Time Series Analysis: Assignment 4

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Part 1

a.

Let y_t follow an AR(2) process: $y_t = a+\phi_1 y_{t-1}+\phi_2 y_{t-1}+\varepsilon_t$, where $\varepsilon_t \sim$ WN. We can rewrite the AR(2) process in the error correction form by subtracting y_{t-1} on both sides, then we obtain $\Delta y_t = a + by_{t-1} + c\Delta y_{t-1} + \varepsilon_t$, where $b := \phi_1 + \phi_2 - 1$, $c := -\phi_2$.

Consider the Augmented Dickey-Fuller (ADF) t-test, where the null hypothesis $H_0 : b = 0$, and the alternative hypothesis $H_a : (-2 <)b < 0$. The intuition is that, under the null hypothesis, there is a unit root in the AR polynomial, whereas under the alternative hypothesis, we observe a stationary AR process characterized by AR roots outside the unit circle. The corresponding t-statistic is $t = \frac{\hat{b}}{\mathrm{SE}(\hat{b})}$, where $\hat{b}$ is the estimate of b. As we conduct a left-sided test, we reject H_0 if the test statistic is small than the critical value at the 5% significance level.

Note that under H_0 the process is $I(1)$, the assumption that explanatory (lagged) variables are stationary is violated, so we can not derive the asymptotic normality of $\hat{b}$. Based on *Exhibit 7.16* from Heij et al, the 5% critical is equal to -2.86 as $n \to \infty$. Therefore, if $t_{obs.} < -2.86$, we reject H_0 at 5% significance level.

b.

Assuming $b = -c - 1$

$$\begin{aligned}
\Delta y_t &= a + by_{t-1} + c\Delta y_{t-1} + \varepsilon_t = a - (c+1)y_{t-1} + c\Delta y_{t-1} + \varepsilon_t \\
y_t - y_{t-1} &= a - cy_{t-1} - y_{t-1} + cy_{t-1} - cy_{t-2} + \varepsilon_t \\
y_t &= a - cL^2 y_t + \varepsilon_t \\
(1 + cL^2)y_t &= a + \varepsilon_t
\end{aligned}$$

The unit root characteristic polynomial is $\phi(L) = 1 + cL^2$, in order to find the stationary process, $\phi(z) = 1 + cz^2 = 0$, we find that $z^2 = \frac{-1}{c}$,

- $c < 0 : z = \pm\sqrt{\dfrac{-1}{c}}, \{y_t\}$ is stationary if $|z| > 1$, $\quad \sqrt{\dfrac{-1}{c}} > 1 \to c > -1$, hence $-1 < c < 0$
- $c > 0 : z = \pm i\sqrt{\dfrac{1}{c}}, \{y_t\}$ is stationary if $|z| > 1$, $\quad \sqrt{\dfrac{1}{c}} > 1 \to c < 1$, hence $0 < c < 1$

Note: when $c = 0$, $y_t = a + \varepsilon_t$, $\{y_t\}$ process is stationary. To conclude, $\{y_t\}$ process is stationary when $|c| < 1$.

c.

Now we take $c = 0$, then $\Delta y_t = a + by_{t-1} + \varepsilon \Rightarrow y_t = a + \varepsilon_t + (b + 1)y_{t-1}$.

$$\begin{aligned}
y_t &= a + \varepsilon_t + (b+1)\left(a + \varepsilon_{t-1} + (b+1)y_{t-2}\right) \\
&= (b+1)^0(a+\varepsilon_t) + (b+1)(a+\varepsilon_{t-1}) + (b+1)^2(a+\varepsilon_{t-2}) + (b+1)y_{t-3}) \\
&= (b+1)^0(a+\varepsilon_t) + (b+1)(a+\varepsilon_{t-1}) + (b+1)^2(a+\varepsilon_{t-2}) + (b+1)^3(a+\varepsilon_{t-3}) + ... \\
&= \sum_{j=0}^{\infty}(b+1)^j(a+\varepsilon_{t-j})
\end{aligned}$$

$$\Rightarrow y_{t+h} = (b+1)^0(a+\varepsilon_{t+h}) + (b+1)^1(a+\varepsilon_{t+h-1}) + ... + (b+1)^h(a+\varepsilon_t) + ...$$
$$= \sum_{j=0}^{\infty}(b+1)^j(a+\varepsilon_{t+h-j})$$

$$\Rightarrow g(h) = \frac{\partial y_{t+h}}{\partial \varepsilon_t} = (b+1)^h$$

Hence, the impulse response function is $g(h) = (b+1)^h$

d.

