My research interest lies in number theory, more precisely in a branch of it called arithmetic geometry. I specialise in algebraic K-theory and p-adic Hodge theory. In the coming year I plan to advance some useful techniques in this subject. This project is supervised by Wiesława Nizioł, currently Professor in the Mathematics Department of the University of Utah.

Dating back to 2000 B.C. number theory often is considered to be the foundation on which the whole mathematical building stands. As such it occupies among the mathematical disciplines an idealised position, similar to the one that mathematics holds among the sciences. Over the centuries many different subjects evolved from it, so that modern number theory has strong ties to such divers fields as complex and real (functional) analysis, (algebraic) topology, algebraic geometry, representation theory and commutative algebra. Indeed, in its ways of reasoning number theory relies strongly on these mentioned areas, however under no obligation to serve needs that do not origin within itself, it is essentially autonomous in setting its goals and thus manages to protect its undisturbed harmony.

In some sense, p-adic Hodge theory gives a connection between subjects in representation theory and algebraic geometry. More precisely, it provides a way to study p-adic Galois representations of local fields of characteristic 0 via decompositions of p-adic cohomology theories.

1 p-adic Representations

Consider a **local field of characteristic** 0. An example would be the p-adic numbers $\mathbb{Q}_p$, for a prime number p, roughly speaking formal power series of the form

$$\sum_{i=-m}^{\infty} a_i p^i$$

where the coeffiecients a_i take values in $\{0, 1, \dots, p-1\}$. Taking the algebraic closure $\overline{K}$ of K, we can consider the **absolute Galois group** of K

$$G_K = \mathrm{Gal}(\overline{K}/K),$$

that is the group of automorphisms of $\overline{K}$ that fixes K. In order to study such groups, one looks at their action on vector spaces, in other words, one studies homomorphisms of the form

$$\rho : G_K \to \mathrm{Aut}(V),$$

where V is a vector space. This is called a **Galois representation**. Many objects arising in number theory are naturally Galois representations. As the name suggests, in p-adic Hodge theory, one studies Galois representations, where V is a finite dimensional vector space over the p-adic numbers. In addition one asks that the representation homomorphisms be continuous. The category of all such representations is denoted by

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K),$$

and p-adic Hodge theory provides a classification in subcategories depending on desirable properties of representations. One has for example crystalline, semi-stable, Hodge-Tate and deRham representations.

In order to understand p-adic Galois representations, one can ask where do they appear naturally and what does this tell us about a possible classification.

2 Cohomology Theory

In algebraic geometry one studies varieties (or more general schemes) which can be thought of as solutions for polynomial equations over a base field. One of the major objectives of arithmetic algebraic geometry is to understand schemes over p-adic fields.

To this end we use auxiliary tools to probe the structure and detect pathologies of schemes. These tools used for probing the structure of a mathematical object can be of different nature such as topological, algebraic or geometric. They are functors which applied to an object give invariants of it characterising major properties. Among them we want to point out **cohomology theories**. In our context, a cohomology theory $H^\bullet$ is a functor that associates to a variety or a scheme a graded algebra satisfying fundamental axioms inspired by topological techniques. According to these axioms a cohomology theory should reflect basic geometric properties of a scheme such as its dimension. Depending on the ultimate goal and object to be studied, different cohomology theories have been developed. We are particularly interested in p-adic cohomology theories, i.e. with coefficients in the p-adic numbers.

Let X be a scheme over the characteristic 0 local field K considered earlier. If we choose the "right" cohomology theory, the graded pieces $H^i(X)$ are finite dimensional p-adic vector spaces coming in addition with an action of G_K on them – they seem thus ideal for our purpose.

3 Period Morphisms

The general strategy is to consider so called **Dieudonné modules**, covariant functors

$$D : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Vec}_K,$$

that associates to a p-adic Galois representation a finite-dimensional (graded or filtered) vector space. They are constructed using **period rings** B with a G_K-action, originally introduced by Fontaine, i.e. for a p-adic representation V set $D(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}$, the elements fixed by G_K. This object obviously doesn't have G_K-action any more, but inherites linear algebraic structures from B. It is clear that $\dim_K D(V) \leq \dim_{\mathbb{Q}_p} V$ and there is a canonical comparison morphism

$$\alpha_V : B \otimes D(V) \to B \otimes V.$$

If this morphism is an isomorphism, we say that V is **B-admissible**.

p-adic Galois representations are classified based on the criterion if they are B-admissible for a certain period ring B. For Galois representations coming from different cohomology theories the following period isomorphisms have been established:

The classical case. For a proper and smooth scheme over the complex numbers $\mathbb{C}$ there is a comparison isomorphism between the algebraic **deRham cohomology** of X over $\mathbb{C}$ and the singular cohomology of $X(\mathbb{C})$, in this case $B = \mathbb{C}$,

$$H^\bullet_{\mathrm{dR}}(X/\mathbb{C}) \cong H^\bullet(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}\,.$$

