Comparison between rigid syntomic and crystalline syntomic cohomology for strictly semistable log schemes with boundary

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Abstract

We introduce rigid syntomic cohomology for strictly semistable log schemes over a complete discrete valuation ring of mixed characteristic $(0, p)$. In case a good compactification exists, we compare this cohomology theory to Nekovář–Nizioł's crystalline syntomic cohomology of the generic fibre. The main ingredients are a modification of Große-Klönne's rigid Hyodo–Kato theory and a generalisation of it for strictly semistable log schemes with boundary.

Résumé

On introduit la cohomologie syntomique rigide pour les schémas logarithmique de réduction strictement semistable sur un anneau de valuation discrète de caractéristique $(0, p)$. Dans le cas de bonne compactification, on compare cette théorie de cohomologie à la cohomologie syntomique cristalline de Nekovář–Nizioł sur la fibre générique. La clé est une modification de la théorie Hyodo–Kato rigide de Große-Klönne et une généralisation de celle-ci aux schémas logarithmique de réduction semistable avec bord.

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Contents

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Introduction

In this article we construct rigid syntomic cohomology for strictly semistable log schemes, and compare it with crystalline syntomic cohomology in the case that there exists a good compactification.

Let K be a p-adic field and $\mathscr{O}_K$ its ring of integers. In general terms, syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X,r)$ is basically defined for semistable (log) schemes X over $\mathscr{O}_K$ and integers $r \in \mathbb{Z}$, as a p-adic analogue of the Deligne–Beilinson cohomology. An important application of syntomic cohomology is, at least conjecturally, the p-adic Beilinson conjecture. It states that the special values of p-adic L-functions are described by the rational part of the special values of L-functions and an arithmetic invariant defined in terms of the p-adic regulator. This conjecture was formulated by Perrin-Riou [37] in the case that X is smooth over $\mathscr{O}_K$. While there is still no precise formulation of such a general conjecture in the semistable case, we may expect a similar picture.

There are two construction of syntomic cohomology. One uses (log) crystalline cohomology, and another uses (log) rigid cohomology of the special fiber of X. We call them crystalline syntomic cohomology and rigid syntomic cohomology respectively. Crystalline syntomic cohomology was originally introduced by Fontaine and Messing [18], and generalized by Kato [28], Nekovář and Nizioł [35] to the logarithmic case. Déglise and Nizioł [14] proved that Nekovář-Nizioł's crystalline syntomic cohomology can be regarded as absolute p-adic Hodge cohomology if the twist r is non-negative. Rigid syntomic cohomology was constructed by Besser [9] for smooth schemes over $\mathscr{O}_K$, and further developed by Bannai [1], as well as Chiarellotto, Ciccioni, and Mazzari [12] as an absolute p-adic Hodge cohomology.

An advantage of rigid syntomic cohomology is that it is purely p-adically analytic. Thus it is useful for computations of p-adic regulators, and should relate directly with p-adic L-functions. Indeed, there are several results concerning the p-adic Beilinson conjecture which use rigid syntomic cohomology [2, 8, 10, 36].

A disadvantage of rigid syntomic cohomology is that the theory of log rigid cohomology often has technical difficulties, since it depends a priori on the choice of local liftings. We address this point in this paper and even construct canonical log rigid complexes analogous to Besser's canonical rigid complexes introduced in [9].

Moreover, the rigid Hyodo–Kato map depends on the choice of a uniformizer of $\mathscr{O}_K$, unlike Beilinson's crystalline Hyodo–Kato map, which was used by Nekovář and Nizioł. Hence their crystalline syntomic cohomology does not depend on such a choice, and moreover extends to a very sophisticated theory for any varieties over K. We remark that the constructions of the crystalline and the rigid Hyodo–Kato map are based on very different techniques. Hence their comparison does not automatically follow from the construction, whereas the comparison of Frobenius and monodromy operator are more straight forward.

Overview of the paper

In Section 1 we construct the rigid Hyodo–Kato complex for strictly semistable log schemes with boundary and log rigid complexes for fine log schemes and log schemes with boundary. In particular, we construct canonical complexes (not only in the derived category) for embeddable objects, in order to clarify the definition of these complexes for simplicial objects and prove important functoriality properties. The notion of log schemes with boundary was introduced in [20] to express compactifications in the sense of log geometry. Although the use of strictly semistable log schemes with boundary makes the construction more involved due to combinatorial difficulties, it pays of as it allows us to compare log rigid and log crystalline cohomology in the appropriate cases.

As mentioned above, in the construction of log rigid cohomology [23] there is usually a choice of local liftings involved. Thus we start in § 1.1 with some technical points which allow us to glue in a canonical way. In § 1.2 we introduce rigid Hyodo–Kato complexes mimicking a construction due to Kim and Hain [29]. This has the advantage that it provides Frobenius and monodromy operator and cup product. We relate this to a construction based on a certain Steenbrink double complex which has a weight filtration and a weight spectral sequence. However, because of the fact that both of these constructions use certain embeddings which behave like admissible liftings, they lack certain functoriality properties. This is why we introduce log rigid complexes for fine log schemes based on a more classical construction in § 1.3. In

§ 1.4 we study how the complexes we constructed previously relate to each other and show functoriality with respect to base change which will be very important for our purposes.

In Section 2 we look at the additional structure of the complexes constructed in Section 1 which is indispensable for the construction of syntomic cohomology. More precisely, we construct in § 2.1 an analogue of the Hyodo–Kato morphism for log rigid cohomology. This is a generalisation of Große-Klönne's construction in [23] to log schemes with boundary. We use a combinatorial modification of his construction to show the functoriality of the rigid Hyodo–Kato map. In § 2.2 we take a closer look at the Frobenius morphism and the monodromy operator. We show first that the Frobenius on the rigid Hyodo–Kato complex, which is induced by local lifts, and the one on the log rigid complexes induced by base change are compatible. Secondly, we show that the monodromy operator on the rigid Hyodo–Kato complex can be interpreted as a boundary map associated to a certain short exact sequence.

Section 3 focuses on syntomic cohomology. In § 3.1 we recall the theory of p-adic Hodge complexes in order to provide some (philosophical) background information on how the different existing syntomic cohomologies fit together. We review crystalline syntomic cohomology in § 3.2 and include some basic constructions concerning crystalline cohomology. In § 3.3 we finally give a definition of rigid syntomic cohomology for strictly semistable log schemes including a cup product on the level of cohomology.

Section 4 is reserved for the comparison of the crystalline syntomic and the rigid syntomic cohomology in the compactifiable case. An essential point is the comparison between log rigid and log crystalline cohomology which is carried out in § 4.1. The comparison of Frobenius, monodromy and Hyodo–Kato morphisms in § 4.2 finally implies the compatibility of the rigid syntomic and the crystalline syntomic cohomology. It follows immediately from the constructions that the cup products on both are compatible.

Notation and conventions

Let $\mathscr{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0, p)$, with fraction field K, perfect residue field k and maximal ideal $\mathfrak{m}_K$. Let π be a uniformiser on $\mathscr{O}_K$. Furthermore, we assume the valuation on K normalised so that $v(p) = 1$. As usual, denote by $\overline{K}$ an algebraic closure of K and by $\mathscr{O}_{\overline{K}}$ the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Let $\mathscr{O}_F = W(k)$ be the ring of Witt vectors of k, F the fraction field of $\mathscr{O}_F$, and F^{nr} its maximal unramified extension. Let e be the absolute remification index of K and denote by $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K and I_K its inertia subgroup. Let σ be the canonical Frobenius on $\mathscr{O}_F$.

Since our construction is based on Große-Klönne's work, weak formal log schemes and dagger spaces will play a crucial part. For an $\mathscr{O}_F$-algebra A, we denote by $\widehat{A}$ the p-adic completion of A. The weak completion of A is the $\mathscr{O}_F$-subalgebra $A^\dagger$ of $\widehat{A}$ consisting of power series $\sum_{(a_1,\dots,a_n)\in\mathbb{N}^n} a_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n}$ for $x_1, \dots, x_n \in A$, where $a_I \in \mathscr{O}_F$, such that there exists a constant $c > 0$ which satisfies the Monsky-Whashnitzer condition

$$c(\mathrm{ord}_p(a_{i_1,\dots,i_n}) + 1) \geq i_1 + \cdots + i_n$$

for any $(i_1, \dots, i_n) \in \mathbb{N}^n$. In other words, $A^\dagger$ is the smallest p-adically saturated subring of $\widehat{A}$ containing A and closed under taking the series $\sum_{i_1,\dots,i_n\geqslant 0} a_{i_1\dots i_n} x_1^{i_1}\cdots x_n^{i_n}$ with $a_{i_1\dots i_n} \in \mathscr{O}_F$ and $x_1, \dots, x_n \in pA^\dagger$. For more details, we refer to [33] and [40] for weakly complete algebras, [31] for weak formal schemes, [19] and [30] for dagger spaces.

We will use a shorthand for certain homotopy limits. Namely, if $f\colon C \to C'$ is a map in the differential graded derived category of abelian groups, we set

$$[\, C \xrightarrow{f} C' \,] := \mathrm{holim}(C \to C' \leftarrow 0).$$

If f is represented by a morphism of complexes, this can be seen as a mapping cone as

$$[\, C \xrightarrow{f} C' \,] \cong \mathrm{Cone}(f)[-1].$$

And we set

$$\begin{bmatrix} C_1 \xrightarrow{\alpha} C_2 \\ \Big\downarrow{\scriptstyle\beta} \qquad \Big\downarrow{\scriptstyle\delta} \\ C_3 \xrightarrow{\gamma} C_4 \end{bmatrix} := [[C_1 \xrightarrow{\alpha} C_2] \xrightarrow{(\beta,\delta)} [C_3 \xrightarrow{\gamma} C_4]],$$

for a commutative diagram (the one inside the large bracket) in the differential graded derived category of abelian groups. Again, if the arrows of the diagram are represented by morphisms of complexes, this can be seen as a double cone as

$$\left[\begin{array}{ccc} C_1 & \xrightarrow{\alpha} & C_2 \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} \\ C_3 & \xrightarrow{\gamma} & C_4 \end{array}\right] \cong \operatorname{Cone}\left(\operatorname{Cone}(C_1 \xrightarrow{(\beta,\alpha)} C_3 \oplus C_2) \xrightarrow{0+\gamma-\delta} C_4\right)[-2]. \tag{0.1}$$

For a ring L, denote by $\mathscr{C}_L^+$ the category of bounded below complexes of L-modules. We denote by $\mathscr{D}_L^+$ the derived dg-category of $\mathscr{C}_L^+$. The complexes we construct will be over F, K, or $F[t]^\dagger$. Denote by $\mathscr{C}_F^+(\varphi, N)$ the category of bounded below complexes of (φ, N)-modules over F. This means, an object in $\mathscr{C}_F^+(\varphi, N)$ is a bounded below complex over F with a σ-semilinear endomorphism φ and an F-linear endomorphism N, such that $N\varphi = p\varphi N$. Denote by $\mathscr{D}_F^+(\varphi, N)$ the derived dg-category of $\mathscr{C}^+(\varphi, N)$.

We assume all schemes to be of finite type. For a scheme $X/\mathscr{O}_K$ denote by X_n for $n \in \mathbf{N}$ its reduction modulo p^n and let X_0 be its special fibre, whearеas X_K is the generic fibre. Furthermore, we denote by

$$i\colon X_0 \hookrightarrow X \quad \text{ and } \quad j\colon X_K \hookrightarrow X$$

the canonical (closed resp. open) immersions.

We will use log structures extensively. We consider all log structures on log schemes as given by a sheaf of monoids with respect to Zariski topology. Thus if we say a log scheme X is fine, it means that Zariski locally X has a chart given by a fine monoid. Note that giving a fine log scheme in our context is equivalent to giving a fine log scheme in the usual sense which is of Zariski type ([39, Cor. 1.1.11]). As we use different log structures, we will make it clear which one we mean in the text. However, we will fix the following notations of log structures on the base. We denote by $\mathscr{O}_K^\varnothing, \mathscr{O}_F^\varnothing, \ldots$ the schemes $\operatorname{Spec}\mathscr{O}_K, \operatorname{Spec}\mathscr{O}_F, \ldots$ with the trivial and by $\mathscr{O}_K^\pi$ the scheme $\operatorname{Spec}\mathscr{O}_K$ with the canonical log structure, i.e. the one associated to the closed point, and finally by $\mathscr{O}_K^0, \mathscr{O}_F^0, \ldots$ the schemes $\operatorname{Spec}\mathscr{O}_K, \operatorname{Spec}\mathscr{O}_F, \ldots$ with the log structure associated to $1 \mapsto 0$. Note that despite our notation the log structures on $\mathscr{O}_K^\pi$ is independent of the choice of π.

Let $\widehat{\mathscr{T}}$, resp. $\mathscr{T}$, be the formal, resp weak formal, log scheme $\operatorname{Spf}\widehat{\mathscr{O}_F[t]}$, resp. $\operatorname{Spwf}\mathscr{O}_F[t]^\dagger$, with the log structure associated to the map $\mathbb{N} \to \mathscr{O}_F[t]^\dagger, 1 \mapsto t$, and T the reduction of $\mathscr{T}$ mod p with the induced log structure. By abuse of notation we denote by $\mathscr{O}_F^0$ (resp. k^0) also the exact closed weak formal log subscheme of $\mathscr{T}$ (resp. T) defined by $t = 0$, and by $\mathscr{O}_K^\pi$ the exact closed weak formal log subscheme of $\mathscr{T}$ defined by $t = \pi$. Unless otherwise stated, we use on $\widehat{\mathscr{T}}$, $\mathscr{T}$, T, $\mathscr{O}_F^0$, $\mathscr{O}_K^\pi$, k^0 the charts $c_{\mathscr{T}}$, c_T, $c_{\mathscr{O}_F^0}$, $c_{\mathscr{O}_K^\pi}$, c_{k^0} induced by the above map. Note that the weak formal log scheme $\mathscr{O}_K^\pi$ is independent of the choice of π, but the exact closed immersion $\mathscr{O}_K^\pi \hookrightarrow \mathscr{T}$ and the chart c_π depend on π. We extend the canonical Frobenius σ on $\mathscr{O}_F$ to $\mathscr{O}_F[t]$ by sending t to t^p. This induces a unique Frobenius on $\mathscr{T}$ which we denote by abuse of notation again by σ. For a weak formal log scheme $\mathscr{X}$ over $\mathscr{O}_F$, we denote by $\widehat{\mathscr{X}}$ the p-adic completion of $\mathscr{X}$, which is a formal log scheme over $\mathscr{O}_F$.

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1 Logarithmic rigid complexes

Große-Klönne showed that (non-logarithmic) rigid cohomology can be computed using certain spaces with overconvergent structure sheaf. This implies immediately important finiteness properties even for non-proper schemes. He showed that this construction can also be carried out for log schemes.

In this section we will recall several different versions of log rigid complexes as introduced by by Große-Klönne in [22]. We use his insight to construct *canonical* log rigid complexes similar to the rigid complexes

Besser considers in the non-logarithmic situation [9]. We hope that using canonical complexes will help us to construct a syntomic regulator map later on.

In the first subsection, we start with some technical preliminaries which will be needed in the subsequent sections to glue local constructions, and to prove functoriality. In § 1.2 we construct a canonical complex for strictly semistable log schemes with boundary, which we call rigid Hyodo–Kato complex, based on a construction by Kim and Hain. It allows for a very explicit definition of Frobenius and monodromy operator. In § 1.3 we define canonical log rigid complexes for more general log schemes. This second construction gives us more flexibility with respect to base change. We compare these complexes and discuss base change properties in § 1.4.

1.1 Preliminaries

We start with a definition and a proposition which contain some topological definitions and facts from [3, § 2.1]. This will be used to glue rigid complexes which are constructed locally.

Definition 1.1. Let $\mathcal{V}$ be an essentially small site. A *base* for $\mathcal{V}$ is a pair $(\mathcal{B}, \theta)$ of an essentially small category $\mathcal{B}$ and a faithful functor $\theta\colon \mathcal{B} \to \mathcal{V}$ which satisfy the following property. For any $V \in \mathcal{V}$ and a finite family of pairs (B_α, f_α) of $B_\alpha \in \mathcal{B}$ and $f_\alpha\colon V \to \theta(B_\alpha)$, there exists a set of objects $B'_\beta \in \mathcal{B}$ and a covering family $\{\theta(B'_\beta) \to V\}$ such that every composition $\theta(B'_\beta) \to V \to \theta(B_\alpha)$ lies in $\mathrm{Hom}(B'_\beta, B_\alpha) \subset \mathrm{Hom}(\theta(B'_\beta), \theta(B_\alpha))$.

We define a covering sieve in $\mathcal{B}$ as a sieve whose image by θ is a covering family in $\mathcal{V}$.

Proposition 1.2. *Covering sieves form a Grothendieck topology in $\mathcal{B}$. Moreover the functor θ is continuous and induces an equivalence of the toposes.*

The following notion will be used to define rigid complexes for simplicial objects.

Definition 1.3. Let $\mathcal{C}_\bullet$ be a simplicial site. For any morphism $\alpha\colon [m] \to [n]$ in the simplex category, let $u_\alpha\colon \mathcal{C}_m \to \mathcal{C}_n$ be the functor which induces the morphism of sites corresponding to α. The *total site* $\mathcal{C}_\bullet^{\mathrm{tot}}$ of $\mathcal{C}_\bullet$ is the site whose objects are pairs (n, U), with $n \in \mathbf{N}$ and $U \in \mathcal{C}_n$, whose morphisms are pairs $(\alpha, f)\colon (n, U) \to (m, V)$ consisting of a morphism $\alpha\colon [m] \to [n]$ in the simplex category and a morphism $f\colon U \to u_\alpha(V)$ in $\mathcal{C}_n$, and whose coverings are families $\{(\mathrm{id}, f)\colon (n, U_i) \to (n, U)\}$ such that $\{U_i \to U\}$ is a covering family in $\mathcal{C}_n$.

Next we recall and introduce some notions for log geometry.

Definition 1.4. For a fine log scheme S, let LS_S be the category of fine log schemes of finite type over S. For a fine weak formal log scheme $\mathcal{S}$, let $\mathsf{LS}_\mathcal{S}$ be the category of fine weak formal log schemes over $\mathcal{S}$. We regard LS_S and $\mathsf{LS}_\mathcal{S}$ as sites for the Zariski topology. Namely a covering family of an object Y in LS_S or $\mathsf{LS}_\mathcal{S}$ is a family of strict open immersions $U_h \hookrightarrow Y$ with $\bigcup_h U_h = Y$.

When we say immersion in the current context, we mean the following definition.

Definition 1.5. A morphism $f\colon X \to Y$ of (weak formal) log schemes is an *immersion* if f can be factored as $f = i \circ j$ by a closed immersion $j\colon X \hookrightarrow Z$ and a strict open immersion $i\colon Z \to Y$. This is clearly equivalent to the condition that f is an immersion on the underlying schemes and the induced morphism of monoids associated to the log structures $f^*\mathcal{M}_Y \to \mathcal{M}_X$ is surjective. We say that an immersion f as above is *strict* if $f^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

Definition 1.6. Let $i\colon Y \hookrightarrow Y'$ be an immersion of fine (weak formal) log schemes. An *exactification* of i is a factorisation $f = h \circ i$ by an exact closed immersion $i\colon Y \hookrightarrow Y''$ and a log étale morphism $h\colon Y'' \to Y'$.

Let S be a fine log scheme over k, $\mathcal{S}$ a fine weak formal log scheme over $\mathcal{O}_F$, and $S \hookrightarrow \mathcal{S}$ a closed immersion over $\mathrm{Spwf}\,\mathcal{O}_F$.

Note that according to the proof of [27, Prop. 4.10 (1)] an exactification of i exists if i admits a chart. In general an exactification of an immersion is not unique. However one can show that the tubular neighbourhoods on exactifications are canonically isomorphic to each other.

Lemma 1.7. *Let $Y \hookrightarrow \mathcal{Y}$ be an immersion of a fine log scheme Y over S into a fine weak formal log scheme $\mathcal{Y}$ over $\mathcal{S}$. Assume that there are exactifications $Y \hookrightarrow \mathcal{Y}_i \to \mathcal{Y}$ for $i = 1, 2$. Then there is a canonical isomorphism $\iota\colon]Y[_{\mathcal{Y}_1} \cong]Y[_{\mathcal{Y}_2}$. Moreover, if we let $\omega^\bullet_{\mathcal{Y}_i/\mathcal{S},\mathbf{Q}}$ for $i = 1, 2$ be the complexe of sheaves on $\mathcal{Y}_{i,\mathbf{Q}}$ given by tensoring the log de Rham complexe of $\mathcal{Y}_i$ over $\mathcal{S}$ with $\mathbf{Q}$, then there is a canonical isomorphism $\iota^*\omega^\bullet_{\mathcal{Y}_2/\mathcal{S},\mathbf{Q}}|_{]Y[_{\mathcal{Y}_2}} \cong \omega^\bullet_{\mathcal{Y}_1/\mathcal{S},\mathbf{Q}}|_{]Y[_{\mathcal{Y}_1}}$.*

Proof. The first statement follows exactly in the same way as [22, Lem. 1.2] by taking an exactification $Y \hookrightarrow \mathcal{Y}' \to \mathcal{Y}_1 \times_{\mathcal{Y}} \mathcal{Y}_2$ of the diagonal embeeding. Since the projection $\mathcal{Y}' \to \mathcal{Y}_i$ is strict on neighbourhoods of Y, the second statement follows, too. □

The above lemma implies that the tubes glue to give a canonical dagger space. Thus the following definition makes sense.

Definition 1.8. Let $i\colon Y \hookrightarrow \mathcal{Y}$ be an immersion of a fine log scheme Y over S into a fine weak formal log scheme $\mathcal{Y}$ over $\mathcal{S}$. We define the log tube $]Y[^{\log}_{\mathcal{Y}}$ of i and a complex of sheaves $\omega^\bullet_{\mathcal{Y}/\mathcal{S},\mathbf{Q}}$ on $]Y[^{\log}_{\mathcal{Y}}$ as follows:

(i) If i admits a chart, there exists an exactification $Y \hookrightarrow \mathcal{Y}' \to \mathcal{Y}$. We set $]Y[^{\log}_{\mathcal{Y}} :=]Y[_{\mathcal{Y}'}$ and let $\omega^\bullet_{\mathcal{Y}/\mathcal{S},\mathbf{Q}}$ be the complex of sheaves on $]Y[^{\log}_{\mathcal{Y}}$ given by tensoring the log de Rham complex of $\mathcal{Y}'$ over $\mathcal{S}$ with $\mathbf{Q}$, which is independent of the choice of an exactification by Lemma 1.7.

(ii) In general, take an open covering $\{U_h\}_h$ of Y and $\{\mathcal{U}_h\}_h$ of $\mathcal{Y}$, such that i induces an immersion $U_h \hookrightarrow \mathcal{U}_h$ which admits a chart for each h. Then we can define $]U_h[^{\log}_{\mathcal{U}_h}$ and $\omega^\bullet_{\mathcal{U}_h/\mathcal{S},\mathbf{Q}}$ as above, and they naturally glue along $]U_h \cap U_{h'}[^{\log}_{\mathcal{U}_h \cap \mathcal{U}_{h'}}$. The dagger space and the complex of sheaves obtained by gluing are independent of the choice of an open covering, and we denote them by $]Y[^{\log}_{\mathcal{Y}}$ and $\omega^\bullet_{\mathcal{Y}/\mathcal{S},\mathbf{Q}}$.

As we will work with log schemes with boundary, we recall in the following definitions taken from [20, Def. 1.5] and [22, 1.6]. For more details see [20].

Definition 1.9. As before, let S be a fine log scheme over k, and $\mathcal{S}$ a fine weak formal log scheme over $\mathcal{O}_F$.

(i) A *log scheme with boundary* is a strict open immersion of log schemes $i\colon (Z, \mathcal{M}_Z) \hookrightarrow (\overline{Z}, \mathcal{M}_{\overline{Z}})$ such that $\mathcal{M}_{\overline{Z}} \to i_*\mathcal{M}_Z$ is injective, $\mathcal{M}^{\mathrm{gp}}_{\overline{Z}} \to (i_*\mathcal{M}_Z)^{\mathrm{gp}}$ is an isomorphism, and Z is schematically dense in $\overline{Z}$. To simplify notations we often write $(Z, \overline{Z})$ for $i\colon (Z, \mathcal{M}_Z) \hookrightarrow (\overline{Z}, \mathcal{M}_{\overline{Z}})$.

(ii) An *S-log scheme with boundary* is a log scheme with boundary $(Z, \overline{Z})$ together with a morphism of log schemes $Z \to S$.

We define the notions of weak formal log schemes with boundary and weak formal $\mathcal{S}$-log schemes with boundary in a similar manner.

Definition 1.10. An S-log scheme with boundary $(Z, \overline{Z})$ is *fine* (resp. *of finite type*) if Z and $\overline{Z}$ are fine (resp. of finite type over k).

We define the notion of fine weak formal $\mathcal{S}$-log schemes with boundary in a similar manner.

Definition 1.11. Let $(Z, \overline{Z})$ a fine S-log scheme with boundary with $Z = (Z, \mathcal{M}_Z)$, $\overline{Z} = (\overline{Y}, \mathcal{M}_{\overline{Z}})$, $j\colon Z \hookrightarrow \overline{Z}$, $f\colon Z \to S$. A *chart* for $(Z, \overline{Z})$ is a triple consisting of a chart $\alpha\colon P_{\overline{Z}} \to \mathcal{M}_{\overline{Z}}$ for $\overline{Z}$, a chart $\beta\colon Q_S \to \mathcal{M}_S$ for S, and a homomorphism $\gamma\colon Q \to P^{\mathrm{gp}}$, such that $\alpha^{\mathrm{gp}} \circ \gamma_{\overline{Z}}\colon Q_{\overline{Z}} \to \mathcal{M}_{\overline{Z}}$ coincides with the composition $Q_{\overline{Z}} \xrightarrow{\beta} j_+f^{-1}\mathcal{M}_S \to j_+\mathcal{M}_Z \to \mathcal{M}^{\mathrm{gp}}_{\overline{Z}}$, where j_+ is the sheaf theoretic push forward, and the third map is given by [20, Lem. 1.3].

If a chart β for S is fixed, we call the pair (α, γ) a chart for $(Z, \overline{Z})$ extending β.

The following contains part of [20, Def. 2.1], as well as slight modification thereof.

