INTEGRAL p-ADIC COHOMOLOGY GAUS-SEMINAR HEIDELBERG

by

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Abstract. — This talk is a sequel to a talk by Johannes Sprang that he gave in Heidelberg during the winter term of last year. I will explain developments that have taken place since then in our joint project with Atsushi Shiho. Under certain conditions of resolutions of singularities in positive characteristic, we construct a "good" integral p-adic cohomology theory for open and singular varieties by using a version of Voevodsky's h-topology. I will explain the construction and clarify in which sense our cohomology is a "good" p-adic cohomology theory. If time permits, I will also touch on the question why a similar approach does not work in full generality without resolutions of singularities.

Résumé **(Cohomologie p-adique à coëffcients entiers).** — Cet exposé est une suite à un exposé par Johannes Sprang qu'il a donné à Heidelberg dans le semestre d'hiver de l'année précedente. Je vais expliquer les développements depuis dans notre projet joint avec Atsushi Shiho. Sous certaines conditions de résolutions de singularités en caractéristique positive, on construit une "bonne" théorie de cohomologie p-adique à coefficients entiers pour les variétés ouvertes et singulières en utilisant une version de la topologie h de Voevodsky. Je vais expliquer la construction et préciser en quel sens notre cohomologie est une "bonne" théorie de cohomologie p-adique. Si le temps le permet, j'aborderai également la question de savoir pourquoi une approche similaire ne fonctionne pas en toute généralité sans résolutions de singularités.

Contents

1. The search for a good integral p-adic cohomology theory

Thank you very much for the invitation. It's my first time in Heidelberg and it is a pleasure to be here.

About a year ago, you had Johannes Sprang as a speaker in the seminar, an my talk today is in some sense a continuation of his, and an update on our joint results with Atsushi Shiho. Johannes is sending greeting to all of you and in particular thanks to Alexander Schmidt for the discussions you had last time, as they were very helpful for our progress. This is also related to the talk by Alberto Merici just before Christmas break. So I want to recall some main points of Johannes' talk here and go from there.

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The famous Weil conjectures can be seen as a starting point for the study of p-adic cohomology theories. They concern the Zeta function of varieties in positive characteristic – which encodes the number of points on that variety. Weil has suggested to use a suitable cohomology theory to solve these conjectures for proper and smooth varieties over a field k of characteristic p. Such a cohomology is called "Weil cohomology".

1.1. Weil cohomology. — For a perfect field k of characteristic $p > 0$ a Weil cohomology is a contravariant functor

$$H^* : \{\text{ proper smooth } k\text{-varieties }\} \longrightarrow \{\text{ graded } K\text{-algebras }\},$$

where K is a field of positive characteristic, such that certain axioms are satisfied: finiteness, Poincaré duality, Künneth formula, cycle classes,...

1.1.1. Example. — For a prime $\ell \neq p$, the ℓ-adic étale cohomology is an example. With this the Weil conjectures were solved. There is even an "integral" version:

$$H^*_{\text{ét}}(X, \mathbb{Z}_\ell) := \varprojlim H^*_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}/\ell^n\mathbb{Z}) \longrightarrow \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^*_{\text{ét}}(X, \mathbb{Z}_\ell).$$

This is only well-behaved for $\ell \neq p$!

1.1.2. Question. — Is there a good cohomology theory with p-adic coefficients, that is $\ell = p$? And if so, why should we bother?

— to study p-torsion phenomena.

— for computability reasons (efficient algorithms for counting points, e.g. Kedlaya [**11**]).

1.2. A forest of p-adic cohomolog theories. — Let Var_k denote the category of k-varieties (reduced separated scheme of finite type over k), and Sm_k the category of smooth k-varieties. In the following we will consider cohomology theories with coefficients in

$$W(k) := \text{ Witt vectors of } k \qquad\qquad K := \mathrm{Quot}(W(k))$$

1.2.1. Rigid cohomology. — $H^*_{\text{rig}}(-/K) : \mathsf{Var}_k \longrightarrow \{K\text{-vector spaces}\}$
(rigid)) analytic approach, computes by a certain (overconvergent) de Rham complex (Berthelot, [**4**])
Pro: well-behaved (finite dimensional) for all k-varieties
Contra: not integral! K-vector spaces

