Research Proposal

Introduction

Mathematics is commonly considered to be the foundation on which science stands. It provides not only a language to describe phenomena that are observed in nature but also tools to develop new technologies.

"Ich behaupte aber, daß in jeder besonderen Naturlehre nur so viel eigentliche Wissenschaft angetroffen werden könne, als darin Mathematik anzutreffen ist.–
I claim that in every natural teaching there can be found only as much science as can be found mathematics."
–Immanuel Kant: Metaphysische Anfangsgründe der Naturwissenschaft, A VIII – (1786)

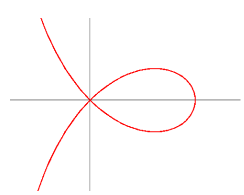
The main stimulus for many mathematicians to study their field of choice however was the aesthetic side of it – the beauty of concepts derived with pure rigour. The quest for a greater understanding of the mathematical universe was and is to this day often a greater force towards the progress of the field as is the necessity for more and more precise mathematical tools in natural sciences such as physics, chemistry and computer science.

On the other hand, an abundance of mathematical tools and theories, make it possible for natural scientists to recognize structures and regularities which they encounter in their research in order to express phenomena in a formal language. It is clear that the deeper the understanding of mathematics is, the greater the possibility that scientists find exactly the theory they need. For example, the formulation of quantum mechanics is based on representation theory of linear groups, general relativity can be considered as a special case of differential geometry and linear algebra is not only used in computer simulations but also in engineering. This is of course only a very small sample out of the numerous sectors of mathematics.

My research interest lies in p-adic Hodge theory, a sector of arithmetic geometry. Arithmetic, or number theory, is one of the oldest and purest branches of mathematics. Early number theorists like the 3rd century mathematician Diophantus of Alexandria, considered polynomial equations such as

$$y^2 = x^3 + x^2$$

and studied their solutions with algebraic methods. Later it became apparent that this problem can also be tackled with geometric methods. The solutions of such equations can be pictured as curves in the plane. The curve $y^2 = x^3 + x^2$ is for example a node of the form



This can be generalised to two dimensional objects, such as spheres or other surfaces, or to even higher dimensional objects that escape the human imagination. Objects given as solutions of polynomial equations are called affine varieties. One of the main objects of algebraic geometry is to understand such varieties and similar objects which can in fact be quite complicated.

We use auxiliary tools to detect pathologies of a variety. Yet different tools provide different results, focus on different properties or apply different criteria and we have to try to reconcile them and therefore understand a possible connection between them.

One method to study a variety X is to look at its sub-varieties – smaller dimensional parts of a variety that are varieties themselves, for example points on a curve or lines on a sphere. These sub-objects are also known as cycles. The data derived by this can be encoded in a mathematical structure called the higher Chow groups $\mathrm{CH}^n(X, i)$ first introduced by Bloch [2] which can be combined to the higher Chow ring. Although it is difficult to compute Chow rings explicitly, they are extremely useful as they encode deep information about the objects studied and produce highly non-trivial results. One advantage of using this tool is that one can manipulate cycles relatively easily, in particular one obtains a multiplication and addition. The structure of the higher Chow ring is an example of a so-called cohomology theory.

Another method to study a variety is to study it locally. This means to look at small neighbourhoods of points. For example if one zooms in close enough on a sphere, we get the impression that it is flat and not curved. This allows us to have flat maps of small parts of the earth although the earth itself is a sphere. Every variety comes with a number of such local data which are called sheaves, and we have to select the appropriate data for our purpose and encode it in a mathematical theory which is easier to handle. This is done via another type of cohomology theory, namely sheaf cohomology $\mathrm{H}^n(X, \mathscr{F})$.

In both cases described here, one obtains a similar mathematical structure with addition and multiplication, which we call a ring. However as they are derived by different methods, it is natural to enquire about their relation. This question of equivalences is the main question that I address here.

Current research

Ideally one wants to find a relationship that can be described via a set of predictable rules, that respects natural structures and is unique. Such a theory is called a theory of higher cycle classes. It is a dictionary

$$\eta^{in} : CH^i(X, n) \to \mathrm{H}^{2i-n}(X, \mathscr{F})$$

that allows to pass from Chow groups into appropriate sheaf cohomology groups.

There are two main aspects to bear in mind when contemplating this question:

- What kind of varieties X do we want to study? This will have an impact on what sheaf cohomology theory we choose as the target of the dictionary η^{in}. The basic set-up is to require the varieties to be smooth, which is a technical way to require it to have only reasonable irregularities. One has then to distinguish between proper and open varieties. In other words, between varieties which are "bounded", for example a sphere, and such ones which are "unbounded", such as a plane. In the first case, sheaves can be controlled much more easily, whereas in the second case, one has to find a way to control them artificially.

- What coefficients should the Chow groups and the sheaf cohomology groups related by the dictionary have? A priori, in both cases the objects occurring are terms with integer coefficients – coefficients in $\mathbb{Z}$. If we allow rational coefficients – coefficients in $\mathbb{Q}$ – we gain more flexibility to manipulate the objects, but we do so at the loss of information.

The state of the art at the moment is that there are theories of cycle classes which are either suited for the study of open and proper varieties at the same time, but only with rational and not with integral coefficients **or** theories with integral coefficients but only suitable for proper varieties. In other words, at the moment we lack an integral theory of higher cycle classes for open varieties.

Denis Petrequin solved the question for open and proper varieties with rational coefficients by defining so-called rigid cycle classes [8]. The name comes from the rigid cohomology groups which are the target for Petrequin's classes and were introduced by Berthelot in [1] as a rational cohomology theory for smooth varieties. Michel Gros constructed in [4] integral cycle classes into crystalline cohomology groups which are in the case of proper varieties compatible with Petrequin's cycle classes. Crystalline cohomology can be calculated as sheaf cohomology using Illusie's de Rham-Witt complex [5]. In both approaches, the authors first define Chern classes, i.e. they establish a dictionary between K-theory groups, which can be seen as a part of the Chow groups, and their cohomology groups of choice, that is rigid or crystalline cohomology respectively.

It thus stands to reason to use a similar approach in the case of integral cycle classes for open varieties. An appropriate cohomology theory to use is the overconvergent cohomology of Davis, Langer and Zink [3] which was shown by them to be a good integral model for rigid cohomology in the case of an open variety. Incidentally, it is calculated by a sub-complex of the de Rham-Witt complex used in crystalline cohomology. In fact, we were able to construct integral Chern classes from K-theory groups into overconvergent cohomology groups

$$c_{ij}^{\mathrm{sc}} : K_j(X) \to \mathrm{H}^{2i-j}(X, W^{\dagger}\Omega).$$

This indicates that it should be possible to construct integral cycle classes η^{in} as described above for open varieties, but new techniques will be necessary since there are obstacles that did not exist in the previous cases.

Under the supervision of Wiesława Nizioł, Professor of Mathematics at the University of Utah, I plan to work on this question. The reason I am interested in this is that it could ultimately lead to a generalisation of important comparison theorems in p-adic Hodge theory, or more precisely of uniqueness statements concerning them [7].

References

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