Definition 1.12. Let $f\colon (Z, \overline{Z}) \to (Z', \overline{Z}')$ be a morphism of S-log schemes with boundary.

(i) We say that f is a *boundary closed immersion* (resp. *boundary exact closed immersion)* if $\overline{Z} \to \overline{Z}'$ is a closed immersion (resp. an exact closed immersion) of log schemes, and if for any open neighbourhood $U \subset Z'$ of Z, there exists an open neighbourhood $\overline{U} \subset \overline{Z}'$ of $\overline{Z}$ such that U is schematically dense in $\overline{U}$.

(ii) We say that f is a *boundary strict open immersion* if $\overline{Z} \to \overline{Z}'$ is a strict open immersion of log schemes and if $Z = \overline{Y} \times_{\overline{Z}'} Y'$.

(iii) We say that f is a *boundary immersion* if j can be factored as $f = i \circ j$ by a boundary closed immersion $j\colon (Z, \overline{Z}) \to (Y, \overline{Y})$ and a boundary strict open immersion $i\colon (Y, \overline{Y}) \to (Z', \overline{Z}')$.

(iv) We say that f is a *first order thickening* if $\overline{Z} \to \overline{Z}'$ is an exact closed immersion defined by a square zero ideal in $\mathcal{O}_{\overline{Z}'}$.

Definition 1.13. Let $i\colon (Z,\overline{Z}) \hookrightarrow (Z',\overline{Z}')$ be a boundary immersion of S-log schemes with boundary. An *exactification* of i is a factorisation $i = f \circ \iota$ by a boundary exact closed immersion $\iota\colon (Z,\overline{Z}) \hookrightarrow (Y,\overline{Y})$ and a morphism $f\colon (Y,\overline{Y}) \to (Z',\overline{Z}')$ which is log étale as a morphism of log schemes $\overline{Y} \to \overline{Z}'$.

We define the notion of an exactification of a boundary immersion of weak formal $\mathcal{S}$-log schemes with boundary in a similar manner.

Definition 1.14. Let $\overline{\mathsf{LS}}_S$ be the category of fine S-log schemes with boundary of finite type. Let $\overline{\mathsf{LS}}_{\mathcal{S}}$ be the category of fine weak formal $\mathcal{S}$-log schemes with boundary. We regard $\overline{\mathsf{LS}}_S$ and $\overline{\mathsf{LS}}_{\mathcal{S}}$ as sites for the Zariski topology. Namely, a covering family of an object $(Z,\overline{Z})$ in $\overline{\mathsf{LS}}_S$ or $\overline{\mathsf{LS}}_{\mathcal{S}}$ is a family of boundary strict open immersions $(U_h,\overline{U}_h) \hookrightarrow (Z,\overline{Z})$ with $\bigcup_h \overline{U}_h = \overline{Z}$.

The following is [20, Lem. 1.2].

Lemma 1.15. *The categories $\overline{\mathsf{LS}}_S$ and $\overline{\mathsf{LS}}_{\mathcal{S}}$ have finite products. The products in $\overline{\mathsf{LS}}_S$ are given by $(Z,\overline{Z}) \times_T (Z',\overline{Z}') = (Z \times_T Z', \overline{Z}\overline{\times}_T\overline{Z}')$, where $\overline{Z}\overline{\times}_T\overline{Z}'$ is the log schematic image of $Z \times_T Z'$ in $\overline{Z} \times_k \overline{Z}'$. The products in $\overline{\mathsf{LS}}_{\mathcal{S}}$ are given in a similar way.*

Next we recall smoothness for S-log schemes with boundary introduced in [20, Def. 2.1]. It is defined as the infinitesimal lifting property and another condition, which ensures that log rigid cohomology is independent of the choice of local lifts.

Definition 1.16. An S-log scheme with boundary $(Z,\overline{Z})$ of finite type is *smooth* if the following conditions hold:

(i) For any first order thickening $i\colon (Y,\overline{Y}) \hookrightarrow (Y',\overline{Y}')$ and any morphism $f\colon (Y,\overline{Y}) \to (Z,\overline{Z})$, locally on $\overline{Y}'$ there exists a morphism $g\colon (Y',\overline{Y}') \to (Z,\overline{Z})$ such that $f = g \circ i$.

(ii) For any boundary exact closed immersion $i\colon (Y,\overline{Y}) \hookrightarrow (Y',\overline{Y}')$ and any morphism $f\colon (Y,\overline{Y}) \to (Z,\overline{Z})$, locally on $\overline{Y}'\overline{\times}_S\overline{Z}$ there exists an exactification $\overline{Y} \hookrightarrow \overline{X} \to \overline{Y}'\overline{\times}_S\overline{Z}$ of the diagonal embedding such that the projection $\overline{X} \to \overline{Y}'$ is strict and log smooth.

We define the notion of smoothness for weak formal $\mathcal{S}$-log schemes with boundary similarly.

Note that if an S-log scheme with boundary $(Z,\overline{Z})$ is smooth, then Z is automatically log smooth over S. To show the functoriality of log rigid cohomology for a certain class of T-log schemes with boundary, we use a stronger condition of smoothness defined in terms of local charts.

Definition 1.17. A weak formal $\mathcal{T}$-log scheme with boundary $(\mathcal{Z},\overline{\mathcal{Z}})$ is *strongly smooth* if locally on $\overline{\mathcal{Z}}$ there exist elements a,b in a monoid P, a chart $(\alpha\colon P_{\overline{\mathcal{Z}}} \to \mathcal{M}_{\overline{\mathcal{Z}}},\ \beta\colon \mathbb{N} \to P^{\mathrm{gp}})$ of $(\mathcal{Z},\overline{\mathcal{Z}})$ extending $c_{\mathcal{T}}$, satisfying the following conditions.

(i) The extension to groups β^{gp} is injective, and its cokernel is p-torsion free.

(ii) The monoid map β satisfies $\beta(1) = b - a$.

(iii) The morphism of weak formal log schemes $\overline{\mathcal{Z}} \to \operatorname{Spwf} \mathscr{O}_F[P]^\dagger$ is classically smooth.

(iv) There is an isomorphism $\mathcal{Z} \cong \overline{\mathcal{Z}}[\frac{1}{\alpha'(a)}]^\dagger$, where $\alpha'\colon P_{\overline{\mathcal{Z}}} \to \mathcal{O}_{\overline{\mathcal{Z}}}$ is induced by α.

Remark 1.18. In the setting of Definition 1.17, let P' be the submonoid of P^{gp} generated by P and $-a$, and endow $\operatorname{Spwf} \mathscr{O}_F[P']^\dagger$ and $\operatorname{Spwf} \mathscr{O}_F[P]^\dagger$ with the log structures defined by P' and P respectively. The above condition implies that there exists a cartesian diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] & \overline{\mathcal{Z}} \arrow[d] \\ \operatorname{Spwf} \mathscr{O}_F[P']^\dagger \arrow[r] & \operatorname{Spwf} \mathscr{O}_F[P]^\dagger, \end{tikzcd}$$

where the vertical arrows are strict and log smooth.

By [20, Thm. 2.5] the above notion is a generalisation of smoothness.

Proposition 1.19. *Strongly smooth weak formal $\mathcal{T}$-log schemes with boundary are smooth.*

We show several useful properties for strongly smooth weak formal log schemes.

Lemma 1.20. *Let $(\mathcal{Z}, \overline{\mathcal{Z}})$ be a strongly smooth $\mathcal{T}$-log scheme with boundary, and $f\colon \overline{\mathcal{Y}} \to \overline{\mathcal{Z}}$ be a smooth morphism. Then the morphism $\mathcal{Y} := \overline{\mathcal{Y}} \times_{\overline{\mathcal{Z}}} \mathcal{Z} \hookrightarrow \overline{\mathcal{Y}}$ defines a weak formal $\mathcal{T}$-log scheme with boundary, which is strongly smooth.*

Proof. Locally on $\overline{\mathcal{Z}}$ and $\overline{\mathcal{Y}}$, we can take a chart $(\alpha\colon P_{\overline{\mathcal{Z}}} \to \mathcal{M}_{\overline{\mathcal{Z}}},\ \beta\colon \mathbb{N} \to P^{\mathrm{gp}})$ of $(\mathcal{Z}, \overline{\mathcal{Z}})$ and elements $a, b \in P$ as in Definition 1.17, and a chart $(\gamma\colon Q_{\overline{\mathcal{Y}}} \to \mathcal{M}_{\overline{\mathcal{Y}}},\ \delta\colon P \to Q)$ of f extending α such that $\gamma^{\mathrm{gp}}\colon P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is injective with p-torsion free cokernel, and

$$\overline{\mathcal{Y}} \to \overline{\mathcal{Z}} \times_{\mathrm{Spwf}\, \mathscr{O}_F[P]^{\dagger}} \mathrm{Spwf}\, \mathscr{O}_F[Q]^{\dagger}$$

is classically smooth (see also [27, Rem. 3.6]). Let Q' be the submonoid of Q^{gp} generated by Q and $-\delta(a)$. Now we have a Cartesian diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d] & \overline{\mathcal{Y}} \arrow[d] \\
\mathrm{Spwf}\, \mathscr{O}_F[Q']^{\dagger} \arrow[r] & \mathrm{Spwf}\, \mathscr{O}_F[Q]^{\dagger}
\end{tikzcd}$$

in which the vertical arrows are strict and log smooth, and hence classically smooth. Thus $\mathcal{Y} \hookrightarrow \overline{\mathcal{Y}}$ is log schematically dense, and defines a weak formal $\mathcal{T}$-log scheme with boundary. Now one can easily see that $(\gamma\colon Q_{\overline{\mathcal{Y}}} \to \mathcal{M}_{\overline{\mathcal{Y}}},\ \delta^{\mathrm{gp}} \circ \beta\colon \mathbb{N} \to Q^{\mathrm{gp}})$ and $\delta(a), \delta(b) \in Q$ satisfy the desired conditions in Definition 1.17. ☐

It is possible to take products of strongly smooth weak formal log schemes.

Lemma 1.21. *For $i = 1, 2$, let $(\mathcal{Z}_i, \overline{\mathcal{Z}}_i)$ be strongly smooth weak formal $\mathcal{T}$-log schemes with boundary. Then $(\mathcal{Z}_1 \times_{\mathcal{T}} \mathcal{Z}_2, \overline{\mathcal{Z}}_1 \overline{\times}_{\mathcal{T}} \overline{\mathcal{Z}}_2)$ is also strongly smooth.*

Proof. We may assume that there are charts $(\alpha_i\colon P_{\overline{\mathcal{Z}}_i} \to \mathcal{M}_{\overline{\mathcal{Z}}_i},\ \beta_i\colon \mathbb{N} \to P_i^{\mathrm{gp}})$ of $(\mathcal{Z}_i, \overline{\mathcal{Z}}_i)$ and elements $a_i, b_i \in P_i$ as in Definition 1.17. Let Q be the image of

$$P_1 \oplus P_2 \to P_1^{\mathrm{gp}} \oplus_{\mathbb{Z}} P_2^{\mathrm{gp}} = \mathrm{Coker}\left(\epsilon\colon \mathbb{Z} \to P_1^{\mathrm{gp}} \oplus P_2^{\mathrm{gp}},\ n \mapsto (\beta_1^{\mathrm{gp}}(n), -\beta_2^{\mathrm{gp}}(n))\right),$$

and let $c, d \in Q$ be the images of $(a_1, a_2), (b_1, b_2) \in P_1 \oplus P_2$ respectively. Then by the construction of products in $\overline{\mathsf{LS}}_{\mathcal{T}}$ we have a chart $\gamma\colon Q_{\overline{\mathcal{Z}}_1 \overline{\times}_{\mathcal{T}} \overline{\mathcal{Z}}_2} \to \mathcal{M}_{\overline{\mathcal{Z}}_1 \overline{\times}_{\mathcal{T}} \overline{\mathcal{Z}}_2}$ which is compatible with projections to $\overline{\mathcal{Z}}_i$ and α_i, and the induced morphism $\overline{\mathcal{Z}}_1 \overline{\times}_{\mathcal{T}} \overline{\mathcal{Z}}_2 \to \mathrm{Spwf}\, \mathscr{O}_F[Q]^{\dagger}$ is classically smooth. Note that if we let P_i' be the submonid of P_i^{gp} generated by P_i and $-a_i$, then $P_1' \oplus_{\mathbb{N}} P_2'$ is the submonoid of $P_1^{\mathrm{gp}} \oplus_{\mathbb{Z}} P_2^{\mathrm{gp}}$ generated by Q and $-(a_1, a_2)$. Thus we have

$$\mathcal{Z}_1 \times_{\mathcal{T}} \mathcal{Z}_2 = (\overline{\mathcal{Z}}_1 \overline{\times}_{\mathcal{T}} \overline{\mathcal{Z}}_2)[\frac{1}{\gamma'(c)}]^{\dagger},$$

where $\gamma'\colon Q_{\overline{\mathcal{Z}}_1 \overline{\times}_{\mathcal{T}} \overline{\mathcal{Z}}_2} \to \mathcal{O}_{\overline{\mathcal{Z}}_1 \overline{\times}_{\mathcal{T}} \overline{\mathcal{Z}}_2}$ is induced by γ. We define $\delta\colon \mathbb{N} \to Q^{\mathrm{gp}}$ by $1 \mapsto d - c$. From the morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d, "(\beta_1^{\mathrm{gp}}, \beta_2^{\mathrm{gp}})"] & \mathbb{Z} \arrow[r] \arrow[d, "\delta^{\mathrm{gp}}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\epsilon"] & P_1^{\mathrm{gp}} \oplus P_2^{\mathrm{gp}} \arrow[r] & Q^{\mathrm{gp}} \arrow[r] & 0,
\end{tikzcd}$$

where the upper horizontal arrows are given by $m \mapsto (m, -m)$ and $(m, n) \mapsto m + n$, we obtain

$$\mathrm{Ker}\, \delta^{\mathrm{gp}} \cong \mathrm{Ker}\, \alpha_1^{\mathrm{gp}} \oplus \mathrm{Ker}\, \alpha_2^{\mathrm{gp}} \qquad \text{and} \qquad \mathrm{Coker}\, \delta^{\mathrm{gp}} \cong \mathrm{Coker}\, \alpha_1^{\mathrm{gp}} \oplus \mathrm{Coker}\, \alpha_2^{\mathrm{gp}}.$$

This shows that $(\mathcal{Z}_1 \times_{\mathcal{T}} \mathcal{Z}_2, \overline{\mathcal{Z}}_1 \overline{\times}_{\mathcal{T}} \overline{\mathcal{Z}}_2)$ is strongly smooth. ☐

The existence of exactifications in the situation of the following lemma is important.

Lemma 1.22. *Let $(\mathcal{Z}, \overline{\mathcal{Z}}) \hookrightarrow (\mathcal{Z}', \overline{\mathcal{Z}}')$ be a boundary immersion in $\overline{\mathsf{LS}}_{\mathcal{T}}$, and assume that $(\mathcal{Z}, \overline{\mathcal{Z}}')$ is strongly smooth. Then locally on $\overline{\mathcal{Z}}'$ there exists an exactification $(\mathcal{Z}, \overline{\mathcal{Z}}) \hookrightarrow (\mathcal{Y}, \overline{\mathcal{Y}}) \to (\mathcal{Z}', \overline{\mathcal{Z}}')$ such that $(\mathcal{Y}, \overline{\mathcal{Y}})$ is strongly smooth.*

Proof. This follows immediately from Lemma 1.20. □

As in the case of fine log schemes, we have the following.

Lemma 1.23. *Let S be a fine log scheme over k, $\mathcal{S}$ a fine weak formal log scheme over $\mathscr{O}_F$, and $S \hookrightarrow \mathcal{S}$ a closed immersion over $\mathscr{O}_F$. Let $(Z, \overline{Z}) \hookrightarrow (\mathcal{Z}, \overline{\mathcal{Z}})$ be a boundary immersion of a fine S-log scheme with boundary into a fine weak formal $\mathcal{S}$-log scheme with boundary. Assume that there are two exactifications $(Z, \overline{Z}) \hookrightarrow (\mathcal{Z}_i, \overline{\mathcal{Z}}_i) \to (\mathcal{Z}, \overline{\mathcal{Z}})$ for $i = 1, 2$. Let $\iota\colon]\overline{Z}[_{\overline{\mathcal{Z}}_1} \hookrightarrow]\overline{Z}[_{\overline{\mathcal{Z}}_2}$ be the canonical isomorphism from Lemma 1.7. Denote by $\omega^\bullet_{(\mathcal{Z}_i, \overline{\mathcal{Z}}_i)/\mathcal{S}, \mathbf{Q}}$ the complexes of sheaves on $\overline{\mathcal{Z}}_{i,\mathbf{Q}}$ given by tensoring the de Rham complex of $(\mathcal{Z}_i, \overline{\mathcal{Z}}_i)$ over $\mathcal{S}$ as in [20, Def. 1.6] with $\mathbf{Q}$. Then we have a canonical isomorphism $\iota^*\omega^\bullet_{(\mathcal{Z}_2, \overline{\mathcal{Z}}_2)/\mathcal{S}, \mathbf{Q}}|_{]\overline{Z}[_{\overline{\mathcal{Z}}_2}} \cong \omega^\bullet_{(\mathcal{Z}_1, \overline{\mathcal{Z}}_1)/\mathcal{S}, \mathbf{Q}}|_{]\overline{Z}[_{\overline{\mathcal{Z}}_1}}$.*

Proof. We may assume that there exists an exactification $\overline{Z} \hookrightarrow \overline{\mathcal{Z}}' \to \overline{\mathcal{Z}}_1 \times_{\overline{\mathcal{Z}}} \overline{\mathcal{Z}}_2$ of the diagonal embedding. Then $p_i\colon \overline{\mathcal{Z}}' \to \overline{\mathcal{Z}}_i$ is smooth and strict on neighbourhoods of $\overline{Z}$. Since shrinking does not change the tube, we may assume that p_i is strict and smooth. If we set $\mathcal{Z}' := p_1^{-1}(\mathcal{Z}_1) \cap p_2^{-1}(\mathcal{Z}_2)$, it is log schematically dense in $\overline{\mathcal{Z}}'$, as in the proof of [20, Prop. 2.6]. Hence we have morphisms $(\mathcal{Z}', \overline{\mathcal{Z}}') \to (\mathcal{Z}_i, \overline{\mathcal{Z}}_i)$ As in the classical case, we obtain $\iota_i\colon]\overline{Z}[_{\overline{\mathcal{Z}}'} \cong]\overline{Z}[_{\overline{\mathcal{Z}}_i}$ and $\iota_i^*\omega^\bullet_{(\mathcal{Z}_i, \overline{\mathcal{Z}}_i)/\mathcal{S}, \mathbf{Q}}|_{]\overline{Z}[_{\overline{\mathcal{Z}}_i}} \cong \omega^\bullet_{(\mathcal{Z}', \overline{\mathcal{Z}}')/\mathcal{S}, \mathbf{Q}}|_{]\overline{Z}[_{\overline{\mathcal{Z}}'}}$. □

Definition 1.24. Let $(Z, \overline{Z}) \hookrightarrow (\mathcal{Z}, \overline{\mathcal{Z}})$ be a boundary immersion of a fine T-log scheme with boundary into a strongly smooth weak formal $\mathcal{T}$-log scheme with boundary. Since locally on $\overline{\mathcal{Z}}$ there exist exactifications as in Lemma 1.22, we can define the complex of sheaves $\omega^\bullet_{(\mathcal{Z}, \overline{\mathcal{Z}})/\mathcal{S}, \mathbf{Q}}$ on $]\overline{Y}[^{\log}_{\overline{\mathcal{Y}}}$ by the gluing procedure explained in Definition 1.8.

1.2 Rigid Hyodo–Kato complexes

In this section, we introduce canonical complexes with Frobenius, monodromy operator and a weight filtration, which compute Große-Klönne's log rigid cohomology. We link it to another construction based on a certain Steenbrink double complex. At the moment, it is nfortunately only available in the case of strictly semistable log schemes with boundary.

Definition 1.25. (i) A *strictly semistable log scheme with boundary* over k^0 is a k^0-log scheme with boundary $(Y, \overline{Y})$, such that Zariski locally on $\overline{Y}$ there exists a chart $(\alpha\colon P_{\overline{Y}} \to \mathcal{M}_{\overline{Y}},\ \beta\colon \mathbb{N} \to P^{\mathrm{gp}})$ extending c_{k^0} of the following form:

- The monoid P equals $\mathbb{N}^m \oplus \mathbb{N}^n$ for some integers $m \geq 1$ and $n \geq 0$, and β is given by the composition of the diagonal map $\mathbb{N} \to \mathbb{N}^m$ and the canonical injection $\mathbb{N}^m \to \mathbb{Z}^m \oplus \mathbb{Z}^n$. In particular the structure morphism of Y extends to a morphism $\overline{Y} \to k^0$ with a chart $\beta'\colon \mathbb{N} \to \mathbb{N}^m \oplus \mathbb{N}^n = P$.
- The morphism of schemes

$$\overline{Y} \to \operatorname{Spec} k \times_{\operatorname{Spec} k[\mathbb{N}]} \operatorname{Spec} k[\mathbb{N}^m \oplus \mathbb{N}^n] = \operatorname{Spec} k[t_1, \ldots, t_m, s_1, \ldots, s_n]/(t_1 \cdots t_m)$$

induced by β' is smooth, and makes the diagram

$$\begin{tikzcd} \overline{Y} \arrow[r] & \operatorname{Spec} k[t_1, \ldots, t_m, s_1, \ldots, s_n]/(t_1 \cdots t_m) \\ Y \arrow[u] \arrow[r] & \operatorname{Spec} k[t_1, \ldots, t_m, s_1^{\pm 1}, \ldots, s_n^{\pm 1}]/(t_1 \cdots t_m). \arrow[u] \end{tikzcd}$$

Cartesian. We denote the category of strictly semistable log schemes with boundary over k^0 by $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$. In addition we regard $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ as a site with respect to the Zariski topology. Namely a covering family of an object $(Y, \overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ is a family of boundary strict open immesions $(U_h, \overline{U}_h) \hookrightarrow (Y, \overline{Y})$ with $\bigcup_h \overline{U}_h = \overline{Y}$.

(ii) A *strictly semistable weak formal log scheme with boundary* over $\mathcal{T}$ is a weak formal $\mathcal{T}$-log scheme with boundary $(\mathcal{Z}, \overline{\mathcal{Z}})$, such that Zariski locally on $\overline{\mathcal{Z}}$ there exists a chart $(\alpha\colon P_{\overline{\mathcal{Z}}} \to \mathcal{M}_{\overline{\mathcal{Z}}},\ \beta\colon \mathbb{N} \to P^{\mathrm{gp}})$ extending $c_{\mathcal{T}}$ of the following form:

- the monoid P equals $\mathbb{N}^m \oplus \mathbb{N}^n$ for some integers $m \geq 1$ and $n \geq 0$, and β is given by the composition of the diagonal map $\mathbb{N} \to \mathbb{N}^m$ and the canonical injection $\mathbb{N}^m \to \mathbb{Z}^m \oplus \mathbb{Z}^n$. In particular the structure morphism of $\mathcal{Z}$ extends to a morphism $\overline{\mathcal{Z}} \to \mathcal{T}$ with a chart $\beta'\colon \mathbb{N} \to \mathbb{N}^m \oplus \mathbb{N}^n = P$.
- The morphism of weak formal schemes

$$\overline{\mathcal{Z}} \to \operatorname{Spwf} \mathcal{O}_F[\mathbb{N}^m \oplus \mathbb{N}^n]^\dagger = \operatorname{Spwf} \mathcal{O}_F[t_1, \ldots, t_m, s_1, \ldots, s_n]^\dagger$$

induced by β' is smooth, and makes the diagram

$$\begin{tikzcd} \overline{\mathcal{Z}} \arrow[r] & \operatorname{Spwf} \mathcal{O}_F[t_1, \ldots, t_m, s_1, \ldots, s_n]^\dagger \\ \mathcal{Z} \arrow[r] \arrow[u] & \operatorname{Spwf} \mathcal{O}_F[t_1, \ldots, t_m, s_1^{\pm 1}, \ldots, s_n^{\pm 1}]^\dagger. \arrow[u] \end{tikzcd}$$

Cartesian. We denote by $\overline{\mathsf{LS}}^{\mathrm{ss}}_{\mathcal{T}}$ the category of strictly semistable weak formal log schemes with boundary over $\mathcal{T}$.

A strictly semistable log scheme in the sense of [22, § 2.1] can be regarded as a strictly semistable log scheme with boundary with $Y = \overline{Y}$. Similarly, a strictly semistable weak formal log scheme can be regarded as a strictly semistable weak formal log scheme with boundary with $\mathcal{Z} = \overline{\mathcal{Z}}$. Note that for an object $(Y, \overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$, the log schemes Y and $\overline{Y}$ are log smooth over k^0. For an object $(\mathcal{Z}, \overline{\mathcal{Z}})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{\mathcal{T}}$, the weak formal log schemes $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are log smooth over $\mathcal{T}$.

The following definition will make some combinatorical considerations later on easier.

Definition 1.26. (i) Let $(Y, \overline{Y})$ be an object in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ and $D := \overline{Y} \setminus Y$ the reduced closed subscheme, which is a simple normal crossing divisor. A *horizontal component* of D is a Cartier divisor on $\overline{Y}$ whose support is contained in D, and which can not be written as the sum of two non-trivial Cartier divisors. We denote by $\Upsilon_{\overline{Y}}$ the set of irreducible components of $\overline{Y}$, and by Υ_D the set of horizontal components of D. Note that each horizontal component is reduced, and that we have $D = \sum_{\beta \in \Upsilon_D} D_\beta$, where D_β is the Cartier divisor corresponding to β.

(ii) Let $(\mathcal{Z}, \overline{\mathcal{Z}})$ be an object in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{\mathcal{T}}$, and let $\mathcal{D} := \overline{\mathcal{Z}} \setminus \mathcal{Z}$ be the reduced closed weak formal subscheme, which is a relative simple normal crossing divisor over $\mathcal{T}$. Set $\overline{\mathcal{Y}} := \overline{\mathcal{Z}} \times_{\mathcal{T}} \mathcal{O}_F^0$. We denote by $\Upsilon_{\overline{\mathcal{Y}}}$ (resp. $\Upsilon_{\mathcal{D}}$) the set of irreducible components of $\overline{\mathcal{Y}}$ (resp. $\mathcal{D}$).

