Investigating the Austrian mortality rates for company's expansion.

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Tutorial (WG03)

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I Developments since 1970

This paper starts with an analysis of the death rates in Austria and compares it to that of the Netherlands. The measure used is the logarithm of the Central Death Rate:

$$log(\hat{\mu}_{xt}) = \frac{D_{xt}}{E_{xt}} \tag{1}$$

The provided plots in Figure 1 display the logarithm of the central death rate in Austria and The Netherlands, focusing on populations between 25 and 100 years old, during the period spanning from 1970 to 2008. Notably, these plots consistently indicate that males exhibit a higher central death rate when compared to females in both countries and throughout the given time frames. Furthermore, it is evident from the plots that Austria's central death rate has experienced significant improvement over the course of 1970 to 2008, depicted by the changing distance between the two lines, which represent the two different times.

Figure 1: Logarithm of Central Death Rate estimates for Austria and the Netherlands

To calculate the expected life expectancy, we make the assumption that the hazard rate remains constant. This assumption allows us to derive equation (2) based on the work of Dickson et al. (2019). Additionally, we assume that the central death rate serves as an estimator of μ and we impose a maximum age limit of 100 years. The formula for complete expected future lifetime:

$$E[K_x] = e_x + 0.5 = \sum_{k=1}^{100} {}_k p_x + 0.5 \tag{2}$$

with

$${}_1 q_x(t) = 1 - exp(-\mu_{xt}) \tag{3}$$

The estimates for life expectancy is displayed in the table below:

	Austria				The Netherlands			
Gender\Year	1970		2008		1970		2008	
Female	73.42	14.89	82.95	20.78	76.53	16.52	82.26	20.45
Male	66.50	11.69	77.59	17.53	70.84	13.58	78.32	17.25

Note: Life expectancy of a newborn is in green

The table presents the life expectancy data for Austria and The Netherlands for the years 1970 and 2008. The life expectancies are given in years and are provided separately for females and males. The values in green represent the life expectancy of a newborn, while the non-colored rows correspond to the life expectancy of 65 years old.

As shown in the table, in 1970, the life expectancy of newborn females in Austria was 73.42 years, while for males, it was 6.98 years lower at 66.50 years. Similarly, in the Netherlands, newborn females were expected to live 5.7 years longer than newborn males. For the population at 65 years old, females in Austria and the Netherlands lived longer than their male counterparts by 3.2 and 2.95 years, respectively. In 2008, both countries experienced an improvement in life expectancies. In the Netherlands, newborn females and males could live till 82.25 and 78.32. The 65 years old group was seeing an approximately 4 years longer life compared with 38 years ago.

Overall, the table shows that both Austria and The Netherlands experienced an increase in life expectancy for both genders from 1970 to 2008. Additionally, it is noteworthy that in both countries and for both years, females consistently had higher life expectancies compared to males. Even though the gender gap in life expectancy has narrowed over time, in 2008, Austria still had a slightly larger gender gap compared to The Netherlands.

II Current mortality per age

In this section, we estimate the central death rates for females aged 25-90 years in the year 2008 in Austria. We do so on the basis of the Gompertz (1826) model:

$$\mu_{x,2008} = e^{a+bx} \tag{4}$$

We use 3 main approaches to estimate the central death rate as follows:

1. Least squares approach of the normal Gompertz model.

$$min \sum_{x=25}^{90} \left(\frac{D_{x,2008}}{E_{x,2008}} - e^{a+bx} \right)^2 \tag{5}$$

2. Least squares approach of the log Gompertz model.

$$min \sum_{x=25}^{90} \left(ln \frac{D_{x,2008}}{E_{x,2008}} - (a+bx) \right)^2 \tag{6}$$

3. Weighted least squares approach of the Gompertz model

$$min \sum_{x=25}^{90} D_{x,2008} \left(\frac{D_{x,2008}}{E_{x,2008}} - e^{a+bx} \right)^2 \tag{7}$$

Sometimes, there might be 0 deaths in a given year. This might create a problem for the logarithm approach. However in our dataset, there are no 0 deaths.

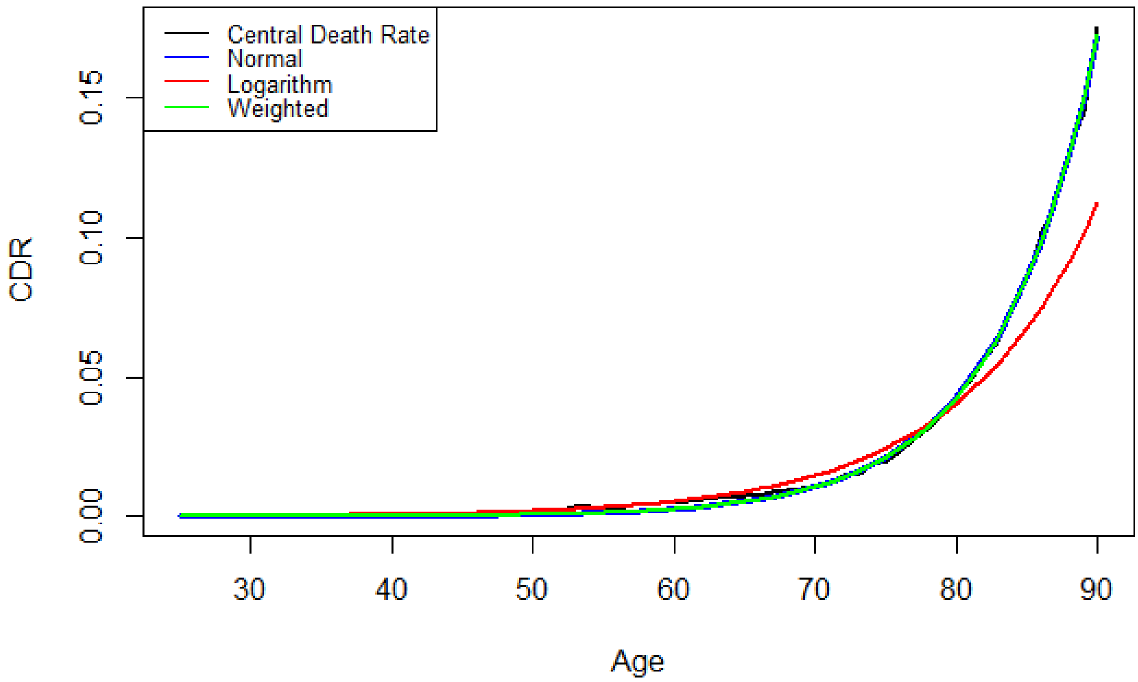


Figure 2: Central Death Rates and its approximations for females (Austria)

Figure 2 shows the central death rate for females aged 25-90 in the year 2008 and its approximations. We notice that the normal least squares and the weighted least squares method underestimate the death rates until age 70, but it is accurate for ages higher than 70. The logarithm approach estimates the death rate well until age 65, but its accuracy is poor for older ages.

Figure 3 represents the logarithms of central death rates and its approximations. We see here the death rates rise with increase in age. As to which approximation method fits the best we notice a similar trend as we discussed above. The normal and weighted least squares method are a poor fit of the data as they tend to underestimate till age 70, but are a good fit for ages 70-90. The logarithm approach fits the data well until age 60, but it tends to overestimate for ages 60-80 and underestimate for ages 80-90.

Overall , the normal least squares and the weighted least squares method can be used to estimate the death rates of older people with greater accuracy and the logarithm approach is better suited to estimate the death rates of younger people.

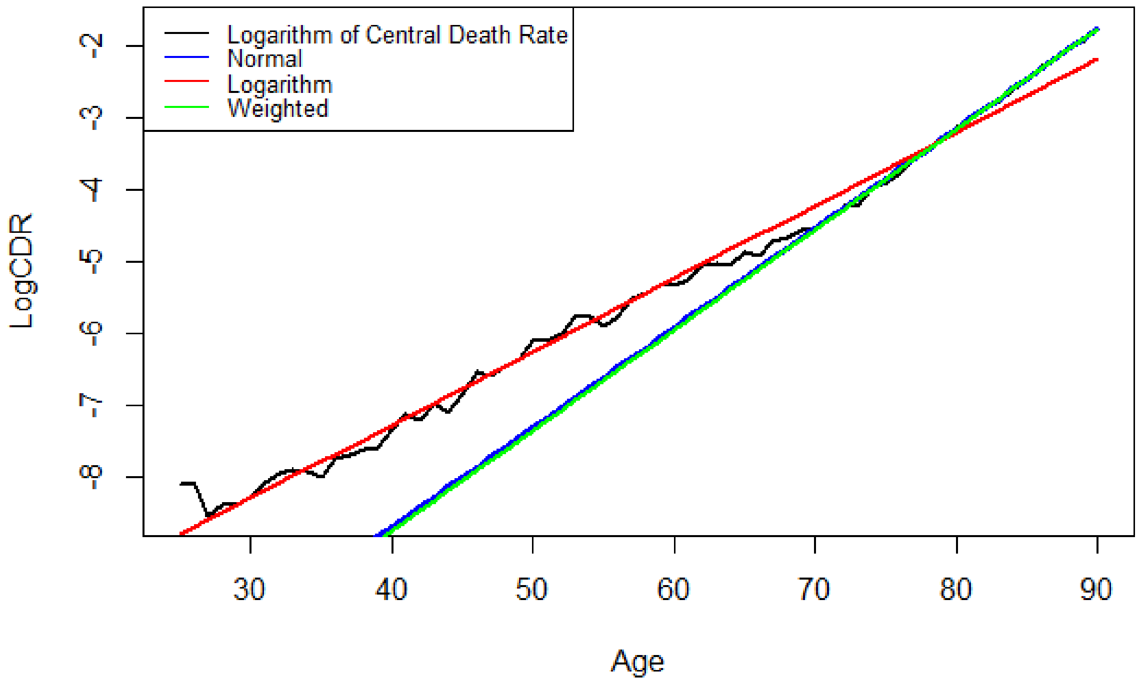


Figure 3: Logarithm of the Central Death Rates and its approximations (Austria)

III Sensitivities in a dynamic model

Now we model the logarithm of mortality rates using the Lee-Carter model (1992), which is defined with the following equation:

$$ln\left(\frac{D_{xt}}{E_{xt}}\right) = \alpha_x + \beta_x \kappa_t + \varepsilon_{xt} \tag{8}$$

where α_x and β_x are age-dependent parameters, and κ_t is a time-dependent parameter. While ε_{xt} is the stochastic error term that is i.i.d. normally distributed with zero mean and constant variance σ_ε^2.

The parameter β_x measures the change of the log of mortality rate relative to the change of κ_t (Lee & Carter,1992). This is the case since $\frac{d\, ln(\frac{D_{xt}}{E_{xt}})}{dt} = \beta_x \frac{d\kappa_t}{dt}$. Meanwhile, the time series parameter κ_t can be considered as the mortality level (Lee & Carter,1992).

The parameter α_x in the Lee-Carter model can be estimated using Maximum Likelihood. By doing so, we get the MLE estimator $\widehat{\alpha_x}$ which equals the mean of the log of mortality $ln\left(\frac{D_{xt}}{E_{xt}}\right)$ over time. Therefore, $\widehat{\alpha_x}$ can be interpreted as the (log) force of mortality at age x (Lee & Carter,1992). Due to the identification problem, parameters β_x and κ_t cannot be estimated with Maximum Likelihood. Instead, we use Singular Value Decomposition (SVD) on a matrix R with the entries:

$$R_{xt} = ln\Big(\frac{D_{xt}}{E_{xt}}\Big) - \widehat{\alpha_x} \tag{9}$$

Each entry of the estimate for this matrix corresponds to the product of estimates of β_x and κ_t. In addition, for identification, we impose three more constraints:

$$1. \quad \sum_t \widehat{\kappa_t} = 0 \tag{10}$$

$$2. \quad \sum_x (\widehat{\beta_x})^2 = 1 \tag{11}$$

$$3. \quad \widehat{\beta_x} \text{ is mostly positive} \tag{12}$$

In this way, we estimated the values of α_x, β_x, and κ_t for males and females in Austria and in the Netherlands. We found that the constraint of $\widehat{\beta_x}$ being mostly positive is not satisfied, so we multiply $\widehat{\beta_x}$ with -1, and do the same for estimates of κ_t. The plots of the estimates can be found in Figure 4.

