From Fibonacci Sequences to Generating Functions

Math HL Internal Assessment

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1 Introduction

During the study of mathematics, there are a number of important sequences that is commonly studied. Fibonacci sequence is usually the first sequence that a learner may meet whose term is determined by not only the previous term (and probably some coefficients), but the previous several terms (recurrence relation). To be more specific, the Fibonacci sequence is defined as follows.

Definition: Fibonacci sequence

The Fibonacci sequence $\{F_n\}$ is given by

$$F_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ F_{n-1} + F_{n-2}, & n \geq 2. \end{cases} \tag{1}$$

Fibonacci sequence is very useful in mathematics, computer science and can be found in nature. The sequence itself is the solution to the Fibonacci Rabbit problem. Additionally, the sequence (or generalized Fibonacci sequences) give solution to many combinatorics problems. For example, the number of ways to walk n steps of stairs with the condition that each time either 1 or 2 steps cam be taken. In computer science, a consequtive pair in Fibonacci sequence can be used to test the worst posibble performance of algorithms like Euclidean Greatest Common Divisior (GCD) algorithm. Fibonacci trees and heaps are also very effective data structures, whose analysis requires the knowledge of Fibonacci sequence.

Despite the simple definition of the Fibonacci sequence, its closed-form expression is not as straightforwad as the expression of artihmatic sequence and geometric sequence. Despite having learnt the Fibonacci sequence for centuries, it was not until the 1800s did human discover an accurate closed-form expression. In this paper, we will study the closed-form expression of the Fibonacci sequence.

Problem

What is F_n? Express F_n in terms of n.

Generalized Problem

The first two terms a_0 and a_1 of an infinite sequence $\{a_n\}$ is given. The following terms are given by the formula $a_n = k_0 + k_1 a_{n-1} + k_2 a_{n-2}$. What is a_n?

2 Background Knowledge and Definition

2.1 Maclaurin series

Having studied sequences and calculus (especially Maclaurin series), we can notice that each infinite sequence can connect with a polynomial with infinite number of terms. To be more specific, we the following definitions and theorems in calculus are used:

Definition: Taylor Series and Maclaurin Series

Suppose that f is a function such that f is n-times differentiable at a point a ($f'(a), f''(a), \cdots f^{(n)}(a)$ exist). Then the **n-th degree Taylor polynomial of f at a** (Spivak, 2008) is defined as:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{2}$$

If f is infinitely differentiable at a and we let $n \to \infty$, we obtain the **Taylor series of f at a**:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{3}$$

The special case where $a = 0$ is called the **Maclaurin series of f**:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k. \tag{4}$$

Most commonly used function has a Maclaurin series. In this essay, the main kind of function that requires the analysis of Maclaurin series is in the form of $f(x) = \dfrac{1}{x-a}$.

Theorem: The Maclaurin Series of 1/(x-a)

The main focus of this essay is not calculus; therefore, the Maclaurin series of $\dfrac{1}{x-a}$ will be given without proof.

$$\frac{1}{x-a} = \sum_{k=0}^{\infty} \frac{1}{a^{k+1}}x^k \quad \text{(radius of convergence:}|x| < |a|\text{)} \tag{5}$$

2.2 Generating functions

Functions, on the other hand, have some ideal properties that can facilitate our understanding. In mathematics, a generating function is a representation of an infinite sequence of numbers as the coefficients of a formal power series (wikipedia, need to change cite sources).

Definition: Generating Function

For a sequence $\{a_i\}$, its generating function is defined as:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \tag{6}$$

Here x is not the variable of a polynomial, but the variable of a formal power series, or the 'placeholder' for the 'variable' position. The function does not have to converge on any given x, though sometimes it does.

Having the generating function defined this way, we can now express some sequence operations on the infinite sequence as operations on the generating function.

