Theoretical and empirical runtime analysis of evolutionary algorithms for the partition problem

Bachelor Thesis of

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract

A short summary of what is going on here.

Deutsche Zusammenfassung

Kurze Inhaltsangabe auf deutsch.

Contents

1. Introduction

This chapter should contain

1. A short description of the thesis topic and its background.

2. An overview of related work in this field.

3. Contributions of the thesis.

4. Outline of the thesis.

2. Preliminaries

2.1. Notations

1. **RLS:** Randomised Local Search

2. **RSH:** Randomised Search Heuristic referring to all analysed Evolutionary algorithms

3. n**:** The input length of the problem

4. w_i**:** The i-th object of the input. If not mentioned otherwise the weights are sorted in non-increasing order so: $w_1 \geq w_2 \geq \ldots \geq w_{n-1} \geq w_n$

5. W**:** The sum of all objects: $W = \sum_{i=1}^{n} w_i$

6. **bin:** When solving Partition a set of numbers is divided into two distinct subsets and in this paper both subsets are referred to as bins

7. b_F**:** The fuller bin (the bin with more total weight)

8. b_E**:** The emptier bin (the bin with less total weight)

9. b_{w_i}**:** The bin containing the object w_i

10. opt**:** The optimal solution for a given partition instance.

11. x**:** A vector $x \in \{0,1\}^n$ describing a solution

3. Content Chapters

3.1. Improving bounds on the RLS and the (1+1) EA

Lemma 3.1. *If $w_1 \geq \frac{W}{2}$ then the RLS and the (1+1) EA reach the optimal value in expected time $\Theta(n \log n)$*

Proof. The optimal solution is putting w_1 in one bin and all other elements in the other bin. So the problem is almost identical to OneMax/ZeroMax. A single bit flip of the first bit can only happen, if the emptier bin has a weight of at most $\frac{W-w_1}{2}$. After this flip the weight of the emptier bin is at least $\frac{W-w_1}{2}$ and therefore another single bit flip of w_1 can only happen before a different bit is flipped. After a different bit has been flipped, the RLS wont flip the first bit again, because it will never result in an improvement. So the run of the RLS can be divided into three phases:

Phase 1: The RLS behaves exactly like OneMax/ZeroMax and flips every bit to the opposite of the first bit (except for the first bit).

Phase 2: The RLS flips only the first bit or bits that do not result in an improvement.

Phase 3: The RLS behaves exactly like ZeroMax/OneMax and flips every bit to the opposite of the first bit (except for the first bit).

The expected length of the first phase is $\mathcal{O}(n)$ because the probability of flipping the first bit is at least $\frac{1}{n} \cdot (1 - \frac{1}{n})^{n-1} \geq \frac{1}{en}$ and therefore the expected time for such a step is at most $\mathcal{O}(\frac{1}{en}^{-1}) = \mathcal{O}(ne) = \mathcal{O}(n)$.
The length of the second phase is $\mathcal{O}(n)$ because the solution is either optimal or there is at least one bit that needs to be flipped for an optimal solution. Since the expected length of Phase 1 is $\mathcal{O}(n)$ the solution produced by the RLS won't be optimal in expectation due to the bound of $\Theta(n \log n)$ for OneMax/ZeroMax. This again results in expected time $\mathcal{O}(n)$.
The length of the third phase is identical to a run of the RLS on OneMax/ZeroMax where flips of the first bit are ignored as if it was already correctly flipped and therefore the expected time is $\Theta(n \log n)$

So the total expected time is $\mathcal{O}(n) + \mathcal{O}(n) + \Theta(n \log n) = \Theta(n \log n)$

The (1+1) EA can do multiple bit flips in a single step so the first bit can be flipped multiple times if the combined moved weight $y \leq b_F - b_E$. **TODO: insert proof for (1+1) EA.** $\qquad\square$

Lemma 3.2. *If $b_F \leq \frac{2}{3} \cdot W$ the approximation ratio is at most $\frac{4}{3}$*

Proof. $\frac{b_F}{opt} \leq \frac{(2/3) \cdot W}{opt} \leq \frac{(2/3) \cdot W}{(1/2) \cdot W} = \frac{4}{3}$, since $opt \geq \frac{W}{2}$ $\qquad\qquad\square$

Corollary 3.3. *If $w_1 \geq \frac{W}{3}$ and w_1 is in the emptier bin, then the approximation ratio is at most $\frac{4}{3}$*

Proof. w_1 is in the emptier bin, so $b_F \leq W - w_1 \leq W - \frac{W}{3} = \frac{2W}{3}$ and with Lemma 3.2 the assumption follows. $\qquad\qquad\square$

Lemma 3.4. *Any object of weight at most v can be moved from b_F to b_E if $b_F - b_E \geq v$*

Proof. $b_F - b_E \geq v \Leftrightarrow b_F \geq b_E + v$, so after moving an object with weight at most v from b_F to b_E, the new weight of b_E is at most the weight of b_F before moving the object, thus the RSH accepts the step. $\qquad\qquad\square$

