Modeling II
Linear Stability Analysis
and
Wave Equations

Nondimensional Equations

From previous lecture, we have a system of nondimensional PDEs:

$$\frac{\partial c}{\partial T} = \frac{\partial}{\partial X}\left(D_c \frac{\partial c}{\partial X}\right) - \frac{\partial}{\partial X}\left(\chi_u \, c \, \frac{\partial u}{\partial X}\right) - \frac{\partial}{\partial X}\left(\chi_v \, c \, \frac{\partial v}{\partial X}\right) + \mu_c \, c(1-c) \quad \textbf{(21.1)}$$

$$\frac{\partial v}{\partial T} = -\gamma \, u \, v + \mu_v \, v(1-v) \quad \textbf{(21.2)}$$

$$\frac{\partial u}{\partial T} = \frac{\partial}{\partial X}\left(D_u \frac{\partial u}{\partial X}\right) + k_c \, c - k_u \, u \quad \textbf{(21.3)}$$

where here the “*” sign has been dropped for convenience.

Parameters

The parameter values:

$$D_c = 10^{-4} \qquad \mu_c = 0.25 \qquad k_c = 0.05$$

$$D_u = 10^{-2} \qquad \mu_v = 0.15 \qquad k_u = 0.3$$

$$\chi_u = 0.05 \qquad \chi_v = 0.035 \qquad \gamma = 8.15$$

Initial conditions:

$$c(X,0) = \exp(-|X|^2/0.01)$$

$$v(X,0) = 1 - \exp(-|X|^2/0.01)$$

$$u(X,0) = \frac{1}{2}\exp(-|X|^2/0.01)$$

Boundary conditions: zero-flux on all boundaries.

Homogeneous Steady States

A homogeneous steady state of a PDE model is a solution that is constant in space in time. For equations (21.1), (21.2), and (21.3) we take

$$\frac{\partial c}{\partial T} = \frac{\partial v}{\partial T} = \frac{\partial u}{\partial T} = 0$$

$$\frac{\partial c}{\partial X} = \frac{\partial v}{\partial X} = \frac{\partial u}{\partial X} = 0$$

leaving

$$0 = \mu_c \, c_{\mathrm{ss}} (1 - c_{\mathrm{ss}}) \quad \textbf{(21.4)}$$

$$0 = -\gamma \, u_{\mathrm{ss}} \, v_{\mathrm{ss}} + \mu_v v_{\mathrm{ss}} (1 - v_{\mathrm{ss}}) \quad \textbf{(21.5)}$$

$$0 = k_c \, c_{\mathrm{ss}} - k_u u_{\mathrm{ss}} \quad \textbf{(21.6)}$$

Homogeneous Steady States

Possibilities of steady state points:

$$c_{\text{ss}} = 0 \longrightarrow u_{\text{ss}} = 0 \text{ and } v_{\text{ss}} = 0 \text{ or } v_{\text{ss}} = 1$$

$$c_{\text{ss}} = 1 \longrightarrow u_{\text{ss}} = \frac{k_c}{k_u} \text{ and } \; v_{\text{ss}} = 0 \text{ or } v_{\text{ss}} = -\frac{\gamma}{\mu_v} u_{\text{ss}} + 1$$

Hence, there are four possible steady state points:

$$\left.\begin{aligned}
(c_{\text{ss}}, v_{\text{ss}}, u_{\text{ss}}) &= (0, 0, 0) \\
(c_{\text{ss}}, v_{\text{ss}}, u_{\text{ss}}) &= (0, 1, 0) \\
(c_{\text{ss}}, v_{\text{ss}}, u_{\text{ss}}) &= \left(1, 0, \frac{k_c}{k_u}\right) \\
(c_{\text{ss}}, v_{\text{ss}}, u_{\text{ss}}) &= \left(1, 1 - \frac{\gamma\, k_c}{\mu_v\, k_u}, \frac{k_c}{k_u}\right)
\end{aligned}\right\} \quad \textbf{(21.7)}$$

Inhomogeneous Perturbations

- It is of interest to determine whether or not the steady states are stable.
- We analyze stability properties by considering the effect of small perturbations.
- To do this, we must look at spatially non-uniform (also called inhomogeneous) perturbations and explore whether they are amplified or attenuated.
- If an amplification occurs, then a situation close to the spatially uniform steady state will destabilize, leading to some new state in which spatial variations predominate.

Inhomogeneous Perturbations

We take the distributions of the variables

$$u(X,T) = u_{\text{ss}} + \tilde{u}(X,T)$$

$$v(X,T) = v_{\text{ss}} + \tilde{v}(X,T)$$

$$c(X,T) = c_{\text{ss}} + \tilde{c}(X,T)$$

where $\tilde{c}, \tilde{v},$ and $\tilde{u}$ are small.