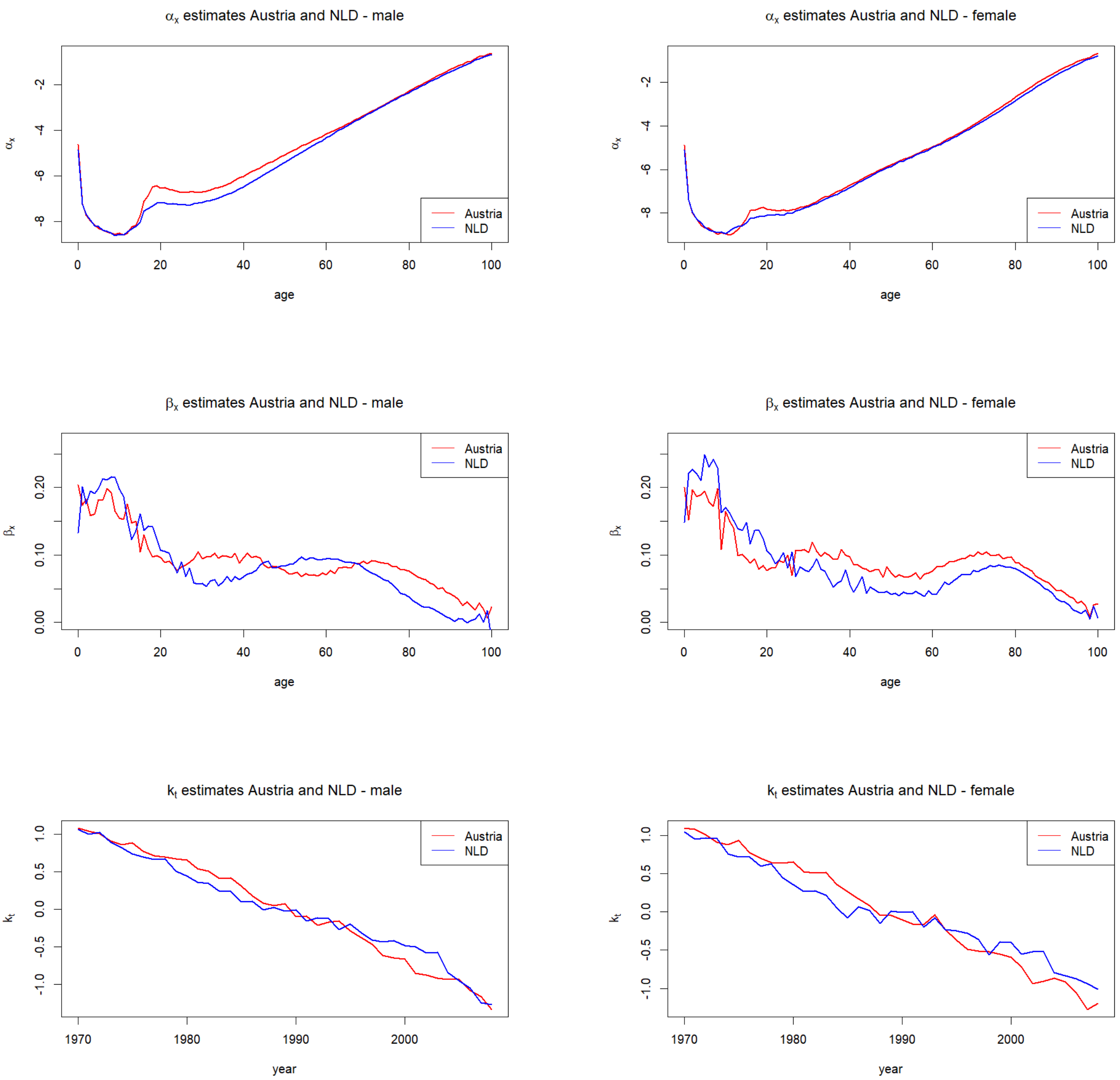


Figure 4: Estimates of α_x, β_x, and κ_t for males and females in Austria and in the Netherlands

We see that the graph of the estimates $\widehat{\alpha}_x$ over the ages does indeed resemble a force of mortality pattern. For both countries, we witness high mortality rates at very early ages, due to child mortality. Also, at the ages of around 20, we see a sudden increase in mortality rates for males. This phenomenon is known as the accident hump. Austria has a much higher male accident hump than the Netherlands, as well as we can observe that the accident hump is present for Austrian females. The estimate of β_x of Austrian and Dutch males and females can be observed to fluctuate in directly opposite ways. But gradually it decreases over age for both countries, implying that the changes in the force of mortality are becoming less sensitive to changes in mortality level κ_t for older people. Finally, the estimate of κ_t generally declines over the years for both countries, which is in accordance with the trend of declining mortality levels in Europe.

IV Simulation of a dynamic model

At this point, we consider using the Lee-Carter model (1992) for predicting the one-year probability of dying for a 65-years old male, for a period of 50 years, with the initial time $t_0 =$ January 1, 2008. We do this by fitting a Random Walk with drift model

$$k_{t+1} = k_t + \theta + \sigma\delta_t, \text{ where } \delta_t \sim N(0,1) \tag{13}$$

We begin by estimating θ and δ parameters using the Method of Moments estimation procedure, using the first two moments of $\Delta K_t := k_{t+1} - k_t$

$$E[\Delta k_t] = E[k_t + \theta + \sigma\delta_t - k_t] \tag{14}$$

$$= \theta + \sigma E[\delta_t | \delta \sim N(0,1)] = \theta \tag{15}$$

$$\implies \hat{\theta} = \frac{1}{T-1}\sum_{t=1}^{T-1} \Delta k_t \tag{16}$$

$$E[(\Delta k_t)^2] = E[(\theta + \sigma\delta_t)^2] = \theta^2 + \sigma^2 \tag{17}$$

$$\implies \hat{\theta}^2 + \hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} (\Delta k_t)^2 \tag{18}$$

$$\implies \hat{\sigma} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T-1} (\Delta k_t)^2 - \hat{\theta}^2} \tag{19}$$

Thus, we can model k_t time series, where the initial value, K_{2008} was previously estimated using 1970 - 2008 data.

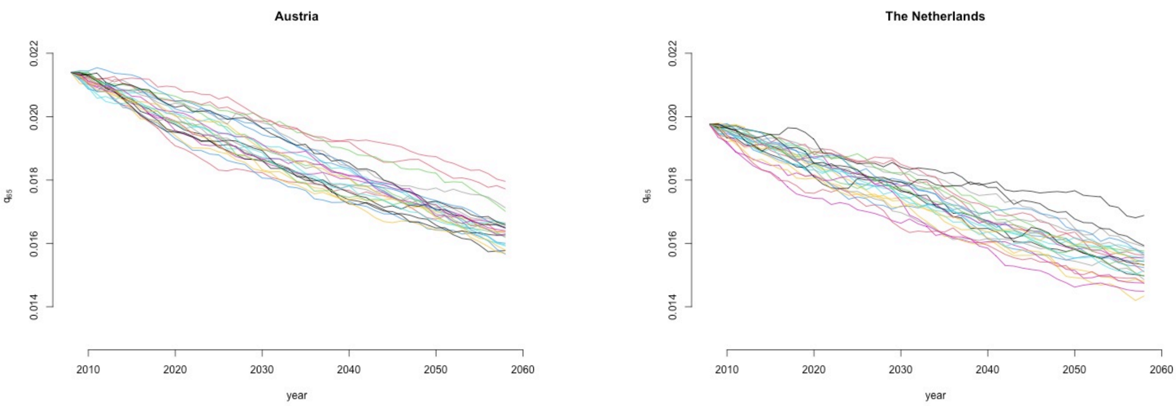


Figure 5: Simulated ${}_1q_{65}$ for males: 2008 - 2058

Figure 5 displays a clear downward-slopping trend for both Austria and The Netherlands. We notice however that in the case of Austria, there is a steeper trend than that in the case of The Netherlands. This aspect is better visible in Figure 6, where we overlap the two simulation processes. At this point, we can say that these findings are in line with the expectations, the two trends indicating a future convergence for the one-year probability of dying for 65 years old males.

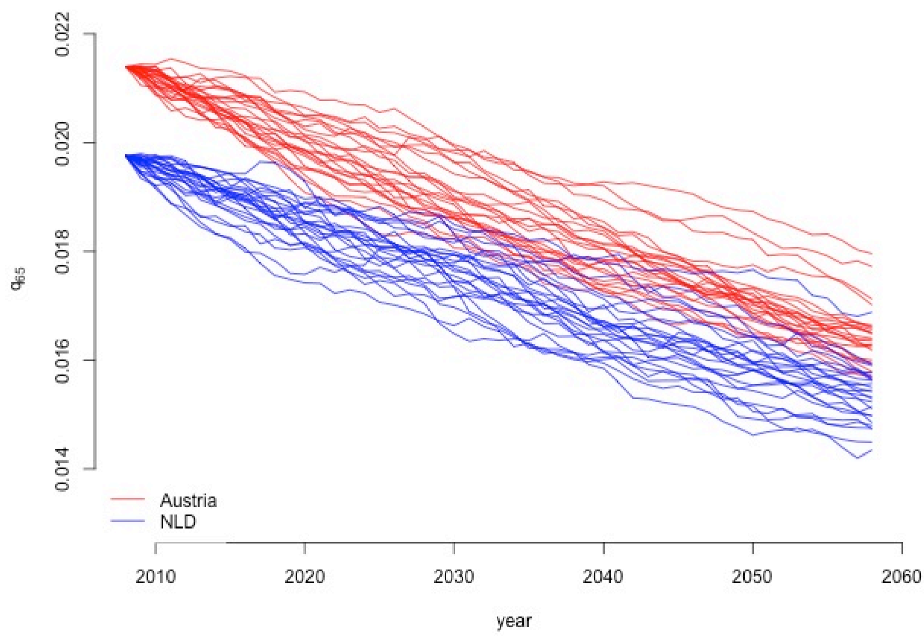


Figure 6: Overlapped simulated ${}_1q_{65}$ for males: Austria and The Netherlands 2008 - 2058

V Uncertainty in predictions

In this section, we assume the maximum age to be 120 years old. We use the Lee-Carter model to simulate one-year probabilities of dying up until the age of 90 years. For ages higher than 90 we want to avoid the "noise" generated by Lee-Carter so use the extrapolation scheme proposed by Kannisto:

$$\mu_{x,t} = H\left[\sum_{i=80}^{90} w_i(x) H^{-1}(\mu_{i,t})\right] \text{, where} \tag{20}$$

$$w_z(x) = \frac{1}{11} + \frac{(z-85)(x-85)}{110};\ H(x) - \frac{1}{1+e^{-x}};\ H^{-1}(x) = -\log(\frac{1}{x} - 1)$$

Using this approach we generate 1000 simulated paths where we compute the following for a female who becomes 65 years old on January 1^{st} 2008: the remaining cohort life expectancy, the expected present value of an annuity paying 1000 EUR at the end of every year and the expected present value of a 1000 EUR death benefit payable at the end of the year of death.

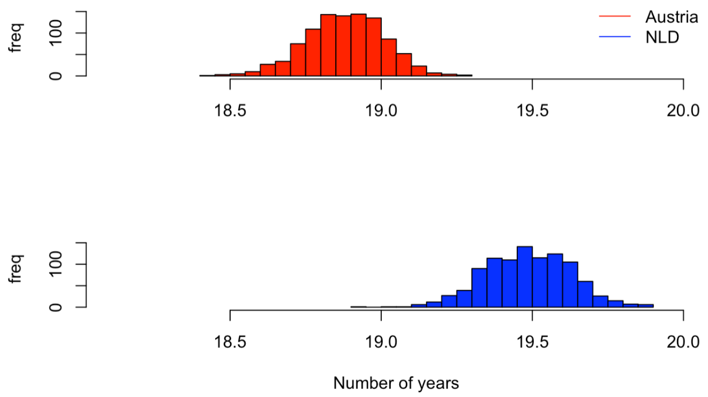


Figure 7: Remaining cohort life expectancy for a 65 y/o female

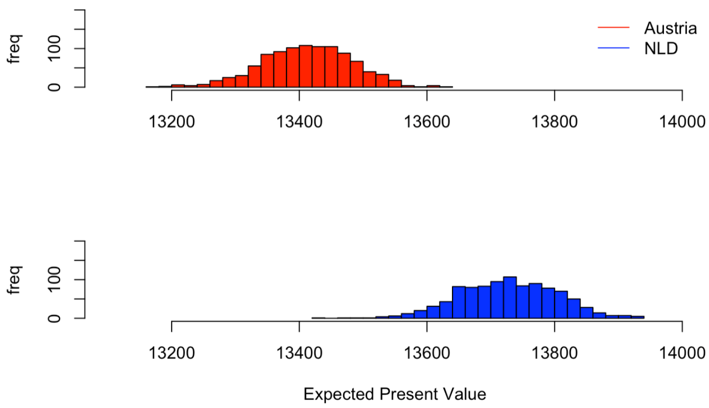


Figure 8: EPV of a 1000 EUR annuity for a 65 y/o female

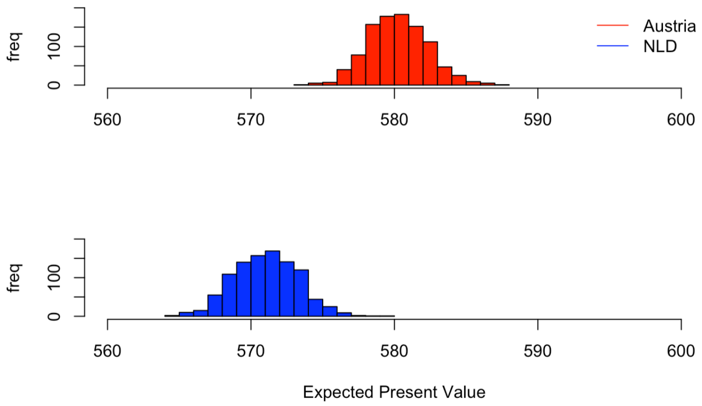


Figure 9: EPV of an 1000 EUR death benefit for a 65 y/o female

Figures 7-9 display the results of the 1000 simulated paths for the three mentioned values, for both Austria and The Netherlands. As expected based on the previously mentioned results, our results show that, on average, the complete remaining cohort life expectancy for 65 years old females is higher in The Netherlands than in Austria. The same results are further reflected in the distribution of the simulations for the annuity and the death benefit. In addition to that, there is a visible indication of a similar spread in results for the two countries.