We introduce now the basic building blocks on which we can construct rigid Hyodo–Kato complexes.

Definition 1.27. (i) A *rigid Hyodo–Kato quadruple* $((Y, \overline{Y}), (\mathcal{Z}, \overline{\mathcal{Z}}), i, \phi)$ consists of an object $(Y, \overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$, an object $(\mathcal{Z}, \overline{\mathcal{Z}})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{\mathcal{T}}$, a strict immersion $i\colon \overline{Y} \hookrightarrow \overline{\mathcal{Z}}$ over $\mathcal{T}$, and an endomorphism $\phi\colon \overline{\mathcal{Z}} \to \overline{\mathcal{Z}}$ and satisfies the following conditions:

- The image of Y under i is a subset of $\mathcal{Z}$. In particular, i induces maps $\Upsilon_{\overline{Y}} \to \Upsilon_{\overline{\mathcal{Y}}}$ and $\Upsilon_D \to \Upsilon_{\mathcal{D}}$, and they are bijective.
- The endomorphism ϕ is a lift of the p-th power Frobenius on the reduction $\overline{Z} := \overline{\mathcal{Z}} \times_{\mathcal{T}} T$, compatible with σ on $\mathcal{T}$, which sends up to units the equations of all irreducible components of $\overline{\mathcal{Y}}$ and $\mathcal{D}$ to their p-th powers.

A morphism $((Y, \overline{Y}), (\mathcal{Z}, \overline{\mathcal{Z}}), i, \phi) \to ((Y', \overline{Y}'), (\mathcal{Z}', \overline{\mathcal{Z}}'), i', \phi')$ of rigid Hyodo–Kato quadruples is a pair (f, F) of morphisms $f\colon (Y, \overline{Y}) \to (Y', \overline{Y}')$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ and $F\colon (\mathcal{Z}, \overline{\mathcal{Z}}) \to (\mathcal{Z}', \overline{\mathcal{Z}}')$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{\mathcal{T}}$ such that $F \circ i = i' \circ f$ and $F \circ \phi = \phi' \circ F'$. We denote by $\mathsf{RQ}_{\mathrm{HK}}$ the category of rigid Hyodo–Kato quadruples.

(ii) A *rigid Hyodo–Kato datum* for an object $(Y, \overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ is a triple $((\mathcal{Z}, \overline{\mathcal{Z}}), i, \phi)$ which makes $((Y, \overline{Y}), (\mathcal{Z}, \overline{\mathcal{Z}}), i, \phi)$ into a rigid Hyodo–Kato quadruple. A morphism of rigid Hyodo–Kato data is a morphism $(id_{(Y,\overline{Y})}, F)$ in $\mathsf{RQ}_{\mathrm{HK}}$. We denote by $\mathsf{RD}_{\mathrm{HK}}(Y, \overline{Y})$ the category of rigid Hyodo–Kato data for $(Y, \overline{Y})$.

(iii) An object $(Y,\overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ is HK-*embeddable* if the category $\mathsf{RD}_{\mathrm{HK}}(Y,\overline{Y})$ is non-empty. We denote by $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$ the full subcategory of $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ consisting of HK-embeddable objects.

The following lemma shows in particular that strictly semistable log schemes with boundary over k^0 are locally HK-embeddable.

Lemma 1.28. *Let $(Y,\overline{Y})$ be an object in $\overline{\mathsf{LS}}^{ss}_{k^0}$ admitting a (global) chart as in Definition 1.25 (i), and assume that $\overline{Y}$ is affine. Then $(Y,\overline{Y})$ is* HK*-embeddable.*

Proof. By [26, Prop. 11.3] there exists a closed immersion $\overline{Y}\hookrightarrow\overline{Z}$ into an affine smooth scheme $\overline{Z}$, which fits to a Cartesian diagram

$$\begin{tikzcd}
\overline{Y} \arrow[r] \arrow[d] & \overline{Z} \arrow[d] \\
\operatorname{Spec} k[t_1,\dots,t_m,s_1,\dots,s_n]/(t_1\cdots t_m) \arrow[r] & \operatorname{Spec} k[t_1,\dots,t_m,s_1,\dots,s_n],
\end{tikzcd}$$

where the vertical arrows are smooth. By [17] we can lift $\overline{Z}$ to an affine smooth $\mathscr{O}_F$-scheme. Let $\overline{\mathcal{Z}}$ be its weak completion. If we take lifts to $\mathcal{O}_{\overline{\mathcal{Z}}}$ of the images of $t_1,\dots,t_m,s_1,\dots,s_n$ in $\mathcal{O}_{\overline{Z}}$, we obtain a smooth morphism $\overline{\mathcal{Z}}\to\operatorname{Spwf}\mathscr{O}_F[t_1,\dots,t_m,s_1,\dots,s_n]^\dagger$. Moreover the infinitesimal lifting property of $\overline{\mathcal{Z}}\to\operatorname{Spwf}\mathscr{O}_F[t_1,\dots,t_m,s_1,\dots,s_n]^\dagger$ implies that there exists a lift to the completion $\hat{\overline{\mathcal{Z}}}$ of $\overline{\mathcal{Z}}$ of the p-th power Frobenius on $\overline{Z}$ which is compatible with σ on $\mathcal{T}$ and which sends t_i and s_j to their p-th powers. Hence by [40, Cor. 2.4.3] we obtain a lift ϕ of the Frobenius to $\overline{\mathcal{Z}}$ of the same form. We endow $\overline{\mathcal{Z}}$ with the log structure defined by the chart $\mathbf{N}^{m+n}\to\mathcal{O}_{\overline{\mathcal{Z}}},(\ell_1,\dots,\ell_{m+n})\mapsto t_1^{\ell_1}\cdots t_m^{\ell_m}s_1^{\ell_{m+1}}\cdots s_n^{\ell_{m+n}}$. Let $\mathcal{D}$ be the closed weak formal subscheme of $\overline{\mathcal{Z}}$ defined by $s_1\cdots s_n=0$, and let $\mathcal{Z}:=\overline{\mathcal{Z}}\setminus\mathcal{D}$. Then $(\mathcal{Z},\overline{\mathcal{Z}})$ and ϕ give a rigid Hyodo–Kato datum for $(Y,\overline{Y})$. ☐

For an object $(\mathcal{Z},\overline{\mathcal{Z}})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{\mathcal{T}}$, we denote by $\overline{\mathcal{Z}}^\infty$ the weak formal log scheme whose underlying weak formal scheme is $\overline{\mathcal{Z}}$ and the log structure is associated to $\mathcal{D}:=\overline{\mathcal{Z}}\setminus\mathcal{Z}$. Let $\overline{\mathcal{Y}}^\infty:=\overline{\mathcal{Z}}^\infty\times_{\operatorname{Spwf}\mathscr{O}_F[t]^\dagger}\operatorname{Spwf}\mathscr{O}_F$ be its exact closed weak formal log subscheme defined by $t=0$. Let

$$\widetilde{\omega}^\bullet_{\overline{\mathcal{Z}}}:=\omega^\bullet_{\overline{\mathcal{Z}}/\mathscr{O}_F^\varnothing},\qquad \widetilde{\omega}^\bullet_{\overline{\mathcal{Z}}^\infty}:=\omega^\bullet_{\overline{\mathcal{Z}}^\infty/\mathscr{O}_F^\varnothing},\qquad\text{and}\qquad \omega^\bullet_{\overline{\mathcal{Y}}}:=\omega^\bullet_{\overline{\mathcal{Y}}/\mathscr{O}_F^0}$$

be the log de Rham complexes of $\overline{\mathcal{Z}}$ over $\mathscr{O}_F^\varnothing$, $\overline{\mathcal{Z}}^\infty$ over $\mathscr{O}_F^\varnothing$, and $\overline{\mathcal{Y}}$ over $\mathscr{O}_F^0$ respectively. Note that in the first two cases we consider $\mathscr{O}_F$ with the trivial log-structure, while in the third case we consider the log-structure associated to $1\mapsto 0$. We set

$$\widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}}:=\widetilde{\omega}^\bullet_{\overline{\mathcal{Z}}}\otimes\mathcal{O}_{\overline{\mathcal{Y}}}.\tag{1.1}$$

Then there is a short exact sequence

$$0\to\omega^\bullet_{\overline{\mathcal{Y}}}[-1]\xrightarrow{\wedge d\log t}\widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}}\to\omega^\bullet_{\overline{\mathcal{Y}}}\to 0.\tag{1.2}$$

Let $\overline{\mathcal{Y}}_{\mathbf{Q}}$ be the generic fiber of $\overline{\mathcal{Y}}$, and $\omega^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ (resp. $\widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$) the complex of sheaves on $\overline{\mathcal{Y}}_{\mathbf{Q}}$ obtained from $\omega^\bullet_{\overline{\mathcal{Y}}}$ (resp. $\widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}}$) by tensoring with $\mathbf{Q}$.

We describe now how to obtain weight filtrations and residue maps. For $j,k\geq 0$ we let

$$P_j\widetilde{\omega}^k_{\overline{\mathcal{Z}}}:=\operatorname{Im}(\widetilde{\omega}^j_{\overline{\mathcal{Z}}}\otimes\widetilde{\omega}^{k-j}_{\overline{\mathcal{Z}}^\infty}\to\widetilde{\omega}^k_{\overline{\mathcal{Z}}})$$

and

$$P_j\widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}}:=P_j\widetilde{\omega}^\bullet_{\overline{\mathcal{Z}}}/(\widetilde{\omega}^\bullet_{\overline{\mathcal{Z}}}\otimes\mathcal{I}_{\overline{\mathcal{Y}}}),$$

where $\mathcal{I}_{\overline{\mathcal{Y}}}$ is the ideal of $\overline{\mathcal{Y}}$ in $\overline{\mathcal{Z}}$.

For a subset $J\subset\Upsilon_{\overline{\mathcal{Y}}}$, taking the residue along $\overline{\mathcal{Y}}_J:=\bigcap_{\alpha\in J}\overline{\mathcal{Y}}_\alpha$, where $\overline{\mathcal{Y}}_\alpha$ is the irreducible component corresponding to α, defines a morphism

$$\operatorname{Res}_J\colon\widetilde{\omega}^k_{\overline{\mathcal{Z}}}\to\widetilde{\omega}^{k-|J|}_{\overline{\mathcal{Z}}}\otimes\mathcal{O}_{\overline{\mathcal{Y}}_J}.$$

Then $\mathrm{Res} := \sum_J \mathrm{Res}_J$ induces an isomorphism

$$\mathrm{Res} : \mathrm{Gr}_j^P \widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \xrightarrow{\sim} \bigoplus_{\substack{J \subset \Upsilon_{\overline{\mathcal{Y}}} \\ |J|=j}} \omega^\bullet_{\overline{\mathcal{Y}}^\infty_{J,\mathbf{Q}}}[-j], \tag{1.3}$$

where $\omega^\bullet_{\overline{\mathcal{Y}}^\infty_{J,\mathbf{Q}}}$ is the complex of sheaves on $\overline{\mathcal{Y}}_{J,\mathbf{Q}}$ obtained from the log de Rham complex of $\overline{\mathcal{Y}}^\infty_J$ over $\mathscr{O}_F$ by tensoring with $\mathbf{Q}$. We define an anti-commutative double complex of sheaves $A^{\bullet,\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ on $\overline{\mathcal{Y}}_{\mathbf{Q}}$ by

$$A^{i,j}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} := \widetilde{\omega}^{i+j+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} / P_j \widetilde{\omega}^{i+j+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$$

with vertical differentials given by

$$A^{i,j}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to A^{i+1,j}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}, \quad (-1)^{j+1} d \colon \widetilde{\omega}^{i+j+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to \widetilde{\omega}^{i+j+2}_{\overline{\mathcal{Y}}_{\mathbf{Q}}},$$

and horizontal differentials given by

$$A^{i,j}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to A^{i,j+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}, \quad \omega \mapsto (-1)^i d\log t \wedge \omega.$$

Let $A^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ be the total complex of $A^{\bullet,\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$. For $k \geq 0$ we set

$$P_k A^{i,j}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} := P_{2j+k+1} \widetilde{\omega}^{i+j+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} / P_j \widetilde{\omega}^{i+j+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}, \tag{1.4}$$

and we let $P_k A^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ be the total complex of $P_k A^{\bullet,\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$.

Lemma 1.29. *The morphism $\widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to A^{\bullet,0}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ defined by $\omega \mapsto \omega \wedge d\log t$ induces a quasi-isomorphism*

$$\omega^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to A^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}}.$$

Proof. Since locally $\overline{\mathcal{Y}}$ is smooth over $\mathrm{Spwf}\, \mathscr{O}_F[t_1, \dots, t_m, s_1, \cdots, s_m]^\dagger / (t_1 \cdots t_m)$, one can show that $\widetilde{\omega}^k_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to A^{k,0}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ induces an isomorphism

$$\omega^k_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \cong \mathrm{Ker}(\widetilde{\omega}^{k+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} / P_0 \widetilde{\omega}^{k+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \xrightarrow{\wedge d\log t} \widetilde{\omega}^{k+2}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} / P_1 \widetilde{\omega}^{k+2}_{\overline{\mathcal{Y}}_{\mathbf{Q}}})$$

for any $k \geq 0$. Thus it suffices to show that the sequence

$$\widetilde{\omega}^k_{\overline{\mathcal{Y}}_{\mathbf{Q}}} / P_0 \widetilde{\omega}^k_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to \widetilde{\omega}^{k+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} / P_1 \widetilde{\omega}^{k+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to \widetilde{\omega}^{k+2}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} / P_2 \widetilde{\omega}^{k+2}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to \cdots$$

is exact. This follows if we show exactness for the sequence

$$\mathrm{Gr}_0^P \widetilde{\omega}^k_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to \mathrm{Gr}_1^P \widetilde{\omega}^{k+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to \mathrm{Gr}_2^P \widetilde{\omega}^{k+2}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to \cdots,$$

which by the isomorphism from (1.3) can be written as

$$\omega^k_{\overline{\mathcal{Y}}^\infty_{\mathbf{Q}}} \to \bigoplus_{\substack{J \subset \Upsilon_{\overline{\mathcal{Y}}} \\ |J|=1}} \omega^k_{\overline{\mathcal{Y}}^\infty_{J,\mathbf{Q}}} \to \bigoplus_{\substack{J \subset \Upsilon_{\overline{\mathcal{Y}}} \\ |J|=2}} \omega^k_{\overline{\mathcal{Y}}^\infty_{J,\mathbf{Q}}} \to \cdots. \tag{1.5}$$

But the map which sends $d\log s_\beta$ to ds_β for all β defines isomorphisms

$$\omega^k_{\overline{\mathcal{Y}}^\infty_{\mathbf{Q}}} \xrightarrow{\cong} \Omega^k_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \qquad \text{and} \qquad \omega^k_{\overline{\mathcal{Y}}^\infty_{J,\mathbf{Q}}} \xrightarrow{\cong} \Omega^k_{\overline{\mathcal{Y}}^\infty_{J,\mathbf{Q}}},$$

where $\Omega^k_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ and $\Omega^k_{\overline{\mathcal{Y}}_{J,\mathbf{Q}}}$ are the sheaves of the non-log differentials. These isomorphisms are compatible with the arrows in (1.5), and the exactness of

$$\Omega^k_{\overline{\mathcal{Y}}^\infty_{\mathbf{Q}}} \to \bigoplus_{\substack{J \subset \Upsilon_{\overline{\mathcal{Y}}} \\ |J|=1}} \Omega^k_{\overline{\mathcal{Y}}_{J,\mathbf{Q}}} \to \bigoplus_{\substack{J \subset \Upsilon_{\overline{\mathcal{Y}}} \\ |J|=2}} \Omega^k_{\overline{\mathcal{Y}}_{J,\mathbf{Q}}} \to \cdots$$

is a classical fact for normal crossing intersections of smooth spaces. $\square$

Definition 1.30. Let ν be the endomorphism of degree $(-1,1)$ on $A^{\bullet,\bullet}_{\overline{Y}_{\mathbf{Q}}}$ such that $(-1)^{i+j+1}\nu$ is the natural projection $A^{i,j}_{\overline{Y}_{\mathbf{Q}}} \to A^{i-1,j+1}_{\overline{Y}_{\mathbf{Q}}}$. We denote the induced endomorphism on $A^{\bullet}_{\overline{Y}_{\mathbf{Q}}}$ by N. If there is a lift of Frobenius ϕ as in Definition 1.27, ϕ induces an endomorphism on $\widetilde{\omega}^{\bullet}_{\overline{Y}_{\mathbf{Q}}}$ preserving the filtration $P_{\bullet}$. Let φ be the endomorphism of degree $(0,0)$ on $A^{\bullet,\bullet}_{\overline{Y}_{\mathbf{Q}}}$ defined by $\frac{\phi}{p^{j+1}}$ on $A^{i,j}_{\overline{Y}_{\mathbf{Q}}}$. Since $\phi(d\log t) = p\cdot d\log t$, we obtain $N\varphi = p\varphi N$. We call φ and N the *Frobenius* and *monodromy operators* respectively.

We will later see that on the level of cohomology the monodromy operator that we just defined can be understood as the boundary map of the short exact seqhence (1.2).

Remark 1.31. Our definition of differentials and monodromy operator on $A^{\bullet,\bullet}$ comes from [34], and differs from those of [32], [22], and [23]. This makes $A^{\bullet}$ into an object of $\mathscr{C}^{+}_{F}(\varphi, N)$. The definition in [22] induces $N\colon (A^{\bullet}, d) \to (A^{\bullet}, -d)$. See [34, Rem. 11.9].

The Steenbrink complex has Frobenius and monodromy operator, and the weight filtration, but it is not a commutative differentially graded algebra. To be able to consider a cup product we use an analogue of a construction due to Kim and Hain [29, pp. 1259–1260].

Definition 1.32. For an object $(\mathscr{Z}, \overline{\mathscr{Z}})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ with a Frobenius lift ϕ as in Definition 1.27, let $\overline{Y}$, $\omega^{\bullet}_{\overline{Y},\mathbf{Q}}$, and $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}$ be as above. Let $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u]$ be the commutative differentially graded algebra on $\overline{Y}_{\mathbf{Q}}$ generated by $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}$ and degree zero elements $u^{[k]}$ for $k \geq 0$ with the relations +

$$du^{[k]} = d\log t \cdot u^{[k-1]} \qquad \text{and} \qquad u^{[0]} = 1.$$

The multiplication is given by

$$u^{[k]} \wedge u^{[\ell]} = \frac{(k+\ell)!}{k!\ell!} u^{[k+\ell]}.$$

We define the Frobenius operator φ on $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u]$ by the Frobenius action on $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}$ induced by ϕ and by $\varphi(u^{[k]}) := p^k u^{[k]}$. We define the monodromy operator N on $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u]$ by $N(u^{[k]}) := u^{[k-1]}$. Then clearly $N\varphi = p\varphi N$ holds. Moreover we have $\varphi(\eta_1 \wedge \eta_2) = \varphi(\eta_1) \wedge \varphi(\eta_2)$ and $N(\eta_1 \wedge \eta_2) = N(\eta_1) \wedge \eta_2 + \eta_1 \wedge N(\eta_2)$ for any sections η_1 and η_2 of $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u]$.

Remark 1.33. Note that $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u]$ does not have a weight filtration. By an analogous construction as in [29, p. 1270], we may also define a commutative differentially graded algebra with φ, N, and the weight filtration, which is quasi-isomorphic to $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u]$.

Lemma 1.34. *The morphism $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u] \to \omega^{\bullet}_{\overline{Y},\mathbf{Q}}$ which sends $u^{[k]} \mapsto 0$ for $k \geq 1$ is a quasi-isomorphism of commutative differentially graded algebras.*

Proof. The compatibility of this morphism with the differentials and the multiplications are straightforward. Note moreover, that as complex, $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u]$ is given as the total complex of a double complex $(\widetilde{\omega}^{i+j}_{\overline{Y},\mathbf{Q}} u^{[-i]})_{i,j}$, with $u^{[-i]} = 0$ for $i > 0$, and horizontal differential ∂_1 and vertical differential ∂_2 defined by

$$\partial^{i,j}_1 \colon \widetilde{\omega}^{i+j}_{\overline{Y},\mathbf{Q}} u^{[-i]} \to \widetilde{\omega}^{i+j+1}_{\overline{Y},\mathbf{Q}} u^{[-i]},\ \eta \cdot u^{[-i]} \mapsto (d\eta)\cdot u^{[-i]} \qquad (\eta \in \widetilde{\omega}^{i+j}_{\overline{Y},\mathbf{Q}})$$
$$\partial^{i,j}_2 \colon \widetilde{\omega}^{i+j}_{\overline{Y},\mathbf{Q}} u^{[-i]} \to \widetilde{\omega}^{i+j+1}_{\overline{Y},\mathbf{Q}} u^{[-i-1]},\ \eta \cdot u^{[-i]} \mapsto (\eta \wedge d\log t)\cdot u^{[-i-1]} \qquad (\eta \in \widetilde{\omega}^{i+j}_{\overline{Y},\mathbf{Q}})$$

By the short exact sequence (1.2), we see that the sequence

$$\cdots \to \widetilde{\omega}^{k-2}_{\overline{Y},\mathbf{Q}} u^{[2]} \xrightarrow{\partial^{-2,k}_2} \widetilde{\omega}^{k-1}_{\overline{Y},\mathbf{Q}} u^{[1]} \xrightarrow{\partial^{-1,k}_2} \widetilde{\omega}^{k}_{\overline{Y},\mathbf{Q}} u^{[0]} \to \omega^{k}_{\overline{Y},\mathbf{Q}} \to 0$$

is exact for any $k \geq 0$. Hence the k-th column of $(\widetilde{\omega}^{i+j}_{\overline{Y},\mathbf{Q}} u^{[-i]})_{i,j}$ is quasi-isomorphic to $\omega^{k}_{\overline{Y},\mathbf{Q}}$. This gives the lemma. ☐

Proposition 1.35. *The composition $\widetilde{\omega}^{\bullet}_{\overline{Y},\mathbf{Q}}[u] \to \omega^{\bullet}_{\overline{Y},\mathbf{Q}} \to A^{\bullet}_{\overline{Y}_{\mathbf{Q}}}$ of the quasi-isomorphisms in Lemma 1.34 and Lemma 1.29 is compatible with φ and N.*

Proof. The compatibility with the Frobenius is clear. The compatibility with the monodromy follows from an easy observation that this morphism sends $\eta \cdot u^{[0]} \mapsto \eta \wedge d\log t \in A^{i,0}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ and $\nu(\eta \wedge d\log t) = 0$ in $A^{i-1,1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} = \widetilde{\omega}^{i+1}_{\overline{\mathcal{Y}},\mathbf{Q}}$ since $\eta \wedge d\log t \in P_1\widetilde{\omega}^{i+1}_{\overline{\mathcal{Y}},\mathbf{Q}}$. □

We will use this quasi-isomorphism $\widetilde{\omega}^{\bullet}_{\overline{\mathcal{Y}},\mathbf{Q}}[u] \xrightarrow{\sim} A^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ for the construction of weight spectral sequence.

Definition 1.36. For a rigid Hyodo–Kato quadruple $((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$, we define the rigid Hyodo–Kato complex by

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z},\overline{\mathcal{Z}})} := \Gamma(]\overline{Y}[_{\overline{\mathcal{Y}}}, \mathrm{Gd}_{\mathrm{an}}\,\widetilde{\omega}^{\bullet}_{\overline{\mathcal{Y}},\mathbf{Q}}[u]),$$

where $\mathrm{Gd}_{\mathrm{an}}$ denotes the generalised Godement resolution for rigid analytic points (c.f. [12, Sec. 3]). It is an object in $\mathscr{C}^{+}_{F}(\varphi,N)$.

Lemma 1.37. *Let* $F\colon ((\mathcal{Z},\overline{\mathcal{Z}}),i,\phi) \to ((\mathcal{Z}',\overline{\mathcal{Z}}'),i',\phi')$ *be a morphism of rigid Hyodo–Kato data for* $(Y,\overline{Y})$*. Then the morphism*

$$F^{*}\colon \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z}',\overline{\mathcal{Z}}')} \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z},\overline{\mathcal{Z}})}$$

induced by F *is a quasi-isomorphism.*

Proof. By Lemma 1.29, $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z}',\overline{\mathcal{Z}}')}$ and $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z},\overline{\mathcal{Z}})}$ are naturally quasi-isomorphic to $\mathrm{R}\Gamma(]\overline{Y}[_{\overline{\mathcal{Y}}'},\omega^{\bullet}_{\overline{\mathcal{Y}}'_{\mathbf{Q}}})$ and $\mathrm{R}\Gamma(]\overline{Y}[_{\overline{\mathcal{Y}}},\omega^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}})$ respectively, which are quasi-isomorphic to each other by [22, Lem. 1.4]. □

Besser's idea of constructing canonical complexes is to take into account all possible available data in a certain situation. In his case this was what he called 'rigid data', whereas the equivalent in our case are rigid Hyodo–Kato data as described above. However, it is not possible to take limits naively. Instead one is lead to consider certain filtered index categories. We describe now the analogues of Besser's definitions in our setting.

Definition 1.38. Let $(Y,\overline{Y})$ be an object in $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$.

(i) Let $\mathrm{PQ}_{\mathrm{HK}}(Y,\overline{Y})$ be the set of all isomorphism classes of quintuples $(f,(Y',\overline{Y}'),(\mathcal{Z}',\overline{\mathcal{Z}}'),i',\phi')$ consisting of a morphism $f\colon (Y,\overline{Y}) \to (Y',\overline{Y}')$ in $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$ and a rigid Hyodo–Kato datum $((\mathcal{Z}',\overline{\mathcal{Z}}'),i',\phi')$ for $(Y',\overline{Y}')$. To simplify notations we often denote elements of $\mathrm{PQ}_{\mathrm{HK}}(Y,\overline{Y})$ by small roman letters like a, and the corresponding objects by $(f^{(a)},(Y^{(a)},\overline{Y}^{(a)}),(\mathcal{Z}^{(a)},\overline{\mathcal{Z}}^{(a)}),i^{(a)},\phi^{(a)})$.

(ii) Let $\mathsf{SET}_{\mathrm{HK}}(Y,\overline{Y})$ be the category whose objects are all finite subsets of $\mathrm{PQ}_{\mathrm{HK}}(Y,\overline{Y})$ and whose morphisms are inclusions.

