Minimax

Computational Intelligence, Lecture 13

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Content

- Homework

Consider the following problem:

Example

Find smallest $x \in \mathbb{R}$, such that $x + y \geq 1$, where $|y| \leq 2$.

In that example we need to find optimal value of x subject to a constraint where another unknown variable is present; the constraint has to be satisfied for the *worst-case scenario*, in this case it is $y = -2$. Solution is $x = 3$

This is closely related to *minimax optimization*

Consider the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x}||, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{1}$$

It is clear that worst-case scenario corresponds to the largest value of $\mathbf{d}^\top \mathbf{y}$, meaning that $\mathbf{y}$ should align with $\mathbf{d}$ and have its maximum possible length p. From that we conclude that $\mathbf{y} = p \frac{\mathbf{d}}{||\mathbf{d}||}$.

Therefor $\mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq h$ becomes:

$$\mathbf{c}^\top \mathbf{x} + p \frac{\mathbf{d}^\top \mathbf{d}}{||\mathbf{d}||} \leq h \tag{2}$$

$$\mathbf{c}^\top \mathbf{x} + p||\mathbf{d}|| \leq h \tag{3}$$

Thus our problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & ||\mathbf{x}||, \\ \text{subject to} \quad & \mathbf{c}^\top \mathbf{x} \leq h - p||\mathbf{d}|| \end{aligned} \tag{4}$$

Consider the following problem, where $\mathbf{x}^*$ is the desired value of $\mathbf{x}$:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & \mathbf{y}^\top \mathbf{D} \mathbf{x} \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{5}$$

This time worst-case scenario corresponds to $\mathbf{y}$ aligned with $\mathbf{D}\mathbf{x}$ and having its maximum possible length p. From that we conclude that $\mathbf{y} = p\frac{\mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||}$. Let us substitute it to $\mathbf{y}^\top \mathbf{D} \mathbf{x}$:

$$p \left(\frac{\mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||} \right)^\top \mathbf{D}\mathbf{x} = p \frac{\mathbf{x}^\top \mathbf{D}^\top \mathbf{D}\mathbf{x}}{||\mathbf{D}\mathbf{x}||} = p \frac{||\mathbf{D}\mathbf{x}||^2}{||\mathbf{D}\mathbf{x}||} = p||\mathbf{D}\mathbf{x}|| \tag{6}$$

Thus our problem becomes:

$$\begin{aligned} \min_{\mathbf{x}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & ||\mathbf{D}\mathbf{x}|| \leq \frac{h}{p} \end{aligned} \tag{7}$$

which is an SOCP.

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{y}|| \leq p \end{aligned} \tag{8}$$

We can rewrite $(\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h$ as:

$$\mathbf{y}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{9}$$

With that we see that the worse case scenario is $\mathbf{y}$ is aligned with $\mathbf{D}(\mathbf{x} - \mathbf{b})$ and has length p:

$$\mathbf{y} = p \frac{\mathbf{D}(\mathbf{x} - \mathbf{b})}{||\mathbf{D}(\mathbf{x} - \mathbf{b})||} \tag{10}$$

Then $\mathbf{y}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \leq h$ becomes:

$$p\frac{(\mathbf{x}-\mathbf{b})^\top \mathbf{D}^\top \mathbf{D}(\mathbf{x}-\mathbf{b})}{||\mathbf{D}(\mathbf{x}-\mathbf{b})||} - \mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \leq h \tag{11}$$

which is the same as:

$$p||\mathbf{D}(\mathbf{x}-\mathbf{b})|| - \mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \leq h \tag{12}$$

$$||\mathbf{D}(\mathbf{x}-\mathbf{b})|| \leq \frac{1}{p}\mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) + \frac{h}{p} \tag{13}$$

which is an SOCP constraint.

And thus we get:

$$\begin{aligned} \min_{\mathbf{x}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & ||\mathbf{D}(\mathbf{x} - \mathbf{b})|| \leq \frac{1}{p}\mathbf{a}^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \frac{h}{p} \end{aligned} \tag{14}$$

which is SOCP.