Using the facts that $c_{\text{ss}}, v_{\text{ss}},$ and u_{ss} are constants and uniform, the temporal and spatial derivatives give

$$\frac{\partial c}{\partial X} = \frac{\partial (c_{\text{ss}} + \tilde{c})}{\partial X} = \frac{\partial \tilde{c}}{\partial X} \qquad \frac{\partial c}{\partial T} = \frac{\partial (c_{\text{ss}} + \tilde{c})}{\partial T} = \frac{\partial \tilde{c}}{\partial T}$$

Inhomogeneous Perturbations

The second-order spatial derivatives:

$$\frac{\partial}{\partial X}\left(D_c\frac{\partial c}{\partial X}\right) = \frac{\partial}{\partial X}\left(D_c\frac{\partial(c_{\mathrm{ss}}+\tilde{c})}{\partial X}\right) = D_c\frac{\partial^2\tilde{c}}{\partial X^2} \quad \textbf{(21.8)}$$

For taxis terms:

$$\begin{aligned}\frac{\partial}{\partial X}\left(\chi_u c\frac{\partial u}{\partial X}\right) &= \frac{\partial}{\partial X}\left(\chi_u(c_{\mathrm{ss}}+\tilde{c})\frac{\partial(u_{\mathrm{ss}}+\tilde{u})}{\partial X}\right)\\ &= \frac{\partial}{\partial X}\left(\chi_u c_{\mathrm{ss}}\frac{\partial\tilde{u}}{\partial X}+\chi_u\tilde{c}\frac{\partial\tilde{u}}{\partial X}\right)\\ &= \chi_u c_{\mathrm{ss}}\frac{\partial^2\tilde{u}}{\partial X^2}+\chi_u\frac{\partial\tilde{c}}{\partial X}\frac{\partial\tilde{u}}{\partial X}+\chi_u\tilde{c}\frac{\partial^2\tilde{u}}{\partial X^2}\end{aligned}$$

Inhomogeneous Perturbations

The terms

$$\frac{\partial \tilde{c}}{\partial X}\frac{\partial \tilde{u}}{\partial X} \quad \text{and} \quad \tilde{c}\,\frac{\partial^2 \tilde{u}}{\partial X^2}$$

are quadratic in the perturbations or their derivatives and consequently are of smaller magnitude than other terms, thus they can be omitted, leaving

$$\frac{\partial}{\partial X}\left(\chi_u c \frac{\partial u}{\partial X}\right) = \chi_u c_{\mathrm{ss}} \frac{\partial^2 \tilde{u}}{\partial X^2} \qquad \textbf{(21.9)}$$

and, similarly

$$\frac{\partial}{\partial X}\left(\chi_v c \frac{\partial v}{\partial X}\right) = \chi_v c_{\mathrm{ss}} \frac{\partial^2 \tilde{v}}{\partial X^2} \qquad \textbf{(21.10)}$$

Inhomogeneous Perturbations

And for the reaction terms:

$$
\begin{aligned}
\mu_c c(1-c) &= \mu_c(c_{\mathrm{ss}} + \tilde{c})(1 - c_{\mathrm{ss}} - \tilde{c}) \\
&= \mu_c\, c_{\mathrm{ss}}(1 - c_{\mathrm{ss}} - \tilde{c}) + \mu_c\, \tilde{c}(1 - c_{\mathrm{ss}} - \tilde{c}) \\
&= \mu_c\, c_{\mathrm{ss}}(1 - c_{\mathrm{ss}}) + \mu_c\, \tilde{c} - 2\mu_c\, c_{\mathrm{ss}}\, \tilde{c} - \mu_c\, \tilde{c}^2 \\
&= \mu_c\, \tilde{c}(1 - 2c_{\mathrm{ss}})
\end{aligned}
$$

$$
-\gamma\, u\, v + \mu_v\, v(1-v) = -\gamma(u_{\mathrm{ss}}\tilde{v} + v_{\mathrm{ss}}\tilde{u}) + \mu_v\, \tilde{v}\,(1 - 2v_{\mathrm{ss}})
$$

$$
k_c\, c - k_u\, u = k_c\, \tilde{c} - k_u\, \tilde{u}
$$

Inhomogeneous Perturbations

Combining all together we rewrite the approximate linearized equations of (21.1) – (21.3) as

$$\frac{\partial \tilde{c}}{\partial T} = D_c \frac{\partial^2 \tilde{c}}{\partial X^2} - \chi_u \, c_{\mathrm{ss}} \frac{\partial^2 \tilde{u}}{\partial X^2} - \chi_v \, c_{\mathrm{ss}} \frac{\partial^2 \tilde{v}}{\partial X^2} + \mu_c \, \tilde{c} \, (1 - 2c_{\mathrm{ss}}) \tag{21.11}$$

$$\frac{\partial \tilde{v}}{\partial T} = -\gamma (u_{\mathrm{ss}} \tilde{v} + v_{\mathrm{ss}} \tilde{u}) + \mu_v \, \tilde{v} \, (1 - 2v_{\mathrm{ss}}) \tag{21.12}$$

$$\frac{\partial \tilde{u}}{\partial T} = D_u \frac{\partial^2 \tilde{u}}{\partial X^2} + k_c \, \tilde{c} - k_u \, \tilde{u} \tag{21.13}$$

which are linear in the quantities $\tilde{c}, \tilde{v},$ and $\tilde{u}$.

