

How does the moment of inertia affect the period of Maxwell's Wheel?

Physics HL Internal Assessment

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Word count: 2193

1 Introduction and background knowledge

In ancient Greece (around 220 B.C.), Archimedes laid a solid foundation for rigid body statics by discovering and formalizing the conditions of equilibrium for levers. People studied Archimedes's work and developed concepts like torque thereafter, but it was not until Isaac Newton (1643-1727) formulated Laws of Motion that physics became well equipped for rigid body dynamics (Farber, 1961). With the assistance of advanced mathematical tools, Leonhard Euler (1707-1783) formalized the dynamics of rigid bodies and derived the concepts of moment of inertia and principal axes (Marquina & Hernández-Gómez, 2016).

Just like inertia (which is determined by mass) in translational motion, the moment of inertia evaluates a body's ability to resist angular acceleration. Generally speaking, the further away the mass is distributed from its axis of rotation, the larger the body's moment of inertia is. Objects with large moment of inertia are good at storing rotational kinetic energy, which makes them suitable for objects like flywheels, whereas those with small moment of inertia can rotate more rapidly.

Maxwell's wheel is commonly used as an instrument to illustrate the conservation of energy and the concept of rotational kinetic energy in physics education (Pecori & Torzo, 1998). It can also be utilized to measure the moment of inertia of a disk-shaped object and even the rolling friction if placed on a soft surface (Chakrabarti, Khaparde, & Kachwala, 2020).

This experiment aims at investigating the significance of the moment of inertia in a Maxwell's wheel experiment. This can help us better understand the rule behind the device and evaluate the pros and cons of Maxwell's Wheel in illustrating physics concepts and principles.

Research question: How does the moment of inertia affect the period of Maxwell's Wheel?

2 Hypothesis and reasoning

As is shown in Figure 1, the Maxwell's wheel mainly consists of three parts: a disk-shaped wheel of radius R , an axle of radius r penetrating the wheel, and a string that swirls around the axle. Aside from friction, two forces are acting on the pendulum: the upward tension from the string and the gravitational force downward. During the entire process, the gravitational potential energy converts to kinetic energy of the wheel.

The downward movement and the upward movement are approximately symmetric, so only the downward process needs algebraic analysis.

Since the wheel is not in equilibrium, it's challenging to analyze the magnitude of tension T . However, the problem can be approached using conservation of energy.

Neglecting friction, it can be assumed that all the gravitational potential energy lost is converted to the kinetic energy. Or more specifically, the sum of the translational kinetic energy and rotational kinetic energy should equal to the loss in gravitational potential energy, as is formally stated in Equation 1.

$$mg\Delta h = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad (1)$$

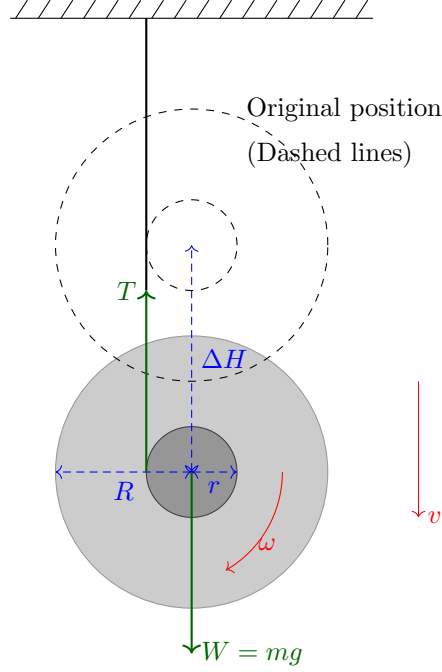


Figure 1: Model of a Maxwell's Wheel

where I is the moment of inertia of the entire wheel.