1.2.2. Crystalline cohomology. — $H^*_{\text{cris}}(-/W(k)) : \mathsf{Var}_k \longrightarrow \{W(k)\text{-modules}\}$
site theoretic approach, can be computed by the de Rham–Witt complex $W\Omega^\bullet_{X/k}$ in certain cases (Berthelot [**3**], Illusie [**10**])
Pro: integral, compares to rigid cohomology after $\otimes_{W(k)}K$,
Contra: only well behaved (finitely generated) for smooth and proper k-varieties

1.2.3. Some variants of crystalline cohomology. —

Log crystalline cohomology : $H^*_{\text{cris}}(-/W(k)) : \{\text{log schemes over } k\} \longrightarrow \{W(k)\text{-modules}\}$
If X is smooth and has a normal crossings compactification $\overline{X}$, consider the de Rham–Witt complex with log poles along $D = \overline{X}\backslash X$: $H^*_{\text{cris}}((X, \overline{X})/W(k)) \cong H^*(\overline{X}, W\Omega^\bullet_{X/k}(\log D)$. (Hyodo–Kato [**9**])
Pro: integral, compares to rigid cohomology after $\otimes_{W(k)}K$,
Contra: (almost) only well behaved (finitely generated) for log smooth proper k-varieties

Overconvergent de Rham–Witt cohomology : $H^*_\dagger(-/W(k)) : \mathsf{Sm}_k \longrightarrow \{W(k)\text{-modules}\}$
Computed via an overconvergent subcomplex $W^\dagger\Omega^\bullet_{X/k} \subset W\Omega^\bullet_{X/k}$ (Davis–Langer–Zink [**5**]) **Pro:** integral, compares to rigid cohomology after $\otimes_{W(k)}K$,
Contra: not finitely generated even modulo torsion (Counterexample: Ertl–Shiho [**6**])

1.2.4. Goal. — Construct a "good" integral p-adic cohomology theory on Var_k, i.e.

— The cohomology groups $H^i_{\text{good}}(X)$ are finitely generates $W(k)$-modules for all $X \in \mathsf{Var}_k$.

— There is a comparison isomorphism with (log) crystalline cohomology, i.e. $H^*_{\text{good}}(X) = H^*_{\text{cris}}(X/W(k))$, for $X \in \mathsf{Var}_k$ (log) smooth and proper varieties.

— There is a rational comparison isomorphism with rigid cohomology, i.e. $H^*_{\text{good}}(X) \otimes_{W(k)} K = H^*_{\text{rig}}(X/K)$ for all $X \in \mathsf{Var}_k$.

— It should have other reasonable properties (Künneth formula, etc.)

2. Construction under resolution of singularities

2.1. Some useful conventions. — The following conventions will be helpful when describing our construction.

2.1.1. Geometric pair $(X, \overline{X})$***.*** — an open immersion $X \hookrightarrow \overline{X}$ in Var_k with dense image, $\overline{X}$ proper – notation: Var_k^{geo}

2.1.2. Normal crossing pair $(X, \overline{X})$***.*** — a geometric pair, such that $\overline{X} \backslash X$ is a simple normal crossing divisor – notation: Var_k^{nc}

2.1.3. Morphism of geometric pairs $f : (X_1, \overline{X}_1) \to (X_2, \overline{X}_2)$***].*** — a morphism $f : \overline{X}_1 \to \overline{X}_2$ in Var_k such that $f(X_1) \subset X_2$

It is called **strict** if $f^{-1}(X_2) = X_1$

It **has property P** if $f : \overline{X}_1 \to \overline{X}_2$ has property P

2.1.4. Weak factorisation. — of a proper birational morphism $f : (X_1, \overline{X}_1) \to (X_2, \overline{X}_2)$ which is an isomorphism on X_2 is a sequence

$$(X_1, \overline{X}_1) = (V_0, \overline{V}_0) \overset{f_1}{\dashrightarrow} \ldots \overset{f_\ell}{\dashrightarrow} (V_\ell, \overline{V}_\ell) = (X_2, \overline{X}_2)$$

where f_i is rational, $f_\ell \circ \ldots \circ f_1 = f$, each composition $f_i \circ \ldots \circ f_1$ is a morphism and induces an isomorphism on X_2, and for each i either f_i or f_i^{-1} is a blow-up along a smooth center Z_i disjoint from X_2 which has normal crossing with $\overline{V}_i \backslash V_i$ (or $\overline{V}_{i-1} \backslash V_{i-1}$.

