Lecture 13: Neural Networks

Applied Machine Learning

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Part 1: An Artifical Neuron

In this lecture, we will learn about a new class of machine learning algorithms inspired by the brain.

We will start by defining a few building blocks for these algorithms, and draw connections to neuroscience.

Review: Components of A Supervised Machine Learning Problem

At a high level, a supervised machine learning problem has the following structure:

$$\underbrace{\text{Training Dataset}}_\text{Attributes + Features} + \underbrace{\text{Learning Algorithm}}_\text{Model Class + Objective + Optimizer} \to \text{Predictive Model}$$

Where does the dataset come from?

Review: Binary Classification

In supervised learning, we fit a model of the form

$$f : \mathcal{X} \to \mathcal{Y}$$

that maps inputs $x \in \mathcal{X}$ to targets $y \in \mathcal{Y}$.

In classification, the space of targets $\mathcal{Y}$ is *discrete*. Classification is binary if $\mathcal{Y} = \{0, 1\}$

Each value of y value is a *class* and we are interested in finding a hyperplane that separates the different classes.

Review: Logistic Regression

Logistic regression fits a model of the form

$$f(x) = \sigma(\theta^\top x) = \frac{1}{1 + \exp(-\theta^\top x)},$$

where

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

is known as the *sigmoid* or *logistic* function.

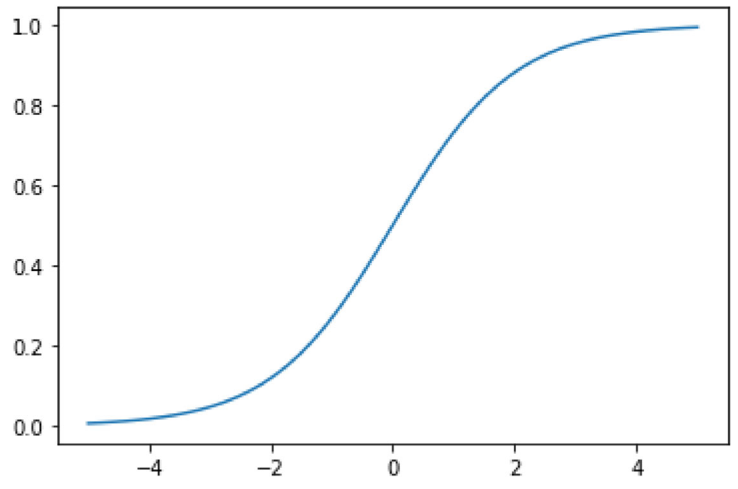
Here is how the logistic function looks like.

```
In [1]: import numpy as np
        from matplotlib import pyplot as plt

        z = np.linspace(-5, 5)
        sigma = 1/(1+np.exp(-z))

        plt.plot(z, sigma)
```

```
Out[1]: [<matplotlib.lines.Line2D at 0x1172c9160>]
```



A Biological Neuron

In order to define an artifical neuron, let's look first a biological one.

TODO: PUT NEURON IMAGE FROM HERE: https://cs231n.github.io/neural-networks-1/ (https://cs231n.github.io/neural-networks-1/)

- Each neuron receives input signals from its dendrites
- It produces output signals along its axon, which connects to the dendrites of other neurons.

An Artificial Neuron: Example

We can imitate this machinery using an idealized artifical neuron.

- The neuron receives signals x_j at dendrites, which are modulated multiplicatively: $w_j \cdot x_j$.
- The body of the neuron sums the modulated inputs: $\sum_{j=1}^d w_j \cdot x_j$.
- These go into the activation function that produces an ouput.

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An Artificial Neuron: Notation

More formally, we say that a neuron is a model $f : \mathbb{R}^d \to [0, 1]$, with the following components:

- Inputs $x_1, x_2, \ldots, x_d$, denoted by a vector x.
- Weight vector $w \in \mathbb{R}^d$ that modulates input x as $w^\top x$.
- An activation function $\sigma : \mathbb{R} \to \mathbb{R}$ that computes the output $\sigma(w^\top x)$ of the neuron based on the sum of modulated features $w^\top x$.

Logistic Regression as an Artifical Neuron

Logistic regression is a model of the form

$$f(x) = \sigma(\theta^\top x) = \frac{1}{1 + \exp(-\theta^\top x)},$$

that can be interpreted as a neuron that uses the *sigmoid* as the activation function.