Finding Eigenvalues

We find eigenvalues by setting the equations (21.11), (21.12), and (21.13) to have no spatial variations, or

$$\frac{d\tilde{c}}{dT} = \mu_c\,\tilde{c}\,(1 - 2c_{\mathrm{ss}}) \qquad \textbf{(21.14)}$$

$$\frac{d\tilde{v}}{dT} = -\gamma\,u_{\mathrm{ss}}\,\tilde{v} - \gamma\,v_{\mathrm{ss}}\,\tilde{u} + \mu_v\,\tilde{v}\,(1 - 2v_{\mathrm{ss}}) \qquad \textbf{(21.15)}$$

$$\frac{d\tilde{u}}{dT} = k_c\,\tilde{c} - k_u\,\tilde{u} \qquad \textbf{(21.16)}$$

Then let

$$\frac{d\tilde{c}}{dT} = F(\tilde{c}, \tilde{v}, \tilde{u}) \qquad \frac{d\tilde{v}}{dT} = G(\tilde{c}, \tilde{v}, \tilde{u}) \qquad \frac{d\tilde{u}}{dT} = H(\tilde{c}, \tilde{v}, \tilde{u})$$

Finding Eigenvalues

Differentiating $F(\tilde{c}, \tilde{v}, \tilde{u}), G(\tilde{c}, \tilde{v}, \tilde{u}),$ and $H(\tilde{c}, \tilde{v}, \tilde{u})$ with respect to $\tilde{c}, \tilde{v},$ and $\tilde{u}$ gives us a Jacobian matrix of the reaction terms:

$$J_{\mathrm{R}} = \begin{bmatrix} \mu_c(1 - 2c_{\mathrm{ss}}) & 0 & 0 \\ 0 & -\gamma u_{\mathrm{ss}} + \mu_v(1 - 2v_{\mathrm{ss}}) & -\gamma v_{\mathrm{ss}} \\ k_c & 0 & -k_u \end{bmatrix} \tag{21.17}$$

Eigenvalues are obtained by taking

$$|J_{\mathrm{R}} - \lambda I| = 0 \tag{21.18}$$

Stability of the Steady States

Steady states in (21.7) are linearly stable if Re $\lambda < 0$ since in this case the perturbations go to zero as time goes to infinity.

Using the parameter values given and substituting into (21.18), we obtain the stability of the steady state:

$(0,0,0)$	gives 2 $\lambda s > 0$ and 1 $\lambda < 0$: unstable
$(0,1,0)$	gives 1 $\lambda > 0$ and 2 $\lambda s < 0$: unstable
$\left(1,0,\dfrac{k_c}{k_u}\right)$	gives 3s $\lambda < 0$: stable
$\left(1,1-\dfrac{\gamma k_c}{\mu_v k_u},\dfrac{k_c}{k_u}\right)$	gives 1 $\lambda > 0$ and 2 $\lambda s < 0$: unstable

Dispersion Relation

Now we consider the full equations (21.11) – (21.13) and differentiate with respect to the second-order spatial derivatives

$$\frac{\partial^2 \tilde{c}}{\partial X^2}, \quad \frac{\partial^2 \tilde{v}}{\partial X^2}, \quad \text{and} \quad \frac{\partial^2 \tilde{u}}{\partial X^2}$$

to get the transport Jacobian

$$J_{\mathrm{T}} = \begin{bmatrix} D_c & -\chi_v c_{\mathrm{ss}} & -\chi_u c_{\mathrm{ss}} \\ 0 & 0 & 0 \\ 0 & 0 & D_u \end{bmatrix} \qquad \textbf{(21.19)}$$

Dispersion Relation

The linearized system of equations (21.11) – (21.13) can now be represented in a compact form

$$\underline{\mathrm{w}}_t = J_{\mathrm{R}}\,\underline{\mathrm{w}} + J_{\mathrm{T}}\,\nabla^2\underline{\mathrm{w}} \qquad \textbf{(21.20)}$$

where

$$\underline{\mathrm{w}} = \begin{bmatrix} \tilde{c} \\ \tilde{v} \\ \tilde{u} \end{bmatrix}$$

to be solved in a domain with zero-flux boundary conditions

$$(\underline{\mathrm{n}} \cdot \nabla)\underline{\mathrm{w}} = 0 \qquad \textbf{(21.21)}$$

Dispersion Relation

To solve the system of equations in (21.20) subject to the boundary conditions, we first define **W(r)** to be the time-independent solution of the spatial eigenvalue problem, defined by

$$\nabla^2 \mathbf{W} + k^2 \mathbf{W} = 0$$
$$(\mathbf{n} \cdot \nabla)\mathbf{W} = 0 \quad \text{for} \quad \mathbf{r} \text{ on } \partial B \qquad \textbf{(21.22)}$$

where k is the eigenvalue. For example, if the domain is 1D, say $0 \leq x \leq L$, then

$$\mathbf{W} \propto \cos(\frac{n\pi x}{L})$$

where n is an integer. This satisfies zero-flux boundary conditions at $x = 0$ and $x = L$.

Dispersion Relation

The eigenvalue in this case is

$$k = \frac{n\pi}{L}$$

So

$$\frac{1}{k} = \frac{L}{n\pi}$$

is a measure of the wavelike pattern: the eigenvalue k is called the <u>wavenumber</u> and $1/k$ is proportional to the wavelength ω:

$$\omega = \frac{2\pi}{k} = \frac{2L}{n}$$

We shall refer to k in this context as the wavenumber.

Dispersion Relation

With finite domains there is a discrete set of possible wavenumbers since *n* is an integer.