A more general case of the previous problem is:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{15}$$

where $\mathbf{H}$ is has an inverse. We start by making substitution:

$$\mathbf{v} = \mathbf{H}\mathbf{y} + \mathbf{f} \tag{16}$$

meaning $\mathbf{y} = \mathbf{H}^{-1}(\mathbf{v} - \mathbf{f})$:

$$(\mathbf{H}^{-1}(\mathbf{v} - \mathbf{f}) - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{17}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}^{-1}\mathbf{f} + \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{18}$$

$$\mathbf{v}^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) - (\mathbf{H}\mathbf{a} + \mathbf{f})^\top \mathbf{H}^{-\top} \mathbf{D}(\mathbf{x} - \mathbf{b}) \leq h \tag{19}$$

We can introduce notation:

$$\mathbf{M} = \mathbf{H}^{-\top}\mathbf{D} \tag{20}$$

$$\mathbf{g} = \mathbf{H}\mathbf{a} + \mathbf{f} \tag{21}$$

With that we can re-write our constraint:

$$\mathbf{v}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) - \mathbf{g}^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \tag{22}$$

$$(\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h \tag{23}$$

And now we formulated type 3 problem as type 2:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{v}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{v} - \mathbf{g})^\top \mathbf{M}(\mathbf{x} - \mathbf{b}) \leq h, \\ & ||\mathbf{v}|| \leq p \end{aligned} \tag{24}$$

Try solving this problem on your own:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\ \text{subject to} \quad & (\mathbf{y} - \mathbf{a})^\top \mathbf{D} (\mathbf{x} - \mathbf{b}) + \mathbf{s}^\top \mathbf{y} + \mathbf{q}^\top \mathbf{x} \leq h, \\ & ||\mathbf{H}\mathbf{y} + \mathbf{f}|| \leq p \end{aligned} \tag{25}$$

MiniMax: Quadratic constraint, type 4: SOLUTION: Part 1

In order to solve this problem, we will make it similar to the problems that we have solved before. We will start by introducing $\mathbf{v}$, wehre $\mathbf{v} = \mathbf{H}\mathbf{y} + \mathbf{f}$; hence, $\mathbf{y} = \mathbf{H}^{-1}\mathbf{v} - \mathbf{H}^{-1}\mathbf{f}$. So the problem now will be.

$$
\begin{aligned}
\min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\
\text{subject to} \quad & (\mathbf{H}^{-1}\mathbf{v} - \mathbf{H}^{-1}\mathbf{f} - \mathbf{a})^\top \mathbf{D}(\mathbf{x} - \mathbf{b}) + \mathbf{s}^\top(\mathbf{H}^{-1}\mathbf{v} - \mathbf{H}^{-1}\mathbf{f}) \\
& + \mathbf{q}^\top \mathbf{x} \leq h, \\
& ||\mathbf{v}|| \leq p
\end{aligned}
\tag{26}
$$

MiniMax: Quadratic constraint, type 4: SOLUTION: Part 2

Modifying the first constraint equation as follows:

$$\begin{aligned}\mathbf{v}^\top \mathbf{H}^{-\top}\mathbf{D}(\mathbf{x}-\mathbf{b}) - \mathbf{f}^\top \mathbf{H}^{-\top}\mathbf{D}(\mathbf{x}-\mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \\ +\mathbf{s}^\top \mathbf{H}^{-1}\mathbf{v} - \mathbf{s}^\top \mathbf{H}^{-1}\mathbf{f} + \mathbf{q}^\top \mathbf{x} \leq h\end{aligned} \tag{27}$$

The goal is to take $\mathbf{v}$ as a common factor in order to be able to find the worst case scenario.
It can be seen that $\mathbf{s}^\top \mathbf{H}^{-1}\mathbf{v}$ is a scaler, so the transposing this part will not change its value.

$$\begin{aligned}\mathbf{v}^\top \mathbf{H}^{-\top}\mathbf{D}(\mathbf{x}-\mathbf{b}) - \mathbf{f}^\top \mathbf{H}^{-\top}\mathbf{D}(\mathbf{x}-\mathbf{b}) - \mathbf{a}^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \\ +\mathbf{v}^\top \mathbf{H}^{-\top}\mathbf{s} - \mathbf{s}^\top \mathbf{H}^{-1}\mathbf{f} + \mathbf{q}^\top \mathbf{x} \leq h\end{aligned} \tag{28}$$