Table 1: Sequence operations vs generating function operations

Sequence Operation	**GF Operation**
Original sequence $(a_0, a_1, a_2, \dots)$	$A(x) = \sum_{n=0}^{\infty} a_n x^n$
Shift right by k: $(0, \dots, a_0, a_1, \dots)$	$x^k A(x)$
Shift left by 1: $(a_1, a_2, a_3, \dots)$	$\dfrac{A(x) - a_0}{x}$
Scale by index: $(0, a_1, 2a_2, 3a_3, \dots)$	$xA'(x)$
Scale by constant c: $(ca_0, ca_1, \dots)$	$cA(x)$
Cumulative sum: $b_n = \sum_{k=0}^{n} a_k$	$\dfrac{A(x)}{1-x}$
Convolution: $c_n = \sum_{k=0}^{n} a_k b_{n-k}$	$A(x)\,B(x)$
Alternating sign: $(a_0, -a_1, a_2, -a_3, \dots)$	$A(-x)$
Difference: $b_n = a_n - a_{n-1}$	$(1-x)A(x)$

(might need to add or modify the table, but now focus on shift)

Proof

Theorem: If $A(x)$ is the generating function of $\{a_i\}$, then $B(x) = x^k A(x)$ is the generating function of $\{b_i\}$ obtained by shifting $\{a_i\}$ to the right by k positions (and inserting k zeros at the beginning).

Proof:

From the definition of generating function, we have

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{7}$$

Therefore we can get that

$$B(x) = x^k A(x) = \sum_{n=0}^{\infty} a_n x^{n+k} = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \cdots \tag{8}$$

We can add shift a by k and k zero term before the sequence to make a new sequence b. That is to say, $\{b_i\}$ is defined in the following way:

$$b_i = \begin{cases} 0, & i < k \\ a_{i-k}, & i \geq k \end{cases} \tag{9}$$

We can notice that the generating function of $\{b_i\}$ is the same as $B(x)$.

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^{n+k} &= \sum_{n=0}^{k-1} 0 \cdot x^n + \sum_{n=k}^{\infty} a_{n-k} x^n \\ &= \sum_{n=k}^{\infty} b_n x^n = B(x). \end{aligned} \tag{10}$$

The definition of Fibonacci sequence is given by Equation 1.

We noticed that F_{n-1} and F_{n-2} are the shift of sequence that can be described by multiplying the generating function $G_0(x)$ by x^k. We can list $G_0(x), xG_0(x), x^2G_0(x)$ and their corresponding sequence as Table 2.

Generating Function	The coefficient of the k-th term							
	0	1	2	3	$\cdots$	n	$n+1$	$n+2$
$G_0(x)$	F_0	F_1	F_2	F_3	$\cdots$	F_n	F_{n+1}	F_{n+2}
$xG_0(x)$	0	F_0	F_1	F_2	$\cdots$	F_{n-1}	F_n	F_{n+1}
$x^2G_0(x)$	0	0	F_0	F_1	$\cdots$	F_{n-2}	F_{n-1}	F_n

Table 2: Shifted GF and Fibonacci Sequence

It can be noticed that for every vertical column with $n \geq 2$, it holds that the sum of the lower two terms equals the upper term, by the definition of Fibonacci sequence. The terms before 2 can be manipulated easily. By forcing filling up the 1 terms, we can get three equations as is shown in Equation 11.

$$G_0(x) = \sum_{n=0}^{\infty} F_n x^n = F_0 x^0 + F_1 x^1 + F_2 x^2 + F_3 x^3 \cdots \tag{11a}$$

$$xG_0(x) = 0 + \sum_{n=0}^{\infty} F_n x^{n+1} = 0x^0 + F_0 x^1 + F_1 x^2 + F_2 x^3 \cdots \tag{11b}$$

$$x + x^2G_0(x) = 1 + \sum_{n=0}^{\infty} F_n x^{n+2} = 0x^0 + 1x^1 + F_0 x^2 + F_1 x^3 \cdots \tag{11c}$$

Therefore, by substracting $G_0(x)$ with $xG_0(x)$ and $x^2G_0(x)$, we can get Equation 12, which can be further manipulated to get Equation 13 and Equation 14.

$$\begin{aligned} & G_0(x) - xG_0(x) - x^2G_0(x) - x \\ =& (F_0 - 0 - 0)x^0 + (F_1 - F_0 - 1)x^1 + (F_2 - F_1 - F_0)x^2 + \\ & (F_3 - F_2 - F_1)x^3 + \cdots \\ =& 0 \end{aligned} \tag{12}$$

$$(1 - x - x^2)G_0(x) = x \tag{13}$$

$$G_0(x) = \frac{x}{1 - x - x^2} \tag{14}$$

We can therefore conclude that Equation 14 is the generating function of Fibonacci sequence. However, a generating function in the factor form (instead of polynomial) does not help us understand the sequence. We can therefore use Maclaurin series to represent the GF in polynomial form.