Moreover, using the definition of velocity and angular velocity, we can derive identities 2, 3, and 4.

$$\frac{d\Delta H}{dt} = v \quad (2)$$

$$\omega r = v \quad (3)$$

$$\Delta H(0) = 0 \quad (4)$$

Therefore, after replacing ω in Equation 1 with Equation 3, we can get Equation 5, whose terms can be moved and combined and v can be replaced using Equation 2 to derive Equation 6.

$$mg\Delta H = \frac{1}{2}mv^2 + \frac{1}{2}\frac{I}{r^2}v^2 \quad (5)$$

$$\frac{2mg\Delta H}{m + \frac{I}{r^2}} = \left(\frac{d\Delta H}{dt}\right)^2 \quad (6)$$

Moving $d\Delta H/dt$ to the left side and make its index 1, a simple ODE (Ordinary Differential Equation) for $\Delta H(t)$ is obtained, as is shown in Equation 7.

$$\frac{d\Delta H}{dt} = \sqrt{\frac{2mg}{m + \frac{I}{r^2}}} \Delta H^{0.5} \quad (7)$$

By letting coefficient k be $\sqrt{\frac{2mg}{m+\frac{I}{r^2}}}$, the ODE can be solved by separating variables and integrating the both sides, as is shown in Equation 8.

$$\begin{aligned}\frac{d\Delta H}{dt} &= k\Delta H^{0.5} \\ \Delta H^{-0.5}d\Delta H &= kdt \\ \int \Delta H^{-0.5}d\Delta H &= \int kdt \\ 2\Delta H^{0.5} &= kt + C\end{aligned}\tag{8}$$

With the help of initial condition Equation 4, we know that $C = 0$.

$$\begin{aligned}2\Delta H^{0.5} &= kt \\ \Delta H &= \left(\frac{k}{2}\right)^2 t^2\end{aligned}\tag{9}$$

Therefore the ODE is solved, and the solution is Equation 10.

$$\Delta H(t) = \frac{k^2}{4}t^2 = \frac{mg}{2m + \frac{2I}{r^2}}t^2\tag{10}$$

The downward movement terminates when $\Delta H(t) = l$, where l is the length of the string. Therefore, the time for downward movement t_{down} is calculated as Equation 11.

$$\begin{aligned}\Delta H(t_{down}) &= \frac{k^2}{4}t_{down}^2 = l \\ t_{down} &= 2\sqrt{\frac{l}{k^2}}\end{aligned}\tag{11}$$

Since the downward and upward movements are symmetric, the entire period of the Maxwell's Wheel can be formulated in Equation 12.

$$T = 2t_{down} = 4\sqrt{\frac{l}{k^2}}\tag{12}$$

When both sides are squared, we can get Equation 13, which can be further simplified by separating terms with and without I to get Equation 14.

$$T^2 = 16\frac{l}{k^2} = 8l\frac{(m + \frac{I}{r^2})}{mg}\tag{13}$$

$$T^2 = \frac{8l}{mgr^2}I + \frac{8l}{g}\tag{14}$$

From this formula, the hypothesis can be derived: **T increases as I increases. T^2 and I has a linear relationship.** When T^2 is plotted against I , a straight line with a positive gradient and positive y-intercept is expected.

3 Experiment design

3.1 Variables

- Independent variable: Moment of inertia (manipulated by changing the distance from the magnets to the pivot: $r' = 2.5 \text{ cm}, 3.5 \text{ cm}, 4.5 \text{ cm}, 5.5 \text{ cm}, 6.5 \text{ cm}, 7.5 \text{ cm}, 8.5 \text{ cm}$).
- Dependent variable: “Period” of the Maxwell’s Wheel (Time between two lowest positions).
- Controlled variables: The material, mass and size of the plate and magnets. The length of the string. The mass, radius and length of the axle, etc. They are listed in Table 1.

Table 1: Controlled variables

Variable	Value	Reason to control	Method to control
Total mass of the wheel m	$\approx 400 \text{ g}$	To ensure the gradient is not affected.	Use the same set of magnets, plate and axle.
Material of the wheel	Acrylic + NdFeB magnet	To ensure the mass distribution is even and consistent.	Use the same set of magnets and plate.
Height of displacement	$\approx 20 \text{ cm}$	Unify the initial kinetic energy and length of path.	Determine and mark a release position on the string.
Temperature and humidity	Around 25°C , 40% RH	Avoid the change in air resistance	Conduct the experiment in a short period of time.