We now look for solutions of (21.20) in the form

$$\underline{\mathrm{w}} = \sum_k a_k \, e^{\lambda T} \, \mathbf{W}_k \qquad \textbf{(21.23)}$$

Substituting (21.23) into (21.20) with (21.21) and canceling $e^{\lambda T}$ we get, for each *k*,

$$\begin{aligned} \lambda \, \mathbf{W}_k &= J_{\mathrm{R}} \, \mathbf{W}_k + J_{\mathrm{T}} \, \nabla^2 \mathbf{W}_k \\ &= J_{\mathrm{R}} \, \mathbf{W}_k - J_{\mathrm{T}} \, k^2 \mathbf{W}_k \end{aligned}$$

Dispersion Relation

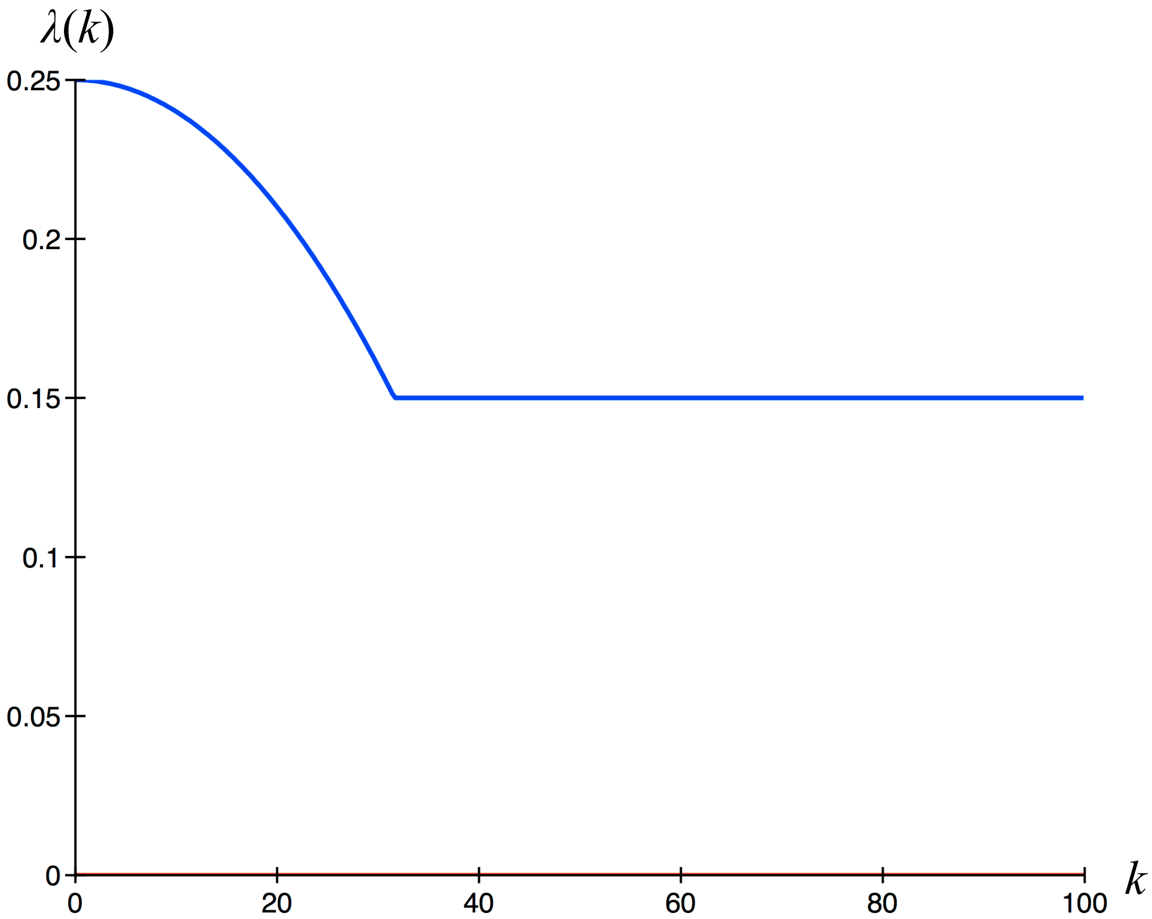
We require nontrivial solutions for $\mathbf{W}_k$ so the now the λ are determined by the roots of

$$|\lambda I - J_{\mathrm{R}} + J_{\mathrm{T}} k^2| = 0 \qquad \textbf{(21.24)}$$

Evaluating the determinant with J_{T} and J_{R} we get the eigenvalues $\lambda(k)$ as functions of the wavenumber k.

