Research Proposal

Dating back to 2000 B.C., number theory is often considered to be the foundation on which mathematics stands. As such, it occupies an idealised position among the mathematical disciplines. Over the centuries many different subjects evolved from it, and for that reason modern number theory has strong ties to such diverse fields as complex and real (functional) analysis, (algebraic) topology, algebraic geometry and commutative algebra.

One of the original philosophical motivations to study mathematics was to understand the natural numbers – positive whole numbers (integers) – using only properties of the numbers themselves. One of the most famous problems concerns the Pythagorean Theorem: find (all) *integer* solutions to the equation

$$x^2 + y^2 = z^2. \tag{1}$$

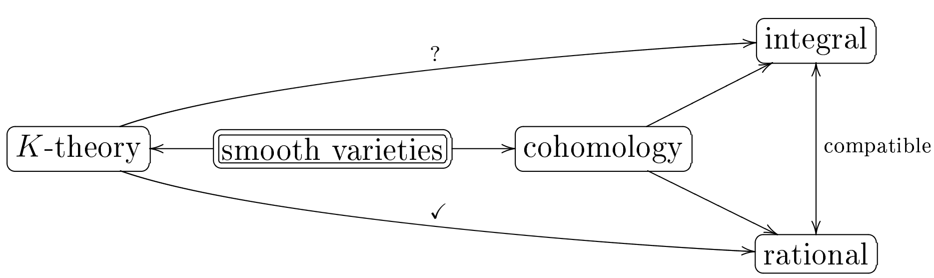
This is an example of a Diophantine equation and we can readily find solutions: The trivial one $(1, 0, 1)$, but also $(3, 4, 5)$. It is a different matter if we alter this equation just slightly, for example by raising the powers:

$$x^3 + y^3 = z^3. \tag{2}$$

Finding a non-trivial solution to this equation is much more challenging. In fact, there is not a single one.

My research interest lies in a branch of number theory called arithmetic geometry. I specialise in algebraic K-theory and p-adic Hodge theory. In the coming year, I plan to advance techniques in this subject. This project is supervised by Wiesława Nizioł, Professor of Mathematics at the University of Utah.

The main question I plan to address in this research proposal is the relationship between two tools designed to study geometric objects, namely to relate K-theory and integral cohomology via Chern classes for smooth open varieties. While cohomology theories occur naturally in the study of geometric objects but are difficult to compare and interpret, K-theory is in its very nature much easier to handle, Chern classes give a method to gain a deeper insight in pathologies and properties of geometric objects. In particular, we are interested in integral Chern classes, as opposed to rational Chern classes, on open varieties which have yet to be defined due to erratic behavior of cycles on the boundary of open varieties.



In what follows I will describe three topics that will provide the ingredients to achieve the construction of Chern classes for open varieties.

1 Geometric Objects over Local Fields

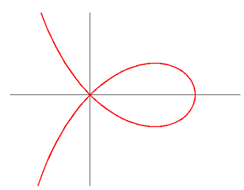
To tackle problems like the one described by example (2), it becomes apparent that the traditional purely number theoretic methods are not sufficient. One important step was to enlarge the set of numbers to be considered. This gives rise to **algebraic number fields** and **function fields**, called **global fields**. In the study of objects like the set of solutions to a Diophantine equation over a global field, it makes often sense to consider the arising questions "locally", that is for each prime number separately. This is done using **local fields**.

A typical example are the p-adic numbers $\mathbb{Q}_p$. One obtains the real numbers $\mathbb{R}$ by completing the rationals $\mathbb{Q}$ with limits of converging sequences with respect to the absolut value. Replacing the usual absolut value by the p-adic one means that now a sequence converges if the differences of its consecutive members are divisible by higher and higher powers of p. For $p = 3$ a sequence of the form

$$\{2\ ,\ 2+3\ ,\ 2+3+4\cdot 3^2\ ,\ 2+3+4\cdot 3^2+5\cdot 3^3\ ,\ldots\}$$

converges since the differences of the numbers in the sequence are more and more divisible by 3. The set of numbers that we obtain from $\mathbb{Q}$ by adding all the limits of sequences of this form are called the p-adic numbers $\mathbb{Q}_p$. The p-adic integers are denoted by $\mathbb{Z}_p$.

At first sight, the questions discussed so far seem purely algebraic, yet in many cases they admit a geometric interpretation. One can think of solutions of polynomial equations as shapes in higher dimensional spaces. For example, the curve $y^2 = x^3 + x^2$ is a node in the plane:



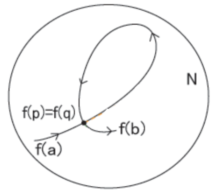
Objects given as solutions of polynomial equations are called **affine varieties**. One of the major objectives of arithmetic algebraic geometry is to understand **algebraic varieties** over p-adic fields. These objects are not as nice as affine varieties, but it turns out that if we zoom in close enough they still look like affine varieties, although as a whole they might be quite complicated.

