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北京航空航天大學
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深度学习与自然语言处理第二次作业

EM 算法与硬币投掷问题

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1 问题描述

一个袋子中三种硬币的混合比例为：s1, s2 与 1-s1-s2 (0<=si<=1), 三种硬币掷出正面的概率分别为：p, q, r。 （1）自己指定系数 s1, s2, p, q, r，生成 N 个投掷硬币的结果（由 01 构成的序列，其中 1 为正面，0 为反面），利用 EM 算法来对参数进行估计并与预先假定的参数进行比较。

2 实验原理

2.1 EM 算法

EM 算法的具体流程如下：

输入：观测变量数据 Y，隐变量数据 Z，联合分布 P(Y, Z| θ)，条件分布 P(Z|Y, θ)

输出：模型参数 θ

1）选择参数 θ 的初始值 θ (0)，开始迭代

2）E 步：记 θ (i)次迭代参数为 θ 的估计值，在第 i+1 次迭代的 E 步，计算

$$\begin{aligned} Q\left(\theta,\theta^{(i)}\right) &= E_z\left[\log P\left(Y,Z\mid\theta\right)\middle|Y,\theta^{(i)}\right] \\ &= \sum_Z \log P\left(Y,Z\mid\theta\right)P\left(Z\mid Y,\theta^{(i)}\right) \end{aligned}$$

3）M 步：求使得 Q 函数极大化的θ值，确定第 i+1 次迭代的参数的估计值θ(i+1)

$$\theta^{(i+1)} = \arg\max_\theta Q\left(\theta,\theta^i\right)$$

4）重复 2, 3 步直至收敛。

2.2 公式推导

给定参数分别为 s_1, s_2, p, q, r ，对于单次抛掷硬币的结果，将该硬币抛掷建模：

$$\begin{aligned} p\left(y_i\mid\theta\right) &= \sum_z p\left(y_i,z\mid\theta\right) \\ &= \sum_z p\left(z\mid\theta\right)p\left(y_j\mid z,\theta\right) \\ &= s_1 p^{y_j}\left(1-p\right)^{1-y_j} + s_2 q^{y_j}\left(1-q\right)^{1-y_j} + \left(1-s_1-s_2\right) r^{y_j}\left(1-r\right)^{1-y_j} \end{aligned}$$

其中 y_j 是第 j 个观测结果 1 或 0；z 是隐变量，表示未观测到的选硬币的结果；$\theta=(s_1,s_2,p,q,r)$ 是模型参数。将观测数据表示为 $Y=(y_1,y_2,...,y_n)^T$，隐变量表示为 $Z=(z_1,z_2,...,z_n)^T$。则观测数据的似然函数可以表示为

$$p(Y|\theta)=\sum_Z p(Z|\theta)p(Y|Z,\theta)$$

$$=\prod_{j=1}^{n}\left[s_1 p^{y_j}(1-p)^{1-y_j}+s_2 q^{y_j}(1-q)^{1-y_j}+(1-s_1-s_2)r^{y_j}(1-r)^{1-y_j}\right]$$

E-Step:确定 Q 函数

设 y_i 来自硬币 A 的概率为 u_{1j}，来自硬币 B 的概率为 u_{2j}，则来自硬币 C 的概率为 $1-u_{1j}-u_{2j}$，第 i 次迭代的参数的估计值是 $\theta^{(i)}=(s_1^{(i)},s_2^{(i)},p^{(i)},q^{(i)},r^{(i)})$，第 i+1 次隐变量：

$$u_{1j}^{(i+1)}=\frac{s_1 p^{y_j}(1-p)^{1-y_j}}{s_1 p^{y_j}(1-p)^{1-y_j}+s_2 q^{y_j}(1-q)^{1-y_j}+(1-s_1-s_2)r^{y_j}(1-r)^{1-y_j}}$$

$$u_{2j}^{(i+1)}=\frac{s_2 q^{y_j}(1-q)^{1-y_j}}{s_1 p^{y_j}(1-p)^{1-y_j}+s_2 q^{y_j}(1-q)^{1-y_j}+(1-s_1-s_2)r^{y_j}(1-r)^{1-y_j}}$$

$$u_{3j}^{(i+1)}=1-u_{1j}^{(i+1)}-u_{2j}^{(i+1)}$$

$$Q(\theta,\theta_i)=\sum_Z p(Z|Y,\theta_i)\log p(Y,Z|\theta)=E_z\left[\log(Y,Z|\theta,\theta^{(i)})\right]$$

$$=\sum_{j=1}^{n}\{u_{1j}^{(i+1)}[\log s_1+y_j\log p+(1-y_j)\log(1-p)]$$

$$+u_{2j}^{(i+1)}[\log s_2+y_j\log q+(1-y_j)\log(1-q)]$$

$$+(1-u_{1j}^{(i+1)}-u_{2j}^{(i+1)})[\log(1-s_1-s_2)+y_j\log r+(1-y_j)\log(1-r)]\}$$