VI References:

Dickson, D., Hardy, M., & Waters, H. (2019). Actuarial Mathematics for Life Contingent Risks (3rd ed., International Series on Actuarial Science). Cambridge: Cambridge University Press. doi:10.1017/9781108784184

Lee, D. & Carter R. (1992). Modeling and Forecasting U. S. Mortality. Journal of the American Statistical Association, Vol. 87, No. 419 https://doi.org/10.2307/2290201

VII Appendix

R codes are given below

assignment 4

```r
rm(list = ls())
setwd("C:/Users/tholo/Downloads")
## Use your own directory name. Mind the forward slashes!
 ##Q1.a
#built the data frame for Austria
Death <- read.table("Deaths_1x1.txt",header=TRUE,skip=2)
Death$Age <- as.character(Death$Age)
Death$Age <- ifelse(Death$Age=="110+","110",Death$Age)
Death$Age <- as.numeric(Death$Age)
head(Death$Age); tail(Death$Age); str(Death)
```

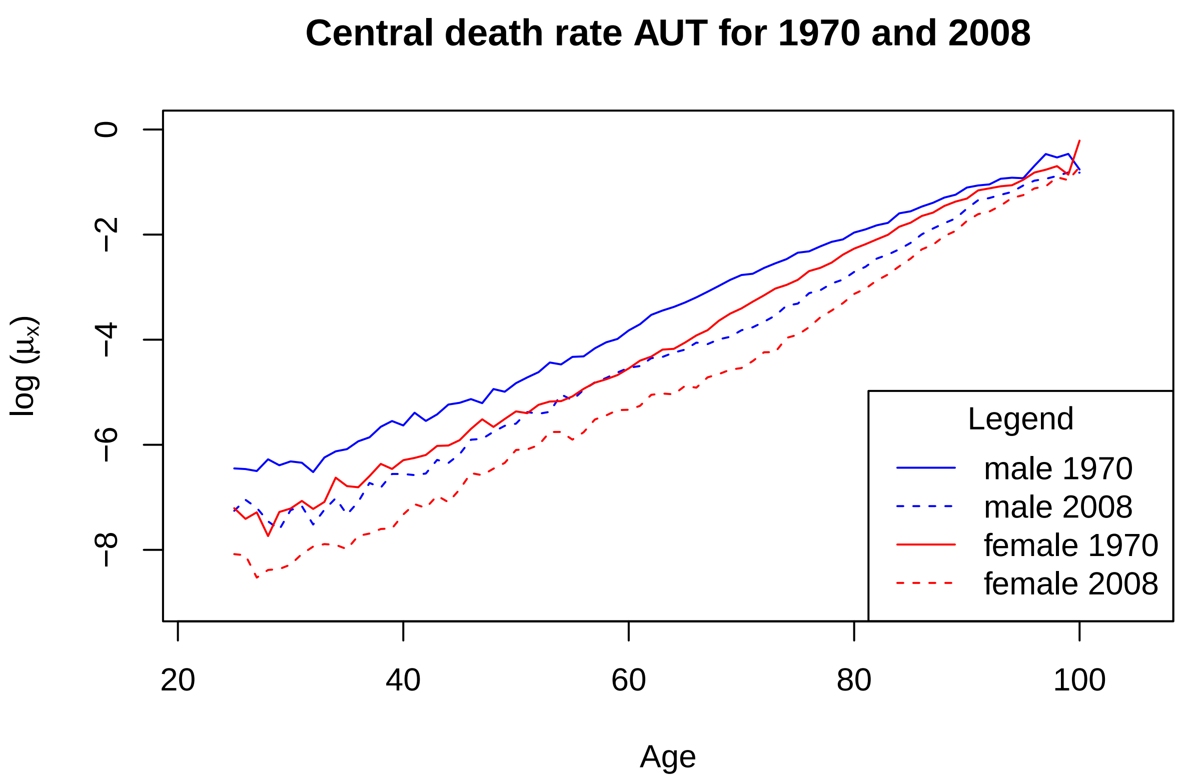
```
## [1] 0 1 2 3 4 5

## [1] 105 106 107 108 109 110

## 'data.frame':    8103 obs. of  5 variables:
##  $ Year  : int  1947 1947 1947 1947 1947 1947 1947 1947 1947 1947 ...
##  $ Age   : num  0 1 2 3 4 5 6 7 8 9 ...
##  $ Female: num  4352 282 210 178 128 ...
##  $ Male  : num  5746 378 239 185 152 ...
##  $ Total : num  10098 660 449 363 280 ...
```

```r
Death <- Death[Death$Year >=1970, ]
total_expos <- read.table("Exposures_1x1.txt", header = TRUE, skip = 2)
expos <- total_expos[total_expos$Year >=1970, ]
l_hat_male <- Death$Male /expos$Male
l_hat_female <- Death$Female/expos$Female
#putting all together in  the same table
df_austria <- cbind(Death, expos$Female, expos$Male, expos$Total, l_hat_male)
df_austria <- cbind(df_austria, l_hat_female)
head(df_austria)
```

```
##      Year Age Female Male Total expos$Female expos$Male expos$Total
## 2554 1970   0   1214 1694  2908     55685.71   58389.48    114075.2
## 2555 1970   1     93  100   193     58977.33   61825.49    120802.8
## 2556 1970   2     53   64   117     60287.95   63472.35    123760.3
## 2557 1970   3     34   52    86     60995.97   63730.08    124726.1
## 2558 1970   4     49   55   104     61059.31   63762.49    124821.8
## 2559 1970   5     27   47    74     63143.75   65938.81    129082.6
##        l_hat_male l_hat_female
## 2554 0.0290120755 0.0218009252
## 2555 0.0016174558 0.0015768771
## 2556 0.0010083131 0.0008791143
## 2557 0.0008159412 0.0005574139
## 2558 0.0008625761 0.0008024984
## 2559 0.0007127820 0.0004275958
```

```r
##Plotting the estimate for log central death rate
plot(25:100, log(df_austria$l_hat_male)[df_austria$Year == 1970 & df_austria$Age >=25 & df_austria$Age
```

```r
     main = "Central death rate AUT for 1970 and 2008",
     type = "l", col = "blue", ylim = c(-9, 0), xlim = c(22, 105),
     xlab = "Age", ylab = expression(paste("log (",mu [x,t], ")")))
points(25:100, log(df_austria$l_hat_male)[df_austria$Year == 2008 & df_austria$Age >=25 & df_austria$Age
       type = "l", col = "blue", lty = 2)
points(25:100, log(df_austria$l_hat_female)[df_austria$Year == 1970 & df_austria$Age >=25 & df_austria$
       type = "l", col = "red")
points(25:100, log(df_austria$l_hat_female)[df_austria$Year == 2008 & df_austria$Age >=25 & df_austria$
       type = "l", col = "red", lty = 2)
legend("bottomright", legend = c("male 1970", "male 2008", "female 1970", "female 2008"), col = c("blue
       title  = "Legend",
       lty = c(1, 2,  1,2 ))
```




```r
#Repeat the procedure for NLD
Death <- read.table("NL_Deaths_1x1.txt",header=TRUE,skip=2)
Death$Age <- as.character(Death$Age)
Death$Age <- ifelse(Death$Age=="110+","110",Death$Age)
Death$Age <- as.numeric(Death$Age)
head(Death$Age); tail(Death$Age); str(Death)
```