Dispersion Relation

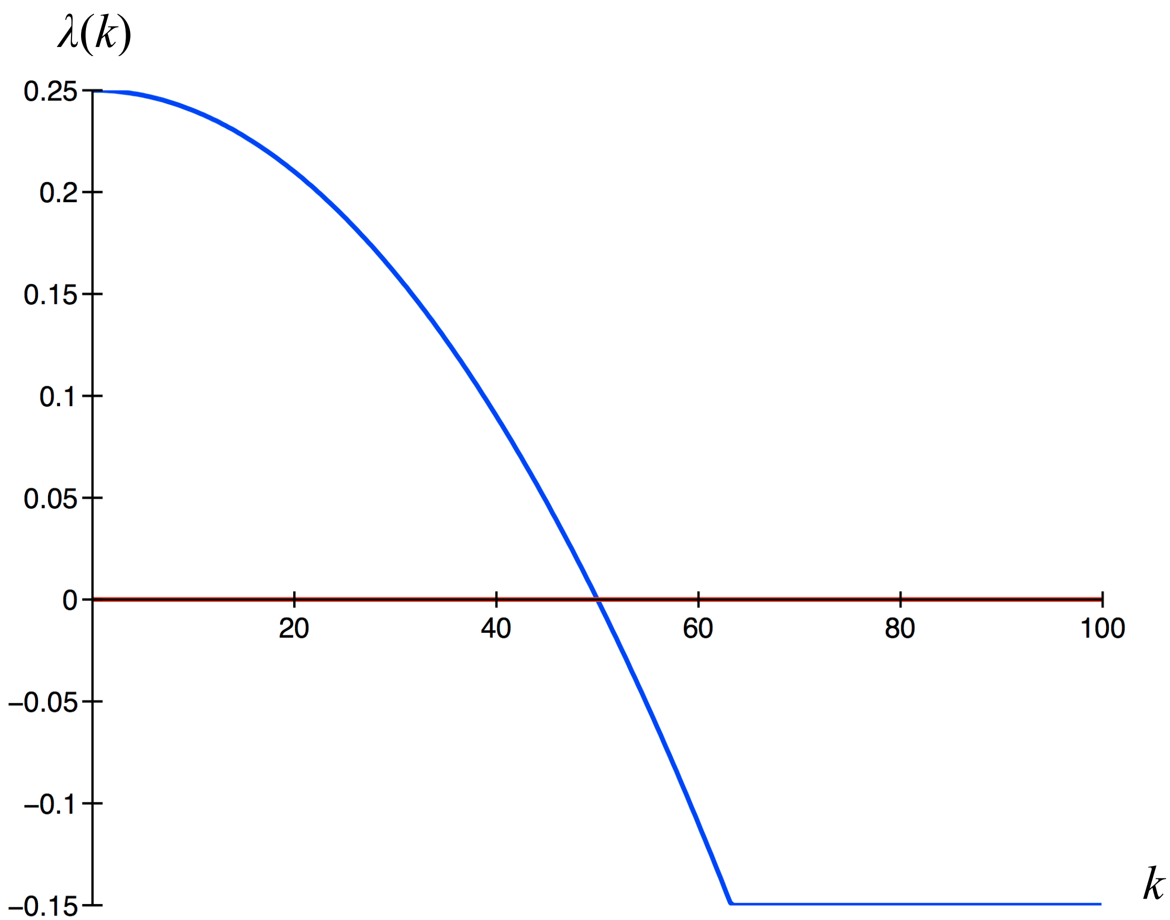
Dispersion relation from steady state (0,0,0)



- Max real part (blue line) of eigenvalues is 0.25
- It means that the perturbation grows with time.
- Imaginary part (red line) is zero.
- With the max real part, there are a range of k where the eigenvalues are positive.

Dispersion Relation

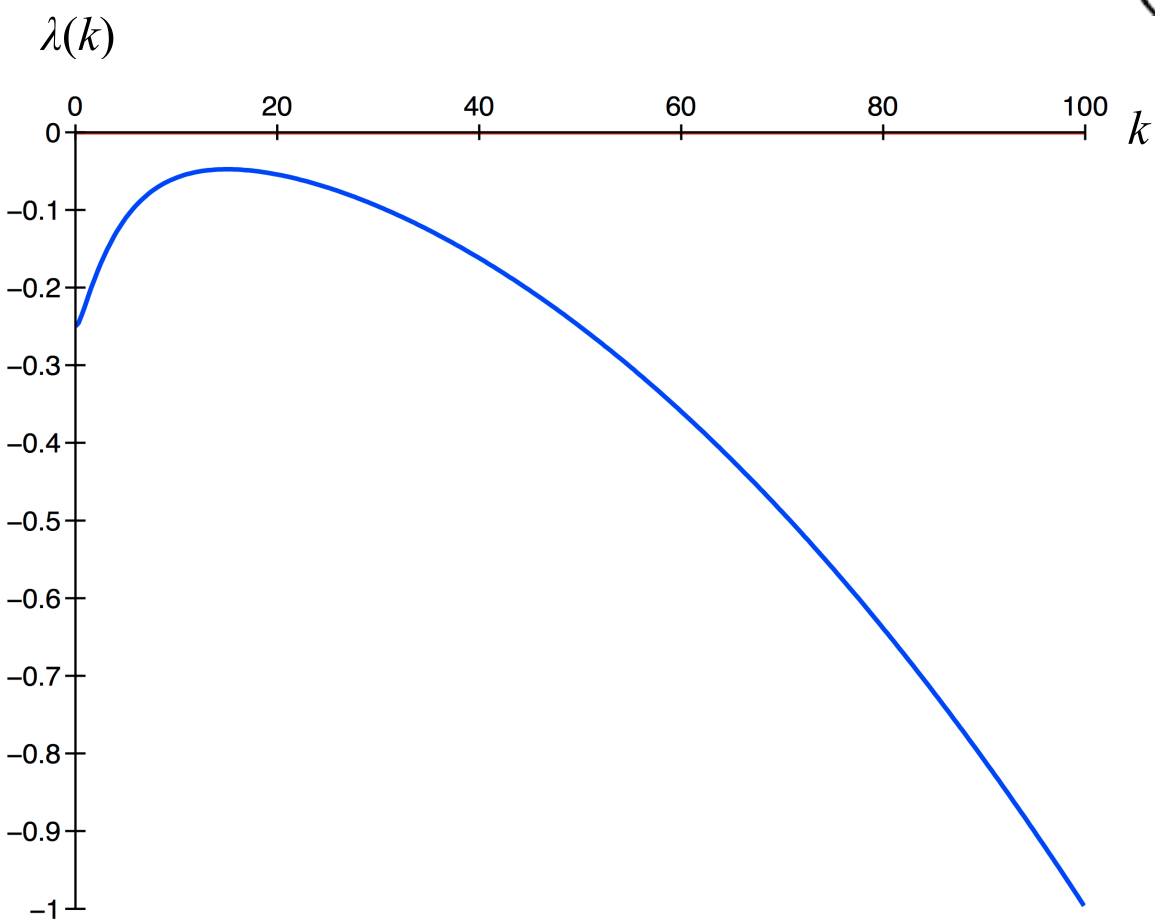
Dispersion relation from steady state (0,1,0)



- Max real part (blue line) of eigenvalues is 0.25.
- The perturbation grows with time.
- Imaginary part (red line) is zero.
- There are a range of k (between k=0 and k≃30) where the eigenvalues are positive.