(iii) Let $\mathsf{SET}^{0}_{\mathrm{HK}}(Y,\overline{Y})$ be the full subcategory of $\mathsf{SET}_{\mathrm{HK}}(Y,\overline{Y})$ whose objects are all subsets which have an element a with $f^{(a)} = \mathrm{id}_{(Y,\overline{Y})}$. Clearly this is a filtered category.

In the following paragraphs, we explain how to define a rigid Hyodo–Kato datum $((\mathcal{Z}_A,\overline{\mathcal{Z}}_A),i_A,\phi_A)$ for $(Y,\overline{Y})$ for an object A in $\mathsf{SET}^{0}_{\mathrm{HK}}(Y,\overline{Y})$. Fix an element $a_0 \in A$ with $f^{(a_0)} = \mathrm{id}_{(Y,\overline{Y})}$. For $\alpha \in \Upsilon_{\overline{Y}^{(a)}}$, let $\Upsilon^{a,\alpha}_{\overline{Y}}$ be the set of elements $\alpha_0 \in \Upsilon_{\overline{Y}}$ such that $f^{(a)}(\overline{Y}_{\alpha_0}) \subset \overline{Y}^{(a)}_{\alpha}$, where $\overline{Y}_{\alpha_0}$ and $\overline{Y}^{(a)}_{\alpha}$ are the irreducible components corresponding to α_0 and α respectively. We regard $\Upsilon^{a,\alpha}_{\overline{Y}}$ as a subset of $\Upsilon_{\overline{Y}^{(a_0)}}$ through the equality $\overline{Y} = \overline{Y}^{(a_0)}$. Since $f^{(a)}(Y) \subset (Y^{(a)})$, we may consider the pull-back $f^{(a)*}D^{(a)}$ of the Cartier divisor $D^{(a)}$. For each $\beta \in \Upsilon_{D^{(a)}}$ and $\beta_0 \in \Upsilon_{D^{(a_0)}} = \Upsilon_D$, let $m^{(a)}_{\beta,\beta_0}$ be the multiplicity of $f^{(a)*}D^{(a)}_{\beta}$ at D_{β_0}, namely we have

$$f^{(a)*}D^{(a)} = \sum_{\beta_0 \in \Upsilon_D} m^{(a)}_{\beta,\beta_0} D_{\beta_0}.$$

Let $\overline{\mathcal{Z}}''_A$ be the blow-up of $\prod_{\mathscr{O}_F}(\overline{\mathcal{Z}}^{(a)})_{a\in A}$ along the ideal

$$\prod_{\substack{a\in A\\ \alpha\in\Upsilon_{\overline{Y}^{(a)}}}} (\mathcal{I}^{(a)}_{\alpha} + \prod_{\alpha_0\in\Upsilon^{a,\alpha}_{\overline{Y}}} \mathcal{I}^{(a_0)}_{\alpha_0}) \times \prod_{\substack{a\in A\\ \beta\in\Upsilon_{D^{(a)}}}} (\mathcal{J}^{(a)}_{\beta} + \prod_{\beta_0\in\Upsilon_D} \mathcal{J}^{(a_0),m^{(a)}_{\beta,\beta_0}}_{\beta_0}),$$

where $\mathcal{I}_{\alpha}^{(a)}$ and $\mathcal{J}_{\beta}^{(a)}$ are the ideals of the inclusions

$$\overline{\mathcal{Y}}_{\alpha}^{(a)} \times \prod_{\mathcal{O}_F}(\overline{\mathcal{Z}}^{(a')})_{a'\in A\setminus\{a\}} \subset \prod_{\mathcal{O}_F}(\overline{\mathcal{Z}}^{(a)})_{a\in A} \qquad \text{and}$$
$$\mathcal{D}_{\beta}^{(a)} \times \prod_{\mathcal{O}_F}(\overline{\mathcal{Z}}^{(a')})_{a'\in A\setminus\{a\}} \subset \prod_{\mathcal{O}_F}(\overline{\mathcal{Z}}^{(a)})_{a\in A}$$

respectively. Let $\overline{\mathcal{Z}}'_A$ be the complement of the strict transforms in $\overline{\mathcal{Z}}''_A$ of $\overline{\mathcal{Y}}_{\alpha}^{(a)} \times \prod_{\mathcal{O}_F}(\overline{\mathcal{Z}}^{(a')})_{a'\in A\setminus\{a\}}$ and $\mathcal{D}_{\beta}^{(a)} \times \prod_{\mathcal{O}_F}(\overline{\mathcal{Z}}^{(a')})_{a'\in A\setminus\{a\}}$ for all $a \in A$, $\alpha \in \Upsilon_{\overline{Y}^{(a)}}$ and $\beta \in \Upsilon_{D^{(a)}}$. Let $\mathcal{T}_A$ be the blow-up of the A-indexed self product $\prod_{\mathcal{O}_F}(\mathcal{T})_{a\in A}$ of $\mathcal{T}$ along the closed weak formal subscheme $\prod_{\mathcal{O}_F}(\mathrm{Spwf}\,\mathcal{O}_F)_{a\in A}$ defined by $t = 0$. Then the diagonal embedding $\mathcal{T} \hookrightarrow \prod_{\mathcal{O}_F}(\mathcal{T})_{a\in A}$ lifts to an embedding $\mathcal{T} \hookrightarrow \mathcal{T}_A$, and there exists a natural morphism $\overline{\mathcal{Z}}'_A \to \mathcal{T}_A$. Let $\overline{\mathcal{Z}}_A := \overline{\mathcal{Z}}'_A \times_{\mathcal{T}_A} \mathcal{T}$. We denote by $\widetilde{\mathcal{E}}_A$ the exceptional divisor on $\mathcal{Z}_A$, and by $\widetilde{\mathcal{D}}_A$ the closed weak formal subscheme of $\mathcal{Z}_A$ defined by the inverse image of

$$\prod_{\substack{a\in A\\ \beta\in\Upsilon_{D^{(a)}}}} (\mathcal{J}_{\beta}^{(a)} + \prod_{\beta_0\in\Upsilon_D} \mathcal{J}_{\beta_0}^{(a_0),m_{\beta,\beta_0}^{(a)}}).$$

Let $\mathcal{E}_A$ and $\mathcal{D}_A$ be the reduced closed weak formal subschemes of $\overline{\mathcal{Z}}_A$ whose underlying spaces are $\widetilde{\mathcal{E}}_A$ and $\widetilde{\mathcal{D}}_A$ respectively. Let $\mathcal{Z}_A := \overline{\mathcal{Z}}_A \setminus \mathcal{D}_A$.

Proposition 1.39. *The closed weak formal subschemes $\mathcal{E}_A$ and $\mathcal{D}_A$ of $\overline{\mathcal{Z}}_A$ are simple normal crossing divisors. The diagonal morphism $\prod f^{(a)}\colon \overline{Y} \to \prod_{\mathcal{O}_F}(\overline{\mathcal{Z}}^{(a)})_{a\in A}$ factors into a strict immersion $i_A\colon \overline{Y} \to \overline{\mathcal{Z}}_A$ and the natural morphism $\overline{\mathcal{Z}}_A \to \prod_{\mathcal{O}_F}(\overline{\mathcal{Z}}^{(a)})_{a\in A}$. We have $i_A(Y) \subset \mathcal{Z}_A$, $\Upsilon_{\overline{Y}} = \Upsilon_{\overline{\mathcal{Y}}}$, and $\Upsilon_D = \Upsilon_{\mathcal{D}}$. Moreover $(\mathcal{Z}_A, \overline{\mathcal{Z}}_A)$ is independent of the choice of a_0 up to canonical isomorphism and functorial on A.*

Proof. Since the construction is local, we may assume that there are commutative diagrams

$$\begin{tikzcd}
\overline{Y}^{(a)} \arrow[r] \arrow[d, "i^{(a)}"] & \mathrm{Spec}\,\dfrac{k[t_{\alpha}^{(a)}, s_{\beta}^{(a)}]_{\alpha\in\Upsilon_{\overline{Y}^{(a)}},\beta\in\Upsilon_{D^{(a)}}}}{(\prod_{\alpha\in\Upsilon_{\overline{Y}^{(a)}}} t_{\alpha}^{(a)})} \arrow[d] \\
\overline{\mathcal{Z}}^{(a)} \arrow[r] & \mathrm{Spwf}\,\mathcal{O}_F[t_{\alpha}^{(a)}, s_{\beta}^{(a)}]^{\dagger}_{\alpha\in\Upsilon_{\overline{Y}^{(a)}},\beta\in\Upsilon_{D^{(a)}}}
\end{tikzcd}$$

whose horizontal arrows are smooth as in Definition 1.25. We denote the image in $\overline{Y}$ of the coordinates appearing on the right hand side of the above diagram again by the same letters. Since $f^{(a)}$ is a morphism of log schemes over $\mathcal{T}$, the homomorphism $f^{(a)\sharp}\colon f^{(a),-1}\mathcal{O}_{\overline{Y}^{(a)}} \to \mathcal{O}_{\overline{Y}}$ sends $\prod_{\alpha\in\Upsilon_{\overline{Y}^{(a)}}} t_{\alpha}^{(a)}$ to $\prod_{\alpha_0\in\Upsilon_{\overline{Y}}} t_{\alpha_0}$, and $t_{\alpha}^{(a)}$ to $\prod_{\alpha_0\in\Upsilon_{\overline{Y}}^{a,\alpha}} t_{\alpha_0}$ up to unit in $\mathcal{O}_{\overline{Y}}$. Moreover since $f(Y) \subset Y^{(a)}$, the morphism $f^{(a)\sharp}$ sends $s_{\beta}^{(a)}$ to $\prod_{\beta_0\in\Upsilon_D} s_{\beta_0}^{m_{\beta,\beta_0}^{(a)}}$ up to a unit in $\mathcal{O}_{\overline{Y}}$. Hence, if we set

$$u_{\alpha}^{(a)} := \frac{t_{\alpha}^{(a)}}{\prod_{\alpha_0\in\Upsilon_{\overline{Y}}^{a,\alpha}} t_{\alpha_0}^{(a_0)}} \qquad \text{and} \qquad v_{\beta}^{(a)} := \frac{s_{\beta}^{(a)}}{\prod_{\beta_0\in\Upsilon_D} s_{\beta_0}^{(a_0),m_{\beta,\beta_0}^{(a)}}},$$

we obtain a smooth morphism

$$\begin{aligned}
\overline{\mathcal{Z}}'_A \quad \to \quad & \mathrm{Spwf}\,\mathcal{O}_F[t_{\alpha}^{(a)}, s_{\beta}^{(a)}, u_{\alpha}^{(a),\pm1}, v_{\beta}^{(a),\pm1}]^{\dagger}_{a\in A,\alpha\in\Upsilon_{\overline{Y}^{(a)}},\beta\in\Upsilon_{D^{(a)}}} \\
&= \mathrm{Spwf}\,\mathcal{O}_F[t_{\alpha_0}^{(a_0)}, s_{\beta_0}^{(a_0)}, u_{\alpha}^{(a),\pm1}, v_{\beta}^{(a),\pm1}]^{\dagger}_{\alpha_0\in\Upsilon_{\overline{Y}},\beta\in\Upsilon_D,a\in A,\alpha\in\Upsilon_{\overline{Y}^{(a)}},\beta\in\Upsilon_{D^{(a)}}}.
\end{aligned}$$

The embedding $\overline{\mathcal{Z}}_A \to \overline{\mathcal{Z}}'_A$ is given by the ideal

$$\sum_{a,a'\in A}\left(\frac{\prod_{\alpha\in\Upsilon_{\overline{Y}^{(a)}}} t_{\alpha}^{(a)}}{\prod_{\alpha'\in\Upsilon_{\overline{Y}^{(a')}}} t_{\alpha'}^{(a')}}\right) = \sum_{a\in A}\left(\prod_{\alpha\in\Upsilon_{\overline{Y}^{(a)}}} u_{\alpha}^{(a)} - 1\right).$$

Consequently we have a smooth morphism

$$\overline{\mathscr{Z}}_A \to \operatorname{Spwf} \frac{\mathscr{O}_F[t_{\alpha_0}^{(a_0)}, s_{\beta_0}^{(a_0)}, u_\alpha^{(a),\pm 1}, v_\beta^{(a),\pm 1}]^\dagger_{\alpha_0\in\Upsilon_{\overline{Y}},\beta\in\Upsilon_D,a\in A,\alpha\in\Upsilon_{\overline{Y}^{(a)}},\beta\in\Upsilon_{D^{(a)}}}}{\sum_{a\in A}(\prod_{\alpha\in\Upsilon_{\overline{Y}^{(a)}}} u_\alpha^{(a)}-1)}.$$

The ideals of $\widetilde{\mathscr{E}}_A\subset\overline{\mathscr{Z}}_A$ and $\widetilde{\mathscr{D}}_A\subset\overline{\mathscr{Z}}_A$ are generated by

$$\prod_{\substack{a\in A\\ \alpha\in\Upsilon_{\overline{Y}^{(a)}}}} t_\alpha^{(a)}\cdot\prod_{\substack{a\in A\\ \beta\in\Upsilon_{D^{(a)}}}} s_\beta^{(a)}\qquad\text{and}\qquad\prod_{\substack{a\in A\\ \beta\in\Upsilon_{D^{(a)}}}} s_\beta^{(a)},$$

and hence by

$$\prod_{\alpha_0\in\Upsilon_{\overline{Y}}} t_{\alpha_0}^{(a_0),|A|}\cdot\prod_{\beta_0\in\Upsilon_D} s_{\beta_0}^{(a_0),m_{\beta_0}}\qquad\text{and}\qquad\prod_{\beta_0\in\Upsilon_D} s_{\beta_0}^{(a_0),m_{\beta_0}}$$

respectively, where $m_{\beta_0}:=\sum_{\substack{a\in A\\ \beta\in\Upsilon_{D^{(a)}}}} m_{\beta,\beta_0}^{(a)}\geq 1$. Thus $\mathscr{E}_A$ and $\mathscr{D}_A$ are simple normal crossing divisors defined by

$$\prod_{\alpha_0\in\Upsilon_{\overline{Y}}} t_{\alpha_0}^{(a_0)}\cdot\prod_{\beta_0\in\Upsilon_D} s_{\beta_0}^{(a_0)}\qquad\text{and}\qquad\prod_{\beta_0\in\Upsilon_D} s_{\beta_0}^{(a_0)}$$

respectively. Therefore there is a natural morphism $\overline{Y}\to\overline{\mathscr{Z}}_A$ sending $u_\alpha^{(a)}$ and $v_\beta^{(\alpha)}$ to units in $\mathscr{O}_{\overline{Y}}$, which is a strict immersion.

If there is another $a_1\in A$ with $f^{(a_1)}=\mathrm{id}_{\overline{Y},D}$, we have

$$\frac{t_\alpha^{(a)}}{\prod_{\alpha_1\in\Upsilon_{\overline{Y}}^{a,\alpha}} t_{\alpha_1}^{(a_1)}}=\frac{u_\alpha^{(a)}}{\prod_{\alpha_1\in\Upsilon_{\overline{Y}}^{a,\alpha}} u_{\alpha_1}^{(a_1)}}\qquad\text{and}\qquad\frac{s_\beta^{(a)}}{\prod_{\beta_1\in\Upsilon_D^{a,\beta}} s_{\beta_1}^{(a_1)}}=\frac{v_\beta^{(a)}}{\prod_{\beta_1\in\Upsilon_D^{a,\beta}} v_{\beta_1}^{(a_1)}}$$

for any $a\in A$, $\alpha\in\Upsilon_{\overline{Y}^{(a)}}$, and $\beta\in\Upsilon_{D^{(a)}}$, where we use the same notation as above with appropriate index changes. This implies that $(\mathscr{Z}_A,\overline{\mathscr{Z}}_A)$ is independent of the choice of a_0. □

We continue to use the notation of the proof of Proposition 1.39. The Frobenius $\phi^{(a)}$ sends up to units $t_\alpha^{(a)}$ and $s_\beta^{(a)}$ to $t_\alpha^{(a),p}$ and $s_\beta^{(a),p}$ respectively. Hence $\prod\phi^{(a)}$ on $\prod_{\mathscr{O}_F}(\overline{\mathscr{Z}}^{(a)})_{a\in A}$ lifts uniquely to $\overline{\mathscr{Z}}_A$, where it sends $u_\alpha^{(a)}$ and $v_\beta^{(a)}$ to $u_\alpha^{(a),p}$ and $v_\beta^{(a),p}$ up to units. We endow $\overline{\mathscr{Z}}_A$ with the log structure associated to $\mathscr{E}_A$. We have obtained a rigid Hyodo–Kato datum $((\mathscr{Z}_A,\overline{\mathscr{Z}}_A),i_A,\phi_A)$ for $(Y,\overline{Y})$. For simplicity, we use the notation

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_A:=\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z}_A,\overline{\mathscr{Z}}_A)}.$$

Definition 1.40. For an object $(Y,\overline{Y})$ in $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$, we define the rigid Hyodo–Kato complex by

$$\widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}):=\varinjlim_{A\in\mathsf{SET}^0_{\mathrm{HK}}(Y,\overline{Y})}\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_A.$$

Let $g\colon(Y,\overline{Y})\to(Y',\overline{Y}')$ be a morphism in $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$. There is a map

$$g^\circ\colon\mathrm{PQ}_{\mathrm{HK}}(Y',\overline{Y}')\to\mathrm{PQ}_{\mathrm{HK}}(Y,\overline{Y})$$

which sends $(f^{(a)},(Y^{(a)},\overline{Y}^{(a)}),(\mathscr{Z}^{(a)},\overline{\mathscr{Z}}^{(a)}),i^{(a)},\phi^{(a)})$ to $(f^{(a)}\circ g,(Y^{(a)},\overline{Y}^{(a)}),(\mathscr{Z}^{(a)},\overline{\mathscr{Z}}^{(a)}),i^{(a)},\phi^{(a)})$. This induces a functor

$$\mathsf{SET}_{\mathrm{HK}}(Y',\overline{Y}')\to\mathsf{SET}_{\mathrm{HK}}(Y,\overline{Y})$$

sending A to $g^\circ(A)$, which we denote again by g°. Note however that g° does not send $\mathsf{SET}^0_{\mathrm{HK}}(Y',\overline{Y}')$ to $\mathsf{SET}^0_{\mathrm{HK}}(Y,\overline{Y})$.

Lemma 1.41. *For $C \in \mathit{SET}^0_{\mathrm{HK}}(Y', \overline{Y}')$ the map $g^\circ\colon C \to B \cup g^\circ(C)$ naturally induces a morphism*

$$g^\circ\colon ((Y, \overline{Y}), (\mathscr{Z}_{B\cup g^\circ(C)}, \overline{\mathscr{Z}}_{B\cup g^\circ(C)}), i_{B\cup g^\circ(C)}) \to ((Y', \overline{Y}'), (\mathscr{Z}_C, \overline{\mathscr{Z}}_C), i_C) \tag{1.6}$$

of rigid Hyodo–Kato quadruples.

Proof. Indeed, g° induces a morphism $\prod_{\mathscr{O}_F}(\overline{\mathscr{Z}}^{(b)})_{b\in B\cup g^\circ(C)} \to \prod_{\mathscr{O}_F}(\overline{\mathscr{Z}}^{(c)})_{c\in C}$. Working locally, we assume without loss of generality that there is a commutative diagram

$$\begin{tikzcd}
\overline{Y}^{(a)} \arrow[r] \arrow[d, "i^{(a)}"] & \operatorname{Spec} \dfrac{k[t_\alpha^{(a)}, s_\beta^{(a)}]_{\alpha\in\Upsilon_{\overline{Y}^{(a)}}, \beta\in\Upsilon_{D^{(a)}}}}{(\prod_{\alpha\in\Upsilon_{\overline{Y}^{(a)}}} t_\alpha^{(a)})} \arrow[d] \\
\overline{\mathscr{Z}}^{(a)} \arrow[r] & \operatorname{Spwf} \mathscr{O}_F[t_\alpha^{(a)}, s_\beta^{(a)}]^\dagger_{\alpha\in\Upsilon_{\overline{Y}^{(a)}}, \beta\in\Upsilon_{D^{(a)}}}
\end{tikzcd}$$

for any $a \in B$ or $a \in C$. Let $b_0 \in B$ and $c_0 \in C$ with $f^{(b_0)} = \mathrm{id}_{(Y,\overline{Y})}$ and $f^{(c_0)} = \mathrm{id}_{(Y',\overline{Y}')}$. Note that for any $c \in C$, $\alpha \in \Upsilon_{\overline{Y}^{(c)}}$, and $\beta \in \Upsilon_{D^{(c)}}$, we have $\Upsilon^{c,\alpha}_{\overline{Y}} = \coprod_{\alpha_0'\in\Upsilon^{c,\alpha}_{\overline{Y}'}} \Upsilon^{c_0,\alpha_0'}_{\overline{Y}}$ and $m^{(c)}_{\beta,\beta_0} = \sum_{\beta_0'\in\Upsilon_{D'}} m^{(c)}_{\beta,\beta_0'} \cdot m^{(c_0)}_{\beta_0',\beta_0}$. Therefore there are equalities

$$\frac{t_\alpha^{(c)}}{\prod_{\alpha_0'\in\Upsilon^{c,\alpha}_{\overline{Y}'}} t^{(c_0)}_{\alpha_0'}} = \frac{t_\alpha^{(c)}}{\prod_{\alpha_0\in\Upsilon^{c,\alpha}_{\overline{Y}}} t^{(b_0)}_{\alpha_0}} \cdot \prod_{\alpha_0'\in\Upsilon^{c,\alpha}_{\overline{Y}'}} \left(\frac{\prod_{\alpha_0\in\Upsilon^{c_0,\alpha_0'}_{\overline{Y}}} t^{(b_0)}_{\alpha_0}}{t^{(c_0)}_{\alpha_0'}} \right),$$

$$\frac{s_\beta^{(c)}}{\prod_{\beta_0'\in\Upsilon_{D'}} s_{\beta_0'}^{(c_0), m^{(c)}_{\beta,\beta_0'}}} = \frac{s_\beta^{(c)}}{\prod_{\beta_0\in\Upsilon_D} s_{\beta_0}^{(b_0), m^{(c)}_{\beta,\beta_0}}} \cdot \prod_{\beta_0'\in\Upsilon_{D'}} \left(\frac{\prod_{\beta_0\in\Upsilon_D} s_{\beta_0}^{(b_0), m^{(c_0)}_{\beta_0',\beta_0}}}{s^{(c_0)}_{\beta_0'}} \right)^{m^{(c)}_{\beta,\beta_0'}},$$

and they belong to $\mathcal{O}^\times_{\overline{\mathscr{Z}}_{B\cup g^\circ(C)}}$. This implies that $\overline{\mathscr{Z}}_{B\cup g^\circ(C)} \to \prod_{\mathscr{O}_F}(\overline{\mathscr{Z}}_c)_{c\in C}$ factors through $\overline{\mathscr{Z}}'_C$, and thus through $\overline{\mathscr{Z}}_C$. The morphism $\overline{\mathscr{Z}}_{B\cup g^\circ(C)} \to \overline{\mathscr{Z}}_C$ defines a morphism (1.6) as desired. $\square$

With this in mind we can now show functoriality of rigid Hyodo–Kato complexes. We denote by g°_* the map $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y', \overline{Y}')_C \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})_{B\cup f^\circ(C)}$ induced by g°.

Proposition 1.42. *For a morphism $g\colon (Y, \overline{Y}) \to (Y', \overline{Y}')$ in $\overline{\mathit{ELS}}^{ss}_{k^0}$, there exists a unique map*

$$g^*\colon \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y', \overline{Y}') \to \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})$$

such that the diagram

$$\begin{tikzcd}
\varinjlim_{C} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y', \overline{Y}')_C \arrow[r, "g^*"] & \varinjlim_{B} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})_B \\
\varinjlim_{(B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y', \overline{Y}')_C \arrow[r, "g^\circ_*"] \arrow[u, "(B,C)\mapsto C"] & \varinjlim_{(B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})_{B\cup g^\circ(C)}, \arrow[u, "(B,C)\mapsto B\cup g^\circ(C)"']
\end{tikzcd} \tag{1.7}$$

where limits are taken over $B \in \mathit{SET}^0_{\mathrm{HK}}(Y, \overline{Y})$ and $C \in \mathit{SET}^0_{\mathrm{HK}}(Y', \overline{Y}')$, commutes. For any two composable morphisms $g\colon (Y, \overline{Y}) \to (Y', \overline{Y}')$ and $h\colon (Y', \overline{Y}') \to (Y'', \overline{Y}'')$, we have $(h \circ g)^* = g^* \circ h^*$.

Proof. The proposition can be shown in the same way as [9, Prop. 4.14]. The existence and uniqueness follows from the fact that the two vertical arrows in diagram (1.7) are isomorphisms (not just quasi-isomorphisms). For two composable morphisms $g\colon (Y, \overline{Y}) \to (Y', \overline{Y}')$ and $h\colon (Y', \overline{Y}') \to (Y'', \overline{Y}'')$, the diagram

$$\begin{tikzcd}
\varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y'',\overline{Y}'')_C \arrow[r, "g^\circ_*"] \arrow[dr, "(gh)^\circ_*"'] & \varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y',\overline{Y}')_{B\cup g^\circ(C)} \arrow[d, "h^\circ_*"] \\
& \varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{A\cup h^\circ(B)\cup (gh)^\circ(C)},
\end{tikzcd} \tag{1.8}$$

where the limits are taken over $A \in \mathsf{SET}^0_{\mathrm{HK}}(Y,\overline{Y})$, $B \in \mathsf{SET}^0_{\mathrm{HK}}(Y',\overline{Y}')$, and $C \in \mathsf{SET}^0_{\mathrm{HK}}(Y'',\overline{Y}'')$, commutes. Then the proposition follows from the fact that each of the maps in diagram (1.8) is isomorphic to the corresponding maps between log rigid complexes, that is, the diagrams

$$\begin{tikzcd}
\varinjlim_{(B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y'',\overline{Y}'')_C \arrow[r, "g^\circ_*"] & \varinjlim_{(B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y',\overline{Y}')_{B\cup g^\circ(C)} \\
\varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y'',\overline{Y}'')_C \arrow[r, "g^\circ_*"] \arrow[u, "(A,B,C)\mapsto(B,C)"] & \varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y',\overline{Y}')_{B\cup g^\circ(C)} \arrow[u, "(A,B,C)\mapsto(B,C)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim_{(A,B)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y',\overline{Y}')_B \arrow[r, "h^\circ_*"] & \varinjlim_{(A,B)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{A\cup h^\circ(B)} \\
\varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y',\overline{Y}')_{B\cup g^\circ(C)} \arrow[r, "h^\circ_*"] \arrow[u, "(A,B,C)\mapsto(A,B\cup g^\circ(C))"] & \varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{A\cup h^\circ(B)\cup (gh)^\circ(C)} \arrow[u, "(A,B,C)\mapsto(A,B\cup g^\circ(C))"']
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim_{(A,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y'',\overline{Y}'')_C \arrow[r, "(gh)^\circ_*"] & \varinjlim_{(A,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{A\cup (gh)^\circ(C)} \\
\varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y'',\overline{Y}'')_C \arrow[r, "(gh)^\circ_*"] \arrow[u, "(A,B,C)\mapsto(A,C)"] & \varinjlim_{(A,B,C)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{A\cup h^\circ(B)\cup (gh)^\circ(C)}. \arrow[u, "(A,B,C)\mapsto(A,C)"']
\end{tikzcd}$$

commute and the vertical arrows are isomorphisms. □

To relate $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z},\overline{\mathcal{Z}})}$ and $\widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ via a zig-zag of quasi-isomorphisms in a functorial way, we need to consider another index category.