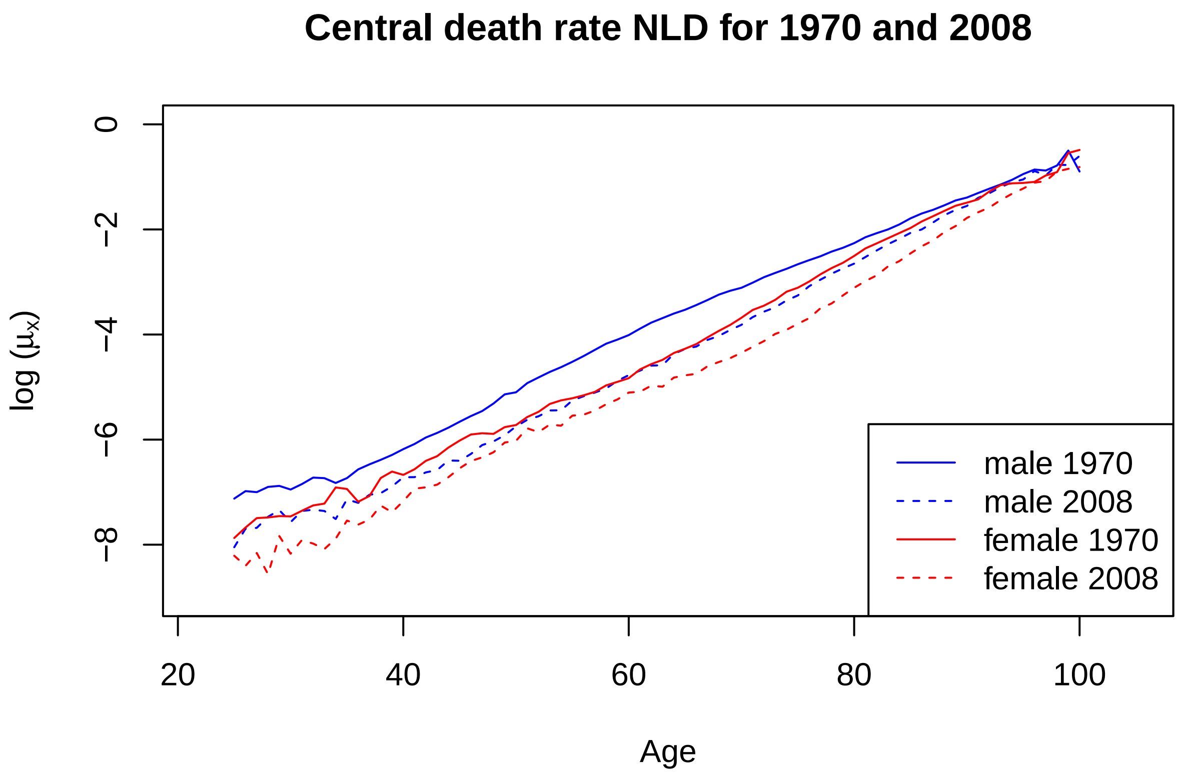
```
## [1] 0 1 2 3 4 5

## [1] 105 106 107 108 109 110

## 'data.frame':    18981 obs. of  5 variables:
##  $ Year  : int  1850 1850 1850 1850 1850 1850 1850 1850 1850 1850 ...
##  $ Age   : num  0 1 2 3 4 5 6 7 8 9 ...
```

```
##  $ Female: num  8735 2292 1009 591 461 ...
##  $ Male  : num  10804 2454 1055 634 521 ...
##  $ Total : num  19539 4746 2064 1225 982 ...
```

```
Death_NLD <- Death[Death$Year >=1970, ]
total_expos <- read.table("NL_Exposures_1x1.txt", header = TRUE, skip = 2)
expos_NLD <- total_expos[total_expos$Year >=1970, ]
l_hat_male <- Death_NLD$Male /expos_NLD$Male
l_hat_female <- Death_NLD$Female/expos_NLD$Female
#putting all together in  the same table
df_nld <- cbind(Death_NLD, expos_NLD$Female, expos_NLD$Male, expos_NLD$Total, l_hat_male, l_hat_female)
head(df_nld)
```

```
##       Year Age Female   Male  Total expos_NLD$Female expos_NLD$Male
## 13321 1970   0 1245.0 1685.0 2930.0         117937.2       123659.2
## 13322 1970   1  166.5  199.0  365.5         117152.8       122630.3
## 13323 1970   2   83.0  113.0  196.0         114187.9       119505.7
## 13324 1970   3   62.0  108.0  170.0         114899.1       120472.4
## 13325 1970   4   55.0   97.5  152.5         116615.7       122437.4
## 13326 1970   5   46.5   81.5  128.0         118939.3       125317.5
##       expos_NLD$Total   l_hat_male l_hat_female
## 13321        241596.4 0.0136261552 0.0105564674
## 13322        239783.1 0.0016227633 0.0014212213
## 13323        233693.6 0.0009455618 0.0007268720
## 13324        235371.5 0.0008964711 0.0005396036
## 13325        239053.1 0.0007963254 0.0004716345
## 13326        244256.8 0.0006503483 0.0003909557
```

```
##Q1. b
##Plotting the estimate for log central death rate
plot(25:100, log(df_nld$l_hat_male)[df_nld$Year == 1970 & df_nld$Age >=25 & df_nld$Age <=100],
     type = "l", col = "blue", ylim = c(-9, 0), xlim = c(22, 105),
     main = "Central death rate NLD for 1970 and 2008",
     xlab = "Age", ylab = expression(paste("log (",mu [x,t], ")")
      )
     )
points(25:100, log(df_nld$l_hat_male)[df_nld$Year == 2008 & df_nld$Age >=25 & df_nld$Age <=100],
       type = "l", col = "blue", lty = 2)
  points(25:100, log(df_nld$l_hat_female)[df_nld$Year == 1970 & df_nld$Age >=25 & df_nld$Age <=100],
       type = "l", col = "red")
points(25:100, log(df_nld$l_hat_female)[df_nld$Year == 2008 & df_nld$Age >=25 & df_nld$Age <=100],
       type = "l", col = "red", lty = 2)
legend("bottomright", legend = c("male 1970", "male 2008", "female 1970", "female 2008"),
       col = c("blue","blue", "red", "red"),
       lty = c(1, 2,  1,2 ))
```




```r
## Q1.c
#1 year dying probabilities for ages 0-110 in year t, for t = 1970,..., 2019
qx_austria_m = 1 - exp(-df_austria$l_hat_male)
qx_austria_f = 1 - exp(-df_austria$l_hat_female)
qx_nld_m = 1- exp(-df_nld$l_hat_male)
qx_nld_f = 1 - exp(-df_nld$l_hat_female)

#1 year survival probabilities for
sx_austria_m <- 1 - qx_austria_m
sx_austria_f <- 1 - qx_austria_f
sx_nld_m <- 1 - qx_nld_m
sx_nld_f <- 1 - qx_nld_f
df_austria <- cbind(df_austria, qx_austria_m, qx_austria_f, sx_austria_m, sx_austria_f)
df_nld  <- cbind(df_nld, qx_nld_f, qx_nld_m, sx_nld_m, sx_nld_f)


life_exp <- function(age, year, max_age){
  kpx_austria_m <- cumprod(df_austria$sx_austria_m[df_austria$Year == year & df_austria$Age >=age &df_au
  kpx_austria_f <- cumprod(df_austria$sx_austria_f[df_austria$Year == year & df_austria$Age >=age &df_au
  kpx_nld_m <- cumprod(df_nld$sx_nld_m[df_nld$Year == year & df_nld$Age >=age &df_nld$Age <max_age])
  kpx_nld_f <- cumprod(df_nld$sx_nld_f[df_nld$Year == year & df_nld$Age >=age &df_nld$Age <max_age])
  complete_life_expectation <- c(sum(kpx_austria_m),sum(kpx_austria_f), sum(kpx_nld_m), sum(kpx_nld_f))
  return(complete_life_expectation)
}

Newborn_le_1970 <- life_exp(0, 1970, 100)
```

```r
cat("New born life expectancies for Austria (1970): \n",
    "Males: " , Newborn_le_1970[1], "\n",
    "Females: ", Newborn_le_1970[2], "\n",
    "New born life expectancies for Netherlands: \n",
    "Males: " , Newborn_le_1970[3], "\n",
    "Females: ", Newborn_le_1970[4], "\n")
```

```
## New born life expectancies for Austria (1970):
##  Males:  66.4996
##  Females:  73.42172
##  New born life expectancies for Netherlands:
##  Males:  70.83473
##  Females:  76.52716
```

```r
Le65_1970 <- life_exp(65, 1970, 100)
cat("65 y/o life expectancies for Austria (1970): \n",
    "Males: " , Le65_1970[1], "\n",
    "Females: ", Le65_1970[2], "\n",
    "65 y/o life expectancies for Netherlands: \n",
    "Males: " , Le65_1970[3], "\n",
    "Females: ", Le65_1970[4], "\n")
```

```
## 65 y/o life expectancies for Austria (1970):
##  Males:  11.69206
##  Females:  14.89145
##  65 y/o life expectancies for Netherlands:
##  Males:  13.57808
##  Females:  16.51402
```

```r
Newborn_le_2008 <- life_exp(0, 2008, 100)
cat("New born life expectancies for  (2008): \n",
    "Males: " , Newborn_le_2008[1], "\n",
    "Females: ", Newborn_le_2008[2], "\n",
    "New born life expectancies for Netherlands: \n",
    "Males: " , Newborn_le_2008[3], "\n",
    "Females: ", Newborn_le_2008[4], "\n")
```

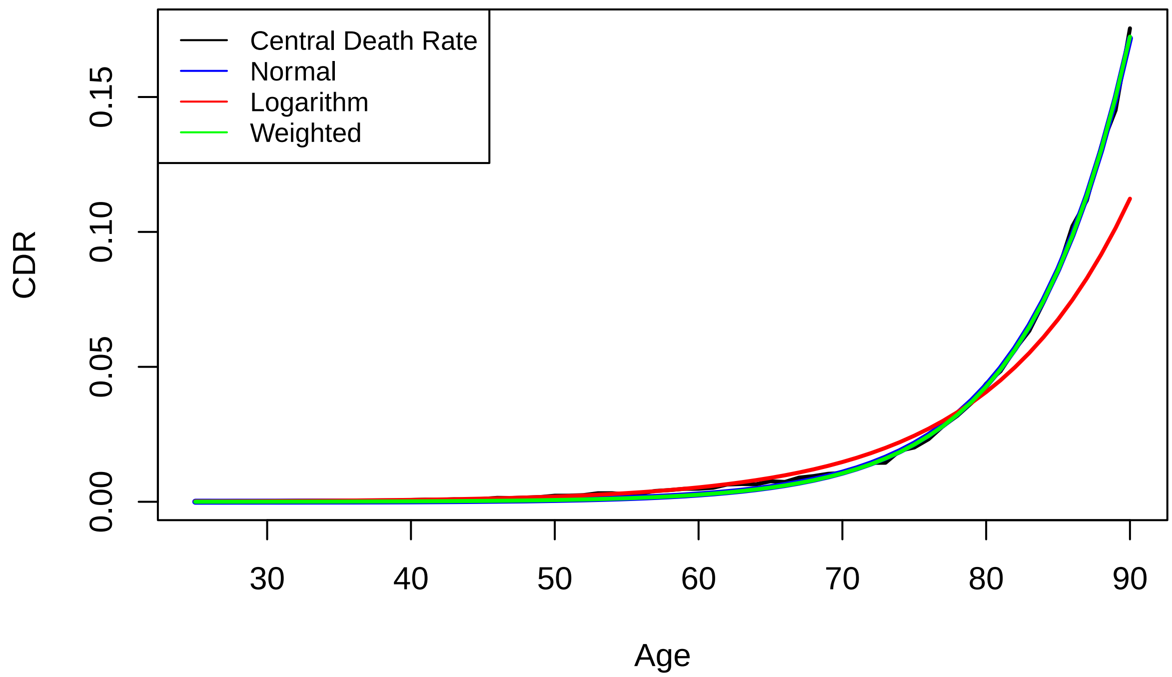
```
## New born life expectancies for  (2008):
##  Males:  77.58179
##  Females:  82.93902
##  New born life expectancies for Netherlands:
##  Males:  78.31703
##  Females:  82.25521
```

```r
Le65_2008 <- life_exp(65, 2008, 100)
cat("65 y/o life expectancies for Austria (2008): \n",
    "Males: " , Le65_2008[1], "\n",
    "Females: ", Le65_2008[2], "\n",
    "65 y/o life expectancies for Netherlands: \n",
    "Males: " , Le65_2008[3], "\n",
    "Females: ", Le65_2008[4], "\n")
```

