n$	-	$3/n$	$4/n$	$5/n$	$10/n$	$50/n$	$100/n$
avg mut/change	2.255	1.554	3.038	3.948	4.883	9.821	49.798	99.814
avg mut/step	2.000	1.000	3.000	4.001	5.000	9.999	49.998	100.001
avg eval count	3,712	3,833	4,195	4,472	5,465	8,282	21,648	29,404
max eval count	39,593	53,450	33,598	42,449	55,717	65,522	149,048	281,857
min eval count	18	13	15	14	25	23	46	17
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The results for the (1+1) EA are similar to the results of the RLS. From mutation rate $2/n$ on the runtime increases with rising mutation rate. The only part that does not fit into the theory of 1 bit flips being superior is the mutation rate $2/n$ performing better than the standard $1/n$. The average number of iterations for the standard (1+1) EA is only slightly higher than for the mutation rate $2/n$, so this might be just due to a too small number of runs of the algorithms. All variants reach an optimal solution within the given limit for the number of iterations.

5.4.3. pmut Comparison

algo type	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut
algo param	3.25	3.00	2.50	2.75	2.25	2.00	1.75	1.50	1.25
avg mut/change	1.682	1.872	2.688	2.150	3.704	6.938	16.352	41.906	107.789
avg mut/step	1.730	1.936	2.918	2.267	4.355	8.463	22.369	70.989	225.029
avg eval count	2,575	2,732	2,734	2,776	2,809	3,165	3,486	4,389	6,151
max eval count	73,911	34,215	75,791	42,620	25,352	31,966	37,725	50,454	55,022
min eval count	33	17	11	0	35	9	5	23	19
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The results for the $pmut_\beta$ operator are even more clear than for the (1+1) EA. With decreasing values for β the amounts of flipped bits per step increases. The performance on the other hand decreases with rising values for β which fits into the theory of one bit flips being better for gemoetric distributed inputs. The only special case that does not support this theory is the value $\beta = 2.5$ performing better than $\beta = 2.75$. Here the same might hold as for the (1+1) EA. The number of repetitions of the algorithm might simply be too small to make the small difference in the performance between the two values visible. The difference in the performance for the $pmut_\beta$ operator is not as drastic as for the (1+1) EA. Only $\beta = 1.25$ performs significantly worse the next best value.

5.4.4. Comparison of the best variants

algo type	RLS_s^S	pmut	EA-SM
algo param	s=2	3.25	$2/n$
avg mut/change	1.483	1.693	2.258
avg mut/step	1.500	1.729	2.001
avg eval count	2,407	2,695	3,421
max eval count	23,155	57,661	52,762
min eval count	19	11	19
fail ratio	0.000	0.000	0.000

For the geometric distribution once again a variant of the RLS performs the best. $pmut_{3.25}$ performs almost equally good with only the (1+1) EA variant performing clearly worse. Here the algorithms are sorted again by their average number of flips per steps. The theory of flipping one bit per step being better seems to be true for this kind of input.
To further confirm the best choice for this kind of input there was another experiment of the best variants. The setup is mostly the same except for having a fixed time limit of 100,000 instead of using $100 \cdot n \ln(n)$ as the limit. The smaller inputs are harder relative to their input size so using $100 \cdot n \ln(n)$ as a bound is too small. The first try was executed with 50,000 as the step limit but there the algorithms performed too bad for $n = 20$. Therefore for the second attempt the step limit was increased to 100,000. The first table lists the number of runs where the different algorithms did not find the optimal solution within the time limit.

fails in 1000 runs	20	50	100	500	1000	5000	10000
RLS	983	951	885	619	425	42	0
RLS_2^S	881	578	171	0	0	0	0
(1+1) EA ($1/n$)	464	132	48	1	1	0	0
(1+1) EA ($2/n$)	149	11	1	0	0	0	0
pmut(3.25)	284	67	27	0	0	0	0
pmut(3.5)	312	91	38	0	1	0	0

For small inputs the geometric distributed input seems to have inputs without a perfect partition because there were many iterations where neither of the algorithms found an optimal solution within the time limit. It is still likely to have a perfect partition even for the small values in comparison to other distributions which follow afterwards. It seems many of the algorithms especially the variants of the RLS seem to be likely to get stuck in a local optimum. The (1+1) EA finds an optimum in most of the runs, so the geometric distributed inputs also seem to be likely to have a perfect partition for small values. They definitely are harder to solve for smaller input sizes as the binomial inputs, but they still have a perfect partition most times. The next table visualises the average number of iterations the algorithms needed for finding an optimal solution if the algorithm managed to do so.

avg	20	50	100	500	1000	5000	10000
RLS	35	78	140	566	904	2119	2188
RLS_2^S	357	2024	5369	4687	3945	2752	2583
(1+1) EA ($1/n$)	21827	20775	14583	9459	7702	4358	3924
(1+1) EA ($2/n$)	18211	11613	7529	5266	4359	3858	3293
pmut(3.25)	22530	16421	11312	5909	5375	2843	2246
pmut(3.5)	24202	17414	11731	6503	5216	2773	2388

The variants of the (1+1) EA and of the *pmut* algorithm seem to take about 20,000 iterations for $n = 20$ if they manage to find the optimal solution. They also perform better and better the bigger the input gets. This is probability caused by the many additional small values that can be used for smaller adjustments to the fitness. Also a really high value does not have as much of an effect, because there are possibly other larger values which cancel each other out, if they are in different bins. The standard (1+1) EA does not only find a perfect partition less often, it also needs more iterations on average if it does. So the (1+1) EA with $p_m = 2/n$ performs indeed better for every input size. The last table again lists the total average number of steps.

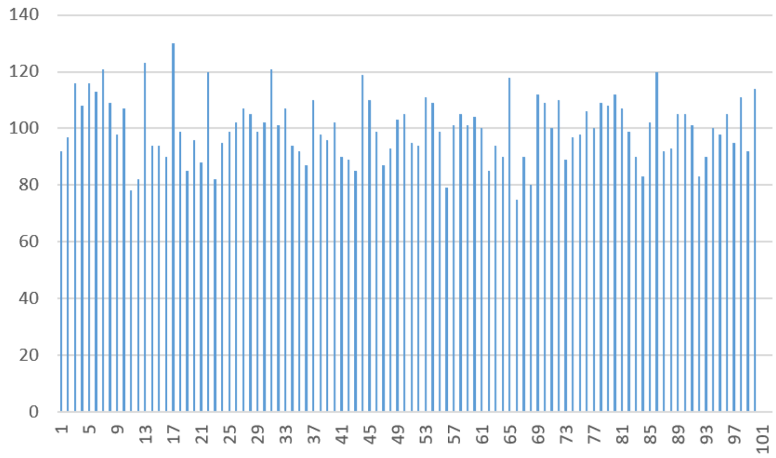
total avg	20	50	100	500	1000	5000	10000
RLS	98300	95103	88516	62115	43019	6230	2188
RLS_2^S	88142	58654	21551	4687	3945	2752	2583
(1+1) EA ($1/n$)	58099	31232	18683	9550	7795	4358	3924
(1+1) EA ($2/n$)	30397	12585	7622	5266	4359	3858	3293
pmut(3.25)	44531	22021	13707	5909	5375	2843	2246
pmut(3.5)	47851	24929	15086	6503	5311	2773	2388

The RLS is only an option if the input is large enough ($n \geq 10,000$). For smaller input sizes especially for $n \leq 100$ choosing the (1+1) EA with mutation rate $2/n$ seems like the best choice. For larger values this (1+1) EA does not find an optimal solution the fastest but is still fast enough to be a viable option. Another rather save option is $pmut_{3.25}$. This algorithm performs worse for $n \leq 100$ but is still good in comparison to the other algorithms. For $n \geq 1000$ $pmut_{3.25}$ starts to outperform the best version of the (1+1) EA and almost all other researched algorithms.