2.1.5. Remark. — Conceptually normal crossing pairs are to geometric pairs, what smooth varieties are to varieties.

2.2. Hypotheses that we need. — Let X be a smooth variety over k. When studying finiteness properties it is common to consider compactifications of X. While by Nagata's compactification theorem every smooth k-variety has a compactification (i.e., a quasi-compact open immersion into a proper k-variety), it might be a rather complicated one. However, under the assumption of resolution of singularities, it is possible to reduce to the case of normal crossing compactifications. Thus we have the following hypotheses:

2.2.1. Strong resolutions of singularities (SR). — For all $X \in \mathsf{Var}_k$ there exists a proper birational morphism $f : X' \to X$, such that X' is smooth, and f is an iso on X_{sm}.

For all proper birational morphisms $f : X' \to X$ in Sm_k there is a sequence of blow-ups along smooth centres

$$X_n \to X_{n-1} \to \cdots \to X$$

that factors through f.

2.2.2. Embedded resolutions of singularities (ER). — For all $(X, \overline{X}) \in \mathsf{Var}_k^{geo}$ with $\overline{X}$ smooth, there is a proper birational morphism $f : (X, \overline{X}') \to (X, \overline{X})$ such that $(X, \overline{X}') \in \mathsf{Var}_k^{nc}$.

2.2.3. Weak factorisation (WF). — For every strict proper birational morphism $(X, \overline{X}') \to (X, \overline{X})$ in Var_k^{nc} which is an iso on X, there exists a weak factorisation.

2.2.4. Embedded resolutions with boundaries (ERB). — For all strict closed immersions of geometric pairs $(Y, \overline{Y}) \to (X, \overline{X})$ such that Y is smooth and $(X, \overline{X}) \in \mathsf{Var}_k^{nc}$, there is a commutative diagram

$$\begin{tikzcd} (Y, \overline{Y}') \arrow[r, hook] \arrow[d] & (X, \overline{X}') \arrow[d] \\ (Y, \overline{Y}) \arrow[r, hook] & (X, \overline{X}) \end{tikzcd}$$

where$(Y, \overline{Y}')$ and $(X, \overline{X}')$ are normal crossing pairs, the horizontal maps are strict closed immersions, and the vertical maps are proper birational morphisms and isos on Y, X.

2.3. Variations of Voevodsky's h-topology. — In this part, we only use Hypothesis 2.2.1 **(SR)**. To make things smooth, we want to consider the topology "generated by blow ups".

2.3.1. Definition. — The *cdp*-topology is the topology generated by completely decomposed proper morphisms $p : Y \to X$.
(Completely decomposed means that for every $x \in X$ there is $y \in p^{-1}(x) \subset Y$, such that for the residue fields $\kappa(x) \xrightarrow{\sim} \kappa(y)$.)

***2.3.2. Lemma* (Suslin–Voevodsky).** — *The cdp-topoology on* Var_k *is generated by blow-ups.*

2.3.3. Definition. — The *cdh*-topology on Var_k is the topology generated by *cdp*-morphisms and Nisnevich morphisms.
(A morphism $p \colon Y \to X$ in Var_k is a Nisnevich morphism if it is étale and completely decomposed.)

One can restric these topologies to Sm_k. The following is not hard to show:

2.3.4. Lemma. — *Under Hypothesis 2.2.1 **(SR)**, the cdp-topoology on* Sm_k *is generated by smooth blow-ups, and the Nisnevich topology on* Sm_k *is generated by smooth Nisnevich morphisms.*

An important result for our construction is the following:

***2.3.5. Proposition* (Ertl–Shiho–Sprang).** — *Let τ be any topology finer than the cdp-topology. The inclusion* $\mathsf{Sm}_k \hookrightarrow \mathsf{Var}_k$ *induces an equivalence of topoi*

$$Sh(\mathsf{Sm}_{k,\tau}) \xrightarrow{\sim} Sh(\mathsf{Var}_{k,\tau}).$$

Proof. — We only sketch the main steps:
- $\mathsf{Sm}_k \hookrightarrow \mathsf{Var}_k$ is fully faithful.
- By Verdier's result it suffices to show that every k-variety has a *cdp*-cover by smooth k-varieties.
- This follows with Hypothesis 2.2.1 **(SR)**.

□

2.4. Improving the topology on smooth k-varieties. — In this part, we need the Hypotheses 2.2.1 **(SR)**, 2.2.2 **(ER)**, and 2.2.4 **(ERB)**.
- We want to embed the objects $X \in \mathsf{Sm}_k$ into objects $(X, \overline{X}) \in \mathsf{Var}_k^{nc}$.
- We want to do something similar with *cdp*-morphisms and Nisnevich morphisms.