We use auxiliary tools comparable to a magnifying glass to detect pathologies of a variety. Different tools however provide different results and we have to try to match them up and therefore understand the connection between the different images we get. This can be compared to the fact that different lenses alter the image of an object studied in different ways; they might, for example, stretch it in different ways. While neither of them gives the "real" image, when put together they allow us to "see" the object. The images we "see" are invariants of the variety characterising its major properties. Cohomology and K-theory are two such tools.

2 Cohomology and K-theory

The methods of K-theory rely on suitably parametrised vector spaces. To a mathematical object such as a variety X is associated a family of (commutative) groups $\{K_i(X)\}_{i\in\mathbb{N}_0}$. As vector spaces have been known for a long time and are well studied, K-theory is in many aspects very powerful and relatively easy to handle.

Alternatively, we can consider subvarieties of an algebraic variety. This can be a curve in a two dimensional variety



Formal linear combinations of irreducible subvarieties are called algebraic cycles. They are closely related to **cohomology**. In our context, a cohomology theory $H^\bullet$ is a functor that associates to a variety X a graded algebra $\{H^n(X)\}_{n\in\mathbb{Z}}$ satisfying fundamental axioms inspired by topological techniques. According to these axioms a cohomology theory should reflect basic geometric properties of a variety such as its dimension. Different cohomology theories have been developed. They can be rational or integral, which refers to the coefficients either being in the base field or in its ring of integers. In this context the main focus is on p-adic cohomology theories, i.e. with coefficients in the p-adic numbers, as they come up naturally in various arithmetic problems.

Berthelot introduced in [1] **rigid cohomology** – a rational cohomology theory for varieties over a field of characteristic p, which is shown by Petrequin [6] to satisfy all the axioms of a Weil cohomology. It is a generalisation of **crystalline cohomology**, an integral cohomology theory, that can be calculated using Illusie's **deRham-Witt complex** [3]. Indeed, for smooth and proper varieties crystalline cohomology is a suitable **integral model** for rigid cohomology. (Here "smooth" is a technical way to require the variety not to have too many irregularities whereas "proper" means that it is complete and not open.) This is not the case for open varieties. The remedy here is, as was shown by Davis, Langer and Zink [2], to consider a subcomplex of the deRham-Witt complex, the **overconvergent deRham-Witt complex**. The resulting cohomology theory is a good integral model for rigid cohomology in the case of an open variety. It is conjectured in [2] that it is rationally compatible with crystalline cohomology for projective varieties.

3 Chern classes

Taking into account the different features of K-theory and cohomology as mentioned, it is desirable to understand the interaction between these mathematical tools:

$$\boxed{K\text{-theory}} \overset{?}{\longleftrightarrow} \boxed{\text{cohomology}}$$

One possible way to do this is via **Chern classes**, group homomorphisms from K-theory to cohomology

$$c_{ij} : K_j(X) \to H^{2i-j}(X).$$

My project concerns **integral** Chern classes for smooth **open** varieties.

We distinguish between **integral** and **rational** Chern classes, depending on whether the Chern class map takes values in an integral or rational cohomology theory. Calculations that arise in arithmetic problems often require not only integral Chern classes, but integral Chern classes compatible with rational ones. An important aspect of the aformentioned compatibility is that the target of an integral Chern class map be a good integral model for the corresponding rational cohomology, which makes them much harder to construct.

Petrequin was able to define in [6] **rational** Chern classes into rigid cohomology for both proper **and** open varieties

$$c_i^{\mathrm{rig}} : K_0(X) \to H^{2i}_{\mathrm{rig}}(X)$$

which are in the case of proper smooth varieties compatible with the **integral** Chern classes for crystalline cohomology.

Given that for a variety being proper is a very strong condition, it is essential to find an analogue to the crystalline Chern class map for more general varieties witch are compatible with the rigid Chern classes. Precisely we want to answer the following question:

Can we define integral Chern classes for open varieties that are compatible with the rigid Chern classes?

I will answer this question by using the overconvergent deRham-Witt cohomology [2], which as mentioned provides indeed a good integral model of rigid cohomology. A well-known method of constructing Chern classes into a certain cohomology theory is based on the **projective space theorem**, whose existence depends on the respective cohomology. In fact, it would be sufficient to have a projective space theorem for the cohomology of a smaller family of spaces, classifying spaces like $\mathrm{BGL}(A)$ for a ring A or alternatively to compute the cohomology of the classifying spaces. This project aims to establish a projective space theorem for overconvergent deRham-Witt cohomology and as a consequence to construct integral Chern classes for smooth open varieties compatible with the rigid ones.

References

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