5.5. Uniform distributed inputs

For the uniform distribution the default values were 1 for the lower bound and 50000 for the upper bound (exclusive). The range was limited to 50000 to reduce the time the algorithms needs to find an optimal solution. The higher the values are with too few values the more likely the input is to not have a perfect partition[BCP01]. This will cause the algorithms to always reach the limit for the number of iterations which drastically increases the time needed for the experiment. The length of the input was 50000.

Figure 5.20.: Distribution of a random uniform input (10000 values between 1 and 100)



5.5.1. RLS Comparison

algo type	RLS_b^B	RLS_s^S	RLS_s^S	RLS_b^B	RLS_s^S	RLS_b^B	RLS
algo param	b=2	s=3	s=4	b=3	s=2	b=4	-
avg mut/change	2.000	1.996	2.476	3.000	1.502	4.000	1.000
avg mut/step	2.000	2.000	2.500	3.000	1.500	4.000	1.000
total avg count	83,118	104,748	105,513	112,223	114,486	121,927	2,443,567
avg eval count	83,118	104,748	105,513	112,223	114,486	121,927	45,834
max eval count	778,110	1,453,252	898,974	1,377,471	915,268	816,633	485,275
min eval count	197	126	45	212	271	155	128
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.447
avg fail dif	-	-	-	-	-	-	1

The picture for the RLS variants on this type of input is not clear. There in no obvious tendency for neither of the variants. The only obvious thing is the RLS being the worst of the RLS variants again. Every variant reaches the optimal solution in every case except for the RLS which only manages for 44.7 % of the inputs. The RLS_2^B seems to be the best variant for these kinds of inputs. The next best variants are the RLS_k^S with $k = 3$ and $k = 4$ which only differ by 1 %.

5.5.2. (1+1) EA Comparison

algo type	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA
algo param	$3/n$	$2/n$	$4/n$	$5/n$	$10/n$	-
avg mut/change	3.102	2.287	4.014	4.937	9.924	1.577
avg mut/step	3.000	2.000	4.000	5.000	10.000	1.000
avg eval count	122,098	122,690	124,634	132,509	183,213	213,186
max eval count	956,375	920,658	1,128,158	1,457,069	1,298,089	2,509,163
min eval count	174	188	265	384	6	111
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000

The (1+1) EA seems to perform better with a lower mutation rate. The vales $p_m = 2/n$ and $p_m = 3/n$ reach an optimal solution equally fast. From then on the speed of convergence decreases with increasing mutation rate. The only exception from this case is the standard (1+1) EA which performs the worst despite having the lowest mutation rate. For the uniform distributed input all variants of the (1+1) EA reach an optimal solution within the step limit as for the previous input types.

5.5.3. pmut Comparison

algo type	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut
algo param	2.50	2.00	2.25	2.75	1.75	3.00	1.50	3.25	1.25
avg mut/change	2.866	8.980	4.204	2.205	27.674	1.933	102.803	1.720	312.822
avg mut/step	2.932	10.108	4.559	2.274	34.643	1.934	158.163	1.729	719.965
avg eval count	117,346	121,090	121,818	126,467	128,188	140,882	142,970	150,311	193,296
max eval count	1,655,807	1,421,071	1,427,930	2,490,695	2,127,979	1,670,194	1,565,473	1,382,253	1,523,513
min eval count	61	186	76	130	357	155	113	226	13
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The optimal value for β seems to be somewhere around 2.0 to 2.5. The values next to this interval start to decrease in both directions, but 1.75 and 2.75 are still relatively close to the performance of the optimal value. The values equally wide apart from 2.25 perform equally good.

5.5.4. Comparison of the best variants

algo type	RLS_b^B	EA-SM	pmut
algo param	b=2	$3/n$	2.25
avg mut/change	2.000	3.109	4.273
avg mut/step	2.000	3.000	4.555
avg eval count	84,884	116,576	124,046
max eval count	741,833	1,176,762	1,159,541
min eval count	52	178	178
fail ratio	0.000	0.000	0.000

For the uniform distributed input the best variant of the RLS once again seems to perform the best. But by looking at the smaller values again this does not hold in general.

fails in 1000 runs	20	50	100	500	1000	5000	10000	50000
RLS_2^S	996	980	892	360	289	14	0	0
RLS_3^S	974	768	645	461	426	54	3	0
RLS_4^S	902	621	541	460	431	53	4	0
(1+1) EA ($2/n$)	885	690	641	510	483	85	5	0
(1+1) EA ($3/n$)	772	596	549	425	449	53	2	0
(1+1) EA ($4/n$)	697	509	453	439	396	44	3	0
pmut (2.5)	840	738	696	553	520	86	10	0

The RLS variants are the most likely to get stuck in a local optimum for $n \leq 100$. The (1+1) EA variants also often do not find an optimal solution, but this happens less frequently. The more values the input has the more likely it is for any of the algorithms to find a perfect partition. Between $n = 100$ and $n = 500$ the performance of the RLS_2^B drastically increases and for $n \geq 500$ this variant of the RLS stays the best variant for the remaining input sizes. Uniform distributed inputs seem to be much less likely to have a perfect partition for the small input sizes which can be explained by the of Borgs *et. al.* coefficient [BCP01].

avg	20	50	100	500	1000	5000	10000	50000
RLS_2^S	364	1540	6759	36358	37324	77210	83500	82738
RLS_3^S	5204	33492	39795	40336	38838	104864	118393	106196
RLS_4^S	17681	39895	38795	39959	38857	98693	108780	107857
(1+1) EA ($2/n$)	36004	41914	40131	42050	38937	108903	133264	122042
(1+1) EA ($3/n$)	32791	38954	38422	40617	41017	103831	112166	111402
(1+1) EA ($4/n$)	39665	39926	40709	39479	41889	99899	127578	110099
pmut (2.5)	39232	36918	37162	39866	40588	118671	144665	128531

The amount of steps needed to find an optimal solution seems to be nearly constant for every algorithm as the number of steps does not strictly increase with n but sometimes even decreases for $n \leq 1000$. This is caused by the number of steps the algorithm was given. For $n \leq 1000$ the time limit was 100,000 and for the bigger values it was $10n\ln(n)$. Interestingly enough the average runtime decreases from $n = 10000$ to $n = 50000$ for most algorithms.