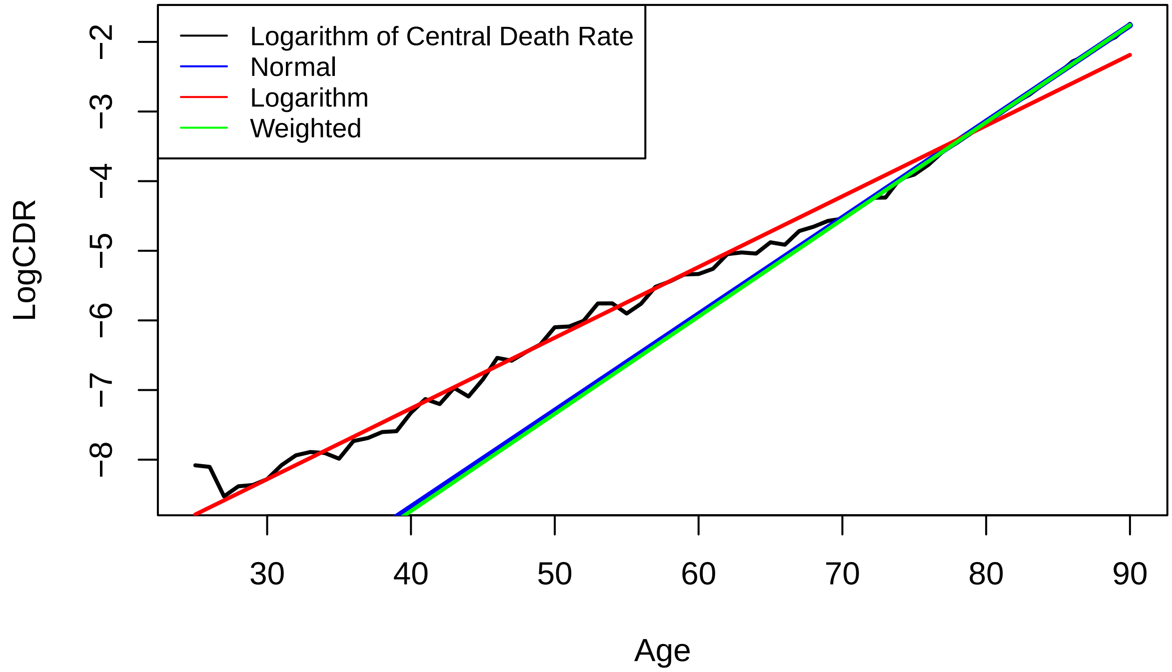
```
## 65 y/o life expectancies for Austria (2008):
##  Males:  17.52249
##  Females:  20.77211
##  65 y/o life expectancies for Netherlands:
```

```
## Males: 17.24932
## Females: 20.43476
```

```r
####################### SECTION 2 ############################################
ages<-c(25:90)


DxtFemale2008<-df_austria$Female[df_austria$Year==2008 & df_austria$Age>=25 & df_austria$Age<=90]
ExtFemale2008<-df_austria$'expos$Female'[df_austria$Year==2008 & df_austria$Age>=25 & df_austria$Age<=9

GompertzLS<-function(p){
  sum((DxtFemale2008/ExtFemale2008 - exp(p[1]+p[2]*ages))^2)
}

Optim1<-optim(c(-10,-1),GompertzLS)
a_hat_ls<-Optim1$par[1]
b_hat_ls<-Optim1$par[2]


GompertzLog<-function(p){
  sum((log(DxtFemale2008/ExtFemale2008) - (p[1]+p[2]*ages))^2)
}

Optim2<-optim(c(0,0),GompertzLog)
a_hat_log<-Optim2$par[1]
b_hat_log<-Optim2$par[2]

GompertzWeighted<-function(p){
  sum(DxtFemale2008 * (DxtFemale2008/ExtFemale2008 - exp(p[1]+p[2]*ages))^2)
}

Optim3<-optim(c(a_hat_log,b_hat_log),GompertzWeighted)
a_hat_w<-Optim3$par[1]
b_hat_w<-Optim3$par[2]


CDR<-DxtFemale2008/ExtFemale2008
LinApprox<-exp(a_hat_ls + b_hat_ls*ages)
LogApprox<-exp(a_hat_log + b_hat_log*ages)
WeightedApprox<-exp(a_hat_w + b_hat_w*ages)

plot(ages,CDR,type="l",col="black",lwd=2,xlab="Age",ylab = "CDR")
lines(ages,LinApprox,type="l",col="blue",lwd=3)
lines(ages,LogApprox,type="l",col="red",lwd=2)
lines(ages,WeightedApprox,type="l",col="green",lwd=2)
legend('topleft', c("Central Death Rate", "Normal", "Logarithm", "Weighted"), cex=.8,col=c("black", "bl
```




```r
plot(ages,log(CDR),type="l",col="black",lwd=2,xlab="Age",ylab = "LogCDR")
lines(ages,log(LinApprox),type="l",col="blue",lwd=3)
lines(ages,log(LogApprox),type="l",col="red",lwd=2)
lines(ages,log(WeightedApprox),type="l",col="green",lwd=2)
legend('topleft', c("Logarithm of Central Death Rate", "Normal", "Logarithm", "Weighted"), cex=.8,
       col=c("black", "blue", "red", "green"), lty=c(1, 1, 1, 1))
```




```
################################## SECTION 3  #######################################
#a. we need the model to be identified, so Lee Carter suggests SVD (page 4/14)

df_austria$Male[df_austria$Male == 0 & df_austria$Year>= 1970 & df_austria$Year<= 2008]<- 1/10
df_austria$`expos$Male`[df_austria$`expos$Male` == 0 & df_austria$Year>= 1970 & df_austria$Year<= 2008]
df_austria$Female[df_austria$Female == 0 & df_austria$Year>= 1970 & df_austria$Year<= 2008]<- 1/10
df_austria$`expos$Female`[df_austria$`expos$Female` == 0 & df_austria$Year>= 1970 & df_austria$Year<= 2

alpha_male_austria <- rep(0, 101)
alpha_female_austria <- rep(0,101)

for(age in 0:100){
  Dxm <-          df_austria$Male[df_austria$Age == age &   df_austria$Year>= 1970 & df_austria$Year<= 2

  Exm <- df_austria$`expos$Male`[df_austria$Age == age &   df_austria$Year>= 1970 & df_austria$Year<= 2

  Dxf <-          df_austria$Female[df_austria$Age == age & df_austria$Year>= 1970 & df_austria$Year<= 2
  Exf <- df_austria$`expos$Female`[df_austria$Age == age & df_austria$Year>= 1970 & df_austria$Year<= 2

  alpha_male_austria[age+1]    <- mean(log(Dxm/Exm))
  alpha_female_austria[age+1] <- mean(log(Dxf/Exf))}



Dxt_male_a <- df_austria$Male[df_austria$Age <101 & df_austria$Year>= 1970 & df_austria$Year<= 2008]
Ext_male_a <- df_austria$`expos$Male`[df_austria$Age <101& df_austria$Year>= 1970 & df_austria$Year<= 2
```

```
Dxt_female_a <- df_austria$Female[df_austria$Age <101 & df_austria$Year>= 1970 & df_austria$Year<= 2008]
Ext_female_a <- df_austria$`expos$Female`[df_austria$Age <101 & df_austria$Year>= 1970 & df_austria$Yea

Rx_male <- matrix(log(Dxt_male_a/Ext_male_a), 101, 39)
Rx_female <- matrix(log(Dxt_female_a/Ext_female_a), 101, 39)
Rx_austria_male <- matrix(rep(0, 101*39), 101, 39)
Rx_austria_female <- matrix(rep(0, 101*39), 101, 39)
for (t in 1:39){
  Rx_austria_male[,t] <- Rx_male[,t] - alpha_male_austria
  Rx_austria_female[,t] <- Rx_female[,t] - alpha_female_austria
}

# Singular value decomposition Austria
#Male
UDV_Austria_male <- svd(Rx_austria_male)
bx_austria_male <- UDV_Austria_male$u[,1] #condition 3 is no satisfied
kt_austria_male <- sqrt(UDV_Austria_male$d[1])*UDV_Austria_male$v[, 1]
sum(bx_austria_male^2)
```

```
## [1] 1
```

```
sum(kt_austria_male)
```

```
## [1] 1.998401e-15
```

```
#Female
UDV_Austria_female <- svd(Rx_austria_female)
bx_austria_female <- UDV_Austria_female$u[,1]
kt_austria_female <- sqrt(UDV_Austria_female$d[1])*UDV_Austria_female$v[,1]
sum(bx_austria_female^2)
```

```
## [1] 1
```

```
sum(kt_austria_female)
```

```
## [1] 4.378442e-15
```

```
###Repeat procedure for NLD
df_nld$Male[df_nld$Male == 0 & df_nld$Year>= 1970 & df_nld$Year<= 2008]<- 1/10
df_nld$`expos_NLD$Male`[df_nld$`expos_NLD$Male` == 0 & df_nld$Year>= 1970 & df_nld$Year<= 2008] <- 1/10
df_nld$Female[df_nld$Female == 0 & df_nld$Year>= 1970 & df_nld$Year<= 2008]<- 1/10
df_nld$`expos_NLD$Female`[df_nld$`expos_NLD$Female` == 0 & df_nld$Year>= 1970 & df_nld$Year<= 2008]<- 1/

alpha_male_nld <- rep(0, 101)
alpha_female_nld <- rep(0,101)

for(age in 0:100){
  Dxm <-             df_nld$Male[df_nld$Age == age &   df_nld$Year>= 1970 & df_nld$Year<= 2008]
  Exm <- df_nld$`expos_NLD$Male`[df_nld$Age == age &   df_nld$Year>= 1970 & df_nld$Year<= 2008]

  Dxf <-         df_nld$Female[df_nld$Age == age & df_nld$Year>= 1970 & df_nld$Year<= 2008]
  Exf <- df_nld$`expos_NLD$Female`[df_nld$Age == age & df_nld$Year>= 1970 & df_nld$Year<= 2008]

  alpha_male_nld[age+1]   <- mean(log(Dxm/Exm))
  alpha_female_nld[age+1] <- mean(log(Dxf/Exf))}
```

```r
Dxt_male_n <- df_nld$Male[df_nld$Age <101 & df_nld$Year>= 1970 & df_nld$Year<= 2008]
Ext_male_n <- df_nld$`expos_NLD$Male`[df_nld$Age <101& df_nld$Year>= 1970 & df_nld$Year<= 2008]
Dxt_female_n <- df_nld$Female[df_nld$Age <101 & df_nld$Year>= 1970 & df_nld$Year<= 2008]
Ext_female_n <- df_nld$`expos_NLD$Female`[df_nld$Age <101 & df_nld$Year>= 1970 & df_nld$Year<= 2008]

Rx_male <- matrix(log(Dxt_male_n/Ext_male_n), 101, 39)
Rx_female <- matrix(log(Dxt_female_n/Ext_female_n), 101, 39)
Rx_nld_male <- matrix(rep(0, 101*39), 101, 39)
Rx_nld_female <- matrix(rep(0, 101*39), 101, 39)
for (t in 1:39){
  Rx_nld_male[,t] <- Rx_male[,t] - alpha_male_nld
  Rx_nld_female[,t] <- Rx_female[,t] - alpha_female_nld
}

# Singular value decomposition Austria
#Male
UDV_nld_male <- svd(Rx_nld_male)
bx_nld_male <- UDV_nld_male$u[,1] #condition 3 is no satisfied
kt_nld_male <- sqrt(UDV_nld_male$d[1])*UDV_nld_male$v[, 1]
sum(bx_nld_male^2)
```

```
## [1] 1
```

```r
sum(kt_nld_male)
```

```
## [1] 3.335006e-15
```

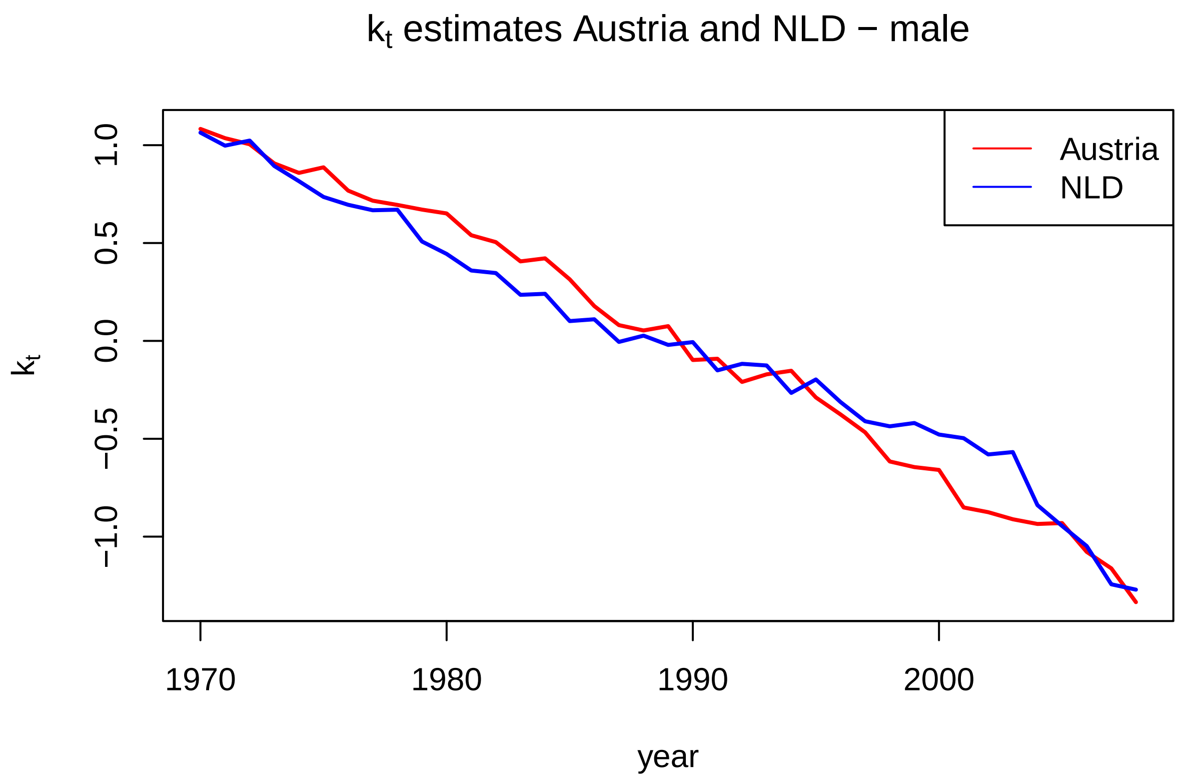
```r
#Female
UDV_nld_female <- svd(Rx_nld_female)
bx_nld_female <- UDV_nld_female$u[,1]
kt_nld_female <- sqrt(UDV_nld_female$d[1])*UDV_nld_female$v[,1]
sum(bx_nld_female^2)
```

```
## [1] 1
```

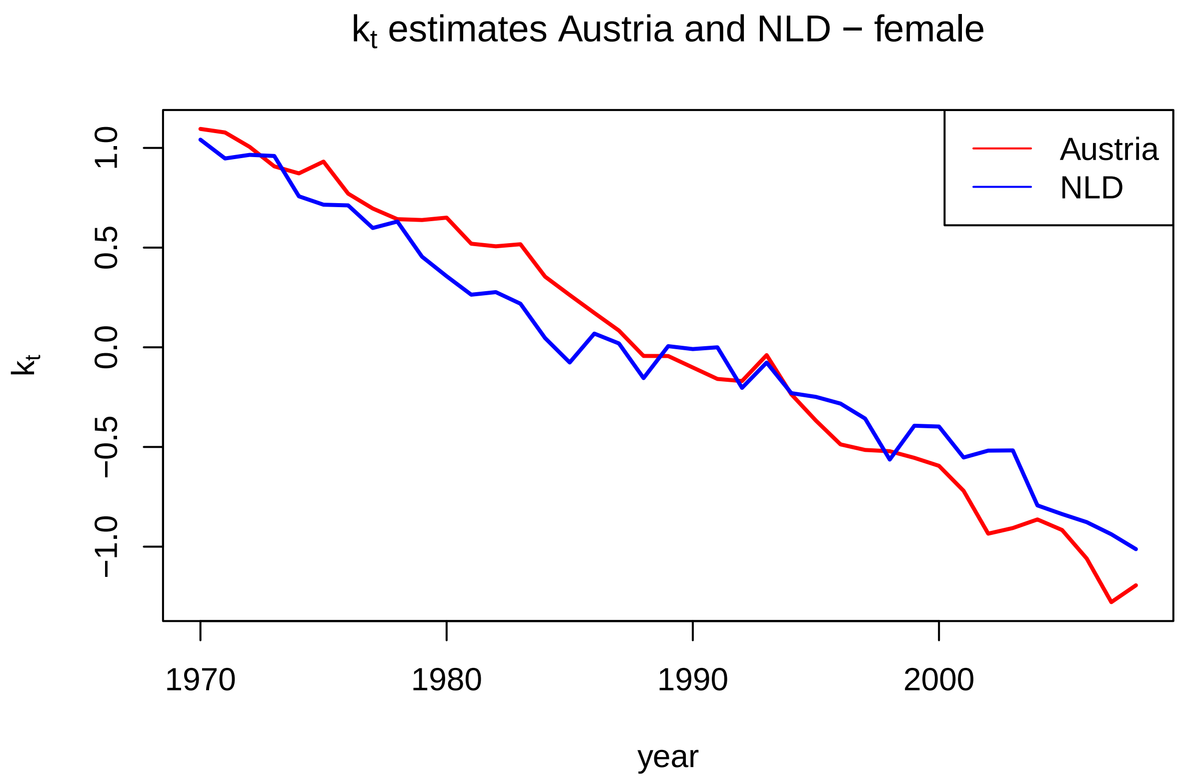
```r
sum(kt_nld_female)
```

```
## [1] -4.809521e-15
```