Consider the regime switching AR(1) model given by:

$$y_t = \alpha_1 + \phi_{11}y_{t-1} + D_t^+(\tau)(\alpha_2 + \phi_{12}y_{t-1}) + \varepsilon_t$$

If we extend this model to k number of thresholds τ_k and write the model with a summation:

$$y_t = \alpha_1 + \phi_{11}y_{t-1} + D_t^+(\tau_1)(\alpha_2 + \phi_{12}y_{t-1}) + ... + D_t^+(\tau_k)(\alpha_{k+1} + \phi_{1(k+1)}y_{t-1})$$
$$= \alpha_1 + \phi_{11}y_{t-1} + \sum_{i=1}^{k} D_t^+(\tau_i)(\alpha_{i+1} + \phi_{1(i+1)}y_{t-1}) + \varepsilon_t$$

for $D_t^+(\tau_i) = 0$ if $t < \tau_i$ and 1 elsewhere.

e.

Now, consider an extension of the model above to a version where each regime k is allowed to contain a specific AR order p_k:

$$y_t = \alpha_1 + \sum_{j=1}^{p_1}\phi_{j1}y_{t-j} + \sum_{i=1}^{k} D_t^+(\tau_i)(\alpha_{1+i} + \sum_{j=1}^{p_{(1+i)}}\phi_{j(1+i)}y_{t-j}) + \varepsilon_t$$

Where the inner summation illustrates the AR(p_k) model with each regime.

f.

The TAR (Threshold Autoregressive) and Markov Switching models are non-linear time series models designed to capture regime shifts. In the TAR model, shifts between regimes are determined by observed thresholds, where dummy variables indicate the current state within the space of outcomes. However, the Markov Switching model has two distinct regimes, indicated by latent state variables that not directly observable, and there are fixed probabilities of a regime change, accommodating unknown states through a flexible, probabilistic framework.

Part 2

The R script imports the monthly data of the Federal Funds Effective Rate (FFER) from July 1, 1954, to January 31, 2024, from the website of Federal Reserve Economic Data(FRED). Any null values in the dataset are replaced with NA and then removed. The script further processes the data to extract rates and dates, ensuring they are converted to the appropriate data types.

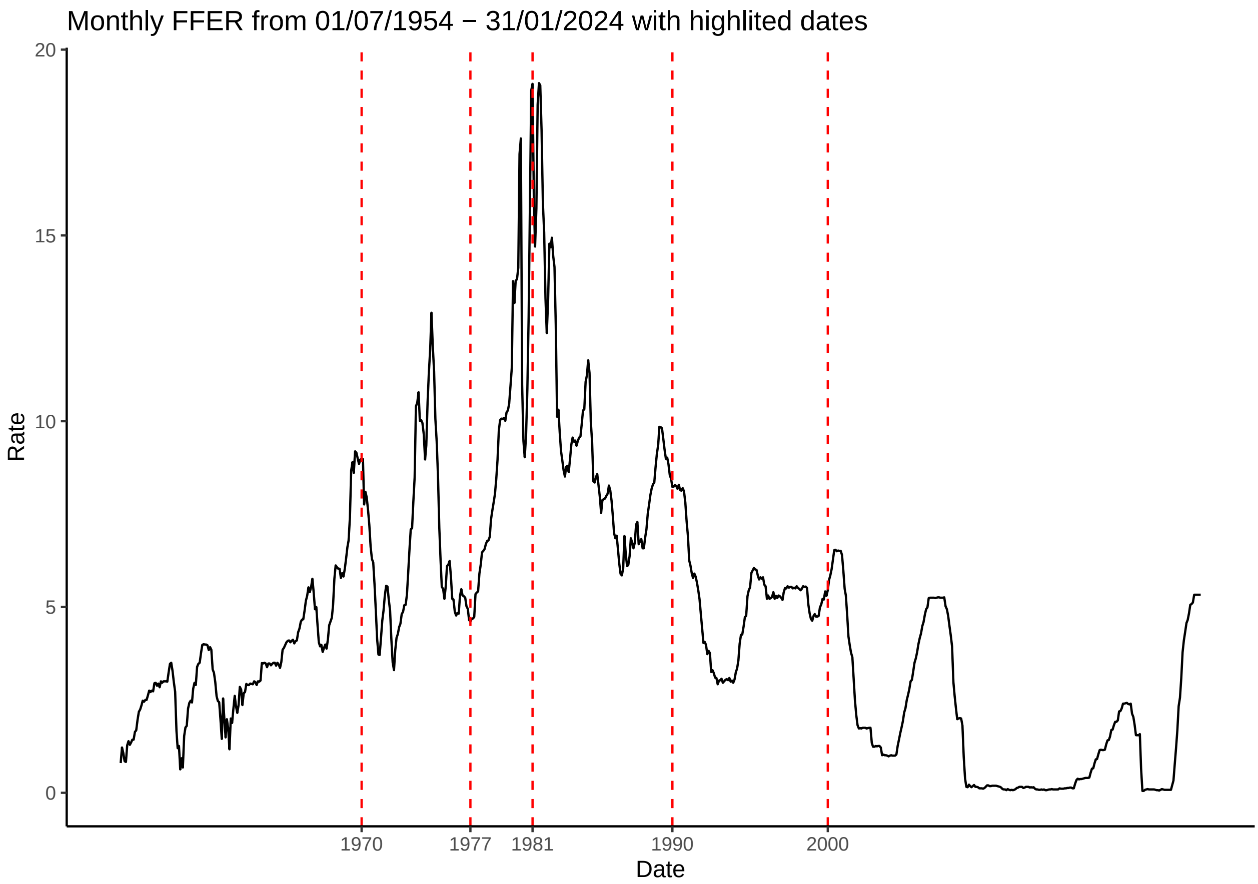
a.