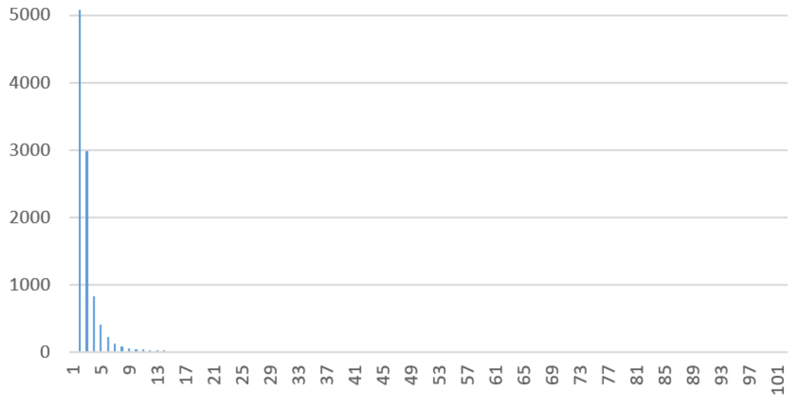
total avg	20	50	100	500	1000	5000	10000	50000
RLS_2^S	99601	98030	89930	59269	55438	82091	83500	82738
RLS_3^S	97535	84570	78627	67841	64893	122198	120801	106196
RLS_4^S	91932	77220	71907	67578	65209	116033	112029	107857
(1+1) EA $(2/n)$	92640	81993	78507	71604	68430	135844	137203	122042
(1+1) EA $(3/n)$	84676	75337	72228	65854	67500	120899	113784	111402
(1+1) EA $(4/n)$	81718	70503	67568	66047	64901	114241	129959	110099
pmut (2.5)	90277	83472	80897	73120	71482	145089	152429	128531

My general advice would be choosing the RLS_2^B for $n \geq 500$ and the (1+1) EA with $p_m = 4/n$ otherwise.

5.6. powerlaw distributed inputs

This distribution has mostly small values, but occasionally it also generates bigger values. The lower the parameter the higher the values get and also the amount of big values increases. For a parameter of $\beta = 2.75$ and a maximum value of 10,000 the distribution looks like in Figure 5.21.

Figure 5.21.: Distribution of a random powerlaw input with $\beta = 2.75$



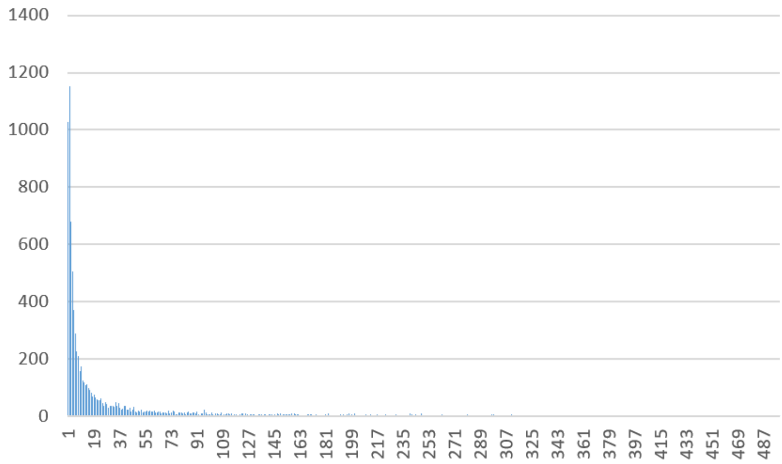
For a value of $\beta = 1.25$ the distribution looks a bit different. There are less small values close to one and instead also big values even over 1000. Figure 5.22 is cropped to get a more clear view for the smaller values. The higher values mostly occurred 0 to 2 times. The highest value 9948 occurred only once.

5.6.1. RLS Comparison

The following table lists the results for the RLS for inputs that are chosen from a powerlaw distribution with $\beta = 2.75$.

algo type	RLS_b^B	RLS_b^B	RLS_s^S	RLS_s^S	RLS_b^B	RLS_s^S	RLS
algo param	b=4	b=3	s=4	s=3	b=2	s=2	-
avg mut/change	4.000	3.000	2.470	1.959	2.000	1.447	1.000
avg mut/step	4.000	3.000	2.501	2.000	2.000	1.501	1.000
avg eval count	196	225	256	275	294	324	399
max eval count	8,523	8,635	10,613	10,930	11,210	12,417	12,562
min eval count	0	0	0	0	0	0	0
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 5.22.: Distribution of a random powerlaw input with $\beta = 1.25$



Due to the many small values and all values being relatively small in general the higher mutation rates perform better. So the algorithms are ranked by the amount of bits they flip in an average step. For this parameter the input is relatively easy as all variants manage to find an optimal solution in 400 steps on average for an input size of 20,000. The next table lists the results for $\beta = 1.25$ and an input size of 10,000.

algo type	RLS_k^S	RLS_k^S	RLS_k^B	RLS_k^B	RLS_k^S	RLS_k^B	RLS
algo param	r=4	r=3	n=3	n=2	r=2	n=4	-
avg mut/change	2.445	1.967	3.000	2.000	1.486	4.000	1.000
avg mut/step	2.500	2.001	3.000	2.000	1.500	4.000	1.000
total avg count	255	272	284	286	330	370	434
max eval count	847	885	1,553	1,149	1,130	1,879	1,911
min eval count	26	25	36	15	12	29	26
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The input is still easy to solve for the RLS variants, but the concept of the more bits flipped the better does not hold any more. So the optimal algorithm parameter for an input from a specific distribution is not fixed for the complete distribution. Instead the optimal value can change even within the distribution if the parameters of the distribution change. Generally the RLS_k^S variants seem to perform good for the different parameters of the distribution especially with increasing value of k.

5.6.2. (1+1) EA Comparison

The first table again shows the results for parameter $\beta = 2.75$

algo type	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA
algo param	$50/n$	$100/n$	$10/n$	$5/n$	$4/n$	$3/n$	$2/n$	-
avg mut/change	49.922	99.873	10.009	5.053	4.105	3.156	2.280	1.534
avg mut/step	49.989	100.017	9.999	4.999	4.005	3.003	2.000	0.999
avg eval count	84	103	111	157	184	208	273	461
max eval count	1,281	1,488	1,946	3,030	3,043	3,283	4,744	7,036
min eval count	0	1	3	1	0	0	2	0
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Here the same rule holds for the RLS to some extent. Until $p_m \leq 50/n$ the speed of convergence increases but at $p_m = 100/n$ the speed decreases again. The optimal value seems to be somewhere around $p_m = 50/n$. The (1+1) variants are generally faster than all RLS variants when comparing the maximum number of iterations. For mutation rates $3/n \leq p_m \leq 100/n$ the (1+1) EA is also faster on average. The next table shows the results for a powerlaw distribution with $\beta = 1.25$.

algo type	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM
algo param	4/n	3/n	2/n	5/n	-	10/n	50/n	100/n
avg mut/change	3.913	3.061	2.255	4.794	1.559	9.489	49.461	99.590
avg mut/step	4.005	2.999	2.001	4.999	1.000	9.997	50.001	100.000
total avg count	243	245	291	299	509	1,248	53,071	99,580
avg eval count	243	245	291	299	509	1,248	53,071	97,933
max eval count	1,107	752	949	1,258	1,701	9,332	530,320	753,937
min eval count	23	4	15	17	50	26	213	121
fails	0	0	0	0	0	0	0	2
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002
avg fail dif	-	-	-	-	-	-	-	1

With this setting the optimal value is shifted to somewhere around $p_m = 4/n$. The higher mutation rates perform drastically slower with $p_m = 100/n$ being 500 times slower than the optimal value. The speed of convergence is sometimes even to slow find an optimal solution in time $10 \cdot n \ln(n)$.