***2.4.1. Lemma* (E–Shiho–Sprang).** — *Under Hypotheses 2.2.1 **(SR)**, 2.2.2 **(ER)**, every* $X \in \mathsf{Sm}_k$ *has an snc-compactification* $\overline{X}$, *i.e.* $(X, \overline{X}) \in \mathsf{Var}_k^{nc}$.
For fixed X, the category of snc-compactifications is filtered, denoted by $\{(X, \overline{X})/X\}$.

2.4.2. Definition. — A good smooth blow-up square is a smooth blow-up square which embeds into nc-pairs

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & X' \arrow[d, "p"] \\ Z \arrow[r, hook, "e"] & X \end{tikzcd} \qquad \begin{tikzcd} (Z', \overline{Z}') \arrow[r, hook] \arrow[d] & (X', \overline{X}') \arrow[d, "p"] \\ (Z, \overline{Z}) \arrow[r, hook, "e"] & (X, \overline{X}) \end{tikzcd}$$

such that all morphisms are strict and p is a blow-up with centre $\overline{Z}$.

***2.4.3. Proposition* (E–Shiho–Sprang).** — *Assume Hypostheses 2.2.1 **(SR)**, 2.2.2 **(ER)**, 2.2.4 **(ERB)**.*
Every smooth blow-up $\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$ *is good.*

Proof. — We only sketch the main steps:

— Take an snc-compactification $(X, \overline{X}_1)$. Let $\overline{Z}_1$ be the closure of Z in $\overline{X}_1$.

— By Hypothesis 2.2.4 **(ERB)** there is a commutative diagram

$$\begin{tikzcd} (Z, \overline{Z}) \arrow[r, hook] \arrow[d] & (X, \overline{X}) \arrow[d] \\ (Z, \overline{Z}_1) \arrow[r, hook] & (X, \overline{X}_1) \end{tikzcd}$$

such that $(Z, \overline{Z}) \hookrightarrow (X, \overline{X})$ is a strict closed immersion of nc-pairs and the vertical morphisms are strict proper birational.

— By setting $\overline{X}' := \mathrm{Bl}_{\overline{Z}}(\overline{X})$ one obtains the desired diagram.

□

2.4.4. Definition. — A good smooth Nisnevich square is a Nisnevich square

$$\begin{tikzcd} Y' \arrow[r, hook] \arrow[d] & X' \arrow[d, "p"] \\ Y \arrow[r, hook, "e"] & X \end{tikzcd}$$

(e an open immersion and p an étale morphism such that the morphism $p^{-1}(X \backslash e(Y)) \to X \backslash e(Y)$ induced by p is an isomorphism) which can be embedded into a finite disjoint union of the following

$$\begin{tikzcd} (Y', \overline{X}') \arrow[r, hook] \arrow[d] & (X', \overline{X}') \arrow[d, "p"] \\ (Y, \overline{X}) \arrow[r, hook, "e"] & (X, \overline{X}) \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ (X, \overline{X}) \arrow[r] & (X, \overline{X}) \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & (X, \overline{X}) \arrow[d] \\ \varnothing \arrow[r] & (X, \overline{X}) \end{tikzcd}$$

where e is the identity in $\overline{X}$ such that the closure of $X \setminus Y$ in $\overline{X}$ is a smooth divisor of $\overline{X}$.

For Nisnevich squares, we only have an embedding result *cdp*-locally (but this is good enough).

***2.4.5. Proposition* (Ertl–Shiho–Sprang).** — *Under Hypotheses 2.2.1 **(SR)**, 2.2.2 **(ER)** and 2.2.4 **(ERB)** given a Nisnevich square in* Var_k