Dispersion Relation

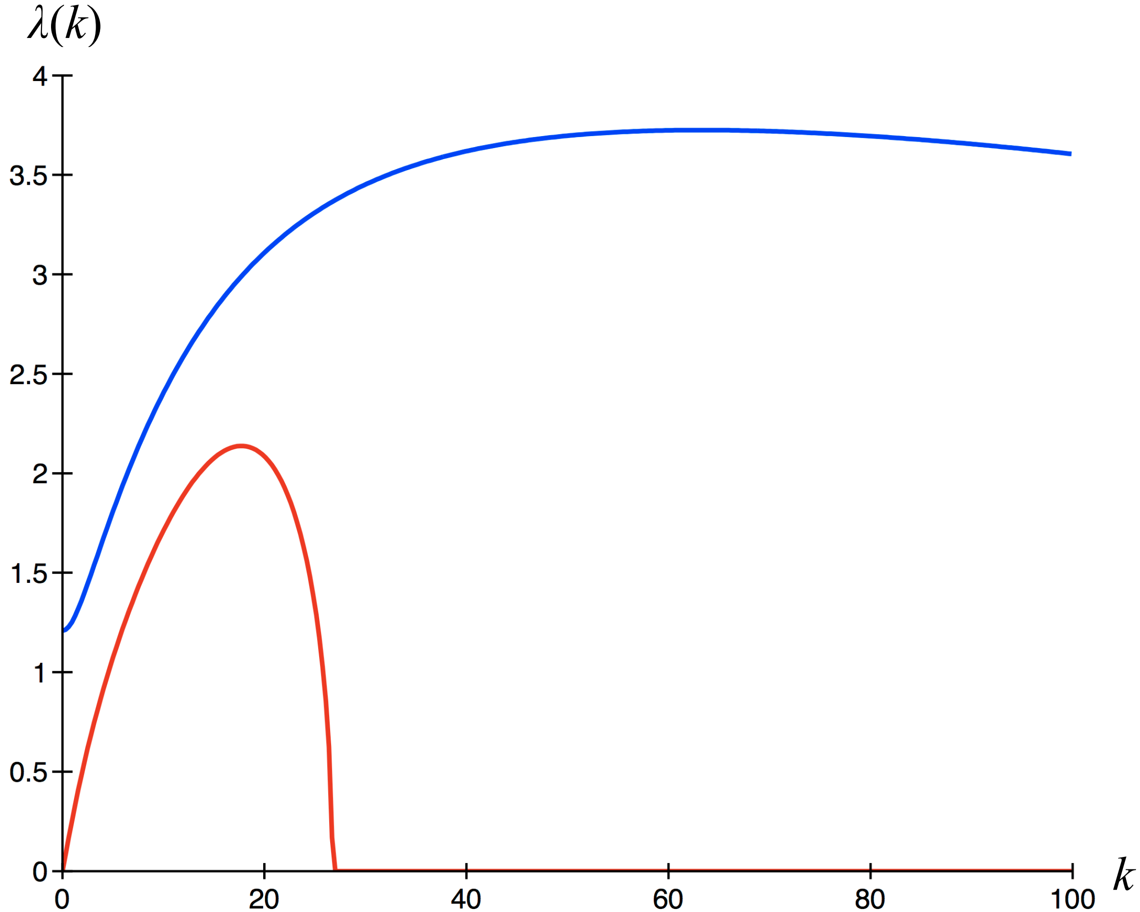
Dispersion relation from steady state $\left(1, 0, \dfrac{k_c}{k_u}\right)$



- Max real part (blue line) of eigenvalues is -0.0477.
- The perturbation is damped away.
- Imaginary part (red line) is zero.