5.6.3. pmut Comparison

The first table again shows the results for parameter $\beta = 2.75$

algo type	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut
algo param	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25
avg mut/change	173.565	62.036	20.819	7.619	4.175	2.729	2.191	1.870	1.672
avg mut/step	370.551	101.051	28.236	9.183	4.338	2.894	2.257	1.939	1.734
total avg count	2,090	2,098	2,137	2,181	2,209	2,240	2,252	2,273	2,289
avg eval count	110	118	156	200	229	260	272	293	309
max eval count	51,062	36,295	31,245	28,794	27,556	27,371	26,718	27,302	27,065
min eval count	0	0	0	0	0	0	0	0	0
fail ratio	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
avg fail dif	176	176	176	176	176	176	176	176	176

For pmut the same holds as for the RLS. The more bits the algorithms flips on average the better the performance on average. Surprisingly the performance in the worst runs behaves inverted. The fewer bits the algorithm flips on average the more stable the search becomes. This might be caused by the really large amount of bits flipped for the lower values.

algo type	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut
algo param	1.50	1.75	2.00	2.25	1.25	2.50	2.75	3.00	3.25
avg mut/change	31.034	12.655	6.445	3.686	70.708	2.643	2.147	1.863	1.676
avg mut/step	70.574	22.234	8.730	4.286	222.822	2.894	2.287	1.943	1.731
total avg count	171	179	193	215	222	246	269	292	306
max eval count	590	475	705	775	1,207	705	894	1,048	1,003
min eval count	16	28	19	16	31	7	21	24	36
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The optimal value here seems to be somewhere around $\beta = 1.5$, so only lightly smaller in comparison to the (1+1) EA where the optimal value almost change from one side of the spectrum to the other. Here the inverted stability of the search does not occur. The variants that take longer on average tend to also take longer in their worst runs.

5.6.4. Comparison of the best variants

The first table again shows the results for parameter $\beta = 2.75$

algo type	pmut	EA-SM	RLS_b^B
algo param	1.25	$50/n$	b=4
avg mut/change	216.029	49.921	4.000
avg mut/step	370.312	50.022	4.000
avg eval count	56	84	181
max eval count	2,616	1,244	3,048
min eval count	0	1	0
fail ratio	0.000	0.000	0.000

The ranking follows the amount of bits the algorithms flip on average per step. $pmut_{1.25}$ manages to find the solution in just 56 iterations on average. The (1+1) EA with $p_m = 50/n$ is slower than $pmut_{1.25}$ but instead has a lower value for the maximum number of iterations. Both options seem fine. Even the RLS_4^B is still very fast for the powerlaw distributed input with $\beta = 2.75$. For $\beta = 1.25$ the results are a bit different.

algo type	pmut	RLS_k^S	EA-SM
algo param	1.50	r=4	4/n
avg mut/change	30.913	2.438	3.917
avg mut/step	70.577	2.501	4.000
avg eval count	178	251	251
max eval count	629	778	1,018
min eval count	17	19	13
fail ratio	0.000	0.000	0.000

The RLS_4^S now performs equally good as the (1+1) EA variant with $p_m = 4/n$, but is still slower than $pmut_{1.5}$. As the first inputs were less difficult to solve than the inputs with $\beta = 1.25$ the second value was chosen for the evaluation of smaller input sizes.

fails in 1000 runs	20	50	100	500	1000	5000	10000	50000
RLS_3^S	822	400	87	0	0	0	0	0
RLS_4^S	806	389	80	0	0	0	0	0
(1+1) EA ($3/n$)	803	364	51	0	0	0	0	0
(1+1) EA ($4/n$)	803	359	49	0	0	0	0	0
pmut (1.5)	802	353	41	0	0	0	0	0
pmut (1.75)	804	354	41	0	0	0	0	0

The RLS is once again the algorithm that is the most likely to be stuck in a local optimum. Compared to the other algorithms it is not as drastic as for the binomial input for example. Only for $n < 500$ the algorithms do not find a global optimum in every run. The setting of the parameter almost doesn't affect the amount of runs without an optimal result. The main differences are between the different algorithms themselves. This type of input is probably easy to solve if it has a perfect partition. The two stopping conditions where a step limit and finding a perfect partition or a partition with difference of one between the two bin for uneven n. So in 80% of the runs the algorithms might have found an optimal solution, but the stopping conditions did not trigger as the solutions where not close to a perfect partition.

avg	20	50	100	500	1000	5000	10000	50000
RLS_3^S	523	2517	1237	139	158	226	279	509
RLS_4^S	1291	1976	1935	141	151	203	246	426
(1+1) EA ($3/n$)	554	1849	1218	152	168	210	248	401
(1+1) EA ($4/n$)	382	1107	633	175	190	223	252	359
pmut (1.5)	667	487	164	147	148	163	178	206
pmut (1.75)	465	665	229	135	135	160	179	244

Looking at the time the algorithms needed on average the runs that hit the step limit could have possibly been no failed runs. The easiest are inputs with size $n = 500$. For smaller values of n the algorithms sometimes fail and even in a good run they need more iterations to find an optimal solution. Due to the increasing size of the input the algorithms need more time for the bigger values.

total avg	20	50	100	500	1000	5000	10000	50000
RLS_3^S	82293	41510	9829	139	158	226	279	509
RLS_4^S	80850	40107	9780	141	151	203	246	426
(1+1) EA ($3/n$)	80409	37576	6256	152	168	210	248	401
(1+1) EA ($4/n$)	80375	36609	5502	175	190	223	252	359
pmut (1.5)	80332	35615	4258	147	148	163	178	206
pmut (1.75)	80491	35829	4320	135	135	160	179	244

$pmut_{1.75}$ is not only the best variant for the bigger values of n but also for smaller inputs as well. It is the least likely to be stuck in a local optimum, and it is also the fastest if it reaches a global optimum.

5.7. Eqiuvalent of linear functions for PARTITION

The input of this subsection is more or less equivalent to linear functions. All values except the last follow any distribution whereas the last value is the sum of all other values. The optimal solution is therefore the 000...01 or the 111...01 string. So the best solution is almost identical to a linear function with positive weights that is maximised/minimised depending on the value of the last bit.
For linear functions the mutation rate of 1/n is proven to be optimal for the (1+1) EA [Wit13]. This should also hold for this input. The RLS variants should also perform worse than the standard RLS. The higher the value for β the better the $pmut_\beta$ mutation should perform. Flips of the first bits could decrease the runtime, depending on how often they happen. By doing some testing with various algorithm variants of the RLS and the (1+1) EA it looked like the last bit was only flipped at most once for every input. There was only one case where it was flipped twice, but it was never flipped more than twice per run. The average number of flips was also mostly closer to zero than to one.
The experiments where conducted with the variant where the smaller values are all one. So for every run of each algorithm the input was $[1, 1, \dots, 1, n-1]$. An input like this takes less time for every algorithm, but the results are mostly the same. For some experiments not all 1000 repetitions were executed as there was a clear tendency which of the algorithms performs better.