Now $\mathbf{v}^\top$ can be taken as a common factor

MiniMax: Quadratic constraint, type 4: SOLUTION: Part 3

$$\mathbf{v}^\top \mathbf{H}^{-\top}(\mathbf{D}(\mathbf{x}-\mathbf{b})+\mathbf{s}) - (\mathbf{H}^{-1}\mathbf{f}+\mathbf{a})^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) \\ -\mathbf{s}^\top \mathbf{H}^{-1}\mathbf{f} + \mathbf{q}^\top \mathbf{x} \leq h \tag{29}$$

To make things clear, lets simplify long expressions with new variables:

$$\mathbf{g} = \mathbf{H}^{-\top}(\mathbf{D}(\mathbf{x}-\mathbf{b})+\mathbf{s}) \tag{30}$$

$$k = (\mathbf{H}^{-1}\mathbf{f}+\mathbf{a})^\top \mathbf{D}(\mathbf{x}-\mathbf{b}) - \mathbf{s}^\top \mathbf{H}^{-1}\mathbf{f} + \mathbf{q}^\top \mathbf{x} \tag{31}$$

So the problem now becomes:

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y}} \quad & ||\mathbf{x}-\mathbf{x}^*||, \\ \text{subject to} \quad & \mathbf{v}^\top \mathbf{g} - k \leq h, \\ & ||\mathbf{v}|| \leq p \end{aligned} \tag{32}$$

The worst case scenario happens when $\mathbf{v}$ becomes aligned with $\mathbf{g}$ and reaches its maximum possible length p. So it's possible to merge the two constraints into one constraint and turn the "min-max" problem into an only "min" problem as follows:

$$\mathbf{v} = \frac{\mathbf{g}}{||\mathbf{g}||} p \tag{33}$$

Now substituting back into the constraint:

$$\frac{\mathbf{g}^\top \mathbf{g}}{||\mathbf{g}||} p - k \leq h \tag{34}$$

$$||\mathbf{g}|| p - k \leq h \tag{35}$$

$$||\mathbf{g}|| \leq \frac{h}{p} + \frac{k}{p} \tag{36}$$

MiniMax: Quadratic constraint, type 4: SOLUTION: Part 5

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & ||\mathbf{x} - \mathbf{x}^*||, \\
\text{subject to} \quad & ||\mathbf{g}|| \leq \frac{h}{p} + \frac{k}{p}
\end{aligned}
\tag{37}
$$

As it can be seen, expression (36) is similar to expression (13). Because **g** contains the variable to be minimized **x**. And that is an SOCP constraint.

Consider the system:

$$\dot{\mathbf{x}} = \mathbf{A}_p\mathbf{x} + \mathbf{B}_p\mathbf{u} \tag{38}$$

where $\mathbf{A}_p$ and $\mathbf{B}_p$ are linear functions of parameters $\mathbf{p}$. We want to stabilize the origin.

Assume we use control law:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^* \tag{39}$$

With that we get:

$$\dot{\mathbf{x}} = (\mathbf{A}_p - \mathbf{B}_p\mathbf{K})\mathbf{x} + \mathbf{B}_p\mathbf{u}^* \tag{40}$$

Let us write Lyapunov function for the system:

$$V = \mathbf{x}^\top \mathbf{S} \mathbf{x} \tag{41}$$
$$\dot{V} = \dot{\mathbf{x}}^\top \mathbf{S} \mathbf{x} + \mathbf{x}^\top \mathbf{S} \dot{\mathbf{x}} = \tag{42}$$
$$= \mathbf{x}^\top \mathbf{S}(\mathbf{A}_p - \mathbf{B}_p \mathbf{K})\mathbf{x} + \mathbf{x}^\top (\mathbf{A}_p - \mathbf{B}_p \mathbf{K})^\top \mathbf{S} \mathbf{x} + \tag{43}$$
$$+ \mathbf{x}^\top \mathbf{S} \mathbf{B}_p \mathbf{u}^* + \mathbf{u}^{*\top} \mathbf{B}_p^\top \mathbf{S} \mathbf{x} \tag{44}$$

