A Rigid Analytic Approach to Hyodo–Kato Theory

Additional Material

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Local construction of our Hyodo–Kato theory

$$
\begin{aligned}
Y &: \text{ semistabel over } k^0 = (\operatorname{Spec} k, 1 \mapsto 0)\\
\mathcal{Z} &: \text{ lift to } \mathcal{S} = (\operatorname{Spwf} W(k)[\![s]\!], 1 \mapsto s) \text{ with a lift of Frobenius}\\
&\Rightarrow \text{ log smooth over } W^{\varnothing} = (\operatorname{Spec} W(k), \mathrm{triv})\\
\mathcal{X} &:= \mathcal{Z} \times V^{\sharp}, \quad V^{\sharp} = (\operatorname{Spec} V, \mathrm{can})\\
\mathcal{Y} &:= \mathcal{Z} \times W(k)^0, \quad W(k)^0 = (\operatorname{Spec} W(k), 1 \mapsto 0)
\end{aligned}
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$\mathfrak{Z}, \mathfrak{X}, \mathfrak{Y}$: associated dagger spaces

Consider the CDGAs

$$\omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}, \qquad \widetilde{\omega}^\bullet_{\mathcal{Y},\mathbb{Q}} := \omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \mathcal{O}_{\mathfrak{Y}}, \qquad \omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u], \qquad \widetilde{\omega}^\bullet_{\mathcal{Y},\mathbb{Q}}[u]$$

degree 0 elements $u^{[i]}$ such that $du^{[i+1]} = -d\log s \cdot u^{[i]}$ and $u^{[0]} = 1$.

- Multiplication: $u^{[i]} \wedge u^{[j]} = \frac{(i+j)!}{i!j!} u^{[i+j]}$
- Frobenius action: $\phi(u^{[i]}) = p^i u^{[i]}$
- Monodromy: $\mathcal{O}$-linear morphism defined by $N(u^{[i]}) = u^{[i-1]}$

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Definition (Rigid Hyodo–Kato cohomology)

$R\Gamma_{\mathrm{HK}}(Y) := R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u])$ with endomorphisms φ and N satisfying $N\varphi = p\varphi N$.

$$\begin{array}{ccccc} R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u]) & \longrightarrow & R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[\![u]\!]) & \xrightarrow[u^{[i]}\mapsto 0]{\sim} & R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/\mathcal{S},\mathbb{Q}}) \\ \downarrow \wr & & \downarrow & & \downarrow \\ R\Gamma(\mathfrak{Y}, \widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[u]) & \xrightarrow{\sim} & R\Gamma(\mathfrak{Y}, \widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[\![u]\!]) & \xrightarrow[u^{[i]}\mapsto 0]{\sim} & \Gamma(\mathfrak{Y}, \omega^{\bullet}_{\mathcal{Y}/W^{0},\mathbb{Q}}) \end{array}$$

Set $R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X},\pi) := R\Gamma_{\mathrm{HK}}(Y)$ and $R\Gamma_{\mathrm{dR}}(\mathcal{X}) := R\Gamma(\mathfrak{X}, \omega^{\bullet}_{\mathcal{X}/V^{\sharp},\mathbb{Q}})$.

Definition (Rigid Hyodo–Kato morphism)

For a uniformiser $\pi \in V$ and $q \in \mathfrak{m}\backslash\{0\}$ we define

$$\Psi_{\pi,q} : R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X},\pi) \to R\Gamma_{\mathrm{dR}}(\mathcal{X})$$

by $\omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}} \to \omega^{\bullet}_{\mathcal{Z}/\mathcal{S},\mathbb{Q}} \to \omega^{\bullet}_{\mathcal{X}/V^{\sharp},\mathbb{Q}}$ and $\Psi_{\pi,q}(u^{[i]}) := \frac{(-\log_q(\pi))^i}{i!}$.

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p-adic logarithm

We have a decomposition: $V^\times = \mu(1+\mathfrak{m})$ where $\mu \subset V^\times$ are the $|k^\times|$th roots of unity in K.
log : $V^\times \to V$ defined by

$$\log(v) := -\sum_{n \geqslant 1} \frac{(1-v)^n}{n} \text{ for } v \in (1+\mathfrak{m}),$$

$$\log(u) := 0 \text{ for } u \in \mu$$

A branch of the p-adic logarithm on K is a group homomorphism $K^\times K$ whose restriction to $V^\times$ is log.

For $q \in \mathfrak{m} \setminus \{0\}$: $\log_q : K^\times K$ uniquely defined by $\log_q(q) = 0$.

For a uniformiser π write $q = \pi^m v$ with $m \geqslant 1$ and $v \in V^\times$.
Then $\log_q$ is defined by $\log_q(\pi) := -m^{-1}\log(v)$.

Any branch of the p-adic logartihm can be written in this way.