5.7.1. RLS Comparison

algo type	RLS	RLS_s^S	RLS_s^S	RLS_s^S	RLS_b^B	RLS_b^B	RLS_b^B
algo param	-	s=2	s=3	s=4	b=3	b=2	b=4
avg mut/change	1.000	1.181	1.688	1.865	3.000	1.997	3,997
avg mut/step	1.000	1.500	2.000	2.500	3.000	2.000	3.000
total avg count	90,931	168,311	236,317	307,533	921,030	921,030	921,030
avg eval count	90,931	168,311	236,317	307,533	-	-	-
max eval count	156,854	296,206	498,474	595,831	-	-	-
min eval count	64,941	120,582	158,304	212,193	-	-	-
fail ratio	0.000	0.000	0.000	0.000	1.000	1.000	1.000
avg fail dif	-	-	-	-	36	53	263

As expected the standard RLS reaches an optimal solution the fastest. It also reaches the optimal value for every instance. The RLS_k^S variants need more iterations to find an optimal solution. By looking at the average values more closely it seems like the average number of steps for the RLS_k^S is roughly $25,000 + 70,000k \pm 5,000$. The standard RLS is equivalent to RLS_k^S or RLS_k^B with $k = 1$. So the value of $k = 1$ seems to be optimal for the RLS variants too. The RLS_k^B variants on the other hand do not reach any of the two optimal solutions in any run. This is most likely caused by their very low possibility of flipping only one bit in a single step. They would eventually reach the optimal solution as well, but this would take much longer than for the RLS. The probability of flipping only one bit in a step is $\mathcal{O}(n^{1-k})$ which results in a single bit flip every $\mathcal{O}(n^{k-1})$ steps in expectation. Because the fitness can only improve for OneMax making steps flipping more bits does not harm the fitness. The bound for OneMax is $\mathcal{O}(n \log n)$ and with the previous result the expected number of steps is bounded by $\mathcal{O}(n \cdot \mathcal{O}(n^{k-1}) \cdot \log(n \cdot \mathcal{O}(n^{k-1}))) = \mathcal{O}(n^{k-1+1} \cdot (k-1+1) \cdot \log(n)) = \mathcal{O}(kn^k \cdot \log(n))$ This problem is not equivalent to OneMax, as a flip of the bit with the highest value inverts the fitness function to ZeroMax but the result might still hold as the bound for the standard RLS for this input is the same as for the RLS on OneMax.

5.7.2. (1+1) EA Comparison

algo type	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM
algo param	-	$2/n$	$3/n$	$4/n$	$5/n$	$10/n$	$50/n$	$100/n$
avg mut/change	1.273	1.750	2.334	2.965	3.636	7.288	45.923	95.590
avg mut/step	1.000	2.000	3.000	4.000	5.000	10.000	49.998	100.00
total avg count	230,328	297,602	495,951	860,736	921,030	921,030	921,030	921,030
avg eval count	230,328	297,602	495,951	812,983	-	-	-	-
max eval count	399,393	625,976	839,325	917,029	-	-	-	-
min eval count	162,400	193,796	347,185	635,812	-	-	-	-
fail ratio	0.000	0.000	0.000	0.442	1.000	1.000	1.000	1.000
avg fail dif	-	-	-	1	18	570	2,488	3,115

This experiment was terminated after 224 runs of the algorithms, as the results were already clear enough. For this input the same as for OneMax holds. The static mutation rate $p_m = 1/n$ is the optimal value and the performance of the (1+1) EA decreases with rising mutation rate. Only for $p_m \leq 3/n$ the (1+1) EA managed to find one of the two optimal solutions in $10 \cdot n \ln(n)$ steps every time. With mutation rate $p_m = 4/n$ the (1+1) EA only managed to find the optimal solution in about 55 % of the inputs. The remaining mutation rates did not manage to find an optimal solution in any of the runs. Another interesting fact is the average number of bits flipped in a successful step. For the other inputs the overall average number of bits flipped in any step was mostly the same as for the average value of the successful steps. Here this is not the case. All mutation rates flipped fewer bits in the successful steps than in the average step. The only exception is the standard mutation rate which is caused by the steps where the algorithm would flip no bit. Those steps decrease the number of the average case but not of the successful case as those steps were skipped.

5.7.3. pmut Comparison

algo type	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut
algo param	3.25	3.00	2.75	2.50	2.25	2.00	1.75	1.50	1.25
avg mut/change	1.289	1.359	1.459	1.591	1.813	2.207	2.760	3.604	5.382
avg mut/step	1.731	1.934	2.270	2.907	4.371	8.486	22.299	70.692	224.466
total avg count	145,095	149,664	170,912	181,099	214,361	249,102	301,566	415,413	715,219
avg eval count	145,095	149,664	170,912	181,099	214,361	249,102	301,566	415,413	683,204
max eval count	217,932	223,561	254,330	246,635	365,378	376,768	431,629	735,214	853,181
min eval count	111,061	119,931	120,965	130,174	161,244	180,570	232,166	311,979	492,686
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.135
avg fail dif	-	-	-	-	-	-	-	-	1

The results for the *pmut* operator are pretty similar to the results for the (1+1) EA and the RLS. The parameter $\beta = 3.25$ which flips the least bits on average finds the solution the fastest. The other values for β increase the time needed for finding one of the two optimums with decreasing value for β. All variants find an optimum in every run except for $\beta = 1.25$ which has a much higher value for the number of flipped bits per steps. The average number of bits flipped in a successful mutation is much lower than for the other inputs especially for the lower values for β. For the binomial and geometric input the successful average was around 100 for $\beta = 1.25$ but for the OneMax equivalent it was only at 5.

5.7.4. Comparison of the best variants

algo type	RLS	pmut	EA
algo param	-	3.25	-
avg mut/change	1.000	1.287	1.272
avg mut/step	1.000	1.729	1.000
avg eval count	91,171	143,121	231,082
max eval count	153,143	227,737	446,942
min eval count	65,783	93,602	165,818
fail ratio	0.000	0.000	0.000

The results for this experiment are as expected. All three algorithms find the optimal value within the time limit. The RLS performs better than the (1+1) EA because it does only single bit flips. The $pmut_{3.25}$ performs better than the standard (1+1) EA although flipping more bits on average. This is most likely cause by the few steps where *pmut* flips many bits which increase the average. But *pmut* most likely chooses to flip only one bit more often as the (1+1) EA.
For this comparison neither of the algorithms failed to find one of the two optimal solutions. The following table lists the amount of iterations the algorithms needed to find an optimal solution.

avg	20	50	100	500	1000	5000	10000	50000
RLS	56	187	446	3050	6725	42056	90485	537292
(1+1) EA ($1/n$)	131	438	1059	7435	16963	106410	230310	1379632
pmut (3.25)	82	277	667	4666	10432	66450	142919	846722
pmut (3.5)	77	265	642	4393	9930	62337	137048	801570

The RLS performs the best closely follow by both *pmut* variants. The standard (1+1) EA performs a bit worse than the other three algorithms and approaches $en\ln(n)$ instead of staying close to $n\ln(n)$. In a previous chapter the $\mathcal{O}(n\log n)$ bound was proven for the (1+1) EA and the RLS (Theorem 3.3). This seems to hold in practice at least for the easiest version of this input where the small values are one (see figure 5.23). The scale of this figure is $n\ln n$ which means that a value of 2 on the y-axis means the algorithm needs $2n\ln n$ steps on average to reach the optimal solution.

Another variant of this input are uniform distributed inputs for example. The small values in this case are chosen from $\sim U(1, 49999)$ and the last value again is the sum of all other values. This input is harder because switching multiple small values for a big value increases the fitness but also increases the Hamming distance to the optimum. Looking at figure 5.24 the equivalent to uniform distributed linear functions looks not much harder compared to the variant where all values are one. The graphs are almost identical. It was clear for the RLS because the RLS can't switch elements which leads to the exact same behaviour. But even the other algorithms that are able to switch seem to not harm the hamming distance drastically by switching elements.