Perceptron

Another model of a neuron.

Example

Need to implement a small example. Can probably copy-paste implementation of LR from the LR slides.

Activation Functions

Let's list a few.

Part 2: Artificial Neural Networks

Let's now see how we can connect neurons into networks that form complex models that further mimic the brain.

Review: Artificial Neuron

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Review: Logistic Regression as Neuron

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Neural Networks: Intuition

A neural network is a directed graph in which a node is a neuron that takes as input the outputs of the neurons that are connected to it.

Neural Networks: Layers

A neural network layer is a model $f : \mathbb{R}^d \to \mathbb{R}^p$ that applies p neurons ih parallel to an input x.

$$f(x) = \begin{bmatrix} \sigma(w_1^\top x) \\ \sigma(w_2^\top x) \\ \vdots \\ \sigma(w_p^\top x) \end{bmatrix}.$$

where each w_k is the vector of weights for the k-th neuron. We refer to p as the *size* of the layer.

By combining the w_k into one matrix W, we can write in a more succinct vectorized form:

$$f(x) = \sigma(W \cdot x) = \begin{bmatrix} \sigma(w_1^\top x) \\ \sigma(w_2^\top x) \\ \vdots \\ \sigma(w_p^\top x) \end{bmatrix},$$

where $\sigma(W \cdot x)_k = \sigma(w_k^\top x)$ and $W_{kj} = (w_k)_j$.

Neural Networks: Notation

A neural network is a model $f : \mathbb{R} \to \mathbb{R}$ that consists of a composition of L neural network layers:

$$f(x) = f_L \circ f_{L-1} \circ \dots f_1(x).$$

The final layer f_L has size one (assuming the neural net has one ouput); intermediary layers f_l can have any number of neurons.

The notation $f \circ g(x)$ denotes the composition $f(g(x))$ of functions

Example of a Neural Network

- Let's implement a small neural net in the same that we implemented logistic regression
- Then we just run it

Types of Neural Network Layers

There are many types of neural network layers that can exist. Here are a few:

- Ouput layer: normally has one neuron and special activation function that depends on the problem
- Input layer: normally, this is just the input vector x.
- Hidden layer: Any layer between input and output.
- Dense layer: A layer in which every input is connected ot every neuron.
- Convolutional layer: A layer in which the operation $w^\top x$ implements a mathematical [convolution (https://en.wikipedia.org/wiki/Convolution)](https://en.wikipedia.org/wiki/Convolution).
- Anything else?

Neuroscience Angle

Annything we should say here?

Part 3: Backpropagation

We have defined what is an artificial neural network.

Let's not see how we can train it.

Review: Neural Network Layers

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TODO: Add some kind of image from the previous part of the lecture

Review: The Gradient

The gradient $\nabla_\theta f$ further extends the derivative to multivariate functions $f : \mathbb{R}^d \to \mathbb{R}$, and is defined at a point θ_0 as

$$\nabla_\theta f(\theta_0) = \begin{bmatrix} \frac{\partial f(\theta_0)}{\partial \theta_1} \\ \frac{\partial f(\theta_0)}{\partial \theta_2} \\ \vdots \\ \frac{\partial f(\theta_0)}{\partial \theta_d} \end{bmatrix}.$$

In other words, the j-th entry of the vector $\nabla_\theta f(\theta_0)$ is the partial derivative $\frac{\partial f(\theta_0)}{\partial \theta_j}$ of f with respect to the j-th component of θ.

Review: Gradient Descent

If we want to optimize an objective $J(\theta)$, we start with an initial guess θ_0 for the parameters and repeat the following update until the function is no longer decreasing:

$$\theta_i := \theta_{i-1} - \alpha \cdot \nabla_\theta J(\theta_{i-1}).$$

As code, this method may look as follows:

```
theta, theta_prev = random_initialization()
while abs(J(theta) - J(theta_prev)) > conv_threshold:
    theta_prev = theta
    theta = theta_prev - step_size * gradient(theta_prev)
```

Backpropagation

How do we apply gradient descent to a neural network?

Explain backpropgation

Review: Chain Rule of Calculus

Probably will need to review this at some point.

Example

Let's implement backprop with the simple NN model we had earlier.