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Example: Hyodo–Kato theory for Tate curves

$$\mathcal{Z}_n := \operatorname{Spwf} W[\![s]\!][v_n, w_n]^\dagger/(v_n w_n - s) \quad \text{for } n \in \mathbb{Z} \text{ with log str:}$$

$$\mathbb{N}^2 \to W[\![s]\!][v_n, w_n]^\dagger/(v_n w_n - s); (1,0) \mapsto v_n, (0,1) \mapsto w_n$$

$$\mathcal{Z}_n = \operatorname{Spwf} W[\![s]\!][v_n, w_n]^\dagger/(v_n w_n - s) = \operatorname{Spwf} W[\![s]\!][\frac{t}{s^{n-1}}, \frac{s^n}{t}]^\dagger$$

$$\text{for } t := s^{n-1} v_n = \frac{s^n}{w_n}$$

$\mathcal{Z}^{(r)}$: glue $\mathcal{Z}_1, \ldots, \mathcal{Z}_r$ naturally (along certain open subsets)

$\mathfrak{Z}^{(r)}$: the associated dagger space

$Y^{(r)} \hookrightarrow \mathcal{Z}^{(r)}$ exact close immersion defined by (p, s)

$\phi^{(r)} : \mathcal{Z}^{(r)} \to \mathcal{Z}^{(r)}$ Frobenius defined by $v_n \mapsto v_n^p$, $w_n \mapsto w_n^p$

$\mathcal{X} := \mathcal{Z}^{(r)} \times_{\mathcal{S}, s \mapsto \pi} V^\sharp$

$\mathfrak{X}$: the associated dagger space

- $Y^{(r)}$ is strictly semistable for $r \geqslant 2$, nodal curve for $r = 1$.
- For $a \in \overline{\mathbb{Q}}_p$ with $|a| < 1$, the fibre at $s = a$ in $\mathfrak{Z}^{(r)}$ is the Tate curve over $F(a)$ with period a^r:

$$\mathfrak{Z}^{(r)} \times_{\mathfrak{S}, s \mapsto a} \mathrm{Sp} F(a) \cong F(a)^\times / a^{r\mathbb{Z}}, \quad v_1 \mapsto t \text{ canonical parameter}$$

- $\mathfrak{X}$ is the Tate curve over K with period π^r.

Rigid Hyodo–Kato cohomology $R\Gamma_{\mathrm{HK}}(Y^{(r)})$:
Computed by the ordered Čech complex $\check{C}^\bullet_{\mathrm{HK}}$ of $\omega^\bullet_{\mathcal{Z}^{(r)}/W^\varnothing, \mathbb{Q}}[u]$ associated to the covering $\{\mathcal{Z}_n\}_{n=1}^r$ of $\mathcal{Z}^{(r)}$.

de Rham cohomology $R\Gamma_{\mathrm{dR}}(\mathcal{X}) = R\Gamma(\mathfrak{X}, \omega^\bullet_{\mathcal{X}/V^\sharp, \mathbb{Q}}) = R\Gamma(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}})$:
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Explicitely

Hyodo–Kato cohomology:

$H^1_{\mathrm{HK}}(\mathcal{X}) = H^1_{\mathrm{HK}}(Y^{(r)}) \cong Fe_1^{\mathrm{HK}} \oplus Fe_2^{\mathrm{HK}}$

$\varphi(e_1^{\mathrm{HK}}) = e_1^{\mathrm{HK}}$, $N(e_1^{\mathrm{HK}}) = 0$, $\varphi(e_2^{\mathrm{HK}}) = pe_2^{\mathrm{HK}}$, and $N(e_2^{\mathrm{HK}}) = re_1^{\mathrm{HK}}$

e_1^{HK} and e_2^{HK} represented by the cocycles

$$(0, \ldots, 0, 1) \in \prod_{n=1}^{r} \Gamma(\mathfrak{W}_n, \mathcal{O}_{\mathfrak{W}_n}) \subset \check{C}^1_{\mathrm{HK}},$$

$$(d\log w_1, \ldots, d\log w_r) + (-u^{[1]}, \ldots, -u^{[1]}, u^{[1]}) \in$$
$$\prod_{n=1}^{r} \Gamma(\mathfrak{Z}_n, \omega^1_{\mathcal{Z}_n/W^{\varnothing},\mathbb{Q}}) \oplus \prod_{n=1}^{r} \Gamma(\mathfrak{W}_n, \mathcal{O}_{\mathfrak{W}_n} u^{[1]}) \subset \check{C}^1_{\mathrm{HK}}$$

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$H^1_{\mathrm{dR}}(\mathcal{X}) \cong Ke_1^{\mathrm{dR}} \oplus Ke_2^{\mathrm{dR}}$ with the Hodge filtration

$$F^p H^1_{\mathrm{dR}}(\mathcal{X}) = \begin{cases} Ke_1^{\mathrm{dR}} \oplus Ke_2^{\mathrm{dR}} & \text{if } p \leq 0, \\ Ke_2^{\mathrm{dR}} & \text{if } p = 1, \\ 0 & \text{if } p \geq 2, \end{cases}$$