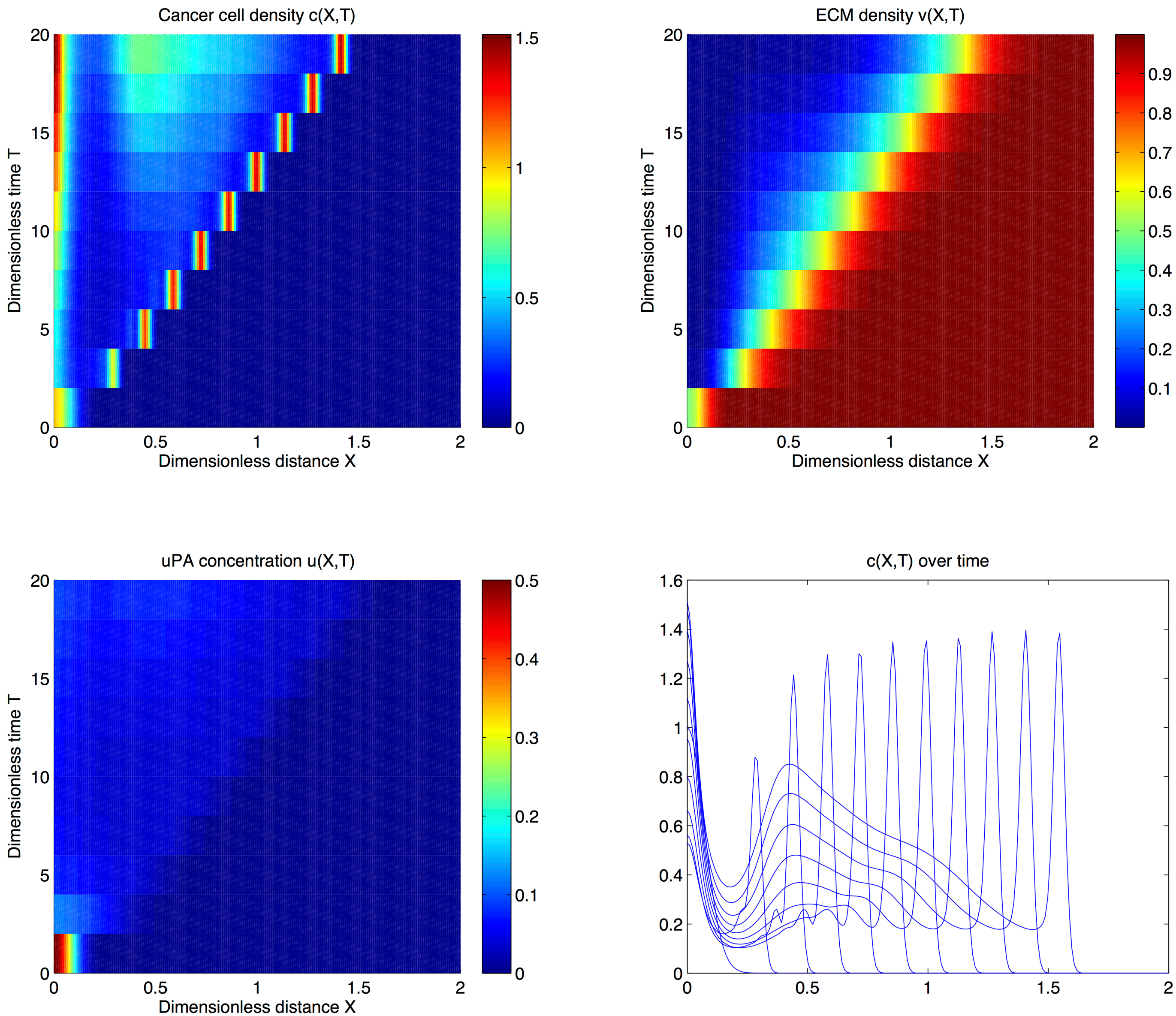
Dispersion Relation

Dispersion relation from steady state $\left(1, 1 - \frac{\gamma k_c}{\mu_v k_u}, \frac{k_c}{k_u}\right)$



- Max real part (blue line) of eigenvalues is 3.725.
- Max imaginary part (red line) is 2.1374.
- The perturbations grows with time.
- Imaginary part creates oscillating solutions.

Simulation Results



1D Wave Equations

An example of a hyperbolic PDE is a 1D wave equation for the amplitude function $u(x,t)$ as

$$\frac{\partial^2 u}{\partial t^2} = D\frac{\partial^2 u}{\partial x^2} \qquad \textbf{(21.25)}$$

$$\text{for} \quad \text{xmin} \leq x \leq \text{xmax}\,, \quad 0 \leq t \leq \text{tend}$$

In order for this equation to be solvable, the following should be provided:

- boundary conditions at x = xmin and at x = xmax
- initial condition $u(x,0) = f_0$
- initial velocity $u_t(x,0) = v_0$

1D Wave Equations

We replace the second derivatives on both sides by their three-point finite central difference approximation as

$$\frac{U_{i,k-1} - 2U_{i,k} + U_{i,k+1}}{\Delta t^2} = D\left(\frac{U_{i-1,k} - 2U_{i,k} + U_{i+1,k}}{\Delta x^2}\right)$$

Multiplying both sides by Δt^2 gives

$$U_{i,k-1} - 2U_{i,k} + U_{i,k+1} = \frac{D\Delta t^2}{\Delta x^2}\left(U_{i-1,k} - 2U_{i,k} + U_{i+1,k}\right)$$

From which we solve for $U_{i,k+1}$:

$$U_{i,k+1} = rU_{i-1,k} + 2(1-r)U_{i,k} + rU_{i+1,k} - U_{i,k-1}$$

(21.26)

1D Wave Equations

where

$$r = \frac{D\Delta t^2}{\Delta x^2}$$

From equation (21.26), if $k = 0$:

$$U_{i,1} = rU_{i-1,0} + 2(1-r)U_{i,0} + rU_{i+1,0} - U_{i,-1}$$

where $U_{i,-1}$ is not given. Therefore, we approximate the initial condition on the derivative (initial velocity) by the central difference as

$$\frac{\partial u}{\partial t} \simeq \frac{U_{i,k+1} - U_{i,k-1}}{2\Delta t} = v_0 \qquad \textbf{(21.27)}$$

1D Wave Equations

or

$$U_{i,-1} = U_{i,1} - 2\Delta t v_0$$

and make use of this to remove $U_{i,-1}$ from equation (21.26)

$$U_{i,1} = rU_{i-1,0} + 2(1-r)U_{i,0} + rU_{i+1,0} - (U_{i,1} - 2\Delta t v_0)$$

Equating the terms with $U_{i,1}$ yields

$$2U_{i,1} = rU_{i-1,0} + 2(1-r)U_{i,0} + rU_{i+1,0} + 2\Delta t v_0$$

or rewrite

$$U_{i,1} = \frac{1}{2}rU_{i-1,0} + (1-r)U_{i,0} + \frac{1}{2}rU_{i+1,0} + \Delta t v_0$$

(21.28)

1D Wave Equations

We use equation (21.28) together with the initial conditions to get $U_{i,1}$ and then go on with equation (21.26) for $k = 1, 2, \ldots$

The following must be taken into account:

- To guarantee stability, $\boldsymbol{r \leq 1}$
- The accuracy of the solution gets better as r becomes larger so that Δx decreases.