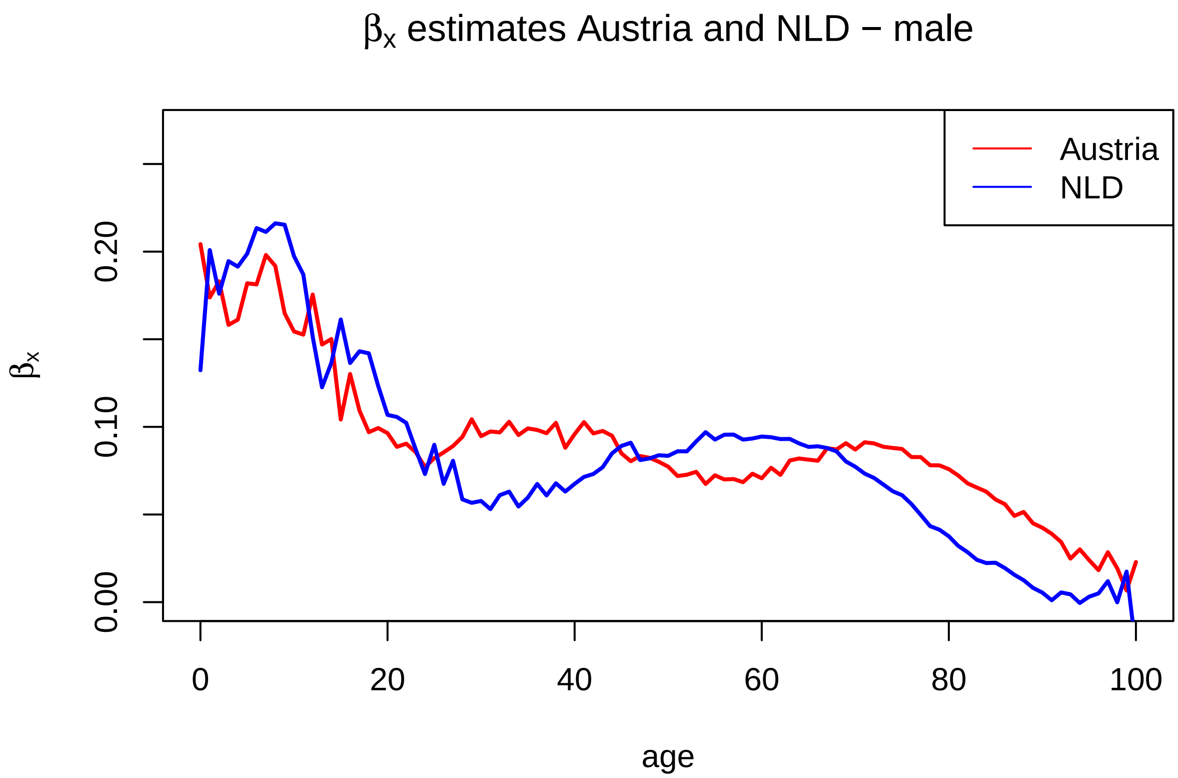
```r
years = c(1970:2008)
ages = c(0:100)

## alpha estimates plots
plot(years, -kt_austria_male, type = "l", col = "red", lwd=2,
     main = expression(paste(k [t]," estimates Austria and NLD - male")),
     ylab = expression(paste(k [t])),
     xlab = "year")
points(years, -kt_nld_male, type = "l", col = "blue", lwd=2)
legend("topright", legend = c("Austria", "NLD"), col = c("red","blue"), lty = c(1, 1))
```



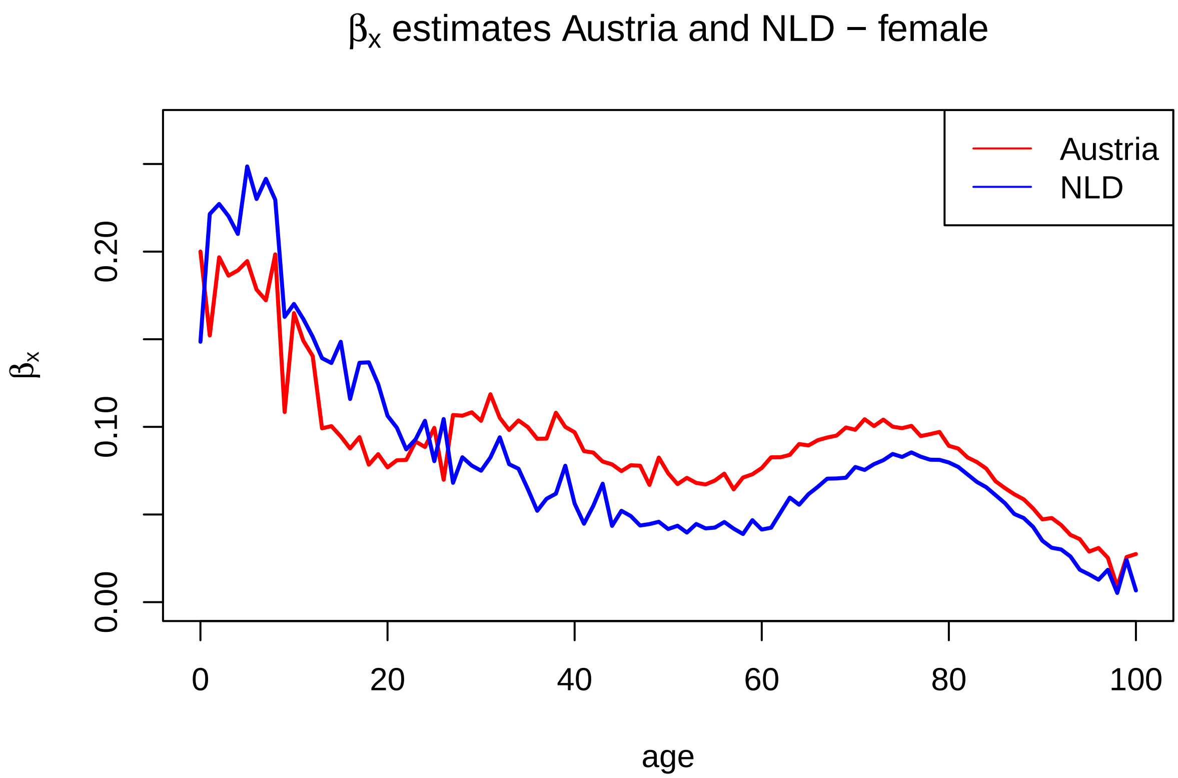

```r
plot(years, -kt_austria_female, type = "l", col = "red",lwd=2,
     main = expression(paste(k [t]," estimates Austria and NLD - female")),
     ylab = expression(paste(k [t])),
     xlab = "year")
points(years, -kt_nld_female, type = "l",lwd=2, col = "blue")
legend("topright", legend = c("Austria", "NLD"), col = c("red","blue"), lty = c(1, 1))
```



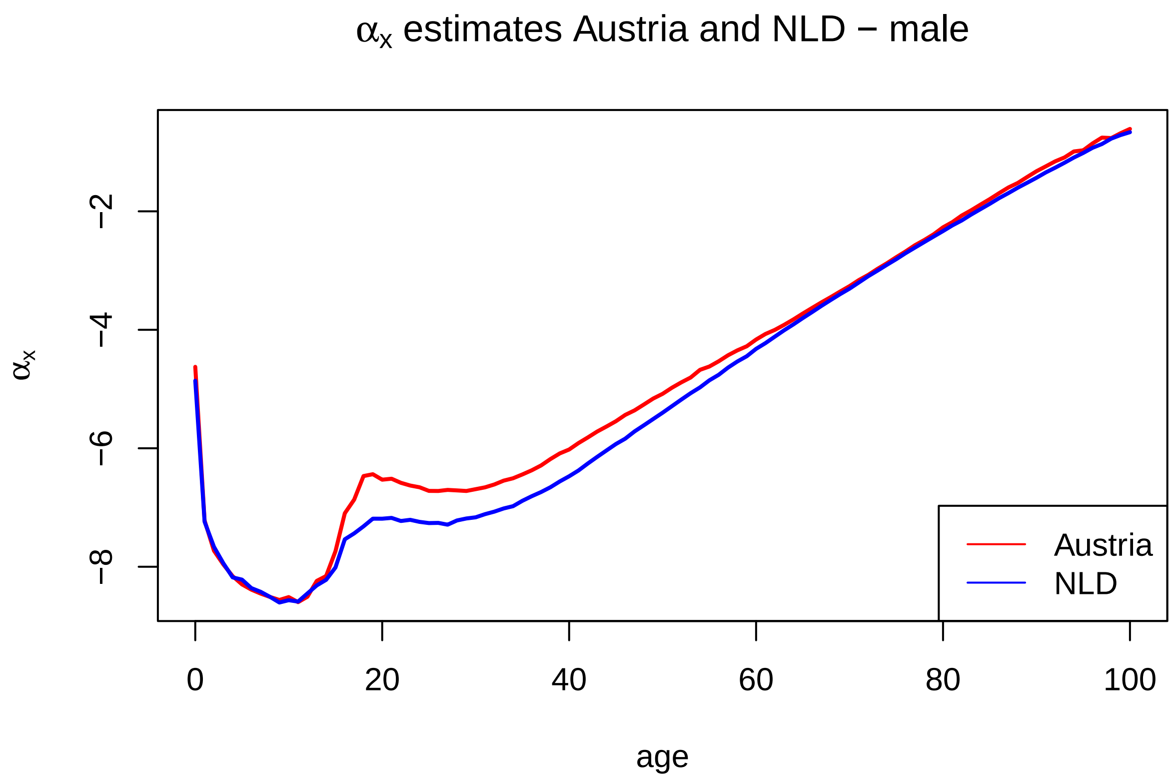

```
##beta estimates plots
plot(ages, -bx_austria_male, type = "l", col = "red", lwd=2,
     main = expression(paste(beta [x,2008]," estimates Austria and NLD - male")),
     ylab = expression(paste(beta [x,2008])),
     xlab = "age",
     ylim = c(0, 0.27))

points(ages, -bx_nld_male, type = "l", lwd=2, col = "blue")
legend("topright", legend = c("Austria", "NLD"), col = c("red","blue"), lty = c(1, 1))
```



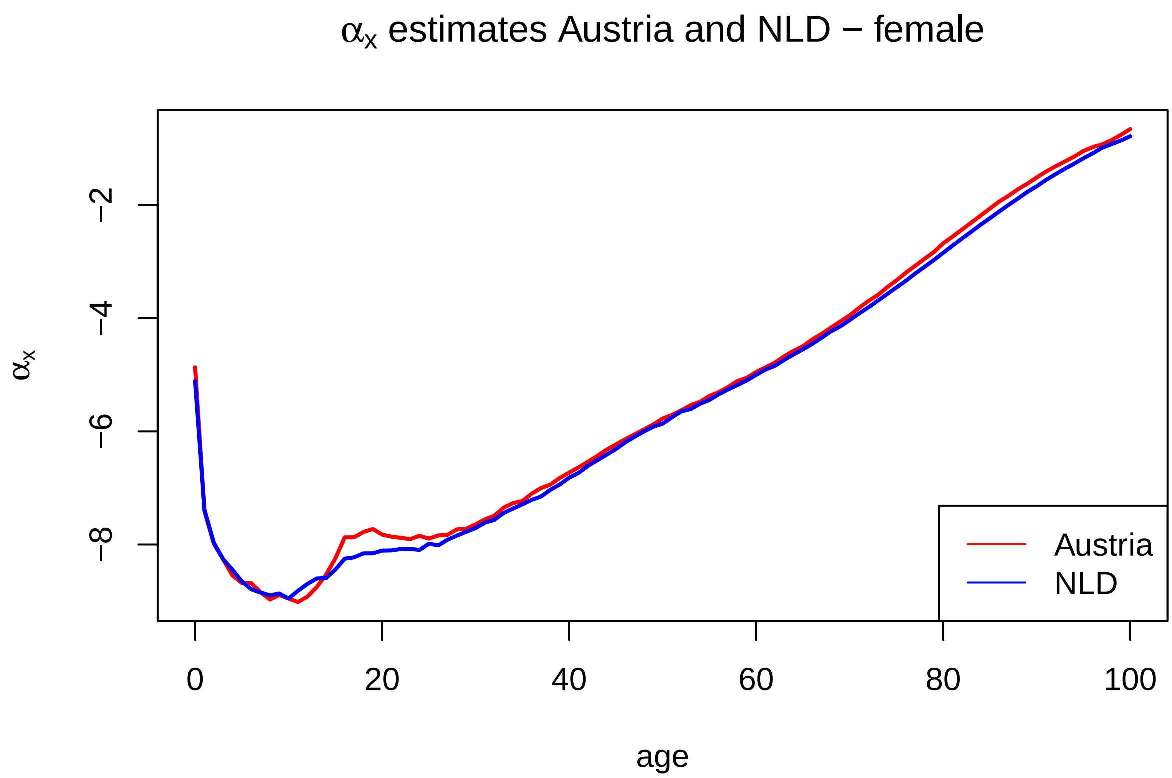

```
plot(ages, -bx_austria_female, type = "l", col = "red", lwd=2,
     main = expression(paste(beta [x,2008]," estimates Austria and NLD - female")),
     ylab = expression(paste(beta [x,2008])),
     xlab = "age",
     ylim = c(0, 0.27))

points(ages, -bx_nld_female, type = "l",lwd=2, col = "blue")
legend("topright", legend = c("Austria", "NLD"), col = c("red","blue"), lty = c(1, 1))
```




```r
## alpha estimates plots
plot(ages, alpha_male_austria, type = "l", col = "red", lwd=2,
     main = expression(paste(alpha [x,2008]," estimates Austria and NLD - male")),
     ylab = expression(paste(alpha [x,2008])),
     xlab = "age")
points(ages, alpha_male_nld, type = "l",lwd=2, col = "blue")
legend("bottomright", legend = c("Austria", "NLD"), col = c("red","blue"), lty = c(1, 1))
```




```r
plot(ages, alpha_female_austria, type = "l", col = "red", lwd=2,
     main = expression(paste(alpha [x,2008]," estimates Austria and NLD - female")),
     ylab = expression(paste(alpha [x,2008])),
     xlab = "age")
points(ages, alpha_female_nld, type = "l",lwd=2, col = "blue")
legend("bottomright",  legend = c("Austria", "NLD"), col = c("red","blue"), lty = c(1, 1))
```