Hodge-Tate representations. Let X be a proper smooth scheme over K and let $\mathbb{C}_K$ denote the completion of the algebraic closure of K which has an action of G_K and $C_K(i)$ the same field where now G_K acts with a twist. Now set $B_{HT} = \oplus_{i \in \mathbb{Z}} \mathbb{C}_K(i)$. Faltings established the following comparison isomorphism in [4] for Hodge cohomology and étale cohomology

$$B_{HT} \otimes_K H^\bullet_{\mathrm{Hodge}}(X/K) \cong B_{HT} \otimes_{\mathbb{Q}_p} H^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p).$$

The deRham period map. For a proper smooth scheme X a period isomorphism for deRham and étale cohomology

$$B_{dR} \otimes_{\overline{K}} H^\bullet_{dR}(X) \cong B_{dR} \otimes_{\mathbb{Q}_p} H^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p),$$

has been established in different ways by Faltings, Nizioł, Tsuji and Beilinson. The non-proper case has been treated by Yamashita.

The crystalline period map. Similar to the above case, there is a period isomorphism for crystalline and étale cohomology

$$B_{cr} \otimes H^\bullet_{cr}(X_0/W(k)) \cong B_{cr} \otimes_{\mathbb{Q}_p} H^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p),$$

where k is the residue field of K and $W(k)$ its ring of Witt vectors.

4 A uniqueness criterion

Since there are several different methods to construct such period isomorphisms, the natural question arises, wether they all define the same morphism, in other words how unique such a period morphism is.

Nizioł established in [9] a criterion to check uniquenes. This method is based on yet another invariant of schemes, K**-theory**. Although cohomology theory comes up more naturally, it is often more powerful and easier to work with K-theory. It is a functor that associates to a scheme X a family of (commutative) groups $\{K_i(X)\}_i \in \mathbb{N}_0$. In lower degrees, there are explicit algebraic definitions of K-groups, beginning with the Grothendieck group of vector bundles as K_0, continuing with H. Bass's definition of K_1 motivated by questions in geometric topology and including J. Milnor's definition of K_2 arising from algebraic number theory. It is therefore desirable to find a connection between these two functors. It is known for some cases that they are related via **Chern classes**, group homomorphisms from K-theory to cohomology

$$c_{ij} : K_j(X) \to H^{2i-j}(X).$$

The crystalline period morphisms are often conjectured to preserve Chern classes of vector bundles. As Nizioł shows it turns out that if one postulates instead that the period morphism should be compatible with higher Chern classes that there is indeed a unique period morphism. Nizioł establishes the uniqueness of the crystalline period morphism in three of the above mentioned cases.

My project is concerned with constructing Chern classes for Beilinson's construction of the crystalline period isomorphism in [2] in order to show that it satisfies as well a uniqueness criterion and is therefore compatible with the other constructions of the period map.

References

[1] Beilinson, Alexander: *p-adic periods and derived deRham cohomology.* arXiv:1102.1294v4 [math.AG], (2011).

[2] Beilinson, Alexander: *On the crystalline period map.* arXiv:1111.3316v1 [math.AG], (2011).

[3] Brinon, Olivier; Conrad, Brian: *CMI Summer School notes on p-adic Hodge theory*, (2009).

[4] Faltings, Gerd: *p-adic Hodge theory*, Journal of the American Mathematical Society 1 (1), (1988).

[5] Faltings, Gerd: *Crystalline cohomology and p-adic Galois representations*, in Igusa, Jun-Ichi: *Algebraic analysis, geometry, and number theory*, Baltimore, Johns Hopkins University Press, pp. 25-80, (1989).

[6] Kahn, Bruno: *Algebraic K-theory, Algebraic Cycles and Arithmetic Geometry.* In Friedlander, Eric M.; Grayson, Daniel R. (editors): *Handbook of K-theory.* Springer-Verlag, Berlin Heidelberg, (2005).

[7] Neukirch, Jürgen: *Algebraische Zahlentheorie.* Springer-Verlag, Berlin (1992).

[8] Nizioł, Wiesława: *Semistable conjecture via K-theory.* Duke Math. J. 141, no.1, 151-178, (2008).

[9] Nizioł, Wiesława: *On uniqueness of p-adic period morphisms.* Pure Appl. Math. Q. 5, no.1 163-212, (2009).

[10] Tsuji, Takeshi: *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case.* Inv. Math. 137, 233-411, (1999).

[11] Yamashita, Go: *p-adic étale cohomology and crystalline cohomology for open varietieswith semi-stablereduction.* http://www.kurims.kyoto-u.ac.jp/~gokun/myworks.html