The AR(p) models for p = 1, 2, 3, 4, 5 were estimated using the full sample. Considering AIC, the preferred model is the AR(5) model, as it has the lowest AIC value, indicating a good balance between model fit and complexity.

lag(s)	AIC
1	1168.156
2	1029.022
3	1012.814
4	1014.393
5	1011.367

b.

We graphed the time series, showcasing the date along the x-axis and FFER along the y-axis. Specifically, we emphasized five distinct dates within our dataset: January 1, 1970, January 1, 1977, January 1, 1981, January 1, 1990, and January 1, 2000.



c.

In part c we consider three options for a set of breakpoints in a threshold AR model:

(i) 1 January 1981

(ii) 1 January 1970 and 1 January 1990

(iii) 1 January 1970, 1 January 1977, 1 January 1981, 1 January 1990, and 1 January 2000.

Six different Threshold Autoregressive (TAR) models are considered using different sets of breakpoints along with order $p = 1$ and $p = 2$. The AIC values for these models are presented in the following table. As we can observe, when modeling the monthly data of the FFER, the preferred TAR model is the one with breakpoint set (iii), containing five breakpoints, and a lag order of $p = 2$.

Options	AIC with 1 lag	AIC with 2 lags
i	1158.037	1029.181
ii	1166.736	1011.705
iii	1142.206	1002.157

d.

In part d we only look at option (i), so with one breakpoint and define a variation of the STAR model, with smooth function:

$$F(\tau, t) = \frac{1}{1 + \exp[-\gamma(t - \tau)]}$$

Where $(t - \tau)$ is the difference in months between the month t and January 1981. Next, we estimate the STAR model with this smooth function over a grid for γ of (0.5, 1, 2, 5, 10), and do this for both AR lags $p = 1$ and $p = 2$ (10 models in total). The AIC values for these models are presented in the following table. As we can observe, the preferred STAR model is the one with $\gamma = 0.5$ and a lag order of $p = 2$.

γ	AIC with 1 lag	AIC with 2 lags
0.5	1166.017	1026.534
1	1164.148	1028.847
2	1160.881	1029.711
5	1158.472	1029.389
10	1158.332	1029.355

e.

Evaluating all TAR and STAR models we have estimated in part c and part d based on AIC, the optimal ones are TAR(2) with breakpoints option (iii) and STAR(2) with $\gamma = 0.5$. Among these two models, STAR(2) with $\gamma = 0.5$ is preferred since it has a lower AIC value. Intuitively, STAR(2) with $\gamma = 0.5$, characterized by its relatively small value of γ, allows for smoother shifts between regimes, so it performs better than the TAR(2) model as expected.

Theoretically speaking, we prefer the Markov Switching model, since in the TAR and STAR models we determine the breakpoints mainly through eyeballing, and it is not always easy to determine the proper threshold. In this case, Markov Switching model is preferred since we model the latent state variable via the conditional probabilities of remaining within or switching between regimes, which is a more flexible modelling strategy. In practice, we may prefer the STAR model, since the shifts of regimes in the STAR model are smoother than the TAR model, and the it is easier to write down the Log-Likelihood of STAR model compared to the Markov Switching model, as in the later model you need to take into account the probability of being in a state rather than a certain state. In all, the choice between these models depends on various factors, each carrying its own set of advantages and disadvantages. It is essential to consider the specific circumstances and requirements of the analysis when deciding which model to use.