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

there is a split cdp-hypercovering $X_\bullet \to X$ *along which the pullback*

$$\begin{tikzcd} Y'_\bullet \arrow[r] \arrow[d] & X'_\bullet \arrow[d] \\ Y_\bullet \arrow[r] & X_\bullet \end{tikzcd}$$

satisfies: For each i*, the square*

$$\begin{tikzcd} Y'_i \arrow[r] \arrow[d] & X'_i \arrow[d] \\ Y_i \arrow[r] & X_i \end{tikzcd}$$

admits a factorisation

$$\begin{tikzcd} Y'_i \arrow[r, equal] & X'_{i,n} \arrow[r] \arrow[d] & X'_{i,n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X'_{i,1} \arrow[r] \arrow[d] & X'_{i,0} \arrow[r, equal] \arrow[d] & X'_i \\ Y_i \arrow[r, equal] & X_{i,n} \arrow[r] & X_{i,n-1} \arrow[r] & \cdots \arrow[r] & X_{i,1} \arrow[r] & X_{i,0} \arrow[r, equal] & X_i \end{tikzcd}$$

for some n *(which may depend on* i*) such that each square*

$$\begin{tikzcd} X'_{i,l} \arrow[r] \arrow[d] & X'_{i,l-1} \arrow[d] \\ X_{i,l} \arrow[r] & X_{i,l-1} \end{tikzcd}$$

is a good smooth Nisnevich square.

2.5. Construction for smooth open k-varieties. — In this part, we need Hypotheses 2.2.1 **(SR)**, 2.2.2 **(ER)**, 2.2.4 **(ERB)**, 2.2.3 **(WF)**.

2.5.1. Log structures. — We will use log structures:

— $k := (\mathrm{Spec}(k), triv)$, $W_n(k) := (\mathrm{Spec}(W_n(k)), triv)$, $W(k) := (\mathrm{Spf}(W(k)), triv)$

— We consider $(X, \overline{X}) \in \mathsf{Var}_k^{nc/geo}$ as a log scheme $\overline{X}$ with log structure induced by $\overline{X}\backslash X$.

2.5.2. Crystalline cohomology theories. — Several versions of crystalline cohomology are the basis for our construction.

— For $X \in \mathsf{Sm}_k$: $W_n\Omega^\bullet_{X/k}$ which computes crystalline cohomology

$$R\Gamma_{\mathrm{cris}}(X/W_n(k)) = R\Gamma(X, W_n\Omega^\bullet_{X/k}).$$

— For $(X, \overline{X}) \in \mathsf{Var}_k^{nc}$: $W_n\omega^\bullet_{(X,\overline{X})/k} = W_n\Omega^\bullet(\log \overline{X}\backslash X)_{\overline{X}/k}$,
which computes log-crystalline cohomology

$$R\Gamma_{\mathrm{cris}}((X, \overline{X})/W_n(k)) = R\Gamma(\overline{X}, W_n\omega^\bullet_{(X,\overline{X})/k}).$$

For $(X, \overline{X}) \in \mathsf{Var}_k^{nc}$ let $A_n^\bullet(X, \overline{X})$ be an explicit complex functorial in $(X, \overline{X})$ representing $R\Gamma(\overline{X}, W_n\omega^\bullet_{(X,\overline{X}/k)})$.

2.5.3. Proposition* (Ertl–Shiho–Sprang).** — *Assume Hypotheses 2.2.1* ***(SR)*, 2.2.2* ***(ER)****, 2.2.3* ***(WF)****. For fixed $X \in \mathsf{Sm}_k$ and varying $(X,\overline{X}) \in \mathsf{Var}_k^{nc}$ all $A_n^\bullet(X,\overline{X})$ are quasi-isomorphic.*

Proof. — The main steps are

— We know that $\{(X,\overline{X})/X\}$ is filtered.

— We only need to show that for a strict proper morphism $(X,\overline{X}) \to (X,\overline{X}')$ the induced morphism $A_n(X,\overline{X}') \xrightarrow{\sim} A_n^\bullet(X,\overline{X})$.

— By weak factorisation we may assume that this is a blow-up with smooth centre.

— We work Zariski-locally, i.e. with affine schemes.

— Then everything lifts and we can compute explicitly.

□

2.5.4. Definition. — Assume Hypothesis 2.2.1 **(SR)**, 2.2.2 **(ER)**, 2.2.3 **(WF)**. For $X \in \mathsf{Sm}_k$ define

$$A_n^\bullet(X) := \varinjlim_{(X,\overline{X})} A_n^\bullet(X,\overline{X}), \qquad A^\bullet(X) := R\varprojlim_n A_n^\bullet(X).$$

With this definition $A_n^\bullet(X) \cong A_n^\bullet(X,\overline{X})$ for all $(X,\overline{X}) \in \mathsf{Var}_k^{nc}$.