The definition of Maclaurin series is given in Equation 4. We need to find the Maclaurin series of $1/(1 - x - x^2)$ to write the GF in polynomial form.

The denominator of the generating function, $1 - x - x^2$, can be factorized by finding its roots.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{1 \pm \sqrt{(-1)^2 + 4}}{2 \times (-1)} \\ &= \frac{-1 \pm \sqrt{5}}{2} \\ &(x_1 = \frac{-1 + \sqrt{5}}{2}, x_2 = \frac{-1 - \sqrt{5}}{2}) \end{aligned} \tag{15}$$

$$1 - x - x^2 = -(x - x_1)(x - x_2) \tag{16}$$

Assuming that λ_1 and λ_2 are constants, and

$$\begin{aligned} \frac{x}{1 - x - x^2} &= \frac{\lambda_1}{x - x_1} + \frac{\lambda_2}{x - x_2} \\ &= -\frac{\lambda_1(x - x_2) + \lambda_2(x - x_1)}{1 - x - x^2} \\ &= -\frac{(\lambda_1 + \lambda_2)x - \lambda_1 x_2 - \lambda_2 x_1}{1 - x - x^2} \end{aligned} \tag{17}$$

Therefore

$$\begin{cases} \lambda_1 + \lambda_2 = -1 \\ \lambda_1 x_2 + \lambda_2 x_1 = 0 \end{cases} \tag{18}$$

We can therefore get

$$\begin{aligned} \lambda_1 &= \frac{x_1}{x_1 - x_2} = \frac{\sqrt{5} - 5}{10} \\ \lambda_2 &= \frac{-x_2}{x_1 - x_2} = \frac{-\sqrt{5} - 5}{10} \end{aligned} \tag{19}$$

Therefore $G_0(x)$ can now be re-written as $\dfrac{\lambda_1}{x - x_1} + \dfrac{\lambda_2}{x - x_2}$.

By the definition of Maclaurin series, we can get that

$$
\begin{aligned}
G_0(x) &= \frac{\lambda_1}{x - x_1} + \frac{\lambda_2}{x - x_2} \\
&= -\sum_{n=0}^{\infty} \frac{\lambda_1}{x_1^{n+1}} x^n - \sum_{n=0}^{\infty} \frac{\lambda_2}{x_2^{n+1}} x^n \\
&= -\sum_{n=0}^{\infty} \frac{\lambda_1 x_2^{n+1} + \lambda_2 x_1^{n+1}}{(x_1 x_2)^{n+1}} x^n \\
&= \sum_{n=0}^{\infty} \frac{\frac{1}{\sqrt{5}} x_2^n - \frac{1}{\sqrt{5}} x_1^n}{(-1)^n} x^n \\
&= \sum_{n=0}^{\infty} \frac{x_2^n - x_1^n}{\sqrt{5}(-1)^n} x^n \\
&= \sum_{n=0}^{\infty} \frac{(\frac{-1-\sqrt{5}}{2})^n - (\frac{-1+\sqrt{5}}{2})^n}{\sqrt{5}(-1)^n} x^n \\
&= \sum_{n=0}^{\infty} \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} x^n
\end{aligned}
\tag{20}
$$

By the definition of the GF, the coefficient of the n-th term is the n-th term of Fibonacci sequence.

$$
F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} \tag{21}
$$

3 Verification

Despite the complicated process of finding the closed term expression, the conclusion can be verified both algebraically and graphically simply.

3.1 Algebraic Verification

It can be noticed that the Golden Ratio is found in the closed form. Let $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$. We have

$$
\begin{aligned}
\varphi^2 &= 1 + \varphi \\
\psi^2 &= 1 + \psi
\end{aligned}
\tag{22}
$$

Let the statement P_k be "$F_k = \dfrac{\varphi^k - \psi^k}{\sqrt{5}}$".

P_0 is true because $F_0 = \dfrac{\varphi^0 - \psi^0}{\sqrt{5}} = 0$.

P_1 is true because $F_1 = \dfrac{\varphi^1 - \psi^1}{\sqrt{5}} = 1$.