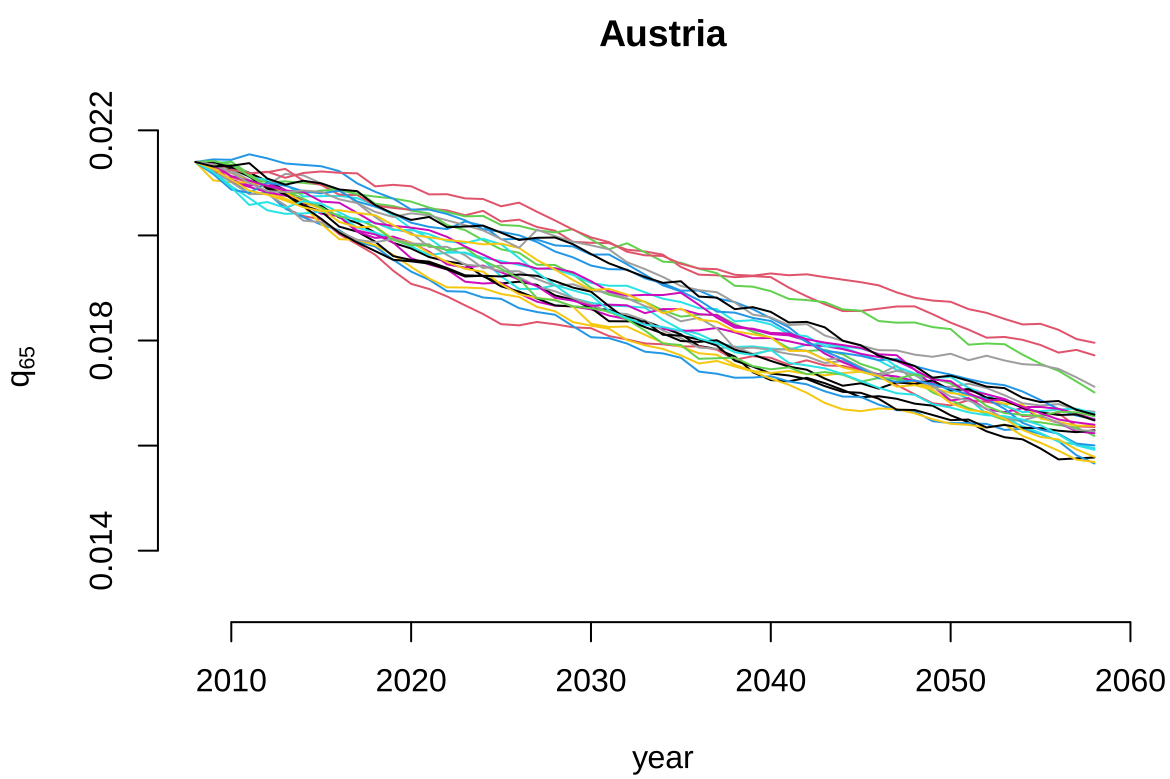


SECTION 4

```
#using the Method of Moments estimation method we compute estimates for Theta and sigma
thetaMA <- mean(diff(kt_austria_male)) ##theta hat for males in austria
thetaFA <- mean(diff(kt_austria_female))
thetaMN <- mean(diff(kt_nld_male))
thetaFN <- mean(diff(kt_nld_female))

sigMA <- sqrt(mean(diff(kt_austria_male)^2) - thetaMA^2)
sigFA <- sqrt(mean(diff(kt_austria_female)^2) - thetaFA^2)
sigMN <- sqrt(mean(diff(kt_nld_male)^2) - thetaMN^2)
sigFN <- sqrt(mean(diff(kt_nld_female)^2) - thetaFN^2)
```

```
sim_qx_austria <- matrix(rep(0, 51, 25), nrow = 51, ncol = 25)
sim_qx_nld <- matrix(rep(0, 51, 25), nrow = 51, ncol = 25)
ktMA <- c(kt_austria_male[39], rep(0, 50))
ktMN <- c(kt_nld_male[39], rep(0, 50))
for (i in 1:25){
  set.seed(i)
  for(t in 2:51){
    ktMA[t] = ktMA[t-1] + thetaMA + sigMA*rnorm(1)
    ktMN[t] = ktMN[t-1] + thetaMN + sigMN*rnorm(1)
  }
  sim_qx_austria[,i] <- 1 - exp(-exp(alpha_male_austria[66]+bx_austria_male[66]*ktMA))
  sim_qx_nld[,i] <- 1 - exp(-exp(alpha_male_nld[66]+bx_nld_male[66]*ktMN))
}
```

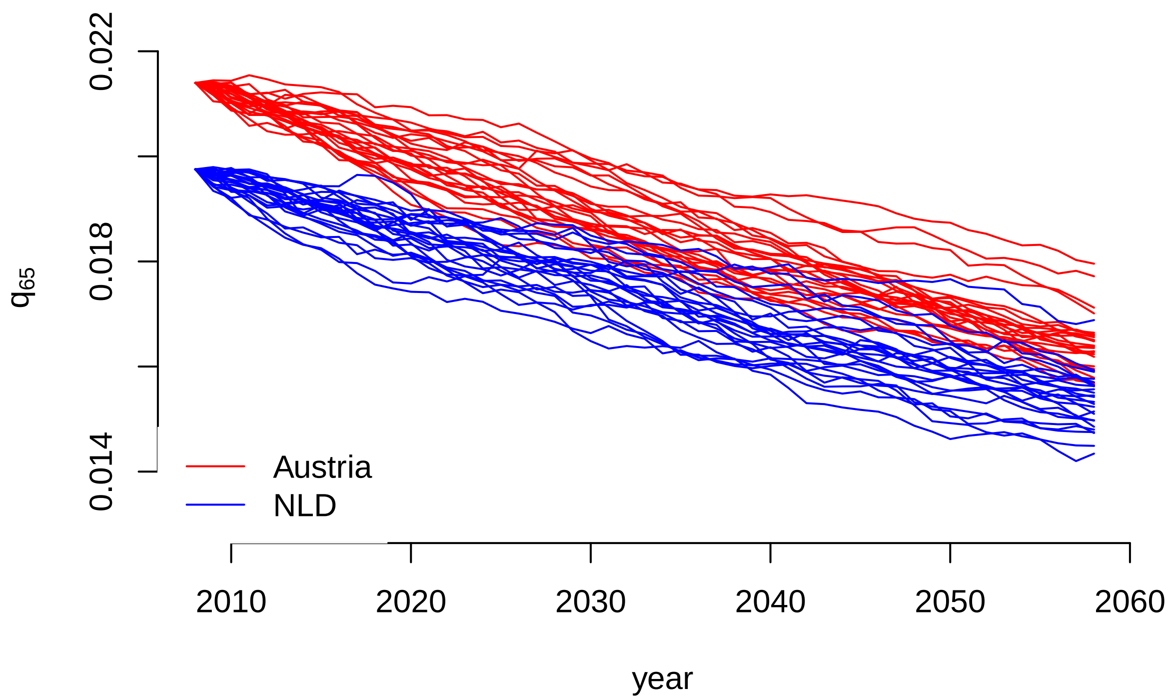
```r
plot(2008:2058, sim_qx_austria[,1], type = "l", ylim = c(0.013, 0.022), xlim = c(2008, 2060),
     main = "Austria",
     ylab = expression(paste(q[65])),
     xlab = "year",
     frame.plot = FALSE)
for(j in 2:25){
  points(2008:2058, sim_qx_austria[, j], type = "l", col = j)
}
```




```r
jpeg(file="plot2.jpeg", height = 480, width = 640)
plot(2008:2058, sim_qx_nld[,1], type = "l", ylim = c(0.013, 0.022), xlim = c(2008, 2060),
      main = "The Netherlands",
     ylab = expression(paste(q[65])),
     xlab = "year",
     frame.plot = FALSE)
for(j in 2:25){
  points(2008:2058, sim_qx_nld[, j], type = "l", col = j)
}
```

```r
plot(2008:2058, sim_qx_austria[,1], type = "l", ylim = c(0.013, 0.022),
     xlim = c(2008, 2060), col = "red",
     main = NULL,
     ylab = expression(paste(q[65])),
     xlab = "year",
     frame.plot = FALSE)
for(j in 2:25){
```

```
  points(2008:2058, sim_qx_austria[, j], type = "l", col = "red")
}
for(i in 1:25){
  points(2008:2058, sim_qx_nld[, i], type = "l", col = "blue")
}
legend("bottomleft",  legend = c("Austria", "NLD"), col = c("red","blue"), lty = c(1, 1), box.col  = "wl
```



Write the functions uses in the scheme proposed by Kannisto:

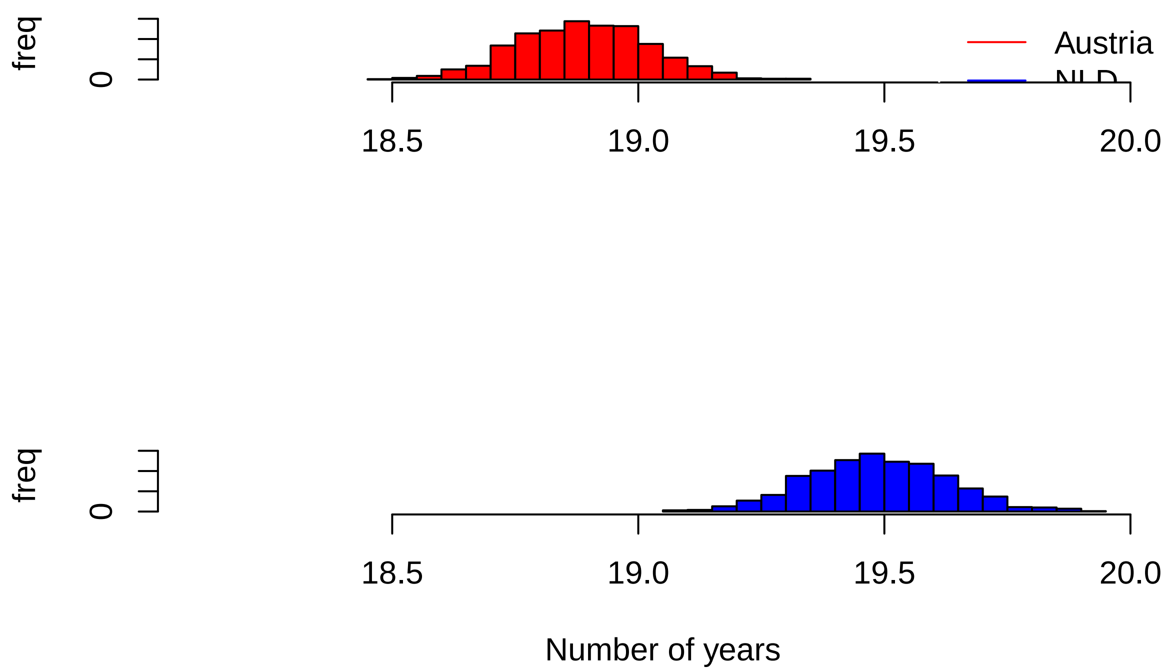
```
w <- function(z, x){
  1/11 + (z-85)*(x-85)/110
}
H <- function(x){
  1/(1+exp(-x))
}
Hinv <- function(x){
  -log(1/x - 1)
}
```

```
ka <- c(kt_austria_female[39], rep(0, 54))
kn <- c(kt_nld_female[39], rep(0, 54))

dprob_female_austria <- function(kt){
  m <- rep(0, 55)
  kta  <- c(kt[1], rep(0, 54))
  for( t in 2:56){
    kta[t] <- kta[t-1] + thetaFA + sigFA*rnorm(1)
```

```r
  }
  for (i in 1:55){
    if(i < 27)
      {m[i] <- exp(alpha_female_austria[65+i]+ bx_austria_female[65+i]*kta[i])}
   else{
      h <- exp(alpha_female_austria[81:91]+ bx_austria_female[81:91]*kta[i])
      m[i] <- H(sum(w(80:90, 64+i)*Hinv(h)))
      }
  }
  return(1 - exp(-m))
}

dprob_female_nld <- function(kt){
  m <- rep(0, 55)
  ktn  <- c(kt[1], rep(0, 54))
  for( t in 2:55){
    ktn[t] <- ktn[t-1] + thetaFN + sigFN*rnorm(1)
  }
  for (i in 1:55){
    if(i < 27)
      {m[i] <- exp(alpha_female_nld[65+i]+ bx_nld_female[65+i]*ktn[i])}
   else{
      h <- exp(alpha_female_nld[81:91]+ bx_nld_female[81:91]*ktn[i])
      m[i] <- H(sum(w(80:90, 64+i)*Hinv(h)))
      }
  }
  return(1 - exp(-m))
  }
```