2.5.5. Proposition* (Ertl–Shiho–Sprang).** — *Assume 2.2.1* ***(SR)*, 2.2.2* ***(ER)****, 2.2.3* ***(WF)****, then $A_n^\bullet(X)$ and $A^\bullet(X)$ are functorial in X.*

Proof. — The main steps are:

— For $f : X \to Y$ in Sm_k consider the category

$$\{\overline{f}/f\} = \{\overline{f} : (X,\overline{X}) \to (Y,\overline{Y}) \mid (X,\overline{X}), (Y,\overline{Y}) \in \mathsf{Var}_k^{nc}\}.$$

— Show this category is non-empty and filtered.

— There are projections, where p_2 is surjective

$$\begin{tikzcd} & \{\overline{f}/f\} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \{(X,\overline{X})/X\} & & \{(Y,\overline{Y})/Y\} \end{tikzcd}$$

— Any extension $\overline{f}$ of f induces a natural morphism $A_n^\bullet(p_2(\overline{f})) \to A_n^\bullet(p_1(\overline{f}))$.

— We obtain a zig-zag $\varinjlim_{\overline{Y}} A_n^\bullet(Y,\overline{Y}) \xleftarrow{\sim} \varinjlim_{\overline{f}} A_n^\bullet(p_2(\overline{f})) \to \varinjlim_{\overline{f}} A_n^\bullet(p_1(\overline{f})) \to \varinjlim_{\overline{X}} A_n^\bullet(X,\overline{X})$

□

2.5.6. Idea. — We may regard $A_n^\bullet$ as a complex of presheaves on Sm_k. and sheafify it with respect to the *cdp*- and *cdh*-topology.

2.5.7. Definition. — Define the following complexes of sheaves on Sm_k

$$a_{cdp}^* A_n^\bullet \qquad\qquad a_{cdh}^* A_n^\bullet$$

$$a_{cdp}^* A^\bullet := R\varprojlim a_{cdp}^* A_n^\bullet \qquad\qquad a_{cdh}^* A^\bullet := R\varprojlim a_{cdh}^* A_n^\bullet$$

The following descent results is one of the most important that make our construction work.

2.5.8. Proposition* (Ertl–Shiho–Sprang).** — *Assume Hyposethes 2.2.1* ***(SR)*, 2.2.2* ***(ER)****, 2.2.4* ***(ERB)****, 2.2.3* ***(WF)****. For every $X \in \mathsf{Sm}_k$ the natural morphisms*

$$A_n^\bullet(X) \to R\Gamma_{cdp}(X, a_{cdp}^* A_n^\bullet) \to R\Gamma_{rh}(X, a_{rh}^* A_n^\bullet)$$

are quasi-isomorphisms.

Proof. — The proof is quite intricate. Here are the main steps:

— By work of Cortinas–Haesemayer–Schlichting–Weibel it suffices to show that A_n satisfies the Mayer–Wietoris property: For a smooth blow-up square
$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$
the induced diagram
$$\begin{tikzcd} A_n^\bullet(Z')] & A_n^\bullet(X') \arrow[l] \\ A_n^\bullet(Z) \arrow[u] & A_n^\bullet(X) \arrow[l] \arrow[u] \end{tikzcd}$$
is homotopy co-cartesian.

— To show this, we take a good compactification of the square and work Zariski locally.

— Then compute explicitly.

— Similarly for Nisnevich squares, except that we use the previous result to work *cdp*-locally.

□

2.6. Extension to k-varieties. — In this part, we need Hypotheses 2.2.1 **(SR)**, 2.2.2 **(ER)**, 2.2.4 **(ERB)**, 2.2.3 **(WF)**.

2.6.1. Idea. — We now want to use the equivalence of topoi $\mathrm{Sh}(\mathsf{Sm}_{k,\tau}) \xrightarrow{\sim} \mathrm{Sh}(\mathsf{Var}_{k,\tau})$ to extend the construction to Var_k.

2.6.2. Definition. — By the above equivalence of topoi

$$a^*_{cdp}A^\bullet_n \qquad\qquad a^*_{cdh}A^\bullet_n$$
$$a^*_{cdp}A^\bullet := R\varprojlim a^*_{cdp}A^\bullet_n \qquad\qquad a^*_{cdh}A^\bullet := R\varprojlim a^*_{cdh}A^\bullet_n$$

define (complexes of) sheaves on Var_k.