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Hyodo–Kato map: $\Psi_q = \Psi_{\pi,q} : H^1_{\mathrm{HK}}(\mathcal{X}) \to H^1_{\mathrm{dR}}(\mathcal{X})$ for $q \in \mathfrak{m} \backslash \{0\}$, given by

$$\Psi_q(e_1^{\mathrm{HK}}) = e_1^{\mathrm{dR}} \qquad \text{and} \qquad \Psi_q(e_2^{\mathrm{HK}}) = e_2^{\mathrm{dR}} - r \log_q(\pi) e_1^{\mathrm{dR}}.$$

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Tate module of an elliptic curve E

$E/\mathbb{Q}$ an elliptic curve, $E(\mathbb{Q}_p) := \{(x,y) \in \mathbb{Q}_p^2 \mid y^2 = f(x)\} \cup \{\infty\}$ with a separable polynomial f with $\deg(f) = 3$, and $O := \infty$.

$\Rightarrow G_{\mathbb{Q}_p}$ acts on E by acting on its homogenous coordinates.

The Tate-module is a free $\mathbb{Z}_p$-module of rank 2 defined as

$$T_p(E) := \varprojlim E_{p^n}(\overline{\mathbb{Q}}_p)$$

where $E_{p^n}(\overline{\mathbb{Q}}_p)$ are the p^n-torsion points $\{P \in E(\overline{\mathbb{Q}}_p) \mid p^n P = O\}$.

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with a continuous action of $G_{\mathbb{Q}_p}$ and $\dim_{\mathbb{Q}_p} V_p(E) = 2$.

Realisation via étale cohomology of E: $H^i_{\text{et}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) = \bigwedge^i_{\mathbb{Q}_p} V_p(E)^*$.

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Tate module of an elliptic curve E

$E/\mathbb{Q}$ an elliptic curve, $E(\mathbb{Q}_p) := \{(x, y) \in \mathbb{Q}_p^2 \,\big|\, y^2 = f(x)\} \cup \{\infty\}$ with a separable polynomial f with $\deg(f) = 3$, and $O := \infty$.

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What is a period?

$Z/\mathbb{Q}$: smooth, projective algebraic variety.

The complex numbers which appear in the image of the pairing

$$H^i_{\mathrm{dR}}(Z_{\mathbb{C}}) \times H_i(Z(\mathbb{C}), \mathbb{C}) \to \mathbb{C}, (\omega, \gamma) \mapsto \int_\gamma \omega.$$

are called periods of Z.

Example

For $G_m := \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$ we have $G_{m.\mathbb{C}} = \mathbb{C}^\times$ the complex plane without 0.
Then

$$H^1_{\mathrm{dR}}(G_{m,\mathbb{C}}) = \mathbb{C}\frac{dt}{t} \quad \text{and} \quad H_1(G_m(\mathbb{C}), \mathbb{C}) = \mathbb{C}S^1.$$

$\int_{S^1} \frac{\mathrm{d}t}{z} = 2\pi i$ is a period for G_m.
The comparison isomorphim

$$H^1(G_m(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} H^1_{\mathrm{dR}}(G_{m,\mathbb{C}}), \quad (S^1)^* \mapsto \left(\int_{S^1} \frac{\mathrm{d}t}{z}\right)^{-1} \frac{dt}{t} = \frac{1}{2\pi i} \frac{dt}{t}.$$

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Why there is no analogue of $2\pi i$ in $\mathbb{C}_p$

An analogue t of $2\pi i$ should be a period of $\mathbb{P}^1$.

But $H^2_{\mathrm{et}}(\mathbb{P}^1_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)^* \cong \mathbb{Q}_p(1)$ so

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So we should have $g(t) = \chi(g) \cdot t$ for all $g \in G_{\mathbb{Q}_p}$.

Fontain (1966):

$$\left\{ x \in \mathbb{C}_p \;\middle|\; g(x) = \chi(g) \cdot x \forall g \in G_{\mathbb{Q}_p} \right\} = 0$$

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How to obtain a manifold from a smooth variety

The GAGA principle

The geometry of projective complex analytic spaces is equivalent to the geometry of projective complex varieties.

X : a complex algebraic variety.
Defined by zero loci of polynomials (locally).

$X(\mathbb{C})$: set of complex points of X.
Any complex polynomial is a holomorphic function $\Rightarrow$ $X(\mathbb{C})$ is a complex analytic space.

- $X(\mathbb{C})$ is a complex manifold iff X is smooth.
- $X(\mathbb{C})$ is compact iff X is proper.

It is much harder to go the other direction.