For some positive integer $k > 1$, assume $\forall i \in \mathbb{N} \cap [0, k-1], P_i$ is true.
Then

$$\begin{cases} F_{k-1} = \dfrac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} \\ F_{k-2} = \dfrac{\varphi^{k-2} - \psi^{k-2}}{\sqrt{5}} \end{cases} \tag{23}$$

Then

$$\begin{aligned} F_k &= F_{k-1} + F_{k-2} \\ &= \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} + \frac{\varphi^{k-2} - \psi^{k-2}}{\sqrt{5}} \\ &= \frac{(\varphi \times \varphi^{k-2} - \psi \times \psi^{k-2})}{\sqrt{5}} + \frac{\varphi^{k-2} - \psi^{k-2}}{\sqrt{5}} \\ &= \frac{(\varphi^{k-2}(1+\varphi) - \psi^{k-2}(1+\psi))}{\sqrt{5}} \\ &= \frac{(\varphi^{k-2}\varphi^2 - \psi^{k-2}\psi^2)}{\sqrt{5}} \\ &= \frac{(\varphi^k - \psi^k)}{\sqrt{5}} \end{aligned} \tag{24}$$

Which means P_k is true.

$$\begin{cases} \forall n \in \{0,1\}, P_n \text{ is true} \\ \forall n \geq 2, (P_0, P_1 \cdots P_{n-1} \text{ is true}) \Rightarrow P_n \end{cases} \tag{25}$$

According to the strong mathematical induction, P_n is true for all $n \in \mathbb{N}$.

3.2 Numerical Verification

We can plot the *derived generating function* (shown in Equation 13) with the first few terms of *generating function by definition* (shown in Equation 11a). We can notice that the latter is a good approximation of the former, as is shown in Figure 1. This suggests that the derived generating function is indeed the correct one.

4 Reflection

This essay discussed the generating functions and utilized it to derive a closed form expression of Fibonacci sequence. It also provided two verifications on the mathematical process. Additionally, this essay revealed the connection between two parts of modern mathematics: combinatorics and analysis (calculus), revealling the interconnection in mathematics. However, the essay lacks discussion on more general cases.

References

Spivak, M. (2008). *Calculus*. Publish or Perish.

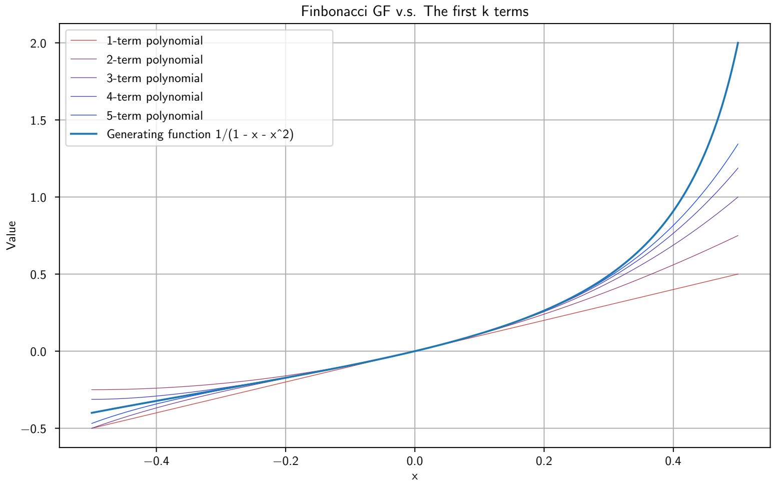

Figure 1: Verification of the generating function.

Appendix

Listing 1: Numerical Verification

```
import numpy as np
import matplotlib.pyplot as plt

def fib(n):
    f = [0, 1]
    for _ in range(2, n+1): f.append(f[-1] + f[-2])
    return f
F = fib(10)
def G(x):
    return x / (1 - x - x**2)

def poly_approx(x, m):
    return sum(F[k] * x**k for k in range(m+1))

plt.figure(figsize=(10,6))
x = np.linspace(-0.5, 0.5, 300)
p = [1, 2, 3, 4, 5]
Gx = G(x)
for pi in p:
    plt.plot(x, poly_approx(x, pi), label=f"{pi}-term polynomial", linewidth = 0.5, color
         = (1-pi/5, 0.2, pi/5))

plt.plot(x, Gx, label="Generating function 1/(1 - x - x^2)")
plt.legend()
plt.xlabel("x")
plt.ylabel("Value")
plt.title("Finbonacci GF v.s. The first k terms")
plt.grid(True)

plt.show()
```