Figure 5.23.: Runtime for the OneMax equivalent with a $n\ln(n)$ scale

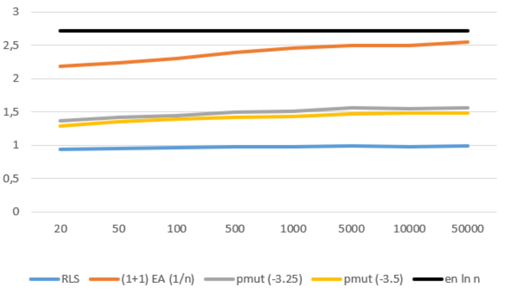


Figure 5.24.: Runtime for the OneMax equivalent with uniform distribution on a $n\ln n$ scale

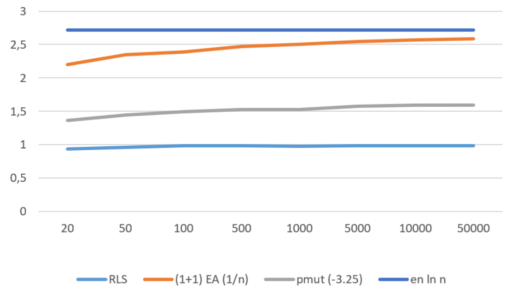


Figure 5.25 shows the average number of steps needed by the RLS_k^S for the equivalent to linear functions with uniform distribution. Lemma 4.2 proved the RLS_k^S needs time $4k+\frac{1+o(1)}{1-o(1)}\cdot kn\ln n$ but the actual expected running time seems to be lower. All RLS_k^S variants in the figure approach the value of $kn\ln n$ from below and for at least $n \leq 50,000$ do not need more time than $kn\ln n$. The RLS_k^B variants on the other hand have a significantly worse performance. Their average runtime is shown in figure 5.26 with a logarithmic y-axis. A value of 2 on the x-axis here means that the algorithm needed n^2 steps on average to find an optimal solution. All RLS_k^B variants needs at least n^2 steps on average, but all variants seem to approach n^k from below. The higher the value of k the less clear the asymptotic runtime appears. Testing higher values for $k = 4$ should reveal a better view on the asymptotic runtime but if the asymptotic runtime is actually close to n^4 then it would take $1000^4 = 10^{12}$ steps for $n = 1000$. With a runtime that high it was out of this thesis time budget to test multiple runs of the RLS_4^B with higher input sizes.

Figure 5.25.: Runtime of the RLS_k^S variants for the OneMax equivalent with uniform distribution on a $n\ln n$ scale

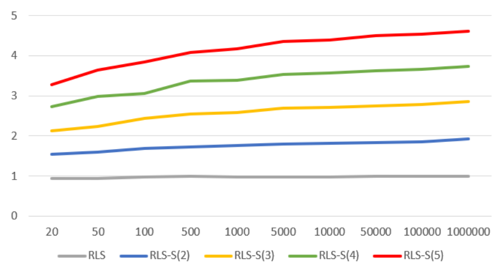
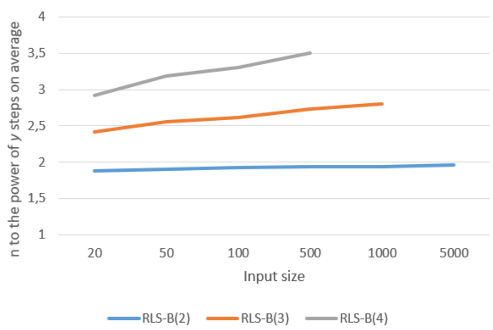


Figure 5.26.: Runtime of the RLS_k^B variants for the OneMax equivalent with uniform distribution on a n^y scale



5.8. Carsten Witt's worst case input

This input is the worst case in put as discussed in the background section. As all researched inputs in this paper contained only integer values this input is adjusted a bit. To prevent the small values to be below zero they are instead normalised to 1. The two big values are scaled by the same factor of $((1/3 + \epsilon/2)/(n-2))^{-1}$. The higher the value for ϵ the more likely the input is to get stuck in the local optima. With increasing ϵ the local optima becomes better and better. For the small values of ϵ there were only a few cases where

some algorithms did not find an optimal solution. To make this effect more visible the value of ϵ was set to $\epsilon = 0.3$.
For $n = 10,000$ this evaluates to $w_1 = w_2 = 5344$ and $W = 9998 \cdot 1 + 2 \cdot 5344 = 20686$. The input then looks like this: $[1, 1, \dots, 1, 1, 5344, 5344]$. The fitness of the local optimum is $f(x) = 2 \cdot 5433 = 10688$. To leave the local optimum the algorithm therefore has to flip at least $5433 + 9998 - 10688 = 4654$ bits as well in the same step. The best fitness is $f(x) = 5344 + 9998/2 = 10343$, which leads to a difference of $f(\mathit{localOptimum}) - f(\mathit{opt}) = 345$ and a approximation ratio of $f(\mathit{localOptimum})/f(\mathit{opt}) = 10688/10343 = 1.033$. This is not really close to the worst case of 4/3 any more but with this setting at least many algorithms are stuck in the local optimum at least once for the 1000 runs.

5.8.1. RLS Comparison

algo type	RLS_b^B	RLS_b^B	RLS_b^B	RLS_s^S	RLS_s^S	RLS_s^S	RLS
algo param	b=4	b=3	b=2	s=4	s=3	s=2	-
avg mut/change	3.997	3.000	1.998	2.344	1.967	1.307	1.000
avg mut/step	4.000	3.000	2.000	2.501	2.000	1.500	1.000
total avg count	4,587	4,844	9,607	11,503	12,841	29,741	114,452
avg eval count	4,587	1,165	6,864	1,387	1,810	2,176	2,376
max eval count	56,029	10,171	75,524	11,060	11,544	12,930	12,023
min eval count	0	0	0	0	0	0	0
fail ratio	0.000	0.004	0.003	0.011	0.012	0.030	0.122
avg fail dif	-	382	401	345	345	345	345

The RLS is by far most likely to get stuck in the local optimum. The general tendency is the more bits the algorithm flips in expectation the more unlikely the local optimum becomes. The only case where this does not hold is the RLS_2^B being better than the RLS_4^S, although the RLS_4^S flips more bits in expectation. This might be caused by the higher probability for flipping only one bit for the RLS_4^S. This causes the algorithm to find the local optimum faster before separating w_1 and w_2. So the ranking of the algorithms is completely inverted compared to the OneMax input. The RLS_2^B and RLS_3^B both had runs where they neither found the global nor one of the two local optima. The algorithms were most likely tricked into the direction of the local optimum and did not manage to leave it. But they were also not fast enough to reach the local optimum because of their low probability to flip only one bit.

5.8.2. (1+1) EA Comparison

algo type	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA
algo param	$100/n$	$50/n$	$10/n$	$5/n$	$4/n$	$3/n$	$2/n$	-
avg mut/change	100.074	49.998	10.024	5.083	4.110	3.130	2.224	1.438
avg mut/step	100.095	50.024	9.995	5.001	4.002	2.999	2.000	1.000
total avg count	64	106	414	855	1,024	4,980	9,252	38,213
avg eval count	64	106	414	855	1,024	1,301	1,899	3,341
max eval count	407	898	4,026	9,068	10,830	10,540	13,135	19,361
min eval count	0	1	0	0	0	0	0	0
fail ratio	0.000	0.000	0.000	0.000	0.000	0.004	0.008	0.038
avg fail dif	-	-	-	-	-	345	345	345

For the (1+1) EA the result is also the inversion of the results for the OneMax equivalent. The higher the mutation rate the faster the algorithms reach a global optimum. This holds at least up to $p_m \leq 100/n$. With mutation rate $p_m \leq 3/n$ the algorithm reaches the worst case at least once in 1000 runs. If the algorithm did not manage to find an optimal solution the fitness was always the same. So there was no run where any algorithm neither found a global nor the local optimum.