We have a similar descent result as before:

2.6.3. Proposition **(Ertl–Shiho–Sprang).** — *Assume Hypotheses 2.2.1* ***(SR)****, 2.2.2* ***(ER)****, 2.2.4* ***(ERB)****, 2.2.3* ***(WF)****. Then for any $X \in \mathsf{Var}_k$*

$$R\Gamma_{cdp}(X, a^*_{cdp}A^\bullet_n) \xrightarrow{\sim} R\Gamma_{cdh}(X, a^*_{cdh}A^\bullet_n),$$
$$R\Gamma_{cdp}(X, a^*_{cdp}A^\bullet) \xrightarrow{\sim} R\Gamma_{cdh}(X, a^*_{cdh}A^\bullet).$$

Proof. — Choose a *cdp*-hypercovering $X_\bullet \to X$ with $X_i \in \mathrm{Sm}_k$. Then there is a commutative diagram
$$\begin{tikzcd} R\Gamma_{cdp}(X, a^*_{cdp}A^\bullet_n) \arrow[r] \arrow[d, "\sim"] & R\Gamma_{rh}(X, a^*_{cdh}A^\bullet_n) \arrow[d, "\sim"] \\ R\Gamma_{cdp}(X_\bullet, a^*_{cdp}A^\bullet_n) \arrow[r, "\sim"] & R\Gamma_{rh}(X_\bullet, a^*_{cdh}A^\bullet_n) \end{tikzcd}$$
where the vertical maps are quasi-isomorphisms because of *cdp*-descent, and the lower horizontal map is a quasi-isomorphism because all maps $R\Gamma_{cdp}(X_i, a^*_{cdp}A^\bullet_n) \xrightarrow{\sim} R\Gamma_{cdh}(X_i, a^*_{cdh}A^\bullet_n)$. □

We may now define:

2.6.4. Definition. — Assume Hypotheses 2.2.1 **(SR)**, 2.2.2 **(ER)**, 2.2.4 **(ERB)**, 2.2.3 **(WF)** For $X \in \mathsf{Var}_k$ we set
$$H^*_{good}(X/W(k)) := H^*R\Gamma_{cdh}(X, a^*_{cdh}A^\bullet).$$

It satisfies the following properties:

2.6.5. Nisnevich descent. — By definition $H^*_{good}(X/W(k))$ satisfies Nisnevich descent, i.e. for a Nisnevich hypercover $X_\bullet \to X$, the induced morphism

$$H^*_{good}(X/W(k)) \longrightarrow H^*_{good}(X_\bullet/W(k))$$

is an isomorphism.

2.6.6. Comparison with crystalline cohomology. — For $(X, \overline{X}) \in \mathsf{Var}_k^{nc}$ we have an isomorphism

$$H^*_{cris}((X, \overline{X})/W(k)) \cong H^*_{good}(X/W(k)).$$

For $X \in \mathsf{Sm}_k$ proper we have an isomorphism

$$H^*_{cris}(X/W(k)) \cong H^*_{good}(X/W(k)).$$

2.6.7. Finite generation. — The cohomology groups $H^n_{good}(X/W(k))$ are finitely generated $W(k)$-modules.

2.6.8. Cohomological dimension. — We have

$$H^n_{good}(X/W(k)) = 0 \text{ for } n < 0, n > 2\dim(X).$$

2.6.9. Comparison with rigid cohomology. — For $X \in \mathsf{Var}_k$ there is a canonical isomorphism

$$H^*_{rig}(X/K) \cong H^*_{good}(X/W(k)) \otimes \mathbb{Q}.$$

2.6.10. Künneth formula. — Let $X_1, X_2 \in \mathsf{Var}_k$, then

$$\bigoplus_{i+j=n} H^i_{good}(X_1/W(k)) \otimes_{W(k)} H^j_{good}(X_2/W(k)) = H^n_{good}(X_1 \times X_2/W(k)).$$

2.6.11. Homotopy invarianc. — From the Künneth formula is follows, that for every $X \in \mathsf{Var}_k$ the natural projection $f : \mathbb{A}^1_X \to X$ induces an isomorphism

$$f^* : H^*_{good}(X/W(k)) \xrightarrow{\sim} H^*_{good}(\mathbb{A}^1_X/W(k)).$$

2.6.12. Chern classes. — The cohomology $H^*_{good}(X/W(k))$ has a theory of Chern classes for vector bundles, compatible with rigid and (log) crystalline Chern classes, i.e. for $X \in \mathsf{Var}_k$ and any locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank, there are elements $c_i^{good}(\mathscr{E}) \in H^{2i}_{good}(X/W(k))$ satisfying the usual properties (normalisation, functoriality, Whitney sum formula).