3.2 Materials

- * 2 Retort stands (height $\approx 50 \text{ cm}$) with clamps
- * 2 Cotton strings (length $\approx 70 \text{ cm}$)
- * 1 Acrylic disc with axle (radius $\approx 10 \text{ cm}$, mass $\approx 110 \text{ g}$)
- * 16 Round magnets (radius $\approx 3.0 \text{ cm}$, mass $\approx 23 \text{ g}$)
- * 1 Electric force gauge (50 N), with compatible data cable
- * 1 Tape rule (range $\geq 50 \text{ cm}$, resolution $\leq 0.1 \text{ cm}$)
- * 1 Electric balance (range $\geq 400 \text{ g}$, resolution $\leq 0.1 \text{ g}$)
- * 1 Vernier caliper (range $\geq 5 \text{ cm}$, resolution $\leq 0.01 \text{ cm}$)
- * 1 Hot glue gun

3.3 Setup diagram

The apparatus is set up by hanging the Maxwell's Wheel between two iron stands using cotton string, as is shown in Figure 2.

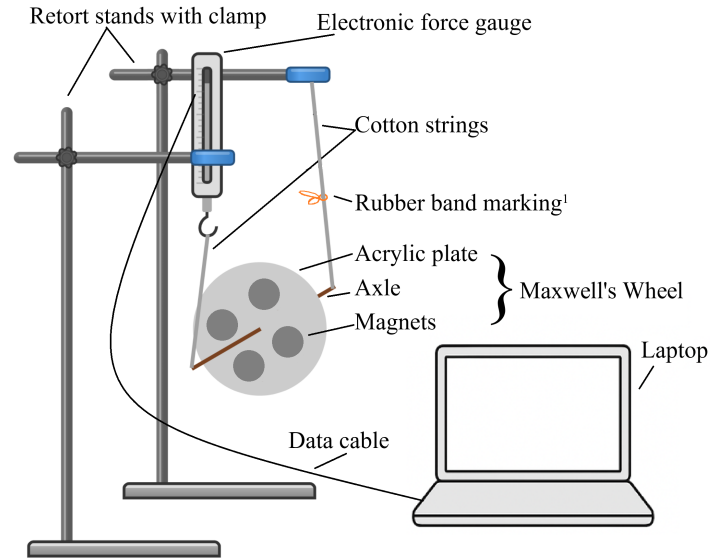


Figure 2: Setup diagram ²

¹ The rough location of rolling up is marked with a rubber band and the precise location is marked with a pen.

² The graph is made with chemix and mspaint.

3.4 Procedure

1. Measure the total mass of the magnets, mass of the acrylic plate, radius of the acrylic plate, radius of the axle and maximum vertical displacement after setting up the apparatus as is shown in Figure 2
2. Find a pair of diameters of the acrylic plane that is perpendicular to each other. This can be done by drawing the perpendicular bisector of an arbitrary pair of perpendicular chords.
3. Calibrate both axis with the assistance of a ruler. Make sure the center of the circle is scaled zero.
4. Attach four pairs of magnets to the plane.
5. Adjust the position of the magnets to make the distance for the magnets' centers of mass are $r = 8.5$ cm. This can be done by placing the magnet (with radius $r' = 3$ cm) between $r = 10$ cm mark and $r = 7$ cm mark, while touching the both marks.
6. Fix the magnets with the glue gun to prevent them from sliding on the plate.

7. Swirl up the axle until it reaches the designated point, start the forcemeter and release the wheel.
8. Record the time for the first and second time where the forcemeter reading reaches local maxima. Take them down as t_1 and t_2 .ⁱ
9. Repeat Steps 7 and 8 for 4 additional trials.
10. Repeat Steps 5 to 9 with $r = 7.5 \text{ cm}, 6.5 \text{ cm}, 5.5 \text{ cm}, 4.5 \text{ cm}, 3.5 \text{ cm}$.