Definition 1.43. Let $((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$ be a rigid Hyodo–Kato quadruple.

(i) Let $\mathrm{PQ}_{\mathrm{HK}}((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$ be the set of all isomorphism classes of morphisms of rigid Hyodo–Kato quadruples whose domain is $((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$. Similarly to above, we often denote elements of this set by small roman letters like a, and the corresponding objects by

$$(f^{(a)},F^{(a)})\colon ((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi) \to ((Y^{(a)},\overline{Y}^{(a)}),(\mathcal{Z}^{(a)},\overline{\mathcal{Z}}^{(a)}),i^{(a)},\phi^{(a)}).$$

(ii) Let $\mathsf{SET}_{\mathrm{HK}}((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$ be the category whose objects are all finite subsets of $\mathrm{PQ}_{\mathrm{HK}}((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$ and whose morphisms are inclusions.

(iii) Let $\mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$ be the full subcategory of $\mathsf{SET}_{\mathrm{HK}}((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$ whose objects are all subsets which contain the element $\mathrm{id}_{((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)}$. Clearly this is a filtered category.

For $A \in \mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$, one can define $(\mathcal{Z}_A,\overline{\mathcal{Z}}_A)$, i_A, ϕ_A, and

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_A := \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z}_A,\overline{\mathcal{Z}}_A)}$$

by the same procedure as before.

Definition 1.44. For a rigid Hyodo–Kato quadruple $((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)$, we define

$$\widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})} := \varinjlim_{A\in\mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_A.$$

Note the difference between the index set over which the limit is taken in Definition 1.40 and Definition 1.44 By a similar argument as in Proposition 1.42, we see that $\widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})}$ is functorial on $\mathsf{RQ}_{\mathrm{HK}}$.

One can see that the diagonal morphism $\prod F^{(a)}\colon \overline{\mathscr{Z}} \to \prod_{a\in A}\overline{\mathscr{Z}}^{(a)}$ lifts to $\overline{\mathscr{Z}} \to \overline{\mathscr{Z}}_A$, which defines a canonical morphism $((\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)\to((\mathscr{Z}_A,\overline{\mathscr{Z}}_A),i_A,\phi_A)$ of rigid Hyodo–Kato data for $(Y,\overline{Y})$ and hence a quasi-isomorphism $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_A \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})}$. Taking the limit over $\mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)$, we obtain a canonical quasi-isomorphism

$$\widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})} \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})}. \tag{1.9}$$

The forgetful map $\Pi\colon \mathrm{PQ}_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)\to \mathrm{PQ}_{\mathrm{HK}}(Y,\overline{Y})$ induces functors

$$\begin{aligned}\Pi\colon \mathsf{SET}_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi) &\to \mathsf{SET}_{\mathrm{HK}}(Y,\overline{Y})\\ \Pi\colon \mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi) &\to \mathsf{SET}^0_{\mathrm{HK}}(Y,\overline{Y}).\end{aligned}$$

For $A\in\mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)$, the canonical projection $\Pi\colon A\to\Pi(A)$ induces a morphism

$$\Delta_A\colon ((Y,\overline{Y}),(\mathscr{Z}_{\Pi(A)},\overline{\mathscr{Z}}_{\Pi(A)}),i_{\Pi(A)},\phi_{\Pi(A)})\to((Y,\overline{Y}),(\mathscr{Z}_A,\overline{\mathscr{Z}}_A),i_A,\phi_A)$$

and hence a quasi-isomorphism $\Delta_A^*\colon \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_A\to\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{\Pi(A)}$. If we take the limit over the index set $\mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)$, we obtain a quasi-isomorphism

$$\Delta\colon \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})} \to \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}). \tag{1.10}$$

One can easily verify that Δ is functorial on $\mathsf{RQ}_{\mathrm{HK}}$.

In conclusion, we obtained the following proposition.

Proposition 1.45. *The log rigid complexes constructed above define functors*

$$\begin{aligned}\overline{\mathsf{ELS}}^{ss}_{k^0} &\to \mathscr{C}^+_F(\varphi,N), && (Y,\overline{Y})\mapsto \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}),\\ \mathsf{RQ}_{\mathrm{HK}} &\to \mathscr{C}^+_F(\varphi,N), && ((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)\mapsto \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})},\\ \mathsf{RQ}_{\mathrm{HK}} &\to \mathscr{C}^+_F(\varphi,N), && ((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)\mapsto \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})}.\end{aligned}$$

If $((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)$ is a rigid Hyodo–Kato quadruple, we have quasi-isomorphisms

$$\widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) \leftarrow \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})} \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})},$$

which are functorial on $\mathsf{RQ}_{\mathrm{HK}}$. Here we regard $\widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ as a functor from $\mathsf{RQ}_{\mathrm{HK}}$ through the forgetful functor $\mathsf{RQ}_{\mathrm{HK}}\to\overline{\mathsf{ELS}}^{ss}_{k^0}$.

We immediately obtain the following fact from Lemma 1.28.

Lemma 1.46. *The category $\overline{\mathsf{ELS}}^{ss}_{k^0}$ together with the natural functor $\overline{\mathsf{ELS}}^{ss}_{k^0}\to\overline{\mathsf{LS}}^{ss}_{k^0}$ is a base for $\overline{\mathsf{LS}}^{ss}_{k^0}$ with respect to the Zarski topology in the sense of Definition 1.25.*

Definition 1.47. Let $\mathfrak{C}_{\mathrm{HK}}$ be the sheafification of the presheaf $(Y,\overline{Y})\mapsto\widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ on $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$ with respect to the Zariski topology. By Proposition 1.2 and Lemma 1.46, we may extend it to a sheaf on $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$. For any object $(Y,\overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$, we define the rigid Hyodo–Kato complex by

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) := \mathrm{R}\Gamma((Y,\overline{Y})_{\mathrm{Zar}},\mathfrak{C}_{\mathrm{HK}}).$$

It is an object in $\mathscr{D}^+_F(\varphi,N)$.

It is possible to extend the above definition to simplicial schemes.

Definition 1.48. Let $(Y_\bullet, \overline{Y}_\bullet)$ be a simplicial object in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$. Then the Zariski sites on the components form a simplicial site $(Y_\bullet, \overline{Y}_\bullet)_{\mathrm{Zar}}$. The restriction of $\mathfrak{C}_{\mathrm{HK}}$ defines a sheaf on the total site $(Y_\bullet, \overline{Y}_\bullet)^{\mathrm{tot}}_{\mathrm{Zar}}$, which we denote again by $\mathfrak{C}_{\mathrm{HK}}$. We define the rigid Hyodo–Kato complex by

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet, \overline{Y}_\bullet) := \mathrm{R}\Gamma((Y_\bullet, \overline{Y}_\bullet)^{\mathrm{tot}}_{\mathrm{Zar}}, \mathfrak{C}_{\mathrm{HK}}).$$

It is an object in $\mathscr{D}^+_F(\varphi, N)$.

Proposition 1.49. *Assume that $(Y, \overline{Y})$ is an object in $\overline{\mathsf{ELS}}^{ss}_{k^0}$. Then the natural morphism*

$$\widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y}) \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})$$

is a quasi-isomorphism.

Proof. Take a rigid Hyodo–Kato datum $((\mathcal{Z}, \overline{\mathcal{Z}}), i, \phi)$ for $(Y, \overline{Y})$. For any object $(U, \overline{U})$ in $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$ where $\overline{U}$ is an open subscheme of $\overline{Y}$ and $U = \overline{U} \cap Y$, we set $D_{\overline{U}} := \overline{U} \cap D$. Let $\overline{\mathcal{Z}}_{\overline{U}}$ be the complement in $\overline{\mathcal{Z}}$ of all irreducible components of $\overline{\mathcal{Y}}$ and $\mathcal{D}$ corresponding to elements in $\Upsilon_{\overline{Y}} \setminus \Upsilon_{\overline{U}}$ and $\Upsilon_D \setminus \Upsilon_{D_{\overline{U}}}$. We set $\mathcal{Z}_U := \overline{\mathcal{Z}}_{\overline{U}} \cap \mathcal{Z}$. Since ϕ sends the equations of irreducible components of $\overline{\mathcal{Y}}$ and $\mathcal{D}$ to their p-th powers up to units, $\overline{\mathcal{Z}}_{\overline{U}}$ is stable under ϕ. We denote by $i_{\overline{U}} \colon \overline{U} \hookrightarrow \overline{\mathcal{Z}}_{\overline{U}}$ and $\phi_{\overline{U}} \colon \overline{\mathcal{Z}}_{\overline{U}} \to \overline{\mathcal{Z}}_{\overline{U}}$ the morphisms induced by i and ϕ. Then $(\mathcal{Z}_U, \overline{\mathcal{Z}}_{\overline{U}}, i_{\overline{U}}, \phi_{\overline{\mathcal{Z}}_{\overline{U}}})$ is a rigid Hyodo–Kato datum for $(U, \overline{U})$, and for any open subset $\overline{V} \subset \overline{U}$, natural inclusions define a morphism $((V, \overline{V}), (\mathcal{Z}_V, \overline{\mathcal{Z}}_{\overline{V}}), i, \phi) \to ((U, \overline{U}), (\mathcal{Z}_U, \overline{\mathcal{Z}}_{\overline{U}}), i, \phi)$. Thus, for any Zariski hypercovering $(U_\bullet, \overline{U}_\bullet)$ of $(Y, \overline{Y})$, we obtain a simplicial object $((U_\bullet, \overline{U}_\bullet), (\mathcal{Z}_{U_\bullet}, \overline{\mathcal{Z}}_{\overline{U}_\bullet}), i_\bullet, \phi_\bullet)$ in $\mathsf{RQ}_{\mathrm{HK}}$. By Proposition 1.45 we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})_{(\mathcal{Z}, \overline{\mathcal{Z}})} & \longleftarrow & \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})_{(\mathcal{Z}, \overline{\mathcal{Z}})} & \longrightarrow & \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(U_\bullet, \overline{U}_\bullet)_{(\mathcal{Z}_{U_\bullet}, \overline{\mathcal{Z}}_{\overline{U}_\bullet})} & \longleftarrow & \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(U_\bullet, \overline{U}_\bullet)_{(\mathcal{Z}_{U_\bullet}, \overline{\mathcal{Z}}_{\overline{U}_\bullet})} & \longrightarrow & \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(U_\bullet, \overline{U}_\bullet),
\end{array}$$

where all horizontal arrows are quasi-isomorphisms. By Čech descent for admissible coverings of dagger spaces, we see that the left vertical arrow is a quasi-isomorphism. Therefore the right vertical arrow is also a quasi-isomorphism, and this implies the proposition. □

Remark 1.50. In general, for an object $(Y, \overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ not necessarily HK-embeddable, one can compute $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})$ as the cohomology of a simplicial dagger space as follows. Let $\{(U_h, \overline{U}_h)\}_{h \in H}$ be an affine Zariski covering of $(Y, \overline{Y})$ with rigid Hyodo–Kato data $((\mathcal{Z}_h, \overline{\mathcal{Z}}_h), i_h, \phi_h)$ for $(U_h, \overline{U}_h)$. For $\mathbf{h} = (h_0, \dots, h_i) \in H^{i+1}$, let $(U_{\mathbf{h}}, \overline{U}_{\mathbf{h}}) := (\bigcap_{j=0}^i U_{h_j}, \bigcap_{j=0}^i \overline{U}_{h_j})$. We set $D := \overline{Y} \setminus Y$, $D_h := \overline{U}_h \setminus U_j$, and $D_{\mathbf{h}} := \overline{U}_{\mathbf{h}} \setminus U_{\mathbf{h}}$. For $j = 0, \dots, i$, let $\overline{\mathcal{Z}}_{\mathbf{h}, h_j}$ be the complement in $\overline{\mathcal{Z}}_{h_j}$ of the irreducible components of $\overline{\mathcal{Y}}_{h_j}$ and $\mathcal{D}_{h_j}$ which correspond to elements in $\Upsilon_{\overline{U}_{h_j}} \setminus \Upsilon_{\overline{U}_{\mathbf{h}}}$ and $\Upsilon_{D_{h_j}} \setminus \Upsilon_{D_{\mathbf{h}}}$ respectively. Set $\mathcal{Z}_{\mathbf{h}, h_j} := \overline{\mathcal{Z}}_{\mathbf{h}, h_j} \cap \mathcal{Z}_{h_j}$. Then the set $A_{\mathbf{h}}$ of quintuples $(\mathrm{id}_{\overline{U}_{\mathbf{h}}}, (U_{\mathbf{h}}, \overline{U}_{\mathbf{h}}), (\mathcal{Z}_{\mathbf{h}, h_j}, \overline{\mathcal{Z}}_{\mathbf{h}, h_j}), i_{h_j}, \phi_{h_j})$, where $j = 0, \dots \dots, i$ varies, is an object in $\mathsf{SET}^0_{\mathrm{HK}}(U_{\mathbf{h}}, \overline{U}_{\mathbf{h}})$. Let $((\mathcal{Z}_{A_{\mathbf{h}}}, \overline{\mathcal{Z}}_{A_{\mathbf{h}}}), i_{A_{\mathbf{h}}}, \phi_{A_{\mathbf{h}}})$ be the rigid Hyodo–Kato datum for $(U_{\mathbf{h}}, \overline{U}_{\mathbf{h}})$ associated to $A_{\mathbf{h}}$ given by the construction after Definition 1.38. As a consequence we obtain a simplcial object $((U_\bullet, \overline{U}_\bullet), (\mathcal{Z}_{A_\bullet}, \overline{\mathcal{Z}}_{A_\bullet}), i_{A_\bullet}, \phi_{A_\bullet})$ in $\mathsf{RQ}_{\mathrm{HK}}$ and quasi-isomorphisms

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(U_\bullet, \overline{U}_\bullet)_{(\mathcal{Z}_{A_\bullet}, \overline{\mathcal{Z}}_{A_\bullet})} \xleftarrow{\sim} \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(U_\bullet, \overline{U}_\bullet)_{(\mathcal{Z}_{A_\bullet}, \overline{\mathcal{Z}}_{A_\bullet})} \xrightarrow{\sim} \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(U_\bullet, \overline{U}_\bullet) \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y}). \tag{1.11}$$

1.3 Logarithmic rigid complexes for fine log schemes

In this section we define log rigid complexes for more general log schemes than in the previous section which also behave well with respect to base change as we will see in the subsequent section. Assume that $S \hookrightarrow \mathcal{S}$ is one of $k^0 \hookrightarrow \mathcal{O}^0_F$, $k^0 \hookrightarrow \mathcal{O}^\pi_K$, or $T \hookrightarrow \mathcal{T}$. Recall that LS_S is the category of fine log schemes of finite type over S, and that $\mathsf{LS}_{\mathcal{S}}$ is the category of fine weak formal log schemes over $\mathcal{S}$. The construction we are about to explain, is in spirit very similar to the one from th previous section but without some of the technical difficulties.

Definition 1.51. (i) A *log rigid triple* $(Y, \mathcal{Y}, i)$ on S over $\mathcal{S}$ consists of an object in Y in LS_S, an object $\mathcal{Y}$ in $\mathsf{LS}_{\mathcal{S}}$ which is smooth over $\mathcal{S}$, and an immersion $i\colon Y \hookrightarrow \mathcal{Y}$ over $\mathcal{S}$. A morphism of log rigid triples $(Y, \mathcal{Y}, i) \to (Y', \mathcal{Y}', i')$ is a pair (f, F) where $f : Y \to Y'$ is a morphism in LS_S and $F : \mathcal{Y} \to \mathcal{Y}'$ is a morphism in $\mathsf{LS}_{\mathcal{S}}$, such that $F \circ i = i' \circ f$. Denote by $\mathsf{RT}_{\mathcal{S}}$ the category of log rigid triples on S over $\mathcal{S}$.

(ii) A *log rigid datum* over $\mathcal{S}$ for an object Y in LS_S is a pair $(\mathcal{Y}, i)$ such that $(Y, \mathcal{Y}, i)$ is a log rigid triple over $\mathcal{S}$. A morphism of log rigid data is a morphism (id_Y, F) in $\mathsf{RT}_{\mathcal{S}}$. Denote by $\mathsf{RD}_{\mathcal{S}}(Y)$ the category of log rigid data over $\mathcal{S}$ for Y.

(iii) An object Y in LS_S is *$\mathcal{S}$-embeddable* if the category $\mathsf{RD}_{\mathcal{S}}(Y)$ is non-empty. We denote by $\mathcal{S}$-ELS_S the full subcategory of LS_S of $\mathcal{S}$-embeddable objects.

The next lemma shows that locally fine log schemes of finite type over S are $\mathcal{S}$-embeddable.

Lemma 1.52. *Let Y be an object in LS_S which is affine and admits a (global) chart extending c_S of the structure morphism $Y \to S$. Then Y is $\mathcal{S}$-embeddable.*

Proof. Let $\mathscr{O}$, $\mathscr{O}_0$, and A be the coordinate rings of $\mathcal{S}$, S, and Y respectively, i.e. $\mathscr{O} = \mathscr{O}_F$, $\mathscr{O}_K$, or $\mathscr{O}_F[t]^\dagger$, and $\mathscr{O}_0 = k$ or $k[t]$. Denote by $\overline{\mathscr{O}}$ the reduction of $\mathscr{O}$ modulo p. Assume that we have a chart $(\alpha\colon P_Y \to \mathcal{M}_Y,\ \beta\colon \mathbb{N} \to P)$ of the structure morphism $Y \to S$ that extends c_S. Let $\alpha'\colon P \to A$ be the map induced by α and $\epsilon\colon \mathbb{N} \to \mathscr{O}$ the map induced by $c_{\mathcal{S}}\colon \mathbb{N}_{\mathcal{S}} \to \mathcal{M}_{\mathcal{S}}$. Take surjections $\gamma\colon \mathscr{O}_0[\mathbb{N}^m] \to A$ and $\delta\colon \mathbb{N}^n \to P$. We set $\mathcal{Y} := \mathrm{Spwf}\, \mathscr{O}[\mathbb{N}^m \oplus \mathbb{N}^n]^\dagger$ and endow $\mathcal{Y}$ with the log structure associated to the map

$$\rho\colon \mathbb{N}^n \oplus \mathbb{N} \to \mathscr{O}[\mathbb{N}^m \oplus \mathbb{N}^n]^\dagger,\ (\mathbf{n}, \ell) \mapsto \epsilon(\ell) \cdot \mathbf{n}.$$

Let $\eta\colon \mathscr{O} \to \overline{\mathscr{O}}$ be the natural surjection. Then we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O} \arrow[r] & \mathscr{O}[\mathbb{N}^m \oplus \mathbb{N}^n]^\dagger \arrow[r, "\xi"] & A \\
\mathbb{N} \arrow[u, "\epsilon"] \arrow[r] & \mathbb{N}^n \oplus \mathbb{N} \arrow[u, "\rho"] \arrow[r, "(\delta,\beta)"'] & P, \arrow[u, "\alpha'"']
\end{tikzcd}$$

where the map ξ is defined by

$$a \cdot \mathbf{m} \cdot \mathbf{n} \mapsto \eta(a) \cdot \gamma(\mathbf{m}) \cdot \alpha' \circ \delta(\mathbf{n})$$

for $a \in \mathscr{O}$, $\mathbf{m} \in \mathbb{N}^m$, and $\mathbf{n} \in \mathbb{N}^n$. The horizontal arrows in the right square are surjective, and hence define a closed immersion $i\colon Y \hookrightarrow \mathcal{Y}$. The left square defines the structure morphism $\mathcal{Y} \to \mathcal{S}$. The smoothness criterion [27, Thm. 3.5] implies immediately that $\mathcal{Y}$ is smooth over $\mathcal{S}$. □

The statement hereafter follows directly from Lemma 1.52.

Lemma 1.53. *The natural functor $\mathcal{S}$-$\mathsf{ELS}_S \to \mathsf{LS}_S$ is a base with respect to Zariski topology in the sense of Definition 1.4.*

Definition 1.54. For a log rigid triple $(Y, \mathcal{Y}, i)$ on S over $\mathcal{S}$, we define the log rigid complex over $\mathcal{S}$ by

$$\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} := \mathrm{R}\Gamma(]Y[^{\log}_{\mathcal{Y}}, \omega^\bullet_{\mathcal{Y},\mathbf{Q}}),$$

where $]Y[^{\log}_{\mathcal{Y}}$ and $\omega^\bullet_{\mathcal{Y},\mathbf{Q}}$ are as in Definition 1.8.

Lemma 1.55. *A morphism $(\mathcal{Y}, i) \to (\mathcal{Y}', i')$ in $\mathsf{RD}_{\mathcal{S}}(Y)$ induces a quasi-isomorphism of log-rigid complexes*

$$\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}'} \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}}.$$

Proof. This is simply a reformulation of [22, Lem. 1.4]. □

Definition 1.56. (i) Let Y be an object in $\mathcal{S}$-ELS_S.

(a) Let $\mathrm{PT}_{\mathcal{S}}(Y)$ be the set of all isomorphism classes of quadruples $(f, Y', \mathcal{Y}', i')$ consisting of a morphism $f\colon Y \to Y'$ in $\mathcal{S}$-ELS_S and a log rigid datum $(\mathcal{Y}', i')$ for Y' over $\mathcal{S}$. As before, to simplify notations we often denote elements of $\mathrm{PT}_{\mathcal{S}}(Y)$ by small roman letters like a, and corresponding objects by $(f^{(a)}, Y^{(a)}, \mathcal{Y}^{(a)}, i^{(a)})$.

(b) Let $\mathsf{SET}_{\mathcal{S}}(Y)$ be the category whose objects are all finite subsets of $\mathrm{PT}_{\mathcal{S}}(Y)$ and whose morphisms are inclusions.

(c) Let $\mathsf{SET}^0_{\mathcal{S}}(Y)$ be the full subcategory of $\mathsf{SET}_{\mathcal{S}}(Y)$ whose objects are all subsets which contain an element a with $f^{(a)} = \mathrm{id}_Y$. This is a filtered category.

(d) For $A \in \mathsf{SET}^0_{\mathcal{S}}(Y)$, we define a log rigid datum $(\mathcal{Y}_A, i_A)$ for Y by $\mathcal{Y}_A := \prod_{\mathcal{S}}(\mathcal{Y}^{(a)})_{a\in A}$ with the diagonal embedding $i_A \colon Y \hookrightarrow \mathcal{Y}_A$. We define the log rigid complex $\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})$ by

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S}) := \varinjlim_{A\in\mathsf{SET}^0_{\mathcal{S}}(Y)} \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}_A}. \tag{1.12}$$

(ii) Let $(Y, \mathcal{Y}, i)$ be a log-rigid triple on S over $\mathcal{S}$.

(a) Let $\mathrm{PT}_{\mathcal{S}}(Y, \mathcal{Y}, i)$ be the set of all isomorphism classes of morphisms of log rigid triples on S over $\mathcal{S}$ whose domain is $(Y, \mathcal{Y}, i)$. Similarly to above, we often denote elements of this set by small roman letters like a, and the corresponding objects by

$$(f^{(a)}, F^{(a)}) \colon (Y, \mathcal{Y}, i) \to (Y^{(a)}, \mathcal{Y}^{(a)}, i^{(a)}).$$

(b) Let $\mathsf{SET}_{\mathcal{S}}(Y, \mathcal{Y}, i)$ be the category whose objects are all finite subsets of $\mathrm{PT}_{\mathcal{S}}(Y, \mathcal{Y}, i)$ and whose morphisms are inclusions.

(c) Let $\mathsf{SET}^0_{\mathcal{S}}(Y, \mathcal{Y}, i)$ be the full subcategory of $\mathsf{SET}_{\mathcal{S}}(Y, \mathcal{Y}, i)$ whose objects are all subsets which contain the element $\mathrm{id}_{(Y,\mathcal{Y},i)}$. This is cearly a filtered category.

(d) For $A \in \mathsf{SET}^0_{\mathcal{S}}(Y, \mathcal{Y}, i)$, we define a log rigid datum $(\mathcal{Y}_A, i_A)$ for Y by $\mathcal{Y}_A := \prod_{\mathcal{S}}(\mathcal{Y}^{(a)})_{a\in A}$ with the diagonal embedding $i_A \colon Y \hookrightarrow \mathcal{Y}^{(a)}$. We define the log rigid complex $\widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}}$ by

$$\widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} := \varinjlim_{A\in\mathsf{SET}^0_{\mathcal{S}}(Y,\mathcal{Y},i)} \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}_A}. \tag{1.13}$$

As in [9, Prop. 4.14] or Proposition 1.42, one sees that the log rigid complexes (1.12) and (1.13) are functorial. Since we make a similar argument, we omit the proof here.