M-Step:极大化θ

$$\theta^{(i+1)}=\arg\max_\theta Q(\theta,\theta^i)$$

$$s_1^{(i+1)}=\frac{1}{n}\sum_{j=1}^{n}u_{1j}^{(i+1)}$$

$$s_2^{(i+1)}=\frac{1}{n}\sum_{j=1}^{n}u_{2j}^{(i+1)}$$

$$p^{(i+1)}=\frac{\sum_{j=1}^{n}u_{1j}^{(i+1)}y_j}{\sum_{j=1}^{n}u_{1j}^{(i+1)}}$$

$$q^{(i+1)}=\frac{\sum_{j=1}^{n}u_{2j}^{(i+1)}y_j}{\sum_{j=1}^{n}u_{2j}^{(i+1)}}$$

$$r^{(i+1)}=\frac{\sum_{j=1}^{n}\left(1-u_{1j}^{(i+1)}-u_{2j}^{(i+1)}\right)y_j}{\sum_{j=1}^{n}\left(1-u_{1j}^{(i+1)}-u_{2j}^{(i+1)}\right)}$$

3 实验设计

1、给定参数$\theta=(s_1,s_2,p,q,r)$数值如下，按照此分布去采样生成N个投掷结果。

参数	s1	s2	p	q	r
数值	0.3	0.2	0.7	0.3	0.9

2、初始假设参数$\theta=(s_1,s_2,p,q,r)=(0.2\quad 0.1\quad 0.3\quad 0.4\quad 0.5)$，N=100。

参数	s1	s2	p	q	r
给定数值	0.3	0.2	0.7	0.3	0.9
初始数值	0.2	0.1	0.3	0.4	0.5
迭代后数值	0.1648	0.0941	0.5985	0.6987	0.7767

```
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```

3、初始假设参数$\theta=(s_1,s_2,p,q,r)=(0.2\quad 0.1\quad 0.3\quad 0.4\quad 0.5)$，N=1000。

参数	s1	s2	p	q	r
给定数值	0.3	0.2	0.7	0.3	0.9
初始数值	0.2	0.1	0.3	0.4	0.5
迭代后数值	0.1613	0.0936	0.6355	0.7306	0.8027

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```

4、初始假设参数 $\theta=(s_1,s_2,p,q,r)=(0.5 \quad 0.4 \quad 0.1 \quad 0.8 \quad 0.4)$，N=100。

参数	s1	s2	p	q	r
给定数值	0.3	0.2	0.7	0.3	0.9
初始数值	0.5	0.4	0.1	0.8	0.4
迭代后数值	0.2950	0.6063	0.3018	0.9396	0.7217

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10 [0.29495659363373217] [0.6063662670525014] [0.30182200420462546] [0.9396236705754024] [0.7217427733556737]
```

5、初始假设参数 $\theta=(s_1,s_2,p,q,r)=(0.5 \quad 0.4 \quad 0.1 \quad 0.8 \quad 0.4)$，N=1000。

参数	s1	s2	p	q	r
给定数值	0.3	0.2	0.7	0.3	0.9
初始数值	0.5	0.4	0.1	0.8	0.4
迭代后数值	0.2763	0.6251	0.3349	0.9477	0.7513

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```

6、初始假设参数 $\theta=(s_1,s_2,p,q,r)=(0.35 \quad 0.15 \quad 0.6 \quad 0.4 \quad 0.8)$，N=100。

参数	s1	s2	p	q	r
给定数值	0.3	0.2	0.7	0.3	0.9
初始数值	0.35	0.15	0.6	0.4	0.8
迭代后数值	0.3378	0.1299	0.7237	0.5379	0.8748

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7、初始假设参数 $\theta=(s_1,s_2,p,q,r)=(0.35 \quad 0.15 \quad 0.6 \quad 0.4 \quad 0.8)$，N=1000。

参数	s1	s2	p	q	r
给定数值	0.3	0.2	0.7	0.3	0.9
初始数值	0.35	0.15	0.5	0.4	0.7
迭代后数值	0.3408	0.1348	0.6925	0.5003	0.8573

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4 结果分析与总结

1、EM 算法对初始化参数敏感，初始值可能导致收敛到局部最优点。

2、收敛速度快，且收敛结果一般在初始值附近。

5 代码链接

见 https://github.com/ErrricCai/DL-NLP/tree/main/HW2

参考：

https://www.cnblogs.com/jiangxinyang/p/9278608.html

https://blog.csdn.net/u010834867/article/details/90762296