3.5 Risk assessment

The experiment is relatively safe with little potential hazard.

It is worth keeping in mind that one should exercise caution when using the magnets, since the strong attraction of the magnets can cause unintended damage to the operator (e.g. their skin may be pinched when the magnets snaps together).

The experiment does not involve ethical issues as no human or animal subjects are involved.

The experiment has little environmental issues as little material is disposed and none of the materials are toxic.

4 Results

4.1 Raw data

The raw data were collected as follows:

- Total mass of the magnets M_m : $179.39 \pm 0.01\text{g}$
- Total mass of the acrylic plate M_a : $111.2 \pm 0.01\text{g}$
- Radius of the acrylic plate R_a : $10.0 \pm 0.05\text{cm}$
- Radius of the axle r : $4.0 \pm 0.1\text{mm}$
- Maximum vertical displacement H : $20 \pm 0.5\text{cm}$
- Other data are presented in Table 2 (where r' denotes the distance from the center of mass of the magnets to the pivot and t_1 and t_2 represent the times for the first and second local maxima respectively).

ⁱThis is done with the assistance of a program. If the first local maximum is not prominent, the second and the third local maxima is used instead.

Table 2: Raw Data

Experiments 1-4					Experiments 5-7				
No.		Trial	$t_1(s)$	$t_2(s)$	No.		Trial	$t_1(s)$	$t_2(s)$
	$r'(cm)$		$\pm 0.01 s$	$\pm 0.01 s$		$r'(cm)$		$\pm 0.01 s$	$\pm 0.01 s$
1	2.5	1	2.28	6.68	5	6.5	1	3.58	9.48
		2	2.25	6.63			2	9.28	15.23
		3	2.48	7.03			3	3.45	9.48
		4	6.73	11.28			4	3.53	9.43
		5	2.33	6.78			5	9.28	15.23
2	3.5	1	2.73	7.23	6	7.5	1	10.43	17.68
		2	7.03	11.53			2	3.93	10.68
		3	2.78	7.63			3	4.08	11.03
		4	2.58	7.38			4	4.23	11.13
		5	2.83	7.33			5	4.03	10.53
3	4.5	1	2.88	8.13	7	8.5	1	4.08	11.78
		2	3.18	8.63			2	11.68	18.33
		3	3.08	8.53			3	11.48	18.38
		4	2.88	7.88			4	12.03	20.58
		5	3.03	8.28			5	11.78	18.63
4	5.5	1	3.28	8.93			6	4.13	11.63
		2	3.23	8.73					
		3	3.23	8.83					
		4	3.18	8.73					
		5	3.48	9.23					

4.2 Processed data

The I_a , the moment of inertia due to the plate, is the same ($5.56 \pm 0.112 \times 10^{-4} \text{kgm}^2$) for each trial, as is given in Section 4.3. Other processed data are listed in Table 3, where I_m is the moment of inertia due to magnets, T is the period, T^2 is the square of the period. The data is plotted in Figure 3.

Table 3: Processed data

Exp.	$I_m(\times 10^{-4} \text{kgm}^2)$	$I_{tot}(\times 10^{-3} \text{kgm}^2)$	$\bar{T}(s)$	$T^2(s^2)$
1	1.12 ± 0.10	0.668 ± 0.020	4.466 ± 0.085	19.9 ± 0.76
2	2.20 ± 1.26	0.776 ± 0.024	4.63 ± 0.18	21.4 ± 1.62
3	3.63 ± 1.62	0.919 ± 0.053	5.28 ± 0.23	27.9 ± 2.38
4	5.43 ± 1.98	1.099 ± 0.031	5.61 ± 0.13	31.5 ± 1.40
5	7.58 ± 2.34	1.314 ± 0.035	5.946 ± 0.065	35.4 ± 0.77
6	10.09 ± 2.70	1.566 ± 0.038	6.87 ± 0.38	47.2 ± 5.15
7	12.96 ± 3.06	1.852 ± 0.042	7.12 ± 0.53	50.7 ± 7.48

4.3 Sample processing

Take the data processing of experiment 1 as an example.

