

# Barrier functions

## Computational Intelligence, Lecture 11

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- Analytic center of linear inequalities
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# LINEAR INEQUALITIES

Consider linear inequality constraints:

$$\mathbf{Ax} \leq \mathbf{b} \quad (1)$$

Remember that we can rewrite it as:

$$\mathbf{a}_i^\top \mathbf{x} \leq b_i \quad (2)$$

$$\mathbf{a}_i^\top \mathbf{x} - b_i \leq 0 \quad (3)$$

Instead of *hard constraints* in (3) we can turn these into a cost function component:

$$J = - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x}) \quad (4)$$

Which is called a *barrier function*.

Let us consider barrier functions  $J = - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x})$ :

- It removes the constraint, but modifies the cost.
- When  $b_i - \mathbf{a}_i^\top \mathbf{x}$  is a very small positive number,  $\log(b_i - \mathbf{a}_i^\top \mathbf{x})$  is a very big negative number, hence the minus sign in front.
- Barrier function does not behave well outside of the domain, when  $b_i - \mathbf{a}_i^\top \mathbf{x} < 0$ .

Hence the following QP:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} && \begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b}, \\ \mathbf{C}(\mathbf{x}) = \mathbf{d}. \end{cases} \end{aligned} \tag{5}$$

...can be approximated as:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x} - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x}), \\ & \text{subject to} && \mathbf{C}(\mathbf{x}) = \mathbf{d} \end{aligned} \tag{6}$$

# ANALYTIC CENTER OF LINEAR INEQUALITIES

We can define *analytic center of linear inequalities* as a minimum of the function  $J = - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x})$ . And that can be solved as a convex optimization:

$$\mathbf{x}_a = \underset{\mathbf{x}}{\operatorname{argmin}} \quad - \sum_{i=1}^n \log(b_i - \mathbf{a}_i^\top \mathbf{x})$$

At the analytic center of linear inequalities the shape of contour lines can be analysed as a local quadratic approximation of the function  $J$ :

$$\mathcal{C} = \{\mathbf{x} : (\mathbf{x} - \mathbf{x}_a)^\top \frac{\partial^2 J}{\partial \mathbf{x}^2} (\mathbf{x} - \mathbf{x}_a) = \epsilon\} \quad (7)$$

where  $\epsilon$  is a small number.

# ILLUSTRATION OF A BARRIER FUNCTIONS



Figure 1: Barrier functions

Pink is the domain. The ellipsoids represent the shape of the hessian  $\frac{\partial^2 J}{\partial \mathbf{x}^2}$  at different points on the domain. Green dot is  $\mathbf{x}_a$ .

# HOMEWORK

Visualize contours of a quadratic program of your choice.



Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2021](https://github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2021)



Check Moodle for additional links, videos, textbook suggestions.