For $A \in \mathsf{SET}_{\mathcal{S}}(Y, \mathcal{Y}, i)$ the diagonal morphism $\prod F^{(a)} \colon \mathcal{Y} \to \mathcal{Y}_A$ defines a morphism of log-rigid data $(\mathcal{Y}, i) \to (\mathcal{Y}_A, i_A)$. Hence after taking the limit we obtain a canonical quasi-isomorphism

$$\widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}}.$$

Similar to the previous section, the forgetful map $\Pi \colon \mathrm{PT}_{\mathcal{S}}(Y, \mathcal{Y}, i) \to \mathrm{PT}_{\mathcal{S}}(Y)$ induces functors

$$\begin{aligned} \Pi : \mathsf{SET}_{\mathcal{S}}(Y, \mathcal{Y}, i) &\to \mathsf{SET}_{\mathcal{S}}(Y), \\ \Pi : \mathsf{SET}^0_{\mathcal{S}}(Y, \mathcal{Y}, i) &\to \mathsf{SET}^0_{\mathcal{S}}(Y). \end{aligned}$$

For $A \in \mathsf{SET}^0_{\mathcal{S}}(Y, \mathcal{Y}, i)$, the canonical projection $\Pi \colon A \to \Pi(A)$ induces a morphism

$$\Delta_A \colon (\mathcal{Y}_{\Pi(A)}, i_{\Pi(A)}) \to (\mathcal{Y}_A, i_A)$$

in $\mathsf{RD}_{\mathcal{S}}(Y)$. If we take the limit over $\mathsf{SET}^0_{\mathcal{S}}(Y, \mathcal{Y}, i)$, we obtain a quasi-isomorphism

$$\Delta \colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} \xrightarrow{\sim} \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S}).$$

To summarize we have the following statement.

Proposition 1.57. *The log-rigid complexes defined above define contravariant functors*

$$\begin{aligned} \mathcal{S}\text{-}\mathsf{ELS}_S \to \mathscr{C}^+_{\mathscr{O}_{\mathbf{Q}}}, &\qquad Y \mapsto \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S}), \\ \mathsf{RT}_{\mathcal{S}} \to \mathscr{C}^+_{\mathscr{O}_{\mathbf{Q}}}, &\qquad (Y, \mathcal{Y}, i) \mapsto \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}}, \\ \mathsf{RT}_{\mathcal{S}} \to \mathscr{C}^+_{\mathscr{O}_{\mathbf{Q}}}, &\qquad (Y, \mathcal{Y}, i) \mapsto \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}}, \end{aligned}$$

where $\mathscr{O}_{\mathbf{Q}}$ is the coordinate ring of $\mathcal{S}$ tensored with $\mathbf{Q}$, i.e. $\mathscr{O}_{\mathbf{Q}} = F, K,$ or $F[t]^{\dagger}$. If $(Y, \mathcal{Y}, i)$ is a log rigid triple on S over $\mathcal{S}$, we have quasi-isomorphisms

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S}) \xleftarrow{\sim} \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}},$$

which are functorial in $\mathsf{RT}_{\mathcal{S}}$.

Again, we extend the above local defintion by gluing as follows.

Definition 1.58. (i) Let $\mathfrak{C}_{\mathcal{S}}$ be the sheafification of the presheaf $Y \mapsto \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})$ on $\mathcal{S}$-ELS_S with respect to the Zariski topology. According to Proposition 1.2 and Lemma 1.53, we extend it to a sheaf on LS_S. For any object Y in LS_S, we define the log rigid complex by

$$\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S}) := \mathrm{R}\Gamma(Y_{\mathrm{Zar}}, \mathfrak{C}_{\mathcal{S}}).$$

It is an object in $\mathscr{D}^+_{\mathscr{O}_{\mathbf{Q}}}$.

(ii) Let $Y_\bullet$ be a simplicial object in LS_S. The Zariski sites on the components form a simplicial site $Y_{\bullet,\mathrm{Zar}}$. The restriction of $\mathfrak{C}_{\mathcal{S}}$ defines a sheaf on the total site $Y^{\mathrm{tot}}_{\bullet,\mathrm{Zar}}$, which we denote again by $\mathfrak{C}_{\mathcal{S}}$. We define the log rigid complex of $Y_\bullet$ by

$$\mathrm{R}\Gamma_{\mathrm{rig}}(Y_\bullet/\mathcal{S}) := \mathrm{R}\Gamma_{\mathrm{rig}}(Y^{\mathrm{tot}}_{\bullet,\mathrm{Zar}}, \mathfrak{C}_{\mathcal{S}}).$$

It is an object in $\mathscr{D}^+_{\mathscr{O}_Q}$.

Proposition 1.59. *Let Y be an object in $\mathcal{S}$-ELS_S. Then the natural morphism*

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})$$

is a quasi-isomorphism.

Proof. This follows from a similar argument as in Proposition 1.49. □

We will now explain a similar construction of canonical log rigid complexes for locally embeddable log schemes with boundary. Recall that we denote by $\overline{\mathsf{LS}}_T$ the category of fine T-log schemes with boundary of finites type and by $\overline{\mathsf{LS}}_{\mathcal{T}}$ the category of fine weak formal $\mathcal{T}$-log schemes with boundary.

Definition 1.60. (i) A *log rigid boundary triple* $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i)$ on T over $\mathcal{T}$ consists of an object $(Z,\overline{Z})$ in $\overline{\mathsf{LS}}_T$, a strongly smooth object $(\mathcal{Z},\overline{\mathcal{Z}})$ in $\overline{\mathsf{LS}}_{\mathcal{T}}$, and a boundary immersion $i\colon (Z,\overline{Z}) \hookrightarrow (\mathcal{Z},\overline{\mathcal{Z}})$. A morphism of log rigid boundary triples $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \to ((Z',\overline{Z}'),(\mathcal{Z}',\overline{\mathcal{Z}}'),i')$ is a pair (f,F) where $f\colon (Z,\overline{Z}) \to (Z,\overline{Z}')'$ is a morphism in $\overline{\mathsf{LS}}_T$, and $F\colon (\mathcal{Z},\overline{\mathcal{Z}}) \to (\mathcal{Z}',\overline{\mathcal{Z}}')$ is a morphism in $\overline{\mathsf{LS}}_{\mathcal{T}}$, such that $F\circ i = i'\circ f$. Denote by $\overline{\mathsf{RT}}_{\mathcal{T}}$ the category of log rigid boundary triples on T over $\mathcal{T}$.

(ii) A *log rigid boundary datum* over $\mathcal{T}$ for an object $(Z,\overline{Z})$ in $\overline{\mathsf{LS}}_T$ is a pair $((\mathcal{Z},\overline{\mathcal{Z}}),i)$ such that $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i)$ is a log rigid boundary triple. A morphism of log rigid boundary data is a morphism $(\mathrm{id}_{(Z,\overline{Z})},F)$ in $\overline{\mathsf{RT}}_{\mathcal{T}}$. Denote by $\overline{\mathsf{RD}}_{\mathcal{T}}(Z,\overline{Z})$ the category of log rigid boundary data over $\mathcal{T}$ for $(Z,\overline{Z})$.

(iii) We call an object $(Z,\overline{Z})$ in $\overline{\mathsf{LS}}_T$ *$\mathcal{T}$-embeddable* if the category $\overline{\mathsf{RD}}_{\mathcal{T}}(Z,\overline{Z})$ is non-empty. An object $(Z,\overline{Z})$ in $\overline{\mathsf{LS}}_T$ is *locally $\mathcal{T}$-embeddable* if it can be covered by embeddable objects. We denote by $\mathcal{T}$-$\overline{\mathsf{ELS}}_T$ (resp. $\mathcal{T}$-$\overline{\mathsf{LELS}}_T$) the full subcategory of $\overline{\mathsf{LS}}_T$ of $\mathcal{T}$-embeddable (resp. locally $\mathcal{T}$-embeddable) objects.

Example 1.61. Let $(Z,\overline{Z})$ be a T-log scheme with boundary. Assume that $\overline{Z} = \operatorname{Spec} A$ is affine, and that there exist a chart $(\alpha\colon P_{\overline{Z}} \to \mathcal{M}_{\overline{Z}}, \beta\colon \mathbb{N} \to P^{\mathrm{gp}})$ extending c_T and elements $a,b\in P$ such that $\beta(1) = b-a$ and $Z = \operatorname{Spec} A[\frac{1}{\alpha'(a)}]$, where $\alpha'\colon P \to A$ is the map induced by α. Then $(Z,\overline{Z})$ is $\mathcal{T}$-embeddable.

Indeed, take surjections $\gamma\colon k[\mathbb{N}^m] \to A$ and $\delta\colon \mathbb{N}^n \to P$. We set $Q := \mathbb{N}^n \oplus \mathbb{N}$ and $B := \mathscr{O}_F[\mathbb{N}^m \oplus \mathbb{N}^n \oplus \mathbb{N}]^\dagger$, and let $\epsilon\colon Q \to B$ be the canonical injection. We define $\rho\colon B \to A$ by

$$\rho(x\cdot \mathbf{m}\cdot \mathbf{n}\cdot \ell) := \overline{x}\cdot \gamma(\mathbf{m})\cdot \alpha\circ\delta(\mathbf{n})$$

for $x\in \mathscr{O}_F$, $\mathbf{m}\in\mathbb{N}^m$, $\mathbf{n}\in\mathbb{N}^n$, and $\ell\in\mathbb{N}$. Take $a',b'\in\mathbb{N}^n$ with $\epsilon(a') = a$ and $\epsilon(b') = b$, and let $\widetilde{a} := (a',0)\in Q$, $\widetilde{b} := (b',0)\in Q$. Let Q' be the submonoid of Q^{gp} generated by Q and $-\widetilde{a}$, and let $\widetilde{t} := (b'-a',1)\in Q'$. We endow $\overline{\mathcal{Z}} := \operatorname{Spwf} B$ and $\mathcal{Z} := \operatorname{Spwf} B[\frac{1}{\epsilon(\widetilde{a})}]^\dagger$ with the log structures associated to $Q \xrightarrow{\epsilon} B$ and $Q' \to B[\frac{1}{\epsilon(\widetilde{a})}]^\dagger$. We define a map $\eta\colon \mathbb{N}\to Q'$ by $1\mapsto \widetilde{t}$. Then the commutative diagram

$$\begin{tikzcd} Q \arrow[r, "\epsilon"] \arrow[d, "{(\delta,0)}"'] & B \arrow[d, "\rho"] \\ P \arrow[r, "\alpha"] & A \end{tikzcd}$$

induces closed immersions $\overline{Z} \hookrightarrow \overline{\mathcal{Z}}$ and $(Z, \overline{Z}) \hookrightarrow (\mathcal{Z}, \overline{\mathcal{Z}})$. Obviously we have $\ker \eta^{\mathrm{gp}} = 0$ and $\mathrm{Coker}\, \eta^{\mathrm{gp}} = \mathbb{Z}^n$ for $\eta^{\mathrm{gp}} \colon \mathbb{Z} \to Q'^{\mathrm{gp}} = \mathbb{Z}^n \oplus \mathbb{Z}$. Thus $(\mathcal{Z}, \overline{\mathcal{Z}})$ is a strongly smooth $\mathcal{T}$-log scheme with boundary.

The following lemma is immediate from the definition of $\mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$.

Lemma 1.62. *The natural functor $\mathcal{T}\text{-}\overline{\mathsf{ELS}}_T \to \mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$ is a base with respect to the Zariski topology in the sense of Definition 1.14.*

Definition 1.63. For a log rigid boundary triple $((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$ on T over $\mathcal{T}$, we define the log rigid boundary complex by

$$\mathrm{R}\Gamma_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} := \mathrm{R}\Gamma(]\overline{Z}[^{\log}_{\overline{\mathcal{Z}}}, \omega^\bullet_{(\mathcal{Z},\overline{\mathcal{Z}}),\mathbf{Q}}),$$

where $\omega^\bullet_{(\mathcal{Z},\overline{\mathcal{Z}}),\mathbf{Q}}$ is as in Definition 1.24.

Lemma 1.64. *A morphism $((\mathcal{Z}, \overline{\mathcal{Z}}), i) \to ((\mathcal{Z}', \overline{\mathcal{Z}}'), i')$ in $\overline{\mathsf{RD}}_{\mathcal{T}}(Z, \overline{Z})$ induces a quasi-isomorphism of log rigid boundary complexes*

$$R\Gamma_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})_{(\mathcal{Z}',\overline{\mathcal{Z}}')} \to R\Gamma_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})}.$$

Proof. This follows similarly to [22, Lem. 1.4]. □

Definition 1.65. (i) Let $(Z, \overline{Z})$ be an object in $\mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$.

(a) Let $\overline{\mathrm{PT}}_{\mathcal{T}}(Z, \overline{Z})$ be the set of all isomorphism classes of quadruples $(f, (Z', \overline{Z}'), (\mathcal{Z}', \overline{\mathcal{Z}}'), i')$ consisting of a morphism $f \colon (Z, \overline{Z}) \to (Z', \overline{Z}')$ in $\mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$ and a log rigid boundary datum $((\mathcal{Z}', \overline{\mathcal{Z}}'), i')$ for $(Z', \overline{Z}')$ over $\mathcal{T}$. For simplicity, we often denote elements of $\overline{\mathrm{PT}}_{\mathcal{T}}(Z, \overline{Z})$ by small roman letters like a, and the corresponding objects by $(f^{(a)}, (Z^{(a)}, \overline{Z}^{(a)}), (\mathcal{Z}^{(a)}, \overline{\mathcal{Z}}^{(a)}), i^{(a)})$.

(b) Let $\overline{\mathsf{SET}}_{\mathcal{T}}(Z, \overline{Z})$ be the category whose objects are all finite subsets of $\overline{\mathrm{PT}}_{\mathcal{T}}(Z, \overline{Z})$ and whose morphisms are inclusions.

(c) Let $\overline{\mathsf{SET}}^0_{\mathcal{T}}(Z, \overline{Z})$ be the full subcategory of $\overline{\mathsf{SET}}_{\mathcal{T}}(Z, \overline{Z})$ whose objects are all subsets which contain an element a with $f^{(a)} = \mathrm{id}_{(Z,\overline{Z})}$. This is a filtered category.

(d) For $A \in \overline{\mathsf{SET}}^0_{\mathcal{T}}(Z, \overline{Z})$, we define a log rigid boundary datum $((\mathcal{Z}_A, \overline{\mathcal{Z}}_A), i_A)$ for $(Z, \overline{Z})$ by $(\mathcal{Z}_A, \overline{\mathcal{Z}}_A) := \prod_{\mathcal{T}}((\mathcal{Z}^{(a)}, \overline{\mathcal{Z}}^{(a)}))_{a\in A}$ with the diagonal embedding $i_A \colon (Z, \overline{Z}) \hookrightarrow (\mathcal{Z}_A, \overline{\mathcal{Z}}_A)$. Note that $(\mathcal{Z}_A, \overline{\mathcal{Z}}_A)$ is strongly smooth by Lemma 1.21. We define the log rigid complex $\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})$ for $(Z, \overline{Z})$ over $\mathcal{T}$ by

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T}) := \varinjlim_{A\in\overline{\mathsf{SET}}^0_{\mathcal{T}}(Z,\overline{Z})} \mathrm{R}\Gamma_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})_{(\mathcal{Z}_A,\overline{\mathcal{Z}}_A)}. \tag{1.14}$$

(ii) Let $((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$ be a log rigid boundary triple on T over $\mathcal{T}$.

(a) Let $\overline{\mathrm{PT}}_{\mathcal{T}}((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$ be the set of all isomorphism classes of morphisms of log rigid boundary triples over $\mathcal{T}$ whose domain is $((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$. Again we often denote elements of this set by small roman letters like a, and the corresponding objects by

$$(f^{(a)}, F^{(a)}) \colon ((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i) \to ((Z^{(a)}, \overline{Z}^{(a)}), (\mathcal{Z}^{(a)}, \overline{\mathcal{Z}}^{(a)}), i^{(a)}).$$

(b) Let $\overline{\mathsf{SET}}_{\mathcal{T}}((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$ be the category of finite subsets of $\overline{\mathrm{PT}}_{\mathcal{T}}((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$ and with inclusions as morphisms.

(c) Let $\overline{\mathsf{SET}}^0_{\mathcal{T}}((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$ be the full subcategory of $\overline{\mathsf{SET}}_{\mathcal{T}}((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$ whose objects are all subsets which contain the element $\mathrm{id}_{(Z,\overline{Z})}$. This is clearly a filtered category.

(d) For $A \in \overline{\mathsf{SET}}^0_{\mathcal{T}}((Z, \overline{Z}), (\mathcal{Z}, \overline{\mathcal{Z}}), i)$, we define a log rigid boundary datum $((\mathcal{Z}_A, \overline{\mathcal{Z}}_A), i_A)$ for $(Z, \overline{Z})$ by $(\mathcal{Z}_A, \overline{\mathcal{Z}}_A) := \prod_{\mathcal{T}}((\mathcal{Z}^{(a)}, \overline{\mathcal{Z}}^{(a)}))_{a\in A}$ with the diagonal embedding $i_A \colon (Z, \overline{Z}) \hookrightarrow (\mathcal{Z}_A, \overline{\mathcal{Z}}_A)$. Note that $(\mathcal{Z}_A, \overline{\mathcal{Z}}_A)$ is strongly smooth by Lemma 1.21. We define the log rigid complex $\widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})}$ by

$$\widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} := \varinjlim_{A\in\overline{\mathsf{SET}}^0_{\mathcal{T}}((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i)} \mathrm{R}\Gamma_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})_{(\mathcal{Z}_A,\overline{\mathcal{Z}}_A)}. \tag{1.15}$$

Again as in [9, Prop. 4.14] or Proposition 1.29 one can see that the log rigid complexes (1.14) and (1.15) are functorial.

On the one hand, the diagonal morphism $\prod F^{(a)}\colon (\mathscr{Z},\overline{\mathscr{Z}}) \to (\mathscr{Z}_A,\overline{\mathscr{Z}}_A)$ for $A\in\overline{\mathsf{SET}}_{\mathcal{T}}((Z,\overline{Z}),(\mathscr{Z},\overline{\mathscr{Z}}),i)$ defines a morphism of log rigid boundary data $(\mathscr{Z},\overline{\mathscr{Z}},i)\to((\mathscr{Z}_A,\overline{\mathscr{Z}}_A),i_A)$ and hence after taking the limit a canonical quasi-isomorphism

$$\widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathscr{Z},\overline{\mathscr{Z}})} \xrightarrow{\sim} \mathrm{R}\Gamma((Z,\overline{Z})/\mathcal{T})_{(\mathscr{Z},\overline{\mathscr{Z}})}.$$

On the other hand, the forgetful map $\Pi\colon \overline{\mathrm{PT}}_{\mathcal{T}}((Z,\overline{Z}),(\mathscr{Z},\overline{\mathscr{Z}},i))\to\overline{\mathrm{PT}}_{\mathcal{T}}(Z,\overline{Z})$ induces a functor

$$\Pi\colon \overline{\mathsf{SET}}^0_{\mathcal{T}}((Z,\overline{Z}),(\mathscr{Z},\overline{\mathscr{Z}}),i)\to\overline{\mathsf{SET}}^0_{\mathcal{T}}(Z,\overline{Z}).$$

For $A\in\overline{\mathsf{SET}}^0_{\mathcal{T}}((Y,\overline{Z}),(\mathscr{Z},\overline{\mathscr{Z}}),i)$, the canonical projection $\Pi\colon A\to\Pi(A)$ induces a morphism

$$\Delta_A\colon ((\mathscr{Z}_{\Pi(A)},\overline{\mathscr{Z}}_{\Pi(A)}),i_{\Pi(A)})\to((\mathscr{Z}_A,\overline{\mathscr{Z}}_A),i_A)$$

in $\overline{\mathsf{RD}}_{\mathcal{T}}(Z,\overline{Z})$. If we take the limit over $\overline{\mathsf{SET}}^0_{\mathcal{T}}((Z,\overline{Z}),(\mathscr{Z},\overline{\mathscr{Z}}),i)$, we obtain a quasi-isomorphism

$$\Delta\colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathscr{Z},\overline{\mathscr{Z}})} \xrightarrow{\sim} \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}).$$

To summarize we have the following statement.

Proposition 1.66. *The log-rigid complexes defined above define contravariant functors*

$$\begin{aligned}
\mathcal{T}\text{-}\overline{\mathsf{ELS}}_T &\to \mathscr{C}^+_F, && (Z,\overline{Z})\mapsto\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}),\\
\overline{\mathsf{RT}}_{\mathcal{T}} &\to \mathscr{C}^+_F, && ((Z,\overline{Z}),(\mathscr{Z},\overline{\mathscr{Z}}),i)\mapsto\widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathscr{Z},\overline{\mathscr{Z}})},\\
\overline{\mathsf{RT}}_{\mathcal{T}} &\to \mathscr{C}^+_F, && ((Z,\overline{Z}),(\mathscr{Z},\overline{\mathscr{Z}}),i)\mapsto\mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathscr{Z},\overline{\mathscr{Z}})},
\end{aligned}$$

If $((Z,\overline{Z}),(\mathscr{Z},\overline{\mathscr{Z}}),i)$ is a log rigid boundary triple on T over $\mathcal{T}$, there are quasi-isomorphisms

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \xleftarrow{\sim} \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathscr{Z},\overline{\mathscr{Z}})} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathscr{Z},\overline{\mathscr{Z}})},$$

which are functorial in $\overline{\mathsf{RT}}_{\mathcal{T}}$.

Definition 1.67. (i) Let $\overline{\mathfrak{C}}_{\mathcal{T}}$ be the sheafification of the presheaf $Y\mapsto\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})$ on $\mathcal{T}\text{-}\overline{\mathsf{ELS}}_T$ with respect to the Zariski topology. By Proposition 1.2 and Lemma 1.62, we may extend it to a sheaf on $\mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$. For any object $(Z,\overline{Z})$ in $\mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$, we define the log rigid boundary complex by

$$\mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) := \mathrm{R}\Gamma((Z,\overline{Z})_{\mathrm{Zar}},\overline{\mathfrak{C}}_{\mathcal{T}}).$$

It is an object in $\mathscr{D}^+_F$.

(ii) Let $(Z_\bullet,\overline{Z}_\bullet)$ be a simplicial object in $\mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$. Then the Zariski sites of the components form a simplicial site $(Z_\bullet,\overline{Z}_\bullet)_{\mathrm{Zar}}$. The restriction of $\overline{\mathfrak{C}}_{\mathcal{T}}$ defines a sheaf on the total site $(Z_\bullet,\overline{Z}_\bullet)^{\mathrm{tot}}_{\mathrm{Zar}}$, which we denote again by $\overline{\mathfrak{C}}_{\mathcal{T}}$. We define the log rigid boundary complex of$(Z_\bullet,\overline{Z}_\bullet)$ by

$$\mathrm{R}\Gamma_{\mathrm{rig}}((Z_\bullet,\overline{Z}_\bullet)/\mathcal{T}) := \mathrm{R}\Gamma((Z_\bullet,\overline{Z}_\bullet)^{\mathrm{tot}}_{\mathrm{Zar}},\overline{\mathfrak{C}}_{\mathcal{T}}).$$

It is an object in $\mathscr{D}^+_F$.

Proposition 1.68. *Let $(Z,\overline{Z})$ be an object in $\mathcal{T}\text{-}\overline{\mathsf{ELS}}_T$. Then the natural morphism*

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})\to\mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})$$

is a quasi-isomorphism.

Proof. This follows similarly as in Proposition 1.49. □

To conclude this section, we introduce a variant of log rigid complex, where we will be able construct a canonical Frobenius endomorphism. Let $\overline{T}=\operatorname{Proj}k[x_1,x_2]$ and $\overline{\mathcal{T}}=\operatorname{Proj}\mathscr{O}_F[x_1,x_2]^\dagger$ be the fine log scheme and fine weak formal log scheme with the log structures associated to the points $x_1=0$ and $x_2=0$. Take embeddings $T\hookrightarrow\overline{T}$ and $\overline{T}\hookrightarrow\overline{\mathcal{T}}$ defined by the equality $t=\frac{x_2}{x_1}$. Then $(T,\overline{T})$ and $(\mathcal{T},\overline{\mathcal{T}})$ are strongly smooth objects in $\overline{\mathsf{LS}}_T$ and $\overline{\mathsf{LS}}_{\mathcal{T}}$ respectively.

Definition 1.69. Let $\overline{\mathsf{LS}}^\sharp_T \subset \overline{\mathsf{LS}}_T$ be the full subcategory of objects $(Z,\overline{Z})$ whose structure morphism $Z \to T$ extends to $\overline{Z} \to \overline{T}$. Let $\overline{\mathsf{LS}}^\sharp_{\mathcal{T}} \subset \overline{\mathsf{LS}}_{\mathcal{T}}$ be the full subcategory of objects $(\mathcal{Z},\overline{\mathcal{Z}})$ whose structure morphism $\mathcal{Z} \to \mathcal{T}$ extends to $\overline{\mathcal{Z}} \to \overline{\mathcal{T}}$. Note that such an extension is unique if it exists.

Definition 1.70. (i) Let $\overline{\mathsf{RT}}^\sharp_{\mathcal{T}} \subset \overline{\mathsf{RT}}_{\mathcal{T}}$ be the full subcategory of log rigid boundary triples $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i)$ with $(Z,\overline{Z}) \in \overline{\mathsf{LS}}^\sharp_T$ and $(\mathcal{Z},\overline{\mathcal{Z}}) \in \overline{\mathsf{LS}}^\sharp_{\mathcal{T}}$.

(ii) For $(Z,\overline{Z})$ in $\overline{\mathsf{LS}}^\sharp_T$, let $\overline{\mathsf{RD}}^\sharp_{\mathcal{T}}(Z,\overline{Z})$ be the full subcategory of $\overline{\mathsf{RD}}_{\mathcal{T}}(Z,\overline{Z})$ consisting of objects $((\mathcal{Z},\overline{\mathcal{Z}}),i)$ with $(\mathcal{Z},\overline{\mathcal{Z}}) \in \overline{\mathsf{LS}}^\sharp_{\mathcal{T}}$.

(iii) Let $\mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T \subset \mathcal{T}\text{-}\overline{\mathsf{ELS}}_T$ be the full subcategory of objects $(Z,\overline{Z})$ such that $\overline{\mathsf{RD}}^\sharp_{\mathcal{T}}(Z,\overline{Z})$ is non-empty. Let $\mathcal{T}\text{-}\overline{\mathsf{LELS}}^\sharp_T \subset \mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$ be the full subcategory of objects which can be covered by objects in $\mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T$.

(iv) For an object $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i)$ in $\overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$, we define the log rigid complex

$$\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} := \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})}.$$

As before we can define the sets $\overline{\mathrm{PT}}^\sharp_{\mathcal{T}}(Z,\overline{Z})$ and $\overline{\mathrm{PT}}^\sharp_{\mathcal{T}}((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i)$, as well as the categories $\overline{\mathsf{SET}}^{\sharp,0}_{\mathcal{T}}(Z,\overline{Z})$ and $\overline{\mathsf{SET}}^{\sharp,0}_{\mathcal{T}}((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i)$, and rigid complexes $\widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})$, $\widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})}$ and $\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})$, $\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z_\bullet,\overline{Z}_\bullet)/\mathcal{T})$.