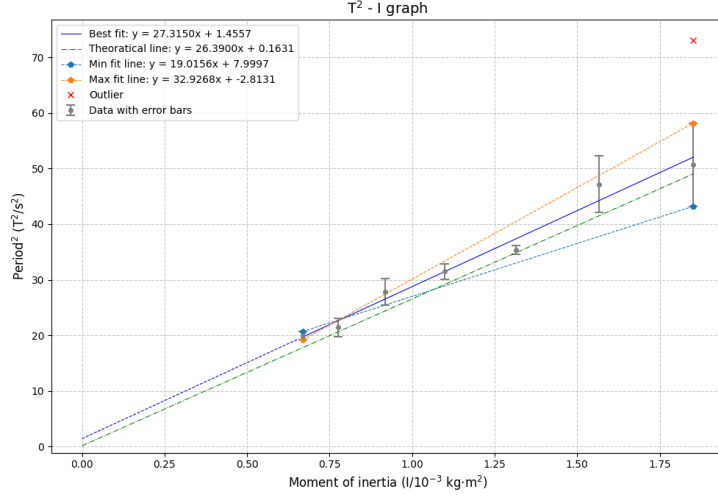


Figure 3: $T^2 - I$ graph

The R , distance from the magnets' center of mass to the pivot, is measured to be $2.5 \pm 0.1\text{cm}$. Therefore the moment of inertia contributed by the magnets is calculated as Equations 15 and 16.

$$\begin{aligned}
 I_m &= M_m \times r'^2 \\
 &= 179.39 \times 10^{-3}\text{kg} \times (2.5 \times 10^{-2})^2\text{m}^2 \\
 &= 1.1 \times 10^{-4}\text{kgm}^2
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 \Delta I_m &= \% \Delta I_m \times I_m \\
 &= \left(\frac{\Delta M}{M} + 2 \frac{\Delta R}{R} \right) \times I_m \\
 &= \pm 9 \times 10^{-6}\text{kgm}^2
 \end{aligned} \tag{16}$$

Furthermore, the moment of inertia contributed by the acrylic disk is calculated as in Equations 17 and 18.

$$\begin{aligned}
 I_a &= \frac{1}{2} M_a \times R_a^2 \\
 &= 111.2 \times 10^{-3}\text{kg} \times (10.0 \times 10^{-2})^2\text{m}^2 \\
 &= 5.56 \times 10^{-4}\text{kgm}^2
 \end{aligned} \tag{17}$$

$$\begin{aligned}
 \Delta I_a &= \% \Delta I_a \times I_a \\
 &= \left(\frac{\Delta M}{M} + 2 \frac{\Delta R}{R} \right) \times I_a \\
 &= \pm 1.12 \times 10^{-5}\text{kgm}^2
 \end{aligned} \tag{18}$$

The total moment of inertia I is calculated as in Equation 19 and the uncertainty is calculated as in Equation 20.

$$\begin{aligned}
I &= I_m + I_a \\
&= 1.1 \times 10^{-4} \text{kgm}^2 + 5.56 \times 10^{-4} \text{kgm}^2 \\
&= 6.68 \times 10^{-4} \text{kgm}^2
\end{aligned} \tag{19}$$

$$\begin{aligned}
\Delta I &= \Delta I_m + \Delta I_a \\
&= \pm(9 \times 10^{-6} \text{kgm}^2 + 1.12 \times 10^{-5} \text{kgm}^2) \\
&= \pm 2.02 \times 10^{-5} \text{kgm}^2
\end{aligned} \tag{20}$$

The force gauge data is recorded and plotted against time in Figure 4. It can be noticed that the reading of the force gauge is noisy but shows clear peaks. With the assistance of a program, prominent peaks are marked in the figure using red dots.