5.8.3. pmut Comparison

algo type	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut
algo param	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25
avg mut/change	192.353	68.078	23.956	8.809	4.373	2.860	2.111	1.770	1.578
avg mut/step	220.467	68.897	22.423	8.578	4.410	2.922	2.271	1.934	1.729
total avg count	41	88	211	493	956	2,272	13,622	16,503	20,333
avg eval count	41	88	211	493	956	1,353	1,670	1,795	1,952
max eval count	212	753	1,704	5,933	8,957	19,703	11,437	11,593	12,032
min eval count	0	0	0	0	0	0	0	0	0
fail ratio	0.000	0.000	0.000	0.000	0.000	0.001	0.013	0.016	0.020
avg fail dif	-	-	-	-	-	345	345	345	345

For *pmut* the result is the exact same as for the (1+1) EA. The lower β the better the performance as more bits are flipped in each step. The algorithm is tricked into the local optima only for $\beta \leq 2.5$. If the algorithm is on the path to the local optimum it is always fast enough to reach it within the time limit.

5.8.4. Comparison of the best variants

algo type	pmut	EA-SM	RLS_b^B
algo param	1.25	$100/n$	b=4
avg mut/change	195.454	99.875	3.998
avg mut/step	224.523	99.980	4.000
avg eval count	43	67	4,458
max eval count	240	521	43,317
min eval count	0	0	0
fail ratio	0.000	0.000	0.000

The $pmut_{1.25}$ and the (1+1) EA with $p_m = 100/n$ perform the best and always find an optimal solution within 600 iterations and even under 100 on the average case. The RLS_4^B performs significantly worse. For the OneMax input $pmut_{1.25}$ flipped 224 bits on average per step, but the average of the successful steps was only 5. Here the average of all steps is again 224, but the average of the successful steps is at 194. The heavy tail here really increases the performance as most of the high values are accepted. In the experiment with different input sizes the mutation rate of $p_m = 100/n$ is ≥ 1 for $n \leq 100$. If the algorithm flips every bit then it won't change its solution. In these cases the mutation rate was then set to $p_m = 1/2$.

fails in 1000 runs	20	50	100	500	1000	5000	10000	50000
RLS_4^B	4	0	3	0	0	0	0	0
(1+1) EA ($100/n$)	0	0	0	0	0	0	0	0
pmut (1.25)	0	0	0	0	0	0	0	0

Only the RLS variant had runs where it did not reach a global optimum. This happened in less than 0.5 % of the inputs for $n = 20$ and $n = 100$. For the other input sizes it also managed to reach a global optimum for all inputs.

avg	20	50	100	500	1000	5000	10000	50000
RLS_4^B	12	26	53	221	442	2166	4194	20690
(1+1) EA ($100/n$)	10	16	24	32	34	47	69	222
pmut (1.25)	7	10	11	19	23	35	43	63

For the lower input sizes the RLS is slower than the remaining algorithm even it manages to find a global optimum.

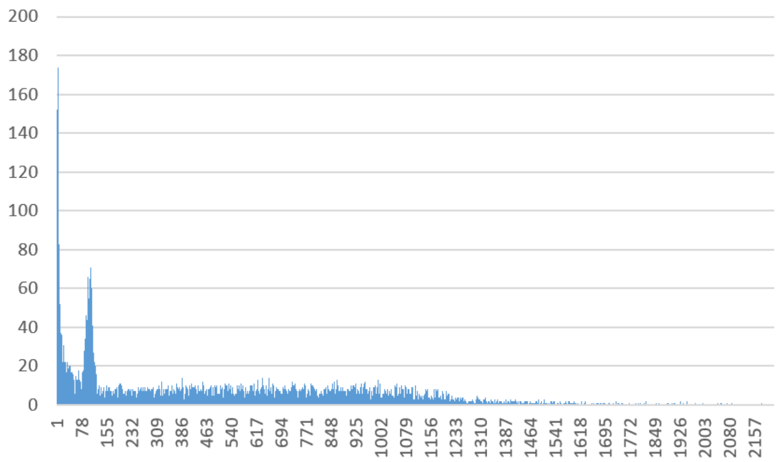
total avg	20	50	100	500	1000	5000	10000	50000
RLS_4^B	412	26	353	221	442	2166	4194	20690
(1+1) EA ($100/n$)	10	16	24	32	34	47	69	222
pmut (1.25)	7	10	11	19	23	35	43	63

The $pmut_{1.25}$ is the best variant closely followed by the (1+1) EA. The RLS version is by far slower than the other to variants for the bigger input sizes. Even for the smaller inputs it is still slower.

5.9. Multiple distributions combined

This subsection covers inputs that do not follow a specific distribution. So instead of sampling from one distribution here instead the value is chosen from a set of distributions. The used distributions were $\sim U(1, 49999)$, $\sim B(10000, 0.1)$, $\sim Geo(0.001)$, $\sim D_{50000}^{1.25}$. Each distribution is chosen with probability 1/8 and with remaining probability 1/2 one value is drawn from every distribution and added together.

Figure 5.27.: Distribution of a mixed and overlapped input with $\sim U(1, 999)$, $\sim B(1000, 0.1)$, $\sim Geo(0.01)$, $\sim D_{1000}^{1.25}$



An input of this distribution is shown in figure 5.27. There are a lot of values close to 0 drawn from the powerlaw (and geometric) distribution. The amount of values then decreases to around 50. From then on the amount of generated numbers rises again until 100, the expected value of the binomial distribution. After the spike caused by the binomial distribution the values look more uniform distributed until 1200 where they fall of again. The higher the numbers get the less often they occur.

5.9.1. RLS Comparison

algo type	RLS	RLS_s^S	RLS_s^S	RLS_s^S	RLS_b^B	RLS_b^B	RLS_b^B
algo param	-	s=2	s=3	s=4	b=2	b=3	b=4
avg mut/change	1.000	1.435	1.833	2.221	2.000	2.999	4.000
avg mut/step	1.000	1.501	1.999	2.501	2.000	3.000	4.000
avg eval count	432	598	830	1,073	2,831	17,483	55,532
max eval count	1,534	2,915	6,576	4,947	18,933	137,135	504,436
min eval count	40	35	81	51	52	103	148
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000

There is a clear preference of algorithms with higher probability to flip only one bit per step similar to the geometric distribution. For geometric distributions all variants had a rather equal runtime but here the RLS_4^B needs more than 100 the iterations on average than the fastest RLS variant. This type of input punishes the worse mutation operators more than geometric inputs but not as extreme as the equivalent to linear functions. Here still every variant reaches an optimal solution in every case as opposed to the linear function equivalent.