Proposition 1.71. *The log rigid complexes above define functors*

$$\begin{aligned}
\mathcal{T}\text{-}\overline{\mathsf{LELS}}^\sharp_T &\to \mathscr{C}^+_F, && (Z,\overline{Z}) \mapsto \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})\\
\mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T &\to \mathscr{C}^+_F, && (Z,\overline{Z}) \mapsto \widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}),\\
\overline{\mathsf{RT}}^\sharp_{\mathcal{T}} &\to \mathscr{C}^+_F, && ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \mapsto \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})},\\
\overline{\mathsf{RT}}^\sharp_{\mathcal{T}} &\to \mathscr{C}^+_F, && ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \mapsto \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})},
\end{aligned}$$

If $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i)$ is an object in $\overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$, there are quasi-isomorphisms

$$\widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \xleftarrow{\sim} \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \xrightarrow{\sim} \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})},$$

which are functorial in $\overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$. If $(Z,\overline{Z})$ is an object in $\mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T$, there is a canonical quasi-isomorphism

$$\widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \xrightarrow{\sim} \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}).$$

1.4 Base change and comparison of rigid complexes

In this subsection we will show that there are canonical morphisms between the canonical rigid complexes $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$, $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F)$, $\mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})$, $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K)$, $\mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})$, and $\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})$ in appropriate situations. We introduce similar morphisms for various auxiliary complexes that appeared in the previous sections and verify the compatibilities of these morphisms. Moreover, we define base change by Frobenius on the complexes $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F)$, $\mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})$, and $\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})$ and show that it is compatible with the above comparison maps.

First we will see the construction of a canonical quasi-isomorphism $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}^0_F)$. The functors

$$\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0} \to \mathsf{LS}_{k^0},\ (Y,\overline{Y}) \mapsto \overline{Y} \qquad \text{and} \qquad \overline{\mathsf{LS}}^{\mathrm{ss}}_{\mathcal{T}} \to \mathsf{LS}_{\mathscr{O}^0_F},\ (\mathcal{Z},\overline{\mathcal{Z}}) \mapsto \overline{\mathcal{Y}} := \overline{\mathcal{Z}} \times_{\mathcal{T}} \mathscr{O}^0_F$$

together induce functors $\mathsf{RQ}_{\mathrm{HK}} \to \mathsf{RT}_{\mathscr{O}^0_F}$ and $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0} \to \mathscr{O}^0_F\text{-}\mathsf{ELS}_{k^0}$. Moreover they induce maps

$$\begin{aligned}
\widehat{\psi}_{\mathrm{HK}}\colon \mathrm{PQ}_{\mathrm{HK}}(Y,\overline{Y}) &\to \mathrm{PT}_{\mathscr{O}^0_F}(\overline{Y}) && \text{for } (Y,\overline{Y}) \in \overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0},\\
\widetilde{\psi}_{\mathrm{HK}}\colon \mathrm{PQ}_{\mathrm{HK}}((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi) &\to \mathrm{PT}_{\mathscr{O}^0_F}(\overline{Y},\overline{\mathcal{Y}},i) && \text{for } ((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi) \in \mathsf{RQ}_{\mathrm{HK}},
\end{aligned}$$

and therefore maps

$$\begin{aligned}
&\widehat{\psi}_{\mathrm{HK}}\colon \mathsf{SET}^0_{\mathrm{HK}}(Y,\overline{Y}) \to \mathsf{SET}^0_{\mathscr{O}_F^0}(\overline{Y}) && A \mapsto \widehat{\psi}_{\mathrm{HK}}(A),\\
&\widetilde{\psi}_{\mathrm{HK}}\colon \mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi) \to \mathsf{SET}^0_{\mathscr{O}_F^0}(\overline{Y},\overline{\mathcal{Y}},i) && A \mapsto \widetilde{\psi}_{\mathrm{HK}}(A),
\end{aligned}$$

such that the diagram

$$\begin{tikzcd}
\mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi) \arrow[r, "\widetilde{\psi}_{\mathrm{HK}}"] \arrow[d] & \mathsf{SET}^0_{\mathscr{O}_F^0}(\overline{Y},\overline{\mathcal{Y}},i) \arrow[d] \\
\mathsf{SET}^0_{\mathrm{HK}}(Y,\overline{Y}) \arrow[r, "\widehat{\psi}_{\mathrm{HK}}"'] & \mathsf{SET}^0_{\mathscr{O}_F^0}(\overline{Y}).
\end{tikzcd}$$

commutes. For $((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi) \in \mathsf{RQ}_{\mathrm{HK}}$ we have a canonical quasi-isomorphism

$$\theta_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}} \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})}$$

by Lemma 1.29.

Proposition 1.72. *There exist canonical isomorphisms in $\mathscr{D}_F^+$*

$$\begin{aligned}
&\widehat{\theta}_{\mathrm{HK}}\colon \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0) \xrightarrow{\sim} \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) && \textit{for } (Y,\overline{Y}) \in \overline{\mathit{ELS}}^{ss}_{k^0},\\
&\widetilde{\theta}_{\mathrm{HK}}\colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}} \xrightarrow{\sim} \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})} && \textit{for } ((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi) \in \mathit{RQ}_{\mathrm{HK}},
\end{aligned}$$

which are functorial on $\overline{\mathit{ELS}}^{ss}_{k^0}$ and $\mathit{RQ}_{\mathrm{HK}}$ respectively, and make the diagram

$$\begin{tikzcd}
\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0) \arrow[d, "\widehat{\theta}_{\mathrm{HK}}"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}} \arrow[d, "\theta_{\mathrm{HK}}"] \\
\widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) & \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})}
\end{tikzcd}$$

commutative.

Proof. On the on hand, the natural projection $\overline{\mathscr{Z}}_A \to \overline{\mathscr{Z}}^{(a)}$ for $A \in \mathsf{SET}^0_{\mathrm{HK}}(Y,\overline{Y})$ and $a \in A$ induces a morphism

$$\overline{\mathcal{Y}}_A := \overline{\mathscr{Z}}_A \times_{\mathcal{T}} \mathscr{O}_F^0 \to \overline{\mathscr{Z}}^{(a)} \times_{\mathcal{T}} \mathscr{O}_F^0 =: \overline{\mathcal{Y}}^{(a)},$$

which together form the diagonal morphism $\overline{\mathcal{Y}}_A \to \prod_{\mathscr{O}_F^0}(\overline{\mathcal{Y}}^{(a)})_{a\in A}$. On the other hand, we also have the diagonal morphism $\overline{\mathcal{Y}}_{\widehat{\psi}_{\mathrm{HK}}(A)} \to \prod_{\mathscr{O}_F^0}(\overline{\mathcal{Y}}^{(a)})_{a\in A}$, where $\overline{\mathcal{Y}}_{\widehat{\psi}_{\mathrm{HK}}(A)}$ denotes the log rigid datum over $\mathscr{O}_F^0$ for $\overline{Y}$ associated to $\widehat{\psi}_{\mathrm{HK}}(A)$. Similarly, for $B \in \mathsf{SET}^0_{\mathrm{HK}}((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)$ we have natural morphisms $\overline{\mathcal{Y}}_B \to \prod_{\mathscr{O}_F^0}(\overline{\mathcal{Y}}^{(b)})_{b\in B} \leftarrow \overline{\mathcal{Y}}_{\widetilde{\psi}_{\mathrm{HK}}(B)}$. They induce a commutative diagram in $\mathscr{C}_F^+$

$$\begin{tikzcd}
\varinjlim_A \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z}_A,\overline{\mathscr{Z}}_A)} \arrow[d, "\sim"'] & \varinjlim_B \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z}_B,\overline{\mathscr{Z}}_B)} \arrow[l, "\sim"'] \arrow[d, "\sim"'] \arrow[dr, "\sim"] & \\
\varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}_A} & \varinjlim_B \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}_B} \arrow[l, "\sim"'] \arrow[dr, "\sim"] & \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})} \arrow[d, "\sim"'] \\
\varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\prod_{\mathscr{O}_F^0}(\overline{\mathcal{Y}}^{(a)})_{a\in A}} \arrow[u, "\sim"] \arrow[d, "\sim"'] & \varinjlim_B \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\prod_{\mathscr{O}_F^0}(\overline{\mathcal{Y}}^{(b)})_{b\in B}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\sim"] \arrow[d, "\sim"'] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}} \\
\varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\mathcal{Y}_{\widehat{\psi}_{\mathrm{HK}}(A)}} \arrow[d, "\sim"'] & \varinjlim_B \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\mathcal{Y}_{\widetilde{\psi}_{\mathrm{HK}}(A)}} \arrow[l, "\sim"'] \arrow[ur, "\sim"] \arrow[d, "\sim"'] & \\
\varinjlim_C \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\mathcal{Y}_C} & \varinjlim_D \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)_{\mathcal{Y}_D} \arrow[l, "\sim"'] \arrow[uur, "\sim"] & ,
\end{tikzcd} \tag{1.16}$$

where A, B, C, D run through the sets $\mathsf{SET}^0_{\mathrm{HK}}(Y, \overline{Y})$, $\mathsf{SET}^0_{\mathrm{HK}}((Y, \overline{Y}), (\mathcal{Z}, \overline{\mathcal{Z}}), i, \phi)$, $\mathsf{SET}^0_{\mathscr{O}^0_F}(\overline{Y})$, and $\mathsf{SET}^0_{\mathscr{O}^0_F}(\overline{Y}, \overline{\mathcal{Y}}, i)$ respectively. The compositions of the left and middle vertical arrows are $\widehat{\theta}_{\mathrm{HK}}$ and $\widetilde{\theta}_{\mathrm{HK}}$. One can see the functoriality of this diagram by similar arguments as in Proposition 1.42. □

By taking the sheafification of the left vertical arrows in diagram (1.16), we obtain the following corollary through the equivalences of topoi induced by $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0} \to \overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$.

Corollary 1.73. *There is a functorial isomorphism in $\mathscr{D}^+_F$*

$$\Theta_{\mathrm{HK}} \colon \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}^0_F) \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})$$

for $(Y, \overline{Y}) \in \overline{\mathsf{LS}}^{ss}_{k^0}$, such that the diagram

$$\begin{tikzcd}
\widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}/\mathscr{O}^0_F) \arrow[r, "\sim"] \arrow[d, "\widehat{\theta}_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}^0_F) \arrow[d, "\Theta_{\mathrm{HK}}"] \\
\widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y}) \arrow[r, "\sim"] & \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y}),
\end{tikzcd}$$

where the horizontal arrows are given by Proposition 1.49 and Proposition 1.59, commutes if $(Y, \overline{Y})$ is HK*-embeddable.*

Next we study the existence of base change maps $\mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F)$ and $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K)$. The functors

$$\mathsf{LS}_{\mathcal{T}} \to \mathsf{LS}_{\mathscr{O}^0_F},\ \mathcal{Z} \mapsto \mathcal{Y} := \mathcal{Z} \times_{\mathcal{T}} \mathscr{O}^0_F,$$
$$\mathsf{LS}_{\mathcal{T}} \to \mathsf{LS}_{\mathscr{O}^\times_K},\ \mathcal{Z} \mapsto \mathcal{X} := \mathcal{Z} \times_{\mathcal{T}} \mathscr{O}^\pi_K$$

together with

$$\mathsf{LS}_T \to \mathsf{LS}_{k^0},\ Z \mapsto Y := Z \times_T k^0$$

induce functors

$$\mathsf{RT}_{\mathcal{T}} \to \mathsf{RT}_{\mathscr{O}^0_F} \text{ and } \quad \mathcal{T}\text{-}\mathsf{ELS}_T \to \mathscr{O}^0_F\text{-}\mathsf{ELS}_{k^0} \quad \text{as well as}$$
$$\mathsf{RT}_{\mathcal{T}} \to \mathsf{RT}_{\mathscr{O}^\times_K} \text{ and } \quad \mathcal{T}\text{-}\mathsf{ELS}_T \to \mathscr{O}^\times_K\text{-}\mathsf{ELS}_{k^0}.$$

For $(Z, \mathcal{Z}, i) \in \mathsf{RT}_{\mathcal{T}}$ there are canonical morphisms

$$\theta_0 \colon \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F)_{\mathcal{Y}} \qquad \text{and} \qquad \theta_\pi \colon \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K)_{\mathcal{X}}.$$

Proposition 1.74. *There exist canonical morphisms*

$$\begin{aligned}
&\Theta_0 \colon \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F), && \Theta_\pi \colon \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K) && \textit{in } \mathscr{D}^+_{F[t]^\dagger} && \textit{for } Z \in \mathsf{LS}_T,\\
&\widehat{\theta}_0 \colon \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) \to \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathscr{O}^0_F), && \widehat{\theta}_\pi \colon \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) \to \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K) && \textit{in } \mathscr{C}^+_{F[t]^\dagger} && \textit{for } Z \in \mathcal{T}\text{-}\mathsf{ELS}_T,\\
&\widetilde{\theta}_0 \colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \to \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathscr{O}^0_F)_{\mathcal{Y}}, && \widetilde{\theta}_\pi \colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \to \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K)_{\mathcal{X}} && \textit{in } \mathscr{C}^+_{F[t]^\dagger} && \textit{for } (Z, \mathcal{Z}, i) \in \mathsf{RT}_{\mathcal{T}},
\end{aligned}$$

which are functorial on LS_T, $\mathcal{T}$-ELS_T and $\mathsf{RT}_{\mathcal{T}}$ respectively, such that for $(Z, \mathcal{Z}, i) \in \mathsf{RT}_{\mathcal{T}}$ the diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F) & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathscr{O}^0_F) \arrow[l, "\sim"'] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathscr{O}^0_F)_{\mathcal{Y}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F)_{\mathcal{Y}} \\
\mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T}) \arrow[u, "\Theta_0"] \arrow[d, "\Theta_\pi"] & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) \arrow[l, "\sim"'] \arrow[u, "\widehat{\theta}_0"] \arrow[d, "\widehat{\theta}_\pi"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\widetilde{\theta}_0"] \arrow[d, "\widetilde{\theta}_\pi"] & \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \arrow[u, "\theta_0"] \arrow[d, "\theta_\pi"] \\
\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K) & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K) \arrow[l, "\sim"'] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K)_{\mathcal{X}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K)_{\mathcal{X}}.
\end{tikzcd}$$

becomes commutative. Here we regard log rigid complexes over $\mathscr{O}^0_F$ and $\mathscr{O}^\pi_K$ as objects in $\mathscr{C}^+_{F[t]^\dagger}$ or $\mathscr{D}^+_{F[t]^\dagger}$ via the maps $F[t]^\dagger \to F,\ t \mapsto 0$ and $F[t]^\dagger \to K,\ t \mapsto \pi$.

Proof. The statement follows by similar arguments as Proposition 1.72 and Corollary 1.73. Note that for $A \in \mathsf{SET}^0_{\mathcal{T}}(Z)$ we have diagonal morphisms

$$\mathcal{Y}_{\widehat{\psi}_0(A)} \to \prod_{\mathcal{O}_F^0} (\mathcal{Y}^{(a)})_{a\in A} = \mathcal{Y}_A := \mathcal{Z}_A \times_{\mathcal{T}} \mathcal{O}_F^0 \quad \text{and} \quad \mathcal{X}_{\widehat{\psi}_\pi(A)} \to \prod_{\mathcal{O}_K^\pi} (\mathcal{X}^{(a)})_{a\in A} = \mathcal{X}_A := \mathcal{Z}_A \times_{\mathcal{T}} \mathcal{O}_K^\pi,$$

where $\widehat{\psi}_0 \colon \mathsf{SET}^0_{\mathcal{T}}(Z) \to \mathsf{SET}^0_{\mathcal{O}_F^0}(Y)$ and $\widehat{\psi}_\pi \colon \mathsf{SET}^0_{\mathcal{T}}(Z) \to \mathsf{SET}^0_{\mathcal{O}_K^\pi}(Y)$ are defined in the obvious way. □

We come to the existence of a canonical morphism $\mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z}) \times_T (T,\overline{T})/\mathcal{T})$. The functors

$$\begin{aligned} &\overline{\mathsf{LS}}_T \to \overline{\mathsf{LS}}^\sharp_T, \quad (Z,\overline{Z}) \mapsto (Z,\overline{Z}^\sharp) := (Z,\overline{Z}) \times_T (T,\overline{T}) \quad \text{and} \\ &\overline{\mathsf{LS}}_{\mathcal{T}} \to \overline{\mathsf{LS}}^\sharp_{\mathcal{T}}, \quad (\mathcal{Z},\overline{\mathcal{Z}}) \mapsto (\mathcal{Z},\overline{\mathcal{Z}}^\sharp) := (\mathcal{Z},\overline{\mathcal{Z}}) \times_{\mathcal{T}} (\mathcal{T},\overline{\mathcal{T}}) \end{aligned}$$

together induce functors $\overline{\mathsf{RT}}_{\mathcal{T}} \to \overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$ and $\mathcal{T}\text{-}\overline{\mathsf{ELS}}_T \to \mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T$. Note that by Lemma 1.21 the functor $\overline{\mathsf{LS}}_{\mathcal{T}} \to \overline{\mathsf{LS}}^\sharp_{\mathcal{T}}$ preserves strong smoothness. For $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}_{\mathcal{T}}$ the natural projections $(Z,\overline{Z}^\sharp) \to (Z,\overline{Z})$ and $(\mathcal{Z},\overline{\mathcal{Z}}^\sharp) \to (\mathcal{Z},\overline{\mathcal{Z}})$ induce a canonical morphism

$$\theta^\sharp \colon \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}}^\sharp)}.$$

Proposition 1.75. *There exist canonical morphisms in* $\mathscr{D}^+_F$

$$\begin{aligned} &\Theta^\sharp \colon \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T}) && \text{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{LELS}}_T, \\ &\widehat{\theta}^\sharp \colon \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T}) && \text{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{ELS}}_T, \\ &\widetilde{\theta}^\sharp \colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}}^\sharp)} && \text{for } ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}_{\mathcal{T}}, \end{aligned}$$

which are functorial on $\mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$, $\mathcal{T}\text{-}\overline{\mathsf{ELS}}_T$ *and* $\overline{\mathsf{RT}}_{\mathcal{T}}$ *respectively, such that for* $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}_{\mathcal{T}}$ *the diagram*

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \arrow[d, "\Theta^\sharp"] & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \arrow[l, "\sim"'] \arrow[d, "\widehat{\theta}^\sharp"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}^\sharp"] & \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[d, "\theta^\sharp"] \\ \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T}) & \widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T}) \arrow[l, "\sim"'] & \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}}^\sharp)} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}}^\sharp)} \end{tikzcd}$$

commutes.

Proof. The statement follows similarly as Proposition 1.72 and Corollary 1.73. Note that for $A \in \overline{\mathsf{SET}}^0_{\mathcal{T}}(Z,\overline{Z})$ we have diagonal morphisms $(\mathcal{Z}_A,\overline{\mathcal{Z}}^\sharp_A) \to \prod_{\mathcal{T}}((\mathcal{Z}^{(a)},\overline{\mathcal{Z}}^{(a),\sharp}))_{a\in A}$ and $(\mathcal{Z}_{\widehat{\psi}^\sharp(A)},\overline{\mathcal{Z}}_{\widehat{\psi}^\sharp(A)}) \to \prod_{\mathcal{T}}((\mathcal{Z}^{(a)},\overline{\mathcal{Z}}^{(a),\sharp}))_{a\in A}$, where $\widehat{\psi}^\sharp \colon \overline{\mathsf{SET}}^0_{\mathcal{T}}(Z,\overline{Z}) \to \overline{\mathsf{SET}}^{\sharp,0}_{\mathcal{T}}(Z,\overline{Z}^\sharp)$ is defined in the obvious manner. □

The next point is the existence of maps $\mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})$ and $\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})$. The functors

$$\begin{aligned} &\overline{\mathsf{LS}}_T \to \mathsf{LS}_T,\ (Z,\overline{Z}) \mapsto Z, && \overline{\mathsf{LS}}^\sharp_T \to \mathsf{LS}_T,\ (Z,\overline{Z}) \mapsto Z, \\ &\overline{\mathsf{LS}}_{\mathcal{T}} \to \mathsf{LS}_{\mathcal{T}},\ (\mathcal{Z},\overline{\mathcal{Z}}) \mapsto \mathcal{Z} && \overline{\mathsf{LS}}^\sharp_{\mathcal{T}} \to \mathsf{LS}_{\mathcal{T}},\ (\mathcal{Z},\overline{\mathcal{Z}}) \mapsto \mathcal{Z} \end{aligned}$$

induce functors $\overline{\mathsf{RT}}_{\mathcal{T}} \to \mathsf{RT}_{\mathcal{T}}$ and $\mathcal{T}\text{-}\overline{\mathsf{ELS}}_T \to \mathcal{T}\text{-}\mathsf{ELS}_T$, as well as $\overline{\mathsf{RT}}^\sharp_{\mathcal{T}} \to \mathsf{RT}_{\mathcal{T}}$ and $\mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T \to \mathcal{T}\text{-}\mathsf{ELS}_T$, since $\overline{\mathsf{LS}}_{\mathcal{T}} \to \mathsf{LS}_{\mathcal{T}}$ preserves smoothness. For $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}_{\mathcal{T}}$ (resp. $\overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$) there is a canonical morphism

$$\theta_b \colon \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \qquad (\text{resp. } \theta^\sharp_b \colon \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})_{\overline{\mathcal{Z}}}).$$

As before we obtain the following result.

Proposition 1.76. *There exist canonical morphisms*

$$\begin{array}{lll}
\Theta_b\colon \mathrm{R\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) & \text{in } \mathscr{D}_F^+ & \text{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{LELS}}_T,\\
\widehat{\theta}_b\colon \widehat{\mathrm{R\Gamma}}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \widehat{\mathrm{R\Gamma}}_{\mathrm{rig}}(Z/\mathcal{T}) & \text{in } \mathscr{C}_F^+ & \text{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{ELS}}_T,\\
\widetilde{\theta}_b\colon \widetilde{\mathrm{R\Gamma}}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \widetilde{\mathrm{R\Gamma}}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} & \text{in } \mathscr{C}_F^+ & \text{for } ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}_{\mathcal{T}},\\
\Theta_b^\sharp\colon \mathrm{R\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) & \text{in } \mathscr{D}_F^+ & \text{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{LELS}}^\sharp_T,\\
\widehat{\theta}_b^\sharp\colon \widehat{\mathrm{R\Gamma}}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \widehat{\mathrm{R\Gamma}}_{\mathrm{rig}}(Z/\mathcal{T}) & \text{in } \mathscr{C}_F^+ & \text{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T,\\
\widetilde{\theta}_b^\sharp\colon \widetilde{\mathrm{R\Gamma}}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \widetilde{\mathrm{R\Gamma}}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} & \text{in } \mathscr{C}_F^+ & \text{for } ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}^\sharp_{\mathcal{T}},
\end{array}$$

which are functorial on $\mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$, $\mathcal{T}\text{-}\overline{\mathsf{LELS}}^\sharp_T$, $\mathcal{T}\text{-}\overline{\mathsf{ELS}}_T$, $\mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T$, $\overline{\mathsf{RT}}_{\mathcal{T}}$, and $\overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$ respectively, such that the diagrams

$$\begin{tikzcd}
\mathrm{R\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \arrow[d, "\Theta_b"] & \widehat{\mathrm{R\Gamma}}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \arrow[l, "\sim"'] \arrow[d, "\widehat{\theta}_b"] & \widetilde{\mathrm{R\Gamma}}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}_b"] & \mathrm{R\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[d, "\theta_b"] \\
\mathrm{R\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) & \widehat{\mathrm{R\Gamma}}_{\mathrm{rig}}(Z/\mathcal{T}) \arrow[l, "\sim"'] & \widetilde{\mathrm{R\Gamma}}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R\Gamma}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}}.
\end{tikzcd}$$

for $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}_{\mathcal{T}}$, and

$$\begin{tikzcd}
\mathrm{R\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \arrow[d, "\Theta_b^\sharp"] & \widehat{\mathrm{R\Gamma}}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \arrow[l, "\sim"'] \arrow[d, "\widehat{\theta}_b^\sharp"] & \widetilde{\mathrm{R\Gamma}}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}_b^\sharp"] & \mathrm{R\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[d, "\theta_b^\sharp"] \\
\mathrm{R\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) & \widehat{\mathrm{R\Gamma}}_{\mathrm{rig}}(Z/\mathcal{T}) \arrow[l, "\sim"'] & \widetilde{\mathrm{R\Gamma}}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R\Gamma}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}}.
\end{tikzcd}$$

for $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$ commutative. Here we regard log rigid complexes of Z over $\mathcal{T}$ as objects in $\mathscr{A}_F^+$ or $\mathscr{D}_F^+$ through the map $F \to F[t]^\dagger$.

Proof. The statement follows by similar arguments as Proposition 1.72 and Corollary 1.73. Note that for $A \in \overline{\mathsf{SET}}^0_{\mathcal{T}}(Z,\overline{Z})$ and $B \in \overline{\mathsf{SET}}^{\sharp,0}_{\mathcal{T}}(Z,\overline{Z})$ we have the diagonal morphisms $\mathcal{Z}_{\widehat{\psi}_b(A)} \to \mathcal{Z}_A$ and $\mathcal{Z}_{\widehat{\psi}^\sharp_b(A)} \to \mathcal{Z}_A$, where $\widehat{\psi}_b\colon \overline{\mathsf{SET}}^0_{\mathcal{T}}(Z,\overline{Z}) \to \mathsf{SET}^0_{\mathcal{T}}(Z)$ and $\widehat{\psi}^\sharp_b\colon \overline{\mathsf{SET}}^{\sharp,0}_{\mathcal{T}}(Z,\overline{Z}) \to \mathsf{SET}^0_{\mathcal{T}}(Z)$ are defined in the obvious way. $\square$

Lastly, we explain the base change by Frobenius. Let $S \hookrightarrow \mathcal{S}$ be one of $k^0 \hookrightarrow \mathscr{O}_F^0$, $k^0 \hookrightarrow \mathscr{O}_K^\pi$, or $T \hookrightarrow \mathcal{T}$.