Corollary 3.5. *The RLS is stuck in a local optima if $b_F - b_E < w_n$ holds and $b_F > opt$.*

Proof. A single bit flip of weight v can only happen if $b_F - b_E \geq v$. If $b_F - b_E < w_n$ there is no weight which satisfies the condition and therefore no single bit flip is possible. Since the RLS can only move one bit at a time and only if it results in an improvement, the RLS is stuck. $\qquad\qquad\square$

Corollary 3.6. *Every object $\leq \frac{W}{3}$ can be moved from b_F to b_E if $b_F \geq \frac{2W}{3}$*

Proof. $b_F \geq \frac{2W}{3} \Rightarrow b_E \leq W - \frac{2W}{3} \leq \frac{W}{3} \Rightarrow b_F - b_E \geq \frac{2W}{3} - \frac{W}{3} = \frac{W}{3}$ and with Lemma 3.4 the assumption follows. $\qquad\qquad\square$

Lemma 3.7. *In expected Time $\mathcal{O}(n \log n)$ the weight of the fuller bin can be decreased to $\leq \frac{2W}{3}$ if every object besides the biggest in the fuller bin is at most $\frac{W}{3}$ and $w_1 \leq \frac{W}{2}$.*

Proof. In expected time $\mathcal{O}(n \log n)$ the RSH can move every object $\leq \frac{W}{3}$ to the emptier bin as long as $b_F \geq \frac{2W}{3}$ due to Corollary 3.6 and results for OneMax. So in expected Time $\mathcal{O}(n \log n)$ the solution can be shifted to w_1 being in one bin and all other objects in the other bin. The RSH will only stop moving the elements if the condition $b_F \geq \frac{2W}{3}$ is no longer satisfied (Corollary 3.6). If $w_1 \geq \frac{W}{3}$ and every object was moved to the bin without w_1, then $b_F = \max\{W - w_1, w_1\} = W - w_1 \leq \frac{2W}{3}$, because $w_1 \leq \frac{W}{2}$. So either the RSH moves all objects to the emptier bin or stops moving objects because $b_F < \frac{2W}{3}$ both resulting in $b_F \leq \frac{2W}{3}$. If w_1 is not in the fuller bin, then the result follows by Corollary 3.3. Now assume $w_1 < \frac{W}{3}$. In this case the RLS will move one object per step to the emptier bin. Each object has weight $< \frac{W}{3}$ and therefore one step can not decrease the weight of the fuller bin from $> \frac{2W}{3}$ to $\leq \frac{W}{3}$. If all objects except the biggest where moved the other bin, the other bin would have a weight of at least $W - w_1 > \frac{2W}{3}$. Therefore the RLS will find a solution with $b_F < \frac{2W}{3}$ before moving all elements from the first to the second bin. ***TODO: insert proof for (1+1) EA*** $\qquad\square$

Lemma 3.8. *The RLS and the (1+1) EA reach an approximation ratio of at most $\frac{4}{3}$ in expected time $\mathcal{O}(n \log n)$ if $w_1 < W/2$*

Proof. If $w_1 + w_2 > \frac{2W}{3}$ after time $\mathcal{O}(n)$ w_1 and w_2 are separated and will remain separated afterwards (Proof by C.Witt [Die05]). From then on the following holds. If w_1 is in the emptier bin, then the result follows directly by Corollary 3.3. Otherwise all elements in the fuller bin except w_1 have a weight of at most $\frac{1}{3}$ and therefore the result follows by Lemma 3.7 and Lemma 3.2. If $w_1 + w_2 \leq \frac{2W}{3}$ the result follows directly by Lemma 3.7 and Lemma 3.2. $\qquad\square$

Corollary 3.9. *The RLS and the (1+1) EA reach an approximation ratio of at most $\frac{4}{3}$ in expected time $\mathcal{O}(n \log n)$*

Proof. This follows directly from Lemma 3.1 and Lemma 3.8 $\qquad\square$

3.2. Binomial distributed input

Lemma 3.10. *A binomial distributed input $\tilde{B}(m,p)$ has an optimal solution with high probability if n is large enough.*

Proof. Sketch:

- The initial distribution is likely rather close to the optimum
- The difference between the bins is probably not more than 10 expected values
- the large values

$\qquad\square$

Lemma 3.11. *With high probability the RLS does not find an optimal solution for an input with distribution $\tilde{B}(m,p)$ if n and m are large enough.*

Proof. Sketch:

- There exists an optimal solution with high probability due to last lemma
- probability for a value to be very low is almost 0 if m is huge
- The RLS only moves one element per step and will step below bF-bE < wn without bF = opt being true
- -> RLS cant make another step and is stuck in a local optimum.

Due to Lemma 3.10 the input has an optimal solution with high probability. $\qquad\square$

4. Conclusion

Summary and outlook.

Bibliography

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Appendix

A. Appendix Section 1

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Figure A.1.: A figure