5.9.2. (1+1) EA Comparison

algo type	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM	EA-SM
algo param	-	$2/n$	$3/n$	$4/n$	$5/n$	$10/n$
avg mut/change	1.479	2.057	2.720	3.444	4.293	9.397
avg mut/step	1.000	1.999	3.001	4.001	5.001	10.000
avg eval count	892	991	1,526	2,708	4,980	91,623
max eval count	3,588	4,475	6,059	13,334	25,079	721,020
min eval count	40	78	99	117	77	84
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000

The results here are pretty similar to the results of the RLS. The lower the mutation rates the faster the algorithm reaches an optimal solution. Even the worst analysed mutation rate managed to find an optimal solution in $10n \ln n$ steps in all 1000 runs.

5.9.3. pmut Comparison

algo type	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut	pmut
algo param	3.25	3.00	2.75	2.50	2.25	2.00	1.75	1.50	1.25
avg mut/change	1.604	1.751	1.977	2.424	3.266	5.140	13.276	31.477	76.065
avg mut/step	1.731	1.935	2.274	2.901	4.382	8.463	22.347	70.468	224.657
avg eval count	574	606	616	670	740	860	1,012	1,297	2,201
max eval count	1,990	2,304	2,109	2,625	2,968	3,283	4,888	6,196	10,991
min eval count	32	70	47	55	67	65	33	88	64
fail ratio	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Here the same holds. The higher mutation rates are less impacted than the higher rates for the RLS and the (1+1) EA.

5.9.4. Comparison of the best variants

algo type	RLS	pmut	EA
algo param	-	3.25	-
avg mut/change	1.000	1.593	1.485
avg mut/step	1.000	1.731	1.000
avg eval count	436	561	917
max eval count	1,556	2,359	4,134
min eval count	46	48	28
fail ratio	0.000	0.000	0.000

The ranking of the algorithm is the same as for the other inputs with a similar preference of low mutation rates. The RLS has the best performance closely follow by $pmut_{3.25}$ and lastly be the standard (1+1) EA.

fails in 1000 runs	20	50	100	500	1000	5000	10000	50000
RLS	997	945	736	51	2	0	0	0
RLS_2^S	972	625	211	0	0	0	0	0
(1+1) EA ($1/n$)	868	436	146	0	0	0	0	0
(1+1) EA ($2/n$)	757	277	81	0	0	0	0	0
pmut (3.0)	781	364	120	0	0	0	0	0
pmut (3.25)	787	417	128	1	0	0	0	0

No unexpected results for the different input sizes. The RLS variants perform the worst for $n \leq 100$. For this input it is also rather hard to find a perfect partition for $n \leq 50$. For the lower values there were probability multiple inputs generated with mostly small values and an uneven number of large values. Even if the amount of large numbers is even there might still be no perfect partition.

avg	20	50	100	500	1000	5000	10000	50000
RLS	60	169	261	425	390	404	436	570
RLS_2^S	412	1889	3552	791	626	591	607	695
(1+1) EA ($1/n$)	19040	15992	9758	1463	883	874	914	1105
(1+1) EA ($2/n$)	20169	16391	8562	1536	1088	1012	1018	1076
pmut (3.0)	26596	17183	8399	1060	606	578	578	691
pmut (3.25)	25871	16012	7669	1229	593	547	553	673

The input gets easier to solve up until $n = 5000$ and from then on gets harder again with increasing input size. The increase for larger input sizes is much smaller than the decrease for the small values.

total avg	20	50	100	500	1000	5000	10000	50000
RLS	99700	94509	73669	5503	589	404	436	570
RLS_2^S	97211	63208	23902	791	626	591	607	695
(1+1) EA ($1/n$)	89313	52619	22933	1463	883	874	914	1105
(1+1) EA ($2/n$)	80601	39550	15969	1536	1088	1012	1018	1076
pmut (3.0)	83924	47328	19391	1060	606	578	578	691
pmut (3.25)	84210	51035	19488	1328	593	547	553	673

This type of inputs are best solved by the (1+1) EA with $p_m = 2/n$ for $n \leq 100$ and also relatively good for $n = 500$. After $n \geq 1000$ the standard RLS becomes the best option and it seems like it stays that way for the remaining input sizes.

5.10. Conclusion of empirical results

There is no clear best algorithm for every input for PARTITION and not even a best parameter for every algorithm. For inputs that are comparable to a linear function/OneMax for all base algorithms the lowest parameters have the lowest runtime. Other instance like the worst case input of C. Witt on the other hand require much higher mutation rates for the optimal performance. Inputs generated from a powerlaw distribution showed that the optimal parameter for every algorithm is not even fixed with a specific distribution. For inputs drawn from $\sim D^{2.75}_{50000}$ the higher mutation rates reached an optimum faster than the lower mutation rate for ever algorithm variant. If the input was drawn from $\sim D^{1.25}_{50000}$ then the fastest mutation rates for the (1+1) EA on $\sim D^{2.75}_{50000}$ distributed inputs then instead become the slowest. So almost no general advice is possible, but a few points still hold for every input type. The first one is the RLS being most likely to be stuck in a local optimum especially for the smaller input size. Even if a variant of the RLS is the fastet for the bigger input sizes it is most likely to be stuck in a local optimum for $n \leq 100$ for most input type. So if the input size $n \leq 100$ choosing the (1+1) EA or *pmut* mutation operator is a better choice. Another noticeable relation is that inputs that require higher mutation rates are general more easy to solve and are also solved very fast by the lower mutation rates. The lower mutation rates are therefore generally a better choice as they are sometimes the faster and are still rather fast if they are not. Only if the algorithm is trapped due to its low mutation rate a higher mutation rate makes a huge difference.

Table 5.1 recaps the results of the previous subsections. For each input and each algorithm the best three variants are listed in this table ordered by their average runtime.

Table 5.2 list my personal preference based on the previous results depending on the distribution and size of the input.

Table 5.1.: Best algorithms variants for all researched inputs

	RLS variants			(1+1) EA variants			*pmut* variants		
	1st	2nd	3rd	1st	2nd	3rd	1st	2nd	3rd
binomial	RLS_2^B	RLS_4^B	RLS_2^S	$3/n$	$4/n$	$2/n$	2.25	2.0	1.75
geometric	RLS_2^S	RLS_3^S	RLS_4^S	$2/n$	$1/n$	$3/n$	3.25	3.0	2.5
polwerlaw	RLS_4^S	RLS_3^S	RLS_3^B	$4/n$	$3/n$	$2/n$	1.5	1.75	2.0
uniform	RLS_2^B	RLS_3^S	RLS_4^S	$3/n$	$2/n$	$4/n$	2.5	2.0	2.25
linear function	RLS	RLS_2^S	RLS_3^S	$1/n$	$2/n$	$3/n$	3.5	3.25	3.0
worst case	RLS_4^B	RLS_3^B	RLS_2^B	$100/n$	$50/n$	$10/n$	1.25	1.5	1.75
combined	RLS	RLS_2^S	RLS_3^S	$1/n$	$2/n$	$3/n$	3.25	3.0	2.75

Table 5.2.: Suggestions for the fastest algorithm on each input depending on the input size

	$n \leq 100$	$100 < n < 500$	$500 \leq n < 10,000$	$10,000 \leq n$
binomial	(1+1) EA $_{3/n}$		RLS_2^B	
geometric	(1+1) EA $_{2/n}$		$pmut_{3.25}$	
uniform	(1+1) EA $_{4/n}$		RLS_2^B	
powerlaw	$pmut_{1.75}$			
linear function	RLS			
worst case	$pmut_{1.25}$			
combined	(1+1) EA $_{2/n}$	$pmut_{1.25}$		RLS

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Appendix

A. Appendix Section 1

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Figure A.1.: A figure