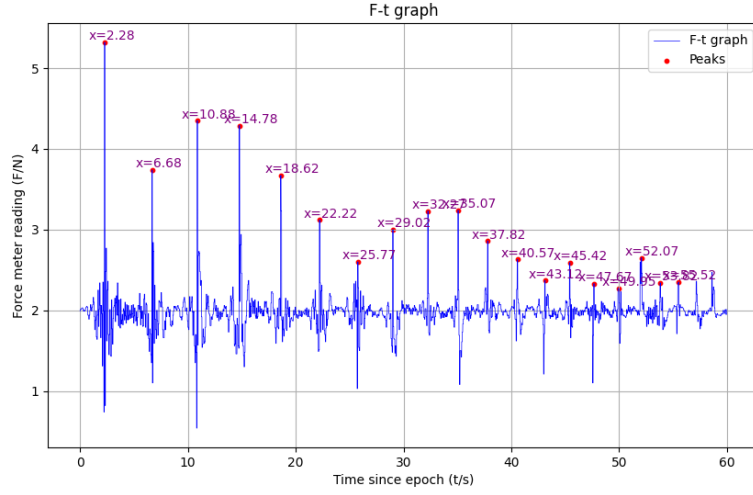


Figure 4: $F - t$ graph sample

The x-index of the first two peaks are taken as t_1 and t_2 of this trial. The “period” of the experiment T_a is calculated as in Equation 21, whose uncertainty is calculated as in Equation 22.

$$\begin{aligned}
T_a &= t_1 - t_0 \\
&= 6.68 \text{ s} - 2.28 \text{ s} \\
&= 4.40 \text{ s}
\end{aligned} \tag{21}$$

$$\begin{aligned}
\Delta T_a &= \Delta t_1 + \Delta t_0 \\
&= (\pm 0.01 \text{ s}) + (\pm 0.01 \text{ s}) \\
&= \pm 0.02 \text{ s}
\end{aligned} \tag{22}$$

Applying this to five trials, we can get the average of the first experiment and the corresponding uncertainties, as is shown in Equations 23 and 24.

$$\begin{aligned}\bar{T} &= \frac{T_a + T_b + T_c + T_d + T_e}{5} \\ &= 4.466 \text{ s}\end{aligned}\tag{23}$$

$$\begin{aligned}\Delta\bar{T} &= \frac{\max\{T_i\} - \min\{T_i\}}{2} \\ &= \frac{4.55 \text{ s} - 4.38 \text{ s}}{2} \\ &= 0.085 \text{ s}\end{aligned}\tag{24}$$

Therefore the square of the period can be calculated as in Equations 25 and 26.

$$\begin{aligned}\bar{T}^2 &= (4.466 \text{ s})^2 \\ &= 19.94 \text{ s}^2\end{aligned}\tag{25}$$

$$\begin{aligned}\Delta\bar{T}^2 &= \frac{\Delta\bar{T}}{\bar{T}} \times 2 \times \bar{T}^2 \\ &= 1.9032\% \times 2 \times 19.94 \text{ s}^2 \\ &= 0.7592 \text{ s}^2\end{aligned}\tag{26}$$

5 Discussion and conclusion

The diagram Figure 3 indicates strong correlation between T^2 and I . The best fit line of $T^2 - I$ has

- a gradient of $(27.3150 \pm 6.9556) \times 10^3 \text{ s}^2\text{kg}^{-1}\text{m}^{-2}$.
- a y-intercept of 1.4557s^2 ⁱⁱ.

As is mentioned in Section 2, the graph is expected to be a straight line very close to the origin. The diagram presents a graph with relatively small error, supporting the initial hypothesis.