The stability condition can be obtained by substituting

$$U_{i,k} = \lambda^k e^{ji\pi/P}, \quad P \text{ is any nonzero integer}$$

into equation (21.26) and applying the Jury test:

1D Wave Equations

to get

$$\lambda = 2r\cos(\pi/P) + 2(1-r) - \lambda^{-1}$$

or

$$\lambda^2 + 2(r(1-\cos(\pi/P)) - 1)\lambda + 1 = 0$$

We need the solution of this equation to be inside the unit circle for stability, which requires:

$$r \leq \frac{1}{1-\cos(\pi/P)}$$

$$r = D\frac{\Delta t^2}{\Delta x^2} \leq 1$$

Exercise 1

Solve the following 1D wave equation:

$$\frac{\partial^2 u}{\partial t^2} = D\frac{\partial^2 u}{\partial x^2}, \qquad D = 0.1$$

over the spatial domain $0 \leq x \leq 1$ and within time interval $0 \leq t \leq 8$, with

- at $x = 0$: $u(0, t) = 0$
- at $x = 1$: $u(1, t) = 0$
- at $t = 0$: $u(x, 0) = x(1 - x)$
- at $t = 0$: $\dfrac{\partial u(x,0)}{\partial t} = 0$

2D Wave Equations

For 2D wave equations such as

$$\frac{\partial^2 u}{\partial t^2} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \textbf{(21.29)}$$

In the same fashion as 1D equations, we apply three-point central difference approximation for the 2D equation:

$$\frac{U_{i,j}^{k-1} - 2U_{i,j}^{k} + U_{i,j}^{k+1}}{\Delta t^2} =$$

$$D\left(\frac{U_{i-1,j}^{k} - 2U_{i,j}^{k} + U_{i+1,j}^{k}}{\Delta x^2} + \frac{U_{i,j-1}^{k} - 2U_{i,j}^{k} + U_{i,j+1}^{k}}{\Delta y^2}\right)$$

2D Wave Equations

which leads to the explicit central difference method:

$$U_{i,j}^{k+1} = r_x(U_{i-1,j}^k + U_{i+1,j}^k) + 2(1 - r_x - r_y)U_{i,j}^k + r_y(U_{i,j-1}^k + U_{i,j+1}^k) - U_{i,j}^{k-1} \quad \textbf{(21.30)}$$

with

$$r_x = D\frac{\Delta t^2}{\Delta x^2} \qquad r_y = D\frac{\Delta t^2}{\Delta y^2}$$

Since $U_{i,j}^{-1}$ is not given when $k = 0$, we approximate the initial condition on the derivative (initial velocity) by the central difference, giving:

$$U_{i,j}^{-1} = U_{i,j}^{1} - 2\Delta t v_0(x_i, y_j)$$

2D Wave Equations

and make use of this to remove $U_{i,j}^{-1}$ from equation (21.30) to get

$$U_{i,j}^{1} = r_x(U_{i-1,j}^{0} + U_{i+1,j}^{0}) + 2(1 - r_x - r_y)U_{i,j}^{0}$$
$$+ \; r_y(U_{i,j-1}^{0} + U_{i,j+1}^{0}) - (U_{i,j}^{1} - 2\Delta t v_0(x_i, y_j))$$

or, rearrange:

$$U_{i,j}^{1} = \frac{1}{2}r_x(U_{i-1,j}^{0} + U_{i+1,j}^{0}) + (1 - r_x - r_y)U_{i,j}^{0}$$
$$+ \; \frac{1}{2}r_y(U_{i,j-1}^{0} + U_{i,j+1}^{0}) + \Delta t v_0(x_i, y_j)$$

(21.31)

2D Wave Equations

Stability for approximation equation (21.31) is guaranteed if and only if:

$$r = \frac{4D\Delta t^2}{\Delta x^2 + \Delta y^2} \leq 1$$

Exercise 2

Solve the following 2D wave equation:

$$\frac{\partial^2 u}{\partial t^2} = D\nabla^2 u\,, \quad D = 0.25$$

over the spatial domain $0 \le x \le 2$, $0 \le y \le 2$, and within time interval $0 \le t \le 8$, with

- at $x = 0$ and $y = 0$: $u(0, y, t) = 0$ and $u(x, 0, t) = 0$
- at $x = 2$ and $y = 2$: $u(2, y, t) = 0$ and $u(x, 2, t) = 0$
- at $t = 0$: $u(x, y, 0) = 0.1*\sin(\pi x)*\sin(\pi y/2)$
- at $t = 0$: $\dfrac{\partial u(x,0)}{\partial t} = 0$

References

(1) Mathematical Biology II: Spatial Models and Biomedical Applications, J.D. Murray, Springer, Third Edition.

(2) Applied Numerical Methods Using MATLAB, Yang Chao Chung and Morris.