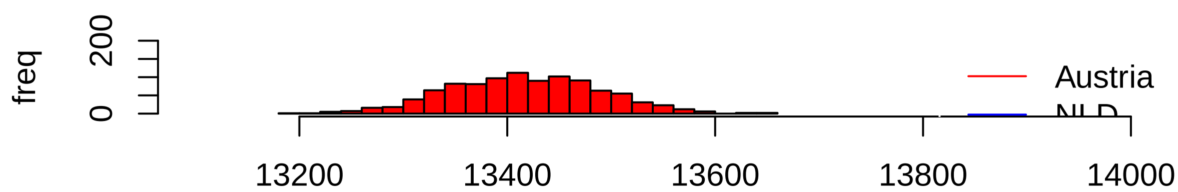
```r
discount_factors <- (1+0.03) ^ - (1:55)
life_expectancy_austria  <- rep(0, 1000)
life_expectancy_nld <- rep (0, 1000)
life_annuity_austria <- rep(0, 1000)
life_annuity_nld <- rep(0, 1000)
death_benefit_austria <- rep(0, 1000)
death_benefit_nld <- rep(0, 1000)


for(j in 1:1000){
  qa <- dprob_female_austria(ka)
  spa <- 1-qa # 1 year survival probabilities 65 y/o female austria
  qn <- dprob_female_nld(kn)
  spn <- 1-qn # 1 year survival probabilities 65 y/o female nld

  life_expectancy_austria[j] <- sum(cumprod(spa))+0.5
  life_expectancy_nld[j] <- sum(cumprod(spn))+0.5
  life_annuity_austria[j] <- 1000*sum(cumprod(spa)*discount_factors)
  life_annuity_nld[j] <- 1000*sum(cumprod(spn)*discount_factors)
  death_benefit_austria[j] <- 1000*sum(cumprod(c(1,spa))[1:55]*qa*discount_factors)
  death_benefit_nld[j] <- 1000*sum(cumprod(c(1,spn))[1:55]*qn*discount_factors)

}
```

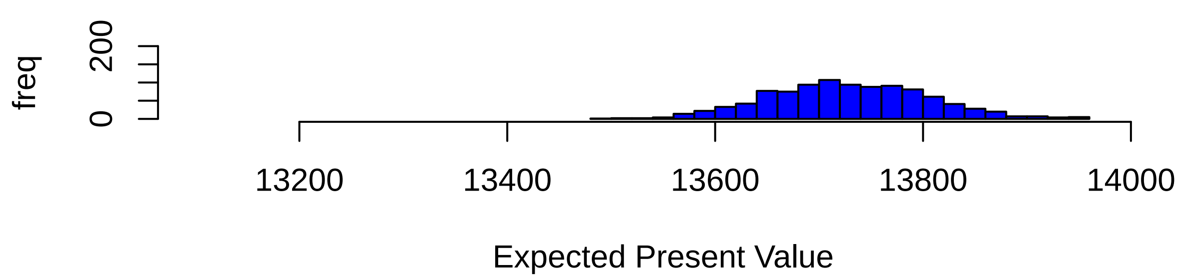
```r
par(mfrow = c(2,1))
hist(life_expectancy_austria, breaks = 25,
     main = NULL,
     xlab = NULL,
     ylab = "freq",
     ylim = c(0, 180),
     xlim = c(18.1, 20),
     col = "red")
 legend ("topright", c("Austria", "NLD"), col = c("red","blue"),
              lty = c(1, 1),  box.col  = "white")

hist(life_expectancy_nld, breaks = 25,
     main = NULL,
     xlab = "Number of years",
     ylab = "freq",
     ylim = c(0, 180),
     xlim = c(18.1, 20),
     col = "blue")
```



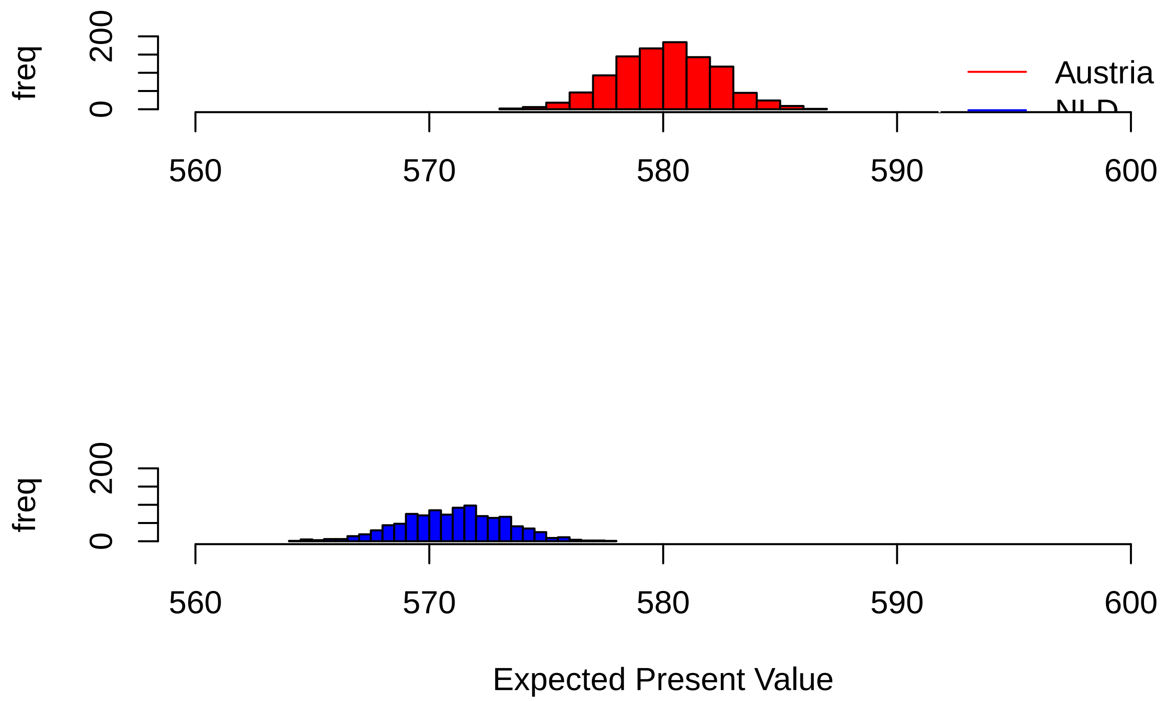

```r
hist(life_annuity_austria, breaks = 20,
     main = NULL,
     xlab = NULL,
     ylab = "freq",
     ylim = c(0, 200),
     xlim = c(13100, 14000),
     col = "red")
```

```
legend ("topright", c("Austria", "NLD"), col = c("red","blue"),
              lty = c(1, 1),  box.col  = "white")
hist(life_annuity_nld, breaks = 20,
     main = NULL,
     xlab = "Expected Present Value",
     ylab = "freq",
     ylim = c(0, 200),
     xlim = c(13100, 14000),
     col = "blue")
```

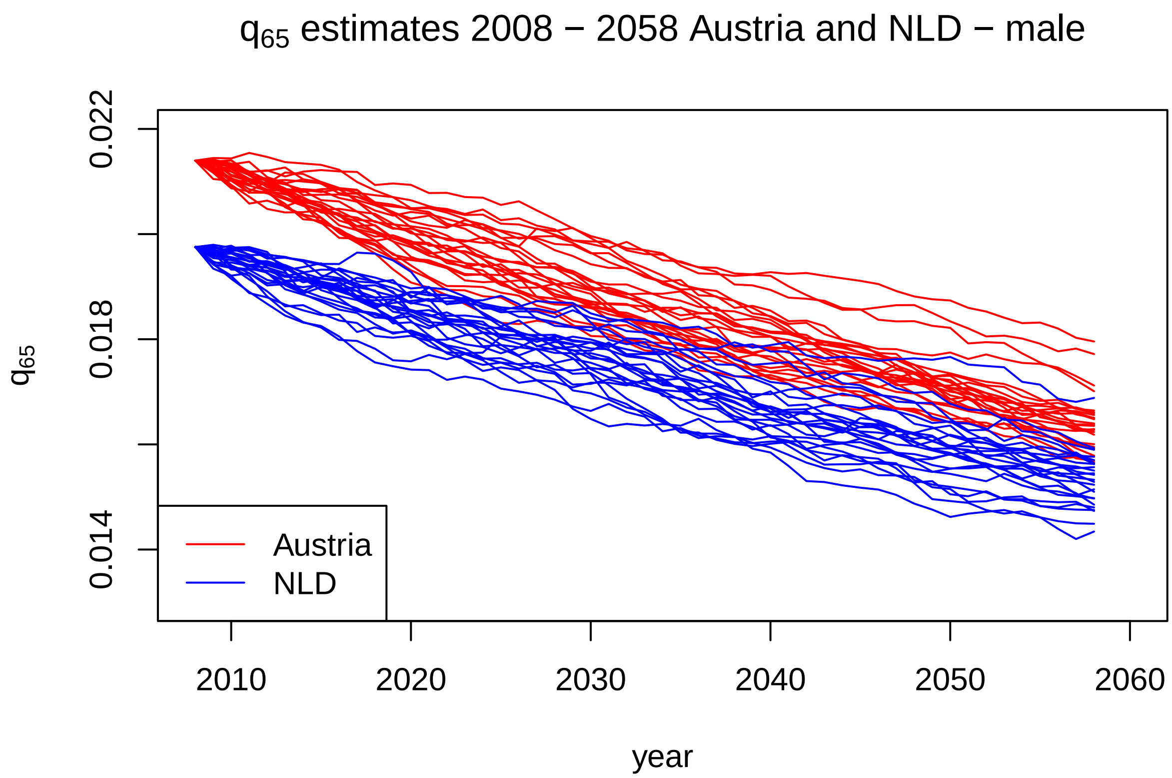





```
hist(death_benefit_austria, breaks = 15,
     main = NULL,
     xlab = NULL,
     ylab = "freq",
    ylim = c(0, 200),
     xlim = c(560, 600),
     col = "red")
legend ("topright", c("Austria", "NLD"), col = c("red","blue"),
              lty = c(1, 1),  box.col  = "white")
hist(death_benefit_nld, breaks = 20,
     main = NULL,
     xlab = "Expected Present Value",
     ylab = "freq",
     ylim = c(0, 200),
     xlim = c(560,600 ),
     col = "blue")
```




```r
plot(2008:2058, sim_qx_austria[,1], type = "l", ylim = c(0.013, 0.022),
     xlim = c(2008, 2060), col = "red",
     main = expression(paste(q[65]," estimates 2008 - 2058 Austria and NLD - male")),
     ylab = expression(paste(q[65])),
     xlab = "year")
for(j in 2:25){
  points(2008:2058, sim_qx_austria[, j], type = "l", col = "red")
}
for(i in 1:25){
  points(2008:2058, sim_qx_nld[, i], type = "l", col = "blue")
}
legend("bottomleft",  legend = c("Austria", "NLD"), col = c("red","blue"), lty = c(1, 1))
```



Write the functions uses in the scheme proposed by Kannisto:

```
w <- function(z, x){
  1/11 + (z-85)*(x-85)/110
}
H <- function(x){
  1/(1-exp(-x))
}
Hinv <- function(x){
  -log(1/x - 1)
}
```