Let us define:

$$a = 2\mathbf{x}^\top \mathbf{S}(\mathbf{A}_p - \mathbf{B}_p \mathbf{K})\mathbf{x} \tag{45}$$
$$\mathbf{b} = 2\mathbf{x}^\top \mathbf{S} \mathbf{B}_p \tag{46}$$

With that we can find Jacobians:

$$\mathbf{a}_x = \frac{\partial a}{\partial \mathbf{p}} \qquad \mathbf{B}_x = \frac{\partial \mathbf{b}}{\partial \mathbf{p}} \tag{47}$$

Thus we get minimax constraint on the Lyapunov function

$$\dot{V} = \mathbf{a}_x^\top \mathbf{p}_t + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{p}_t \tag{48}$$

where $\mathbf{p}_t$ are true values of parameters $\mathbf{p}$. Assuming:

$$\mathbf{p}_t = \mathbf{p} + \mathbf{p}_0 \tag{49}$$

we get:

$$\dot{V} = \mathbf{a}_x^\top (\mathbf{p} + \mathbf{p}_0) + \mathbf{u}^{*\top} \mathbf{B}_x (\mathbf{p} + \mathbf{p}_0) \tag{50}$$

which is a minimax constraint. Let's solve it for the case $||\mathbf{p}|| \leq 1$.

Taking derivative of $\dot{V}$ with respect to $\mathbf{p}$ we get

$$\frac{\partial \dot{V}}{\partial \mathbf{p}} = \mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x \tag{51}$$

this is the direction where the function grow the most. But we know its length is 1, so we conclude that:

$$\mathbf{p} = \frac{\mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x}{||\mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x||} \tag{52}$$

So:

$$\dot{V} = ||\mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x|| + (\mathbf{a}_x^\top + \mathbf{u}^{*\top} \mathbf{B}_x)\mathbf{p}_0 \tag{53}$$

Let's do the same, but for the case when $||\mathbf{G}\mathbf{p}|| \leq 1$:

$$\begin{cases} \dot{V} = \mathbf{a}_x^\top (\mathbf{p} + \mathbf{p}_0) + \mathbf{u}^{*\top} \mathbf{B}_x (\mathbf{p} + \mathbf{p}_0) \leq 0 \\ ||\mathbf{G}\mathbf{p}|| \leq 1 \end{cases} \tag{54}$$

First step is to introduce new variable:

$$\rho = \mathbf{G}\mathbf{p} \tag{55}$$

from which it follows that $\mathbf{p} = \mathbf{G}^{-1}\rho$ ($\mathbf{G}$ should be invertible for the parameters to be bounded). Hence we get:

$$\begin{cases} \dot{V} = \mathbf{a}_x^\top (\mathbf{G}^{-1}\rho + \mathbf{p}_0) + \mathbf{u}^{*\top} \mathbf{B}_x (\mathbf{G}^{-1}\rho + \mathbf{p}_0) \leq 0 \\ ||\rho|| \leq 1 \end{cases} \tag{56}$$

We can find gradient:

$$\frac{\partial \dot{V}}{\partial \mathbf{p}} = \mathbf{a}_x^\top \mathbf{G}^{-1} + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{G}^{-1} \tag{57}$$

We know that length of ρ is bounded, so:

$$\rho = \frac{\mathbf{a}_x^\top \mathbf{G}^{-1} + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{G}^{-1}}{||\mathbf{a}_x^\top \mathbf{G}^{-1} + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{G}^{-1}||} \tag{58}$$

And thus we get SOCP constraint:

$$\dot{V} = ||\mathbf{a}_x^\top \mathbf{G}^{-1} + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{G}^{-1}|| + \mathbf{a}_x^\top \mathbf{p}_0 + \mathbf{u}^{*\top} \mathbf{B}_x \mathbf{p}_0 \leq 0 \tag{59}$$

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github.com/SergeiSa/Computational-Intelligence-Slides-Spring-2021

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