3. What can we do without resolution of singularities?

3.1. Alterations instead of resolutions. — In the previous section it was important to use resolutions of singularities to make varieties smooth and to turn geometric pairs into normal crossing pairs. By de Jong's alteration theorem, this can also be achieved by more general morphisms: alterations. This means that in addition to proper birational morphisms, one also allows finites morphisms. As we will see, finite morphisms don't behave well in positive characteristic, in particular, if one is interested in integral coefficients.

3.1.1. Known results. — Let $X \in \mathsf{Var}_k$.
- — By Nagata, we obtain $(X, \overline{X}) \in \mathsf{Var}_k^{geo}$.
- — By Nakkajima, we obtain a split proper hypercovering $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ by nc-pairs.
- — It is known that $H^*_{rig}(X/K) \cong H^*_{cris}((X_\bullet, \overline{X}_\bullet)/W(k)) \otimes \mathbb{Q}$.

3.1.2. Question. — Is $H^*_{cris}((X_\bullet, \overline{X}_\bullet)/W(k))$ independent of the choice of hypercovering?

No, at least not in general.

3.1.3. Counterexample. — Let $X/\mathbb{F}_p$ be an elliptic curve (then $\overline{X} = X$) and $F : X \to X$ the absolute Frobenius. Consider the associated Čech hypercovering $X'_\bullet \to X$, and set $X_\bullet := (X'_\bullet)_{red}$. Each X_i equals X, $\pi : X_\bullet \to X$ is a split proper hypercovering.

Consider the induced maps

$$F^* : H^1_{cris}(X/W(k)) \xrightarrow{\pi^*} H^1_{cris}(X_\bullet/W(k)) \xrightarrow{H}{}^{1}_{cris}\, (X/W(k))$$

where the second map is the edge map of the spectral sequence

$$E_1^{ij} = H^i_{cris}(X_i/W(k)) \Rightarrow H^1_{cris}(X_\bullet/W(k)).$$

Since $H^1_{cris}(X/W(k))$ has non-trivial slope part, F^* is not an isomorphism. Therefore π^* also not an isomorphism.

This means, that we have to restrict to generically étale hypercoverings.

3.2. Low cohomological degrees. — Here we do have positive results!

- For $H^0_{cris}((X_\bullet, \overline{X}_\bullet)/W(k))$ it is well-known that this is independent of the choice of hypercovering.
- For $H^1_{cris}((X_\bullet, \overline{X}_\bullet)/W(k))$ the independence was shown
  - by Andreatta–Barbieri-Viale for $p \geqslant 3$.
  - by Ertl–Shiho–Sprang for $p \geqslant 2$.

3.3. Counterexamples for higher cohomological degrees. — The above is **not** true for higher cohomological degrees.

3.3.1. Counterexample. — Let $\overline{X} = \mathbb{P}^1_k$, x the coordinate of $\mathbb{A}^1_k \subset \mathbb{P}^1_k$.
For $r \geqslant 1$ choose $a_1, \ldots a_r \in k$ distinct, $n_1, \ldots, n_r$ prime to p.

Let $f : \overline{X}_0 \to \overline{X}$ be the morphism induced by the field extension

$$k(\overline{X}) = k(x) \subseteq k(x)[y]/(y^p - y - \frac{x^{\sum_{i=1}^r n_i}}{\prod_{i=1}^r (x - a_i)^{n_i}}) =: k(\overline{X}_0).$$

This is a finite flat morphism of degree p between proper smooth curves such that $k(\overline{X}_0)/k(\overline{X})$ is a Galois extension with Galois group

$$G = \langle g \rangle \cong \mathbb{Z}/p\mathbb{Z}.$$

Each $P_i = \{x = a_i\}$ is a closed point of $\overline{X}$. The ramification locus of f is $D = \bigcup P_i$.

Let $\overline{X}_i = (\overline{X}_0 \times_{\overline{X}} \cdots \times \overline{X}_0)^{norm} = \coprod^{G^i} \overline{X}_0$ and consider the simplicial scheme

$$\overline{X}_\bullet \to X.$$

With $X := \overline{X} \backslash D$ and $X_\bullet = X \times_{\overline{X}} \overline{X}_\bullet$ we obtain a split proper generically étale hypercovering

$$(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X}).$$

Then

$$H^2_{cris}((X, \overline{X})/W(k)) \to H^2_{cris}((X_\bullet, \overline{X}_\bullet)/W(k))$$

is not an isomorphism.

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