For this assignment, you may work in groups of two or three students. Part 1 is theoretical and part 2 is an application of your results from part 1. For each sub-question, you can earn three points. It is crucial to motivate your answers: No points will be awarded for answers without motivation. Furthermore, reports that are not typeset as a single PDF file in LaTeX will not be considered. Please also make sure to submit your R (and/or Python) codes for question 2, separately, together with a CSV file of the data. For question 2, your code should reproduce all reported results without returning errors. Results that are not returned by your code cannot be graded.

The deadline for this assignment is Monday, March 4, 15:00.

Part 1. (a) Let y_t follow an AR(2) process, that is given in error correction form by:

$$\Delta y_t = a + by_{t-1} + c\Delta y_{t-1} + \varepsilon_t$$

Give the null hypothesis and alternative hypothesis of the Augmented Dickey-Fuller test for this model in terms of the parameter vector $(a, b, c)'$. What is the corresponding test statistic? When is the null hypothesis rejected (what can you say about the critical value at a significance level of 5%)?

(b) Assume that $b = -c - 1$. Give the conditions on a and c that imply a stationary process.

(c) Now suppose that $c = 0$. Give the impulse response function:

$$g(h) = \frac{\partial y_{t+h}}{\partial \varepsilon_t}$$

(d) Consider the regime switching AR(1) model given by:

$$y_t = \alpha_1 + \phi_{11} y_{t-1} + D_t^+(\tau)(\alpha_2 + \phi_{12} y_{t-1}) + \varepsilon_t$$

Extend this model to a situation where there are k number of thresholds τ_k and write the model with a summation.

(e) Now, consider an extension of the model above to a version where each regime is allowed to contain a different number of lags. So each regime has its own specific AR order p_k. Write this model with summations again, for a number of k thresholds.

(f) The Markov switching model and the TAR model are quite similar. Explain how they correspond to and how they differ from each other.

Part 2. In each of the sub-questions below, make sure to program the required log-likelihoods yourself. Besides base R/Python, your code from previous assignments, packages for plotting and optimization, you are not allowed to use any other packages for your final results. Here, base Python refers to any basic operations captured by Numpy and other packages providing operations that translate to R.

Download monthly data of the Federal Funds Effective Rate from July 1, 1954, to January 31, 2024, from the website for Federal Reserve Economic Data (FRED).

(a) Use your code of previous assignments to determine the most suitable order of an AR(p) model for this time series, based on the AIC. Consider the orders $p = 1, 2, 3, 4, 5$.

(b) Suppose we highlight five different dates in our data: 1 January 1970, 1 January 1977, 1 January 1981, 1 January 1990 and 1 January 2000. Plot the time series and draw a vertical line at the five dates.

(c) Now we would like to consider three options for a set of breakpoints in a threshold AR model:

  (i) 1 January 1981

  (ii) 1 January 1970 and 1 January 1990.

  (iii) 1 January 1970, 1 January 1977, 1 January 1981, 1 January 1990 and 1 January 2000.

Furthermore, look at the orders $p = 1$ and $p = 2$ for the TAR(p) model. Choose the best TAR model based on AIC (note that you should consider six different TAR models in total).

(d) Now suppose we only look at option (i), so with one breakpoint and define a variation of the STAR model, with smooth function:

$$F(\tau, t) = \frac{1}{1 + \exp[-\gamma(t - \tau)]}$$

Where $(t - \tau)$ is the difference in months between the month t and January 1981. Estimate the STAR model with this smooth function over a grid for γ of $(0.5, 1, 2, 5, 10)'$. Do this for both AR lags $p = 1$ and $p = 2$ (10 models in total). Report the AICs and use these to choose the optimal model.

(e) Compare all TAR and STAR models you have estimated in this part and make a conclusion about the optimal model. Do you prefer this type of model over a Markov switching model (for example, in terms of interpretation)? Explain why.