The y-intercept is expected to be $\frac{8l}{mgr^2} \approx 0.16\text{s}^2$, but 1.4557s^2 is found. This is almost 9 times the theoretical value. However, considering the large value of T in the dataset, 1.4557s^2 is still less than 10% of the minimum dependent variable. This means it can still be considered an error, whose cause will be discussed in Section 6.

$$\Delta\%k = \frac{|k - k_0|}{k} \times 100\% = \frac{27.3150 - 26.3900}{26.3900} \times 100\% = 3.51\%\tag{27}$$

The gradient is expected to be $\frac{8l}{g} = 26.3900 \times 10^3 \text{ s}^2\text{kg}^{-1}\text{m}^{-2}$, where l is the length for lifting (controlled to be close to 20 cm) and g is the gravitational acceleration near sea level. There is a small 3.51% error (as is shown in Equation

ⁱⁱThe uncertainty is meaningless as the increasing uncertainty of T^2 results in exaggerated uncertainty of the y-intercept

27) between the actual gradient and predicted gradient, and the predicted gradient lies within the uncertainty range of the gradient. The low error indicates high accuracy and precision of this experiment.

Horizontal error bars are too small to be displayed in the graph, showing high precision and accuracy when manipulating the independent variable. Vertical error bars are small when I is small, but becomes larger as I gets larger. That might result from the significant horizontal perturbation getting greater as I increases. Horizontal movements causes the wheel start to swing like a pendulum and makes the motion more chaotic and less trackable.

The best fit line fails to pass through the fifth error bar, which is also coincidentally a very small error bar. That might be due to coincidence, since there are only five trials in each group.

Therefore, we can conclude that the period increases as the moment of inertia increases. The square of the period and the moment of inertia has a linear relationship.

6 Evaluation

The experiment was conducted successfully, gathering sufficient data and supporting the initial hypothesis. The graphs are similar to the graphs in Pecori et al.'s (1998) and Gianino et al.'s (2024) experiments with similar conclusions. However, there are some parts in the experiment that can be improved.

- **Defect in modelling** It is assumed in the hypothesis that the wheel does not swing horizontally and the string is always vertical, but small horizontal movement is inevitable during the experiment, as the extended line of the string does not go through the wheel's center of mass. Using a plumb line to adjust the string before releasing can minimize the horizontal deflection. A more advanced model taking that into consideration can provide a more precise prediction.
- **Energy loss** The model neglected the effect of air resistance and other form of energy loss during the process, which means the energy during the experiment is lower than expected. Considering the damping in the model can make the model more accurate.
- **Screw-threaded axle** For an ideal Maxwell's Wheel, the radius of the small axle should be constant. However, a screw-threaded axle is utilized to ensure better installation of the wheel, making the string swirl around the thread with inconsistent radius. This can be improved by using an axle that has screw thread only in the middle point.
- **String attaching** When attaching the wire to the axle, some adhesive (hot glue) is used to ensure connection stability. This also contributed to inconsistent radius. An axle with a hole for attachment can eliminate the need for adhesive.
- **Imprecise marking on the string** The string is not vertical to the ground, which means the marked distance is not precisely the vertical displacement. Additionally, axle's size can also lead to inconsistent vertical displacement. This might have lead to unexpected large y-intercept

mentioned in Section 5. Using a plumbline to ensure perpendicularity can result in better measurement.

- **Stretching of the string** When the Maxwell’s Wheel reaches the lowest position, the downward movement is changed to an upward one with approximately same magnitude. However, as (Pecori & Torzo, 1998) suggests, this happens because of the stretching of the string. Such process took about 0.16 s in Pecori’s experiment but it could not be measured accurately in the experiment due to different string material and the machine’s sampling frequency. This can lead to longer “period” than expected, and can be compensated using a forcemeter with faster sampling speed.
- **Method of independent variable manipulation** The independent variable is hard to manipulate, as the moment of inertia results from the arithmetical addition of two parts: the acrylic disk and the magnets. The former part is independent of r' , making it impossible to reach very small I . The latter part is proportional to r'^2 , resulting in difficulty to place the magnets on the acrylic plane if even distribution between I is needed. Lighter disk with larger size can make the manipulation easier.
- **Local gravitational acceleration** The gravitational acceleration is dependent on the region and is not always consistent with the g taken. This can result in deviated hypothesis and can be improved by looking up local gravitational acceleration online.
- **Only 7 groups of data and 5 trials for each group** Limited number of trials make the result not convincing enough. It would be better if more independent variables are chosen and more trials are conducted.

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