Base changes by σ define endofunctors on LS_{k^0}, LS_T, $\mathsf{LS}_{\mathscr{O}_F^0}$, $\mathsf{LS}_{\mathcal{T}}$, and hence on $\mathsf{RT}_{\mathcal{S}}$ and $\mathcal{S}$-ELS_S. In each case we denote the base change of Y by $Y^\sigma := Y \times_{S,\sigma} S$ or $\mathcal{Y}^\sigma := \mathcal{Y} \times_{\mathcal{S},\sigma} \mathcal{S}$, and regard them as objects in LS_S and $\mathsf{LS}_{\mathcal{S}}$ via the canonical projection to S and $\mathcal{S}$. For $(Y,\mathcal{Y},i) \in \mathsf{RT}_{\mathcal{S}}$ we have a canonical morphism

$$\theta_\sigma\colon \mathrm{R\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} \to \mathrm{R\Gamma}_{\mathrm{rig}}(Y^\sigma/\mathcal{S})_{\mathcal{Y}^\sigma}.$$

Moreover for $(Z,\overline{Z}) \in \overline{\mathsf{LS}}^\sharp_T$ we define an object $(Z^\sigma,\overline{Z}^\sigma) \in \overline{\mathsf{LS}}^\sharp_T$ as the product $(Z,\overline{Z}) \times_{T,\sigma} (T,\overline{T})$ of fine log schemes with boundary, with structure morphism $Z^\sigma \to T$ given by the canonical projection. Similarly we define $(\mathcal{Z}^\sigma,\overline{\mathcal{Z}}^\sigma) \in \overline{\mathsf{LS}}^\sharp_{\mathcal{T}}$ for $(\mathcal{Z},\overline{\mathcal{Z}}) \in \overline{\mathsf{LS}}^\sharp_{\mathcal{T}}$. These define endofunctors on $\overline{\mathsf{LS}}^\sharp_T$ and $\overline{\mathsf{LS}}^\sharp_{\mathcal{T}}$.

Lemma 1.77. *Let $(\mathcal{Z},\overline{\mathcal{Z}})$ be an object in $\overline{\mathsf{LS}}^\sharp_{\mathcal{T}}$. Then $(\mathcal{Z}^\sigma,\overline{\mathcal{Z}}^\sigma)$ is strongly smooth if $(\mathcal{Z},\overline{\mathcal{Z}})$ is strongly smooth.*

Proof. This follows from local computations. Without loss of generality, we assume that there is a chart $(\alpha\colon P_{\overline{\mathcal{Z}}} \to \mathcal{N}_{\overline{\mathcal{Z}}},\ \beta\colon \mathbf{N} \to P^{\mathrm{gp}})$ for $(\mathcal{Z},\overline{\mathcal{Z}})$ and $a,b \in P$ as in Definition 1.17. Let $\mathcal{U}$ and $\overline{\mathcal{U}}$ be the strict open weak formal log subschemes of $\overline{\mathcal{T}}$ defined by $\mathrm{Spwf}\,\mathscr{O}_F[s]^\dagger$ and $\mathrm{Spwf}\,\mathscr{O}_F[s^{\pm1}]^\dagger$ defined by the equality $s = \frac{x_1}{x_2}$. Then $(\mathcal{T},\overline{\mathcal{T}})$ is covered by $(\mathcal{T},\mathcal{T})$ and $(\mathcal{U},\overline{\mathcal{U}})$, and hence $(\mathcal{Z}^\sigma,\overline{\mathcal{Z}}^\sigma)$ is covered by $(\mathcal{U}_+,\overline{\mathcal{U}}_+) := (\mathcal{Z},\overline{\mathcal{Z}}) \times_{\mathcal{T},\sigma} (\mathcal{T},\mathcal{T})$ and $(\mathcal{U}_-,\overline{\mathcal{U}}_-) := (\mathcal{Z},\overline{\mathcal{Z}}) \times_{\mathcal{T},\sigma} (\mathcal{U},\overline{\mathcal{U}})$. For $\varepsilon \in \{+,-\}$, let Q_ε be the image of

$$P \oplus \mathbf{N} \to P^{\mathrm{gp}} \oplus_{\mathbf{Z},p} \mathbf{Z} := \mathrm{Coker}((\beta^{\mathrm{gp}},\varepsilon p)\colon \mathbf{Z} \to P^{\mathrm{gp}} \oplus \mathbf{Z}).$$

Let $\gamma_\varepsilon\colon Q_{\varepsilon,\overline{\mathcal{U}}_\varepsilon} \to \mathcal{N}_{\overline{\mathcal{U}}_\varepsilon}$ be the morphism induced by α and by

$$\begin{aligned} \mathbf{N}_{\mathcal{T}} \to \mathcal{N}_{\mathcal{T}},\ 1 \mapsto t \quad &\text{if} \quad \varepsilon = +,\\ \mathbf{N}_{\overline{\mathcal{U}}} \to \mathcal{N}_{\overline{\mathcal{U}}},\ 1 \mapsto s \quad &\text{if} \quad \varepsilon = -. \end{aligned}$$

Let $\delta_\varepsilon\colon \mathbf{N} \to Q_\varepsilon^{\mathrm{gp}}$ be the composition of $\mathbf{N} \to P^{\mathrm{gp}} \oplus \mathbf{Z},\ 1 \mapsto (0, \varepsilon 1)$ and the natural surjection $P^{\mathrm{gp}} \oplus \mathbf{Z} \to P^{\mathrm{gp}} \oplus_{\mathbf{Z},p} \mathbf{Z} = Q^{\mathrm{gp}}$. Set

$$\begin{aligned} a_+ &:= (a,0), b_+ := (b,0) \in Q_+,\\ a_- &:= (a,1), b_- := (b,1) \in Q_-. \end{aligned}$$

Since we have $\operatorname{Ker} \delta_\varepsilon^{\mathrm{gp}} \cong \operatorname{Ker} \alpha_\varepsilon^{\mathrm{gp}}$ and $\operatorname{Coker} \delta_\varepsilon^{\mathrm{gp}} \cong \operatorname{Coker} \alpha_\varepsilon^{\mathrm{gp}}$, we see that the chart $(\gamma_\varepsilon, \delta_\varepsilon)$ for $(\mathcal{U}_\varepsilon, \overline{\mathcal{U}}_\varepsilon)$ with $a_\varepsilon, b_\varepsilon \in Q_\varepsilon$ satisfies the conditions in Definition 1.17. $\square$

For $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$ we have a canonical morphism

$$\theta^\sharp_\sigma\colon \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T})_{(\mathcal{Z}^\sigma,\overline{\mathcal{Z}}^\sigma)}.$$

As before we have the following result.

Proposition 1.78. *There exist canonical morphisms*

$$\begin{array}{lll}
\Theta_\sigma\colon \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S}) \to \mathrm{R}\Gamma(Y^\sigma/\mathcal{S}) & \text{in } \mathscr{D}^+_{\mathbf{Q}_p} & \text{for } Y \in \mathsf{LS}_S,\\
\widehat{\theta}_\sigma\colon \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S}) \to \widehat{\mathrm{R}\Gamma}(Y^\sigma/\mathcal{S}) & \text{in } \mathscr{A}^+_{\mathbf{Q}_p} & \text{for } Y \in \mathcal{S}\text{-}\mathsf{ELS}_S,\\
\widetilde{\theta}_\sigma\colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} \to \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y^\sigma/\mathcal{S})_{\mathcal{Y}^\sigma} & \text{in } \mathscr{A}^+_{\mathbf{Q}_p} & \text{for } (Y,\mathcal{Y},i) \in \mathsf{RT}_{\mathcal{S}},\\
\Theta^\sharp_\sigma\colon \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T}) & \text{in } \mathscr{D}^+_{\mathbf{Q}_p} & \text{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{LELS}}^\sharp_T,\\
\widehat{\theta}^\sharp_\sigma\colon \widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T}) & \text{in } \mathscr{A}^+_{\mathbf{Q}_p} & \text{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T,\\
\widetilde{\theta}^\sharp_\sigma\colon \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T})_{(\mathcal{Z}^\sigma,\overline{\mathcal{Z}}^\sigma)} & \text{in } \mathscr{A}^+_{\mathbf{Q}_p} & \text{for } ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}^\sharp_{\mathcal{T}},
\end{array}$$

which are functorial on LS_S, $\mathcal{T}\text{-}\overline{\mathsf{LELS}}^\sharp_T$, $\mathcal{S}\text{-}\mathsf{ELS}_S$, $\mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T$, $\mathsf{RT}_{\mathcal{S}}$, *and* $\overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$ *respectively, and make the following diagram commutative;*

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S}) \arrow[d, "\Theta_\sigma"] & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S}) \arrow[l, "\sim"'] \arrow[d, "\widehat{\theta}_\sigma"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}_\sigma"] & \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S})_{\mathcal{Y}} \arrow[d, "\theta_\sigma"] \\
\mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{S}) & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y^\sigma/\mathcal{S}) \arrow[l, "\sim"'] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y^\sigma/\mathcal{S})_{\mathcal{Y}^\sigma} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{S})_{\mathcal{Y}^\sigma}
\end{tikzcd}$$

for $(Y,\mathcal{Y},i) \in \mathsf{RT}_{\mathcal{S}}$,

$$\begin{tikzcd}
\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \arrow[d, "\Theta^\sharp_\sigma"] & \widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \arrow[l, "\sim"'] \arrow[d, "\widehat{\theta}^\sharp_\sigma"] & \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}^\sharp_\sigma"] & \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[d, "\theta^\sharp_\sigma"] \\
\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T}) & \widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T}) \arrow[l, "\sim"'] & \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T})_{(\mathcal{Z}^\sigma,\overline{\mathcal{Z}}^\sigma)} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T})_{(\mathcal{Z}^\sigma,\overline{\mathcal{Z}}^\sigma)}
\end{tikzcd}$$

for $((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}^\sharp_{\mathcal{T}}$.

Proof. This can be proved similarly as Proposition 1.72 and Corollary 1.73. Note that for $A \in \mathsf{SET}^0_{\mathcal{S}}(Y)$ and $B \in \overline{\mathsf{SET}}^{\sharp,0}_{\mathcal{T}}(Z,\overline{Z})$ we have the diagonal morphisms $\mathcal{Y}_{\widehat{\psi}_\sigma(A)} \to \mathcal{Y}^\sigma_A$ and $(\mathcal{Z}_{\widehat{\psi}^\sharp_\sigma(B)}, \overline{\mathcal{Z}}_{\widehat{\psi}^\sharp_\sigma(B)}) \to (\mathcal{Z}^\sigma_A, \overline{\mathcal{Z}}^\sigma_A)$, where the definition of $\widehat{\psi}_\sigma\colon \mathsf{SET}^0_{\mathcal{S}}(Y) \to \mathsf{SET}^0_{\mathcal{S}}(Y^\sigma)$ and $\widehat{\psi}^\sharp_\sigma\colon \overline{\mathsf{SET}}^{\sharp,0}_{\mathcal{T}}(Z,\overline{Z}) \to \overline{\mathsf{SET}}^{\sharp,0}_{\mathcal{T}}(Z^\sigma,\overline{Z}^\sigma)$ is straight forward. $\square$

For $Y \in \mathsf{LS}_S$, let $\chi\colon Y \to Y^\sigma$ be the realative Frobenius of Y over S. By functoriality we have a morphism in $\mathscr{D}^+_{\mathscr{O}_{\mathbf{Q}}}$

$$\chi^*\colon \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{S}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S}).$$

The Frobenius endomorphism

$$\varphi\colon \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{S}), \tag{1.17}$$

in $\mathscr{D}^+_{\mathbf{Q}_p}$ is the composition of χ^* with Θ_σ. Similarly, for $(Z,\overline{Z}) \in \overline{\mathsf{LS}}^\sharp_T$, let $\chi\colon (Z,\overline{Z}) \to (Z^\sigma,\overline{Z}^\sigma)$ be the morphism over T induced by the absolute Frobenius $(Z,\overline{Z}) \to (Z,\overline{Z})$ and the structure morphism $(Z,\overline{Z}) \to (T,T)$. By functoriality we have a morphism in $\mathscr{D}^+_F$

$$\chi^*\colon \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z^\sigma,\overline{Z}^\sigma)/\mathcal{T}) \to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}),$$

and the composition of χ^* with $\Theta^\sharp_\sigma$ gives the Frobenius endomorphism

$$\varphi\colon \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}),$$

which is a morphism in $\mathscr{D}^+_{\mathbf{Q}_p}$.

It is possible to define a Frobenius endomorphism on $\mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})$ for $(Z,\overline{Z}) \in \overline{\mathsf{LS}}_T$ in the following case. Let $(Z,\overline{Z})$ be an object in $\overline{\mathsf{LS}}_T$, and let $(Z,\overline{Z}^\sharp) = (Z,\overline{Z}) \times_T (T,\overline{T}) \to (Z,\overline{Z})$. If $\Theta^\sharp\colon \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T})$ is a quasi-isomorphism, we can define the Frobenius endomorphism φ on $\mathrm{R}\Gamma_{\mathrm{rig}}(Z,\overline{Z})$ as the composition

$$\mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \xrightarrow[\Theta^\sharp]{\sim} \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T}) \xrightarrow[\varphi']{} \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z}^\sharp)/\mathcal{T}) \xrightarrow[\Theta^{\sharp,-1}]{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}),$$

which is a morphism in $\mathscr{D}^+_{\mathbf{Q}_p}$. The morphism φ is functorial on such $(Z,\overline{Z})$.

Proposition 1.79. *(i) There are equalities*

$$\begin{aligned}
&\Theta_0 \circ \Theta_\sigma = \Theta_\sigma \circ \Theta_0\colon \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathscr{O}^0_F) && \textit{for } Z \in \mathsf{LS}_T,\\
&\widehat{\theta}_0 \circ \widehat{\theta}_\sigma = \widehat{\theta}_\sigma \circ \widehat{\theta}_0\colon \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) \to \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y^\sigma/\mathscr{O}^0_F) && \textit{for } Z \in \mathcal{T}\text{-}\mathsf{ELS}_T,\\
&\widetilde{\theta}_0 \circ \widetilde{\theta}_\sigma = \widetilde{\theta}_\sigma \circ \widetilde{\theta}_0\colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \to \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Y^\sigma/\mathscr{O}^0_F)_{\mathcal{Y}^\sigma} && \textit{for } (Z,\mathcal{Z},i) \in \mathsf{RT}_{\mathcal{T}},\\
&\theta_0 \circ \theta_\sigma = \theta_\sigma \circ \theta_0\colon \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathscr{O}^0_F)_{\mathcal{Y}^\sigma} && \textit{for } (Z,\mathcal{Z},i) \in \mathsf{RT}_{\mathcal{T}},
\end{aligned}$$

where $Y := Z \times_T k^0$, and hence

$$\Theta_0 \circ \varphi = \varphi \circ \Theta_0\colon \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F)$$

for $\overline{\mathsf{LS}}_T$.

(ii) There are equalities

$$\begin{aligned}
&\Theta^\sharp_b \circ \Theta^\sharp_\sigma = \Theta_\sigma \circ \Theta^\sharp_b\colon \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z^\sigma/\mathcal{T}) && \textit{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{LELS}}^\sharp_T\\
&\widehat{\theta}^\sharp_b \circ \widehat{\theta}^\sharp_\sigma = \widehat{\theta}_\sigma \circ \widehat{\theta}^\sharp_b\colon \widehat{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z^\sigma/\mathcal{T}) && \textit{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{ELS}}^\sharp_T,\\
&\widetilde{\theta}^\sharp_b \circ \widetilde{\theta}^\sharp_\sigma = \widetilde{\theta}_\sigma \circ \widetilde{\theta}^\sharp_b\colon \widetilde{\mathrm{R}\Gamma}^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z^\sigma/\mathcal{T})_{\mathcal{Z}^\sigma} && \textit{for } ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}^\sharp_{\mathcal{T}},\\
&\theta^\sharp_b \circ \theta^\sharp_\sigma = \theta_\sigma \circ \theta^\sharp_b\colon \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z^\sigma/\mathcal{T})_{\mathcal{Z}^\sigma} && \textit{for } ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}^\sharp_{\mathcal{T}},
\end{aligned}$$

and hence

$$\Theta^\sharp_b \circ \varphi' = \varphi \circ \Theta^\sharp_b\colon \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})$$

for $\mathcal{T}$-$\overline{\mathsf{LELS}}^\sharp_T$.

(iii) There are equalities

$$\begin{aligned}
&\Theta_b = \Theta_b \circ \Theta^\sharp\colon \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T}) && \textit{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{LELS}}_T,\\
&\widehat{\theta}_b = \widehat{\theta}_b \circ \widehat{\theta}^\sharp\colon \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T}) \to \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T}) && \textit{for } (Z,\overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{ELS}}_T,\\
&\widetilde{\theta}_b = \widetilde{\theta}_b \circ \widetilde{\theta}^\sharp\colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} && \textit{for } ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}_{\mathcal{T}}\\
&\theta_b = \theta_b \circ \theta^\sharp\colon \mathrm{R}\Gamma_{\mathrm{rig}}((Z,\overline{Z})/\mathcal{T})_{(\mathcal{Z},\overline{\mathcal{Z}})} \to \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}} && \textit{for } ((Z,\overline{Z}),(\mathcal{Z},\overline{\mathcal{Z}}),i) \in \overline{\mathsf{RT}}_{\mathcal{T}}.
\end{aligned}$$

Proof. The equality $\widehat{\theta}_0 \circ \widehat{\theta}_\sigma = \widehat{\theta}_\sigma \circ \widehat{\theta}_0$ follows from the commutativity of the diagram

$$\begin{tikzcd}
\varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Z/\mathcal{T})_{\mathcal{Z}_A} \arrow[r] \arrow[d] & \varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Z^\sigma/\mathcal{T})_{\mathcal{Z}_A^\sigma} \arrow[r, "\sim"] \arrow[d] & \varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Z^\sigma/\mathcal{T})_{\mathcal{Z}_{\widehat{\psi}_\sigma(A)}} \arrow[r, "\sim"] \arrow[d] & \varinjlim_C \mathrm{R}\Gamma_{\mathrm{rig}}(Z^\sigma/\mathcal{T})_{\mathcal{Z}_C} \arrow[d] \\
\varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_F^0)_{\mathcal{Y}_A} \arrow[r] \arrow[d, "\sim"] & \varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}_A^\sigma} \arrow[r, "\sim"] \arrow[d, "\sim"] & \varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}_{\widehat{\psi}_\sigma(A)}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \varinjlim_C \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}_C} \arrow[d, "\sim"] \\
\varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_F^0)_{\mathcal{Y}_{\widehat{\psi}_0(A)}} \arrow[r] \arrow[d, "\sim"] & \varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}^\sigma_{\widehat{\psi}_0(A)}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \varinjlim_A \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}_{\widehat{\psi}_0\circ\widehat{\psi}_\sigma(A)}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \varinjlim_C \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}_{\widehat{\psi}_0(C)}} \arrow[d, "\sim"] \\
\varinjlim_B \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_F^0)_{\mathcal{Y}_B} \arrow[r] & \varinjlim_B \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}_B^\sigma} \arrow[r, "\sim"] & \varinjlim_B \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}_{\widehat{\psi}_\sigma(B)}} \arrow[r, "\sim"] & \varinjlim_D \mathrm{R}\Gamma_{\mathrm{rig}}(Y^\sigma/\mathcal{O}_F^0)_{\mathcal{Y}_D},
\end{tikzcd}$$

where A, B, C, D run through the sets $\mathsf{SET}^0_{\mathcal{T}}(Z)$, $\mathsf{SET}^0_{\mathcal{O}_F^0}(Y)$, $\mathsf{SET}^0_{\mathcal{T}}(Z^\sigma)$, and $\mathsf{SET}^0_{\mathcal{O}_F^0}(Y^\sigma)$ respectively, and we set

$$\mathcal{Y}_A := \mathcal{Z}_A \times_{\mathcal{T}} \mathcal{O}_F^0, \qquad \mathcal{Z}_A^\sigma := \mathcal{Z}_A \times_{\mathcal{T},\sigma} \mathcal{T},$$
$$\mathcal{Y}_A^\sigma := \mathcal{Z}_A \times_{\mathcal{T}} \mathcal{O}_F^0 = \mathcal{Y}_A \times_{\mathcal{O}_F^0,\sigma} \mathcal{O}_F^0.$$

This induces $\Theta_0 \circ \Theta_\sigma = \Theta_\sigma \circ \Theta_0$. The other equalities follow similarly. □

2 Frobenius, monodromy and Hyodo–Kato map

As we have mentioned before, the goal of this paper is twofold: in the case of a strictly semistable log scheme over $\mathcal{O}_K^\pi$ with a nice compactification, first to construct a log rigid syntomic cohomology theory, which lends itself to explicit computations, and second to compare this to the log-crystalline syntomic cohomology theory of Nekovář–Nizioł. To accomplish this, we have constructed several canonical log rigid complexes.

- For a strictly semistable log scheme with boundary over k^0, i.e. $(Y, \overline{Y}) \in \overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$, the canonical rigid Hyodo–Kato complex $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y})$ in Definition 1.47.
- For a fine log scheme of Zariski type and of finite type over k^0, i.e. $Y \in \mathsf{LS}_{k^0}$, the canonical log rigid complexes $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_F^0)$ and $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_K^\pi)$ in Definition 1.58.
- For a fine log scheme of Zariski type and of finite type over T, i.e. $Y \in \mathsf{LS}_T$, the canonical log rigid complex $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{T})$ in Definition 1.58.
- For a locally $\mathcal{T}$-embeddable fine T-log scheme with boundary of finite type, i.e. $(Z, \overline{Z}) \in \mathcal{T}\text{-}\overline{\mathsf{LELS}}_T$, the canonical log rigid complex $\mathrm{R}\Gamma_{\mathrm{rig}}((Z, \overline{Z})/\mathcal{T})$ in Definition 1.67.

Each of them caters to a different situation and goal. The canonical complexes $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_F^0)$, $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_K^\pi)$, and $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{T})$ for fine log schemes of Zariski type and of finite type over k^0 have good functoriality properties, especially with respect to base change which is important for us. In particular this allowed us to construct a Frobenius on $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_F^0)$ via base change and the relative Frobenius on Y (c.f. Proposition 1.79).

In the situation of a strictly semistable log scheme with boundary $(Y, \overline{Y})$ over k^0, we will see in § 2.1 how to construct an analogue of the Hyodo–Kato map in terms of a zigzag between $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0)$ and $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_K^\pi)$ (resp. $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_F^0)$ and $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_K^\pi)$) via a canonical log rigid boundary complex $\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet, \overline{P}_\bullet)/\mathcal{T})$ for a certain simplicial locally $\mathcal{T}$-embeddable fine T-log scheme with boundary of finite type coming from $\overline{Y}$ (and Y respectively). The complex $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{T})$ will turn out to be useful in the comparison of Große-Klönne's Hyodo–Kato map with the classical one on log crystalline cohomology.

Consequently we have on the complexes $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{O}_F^0)$ and $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0)$ almost all the structure needed for the construction of a log rigid syntomic cohomology theory. However they unfortunately don't a priori have a monodromy operator. But we can pass to the rigid Hyodo–Kato complex for this.

We have seen, that the construction of the rigid Hyodo–Kato complex, locally needs strict immersions into weak formal log schemes over $\mathcal{T}$, which behave like admissible lifts. This fact accounts for some of the technical difficulties that arise for the rigid Hyodo–Kato complex. Notably, its construction is much more subtle than the construction of the other canonical log rigid complexes. For example, for the rigid Hyodo–Kato complex we have to restrict ourselves to strictly semistable logs schemes with boundary and cannot yet extend it to general fine log schemes of finite type. It also lacks functoriality with respect to base change.

However, there are several good reasons to consider this complex. One is, that it not only provides at least locally an explicite representation of the log rigid complex for a strictly semi stable log scheme with boundary, but also of Frobenius and monodromy operator on it. The reason for this is that it takes advantage of the very concrete construction in [29]. The hope therefore is that it will be a helpful tool to carry out explicite computations for example in the case of a HK-embeddable strictly semistable log scheme with boundary.

Furthermore, as we have seen in Proposition 1.35, it is quasi-isomorphic to a complex coming from a Steenbrink double complex, and this provides us with a weight spectral sequence. While we don't need it in this paper, it can be a powerful tool, so we decided to include it in the outline.

In the last part of this section, we take another look at the Frobenius and monodromy on the rigid Hyodo–Kato complex. While the way they are constructed is convenient for local computations on complexes, they can on the level of cohomology be brought into a form which is more suitable for the comparison with the crystalline Frobenius and monodromy opertor which we will need for our comparison result.

2.1 The rigid Hyodo–Kato map

We will now explain how to construct the rigid Hyodo–Kato map. This is based on the constructions in [22], but modified so that it is functorial, and generalized to strictly semistable log schemes with boundary.

Let $(Y,\overline{Y})$ be an object in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$. We will construct a Hyodo–Kato map $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)\to\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_K^\pi)$ as a morphism in $\mathscr{D}_F^+$. Recall that $\Upsilon_{\overline{Y}}$ and Υ_D are the sets of the irreducible components and the horizontal components of $\overline{Y}$ (Definition 1.26). For $j\in\Upsilon_{\overline{Y}}$, let $\mathcal{L}_j$ be the line bundle on $\overline{Y}$ which corresponds to j as in [22, § 2.1]. Precisely, let $\overline{Y}_j$ be the irreducible component of $\overline{Y}$ which corresponds to j, and let $\mathcal{M}_{\overline{Y},j}$ be the preimage of $\mathrm{Ker}(\mathcal{O}_{\overline{Y}}\to\mathcal{O}_{\overline{Y}_j})$ in $\mathcal{M}_{\overline{Y}}$. Then $\mathcal{M}_{\overline{Y},j}$ is a principal homogeneous space over $\mathcal{O}_{\overline{Y}}^\times$, and its associated line bundle is the dual $\mathcal{L}_j^{-1}$ of $\mathcal{L}_j$. Note that the natural map $\mathcal{M}_{\overline{Y},j}\to\mathcal{O}_{\overline{Y}}$ defines a global section s_j of $\mathcal{L}_j$. For any non-empty subset $J\subset\Upsilon_{\overline{Y}}$, set $\overline{M}_J:=\bigcap_{j\in J}\overline{Y}_j$. We regard it as an exact closed log subscheme of $\overline{Y}$. By abuse of notation, we denote the restriction of $\mathcal{L}_j$ to $\overline{M}_J$ again by $\mathcal{L}_j$. For $j\in J$, set

$$\overline{V}_J^j:=\operatorname{Spec}\operatorname{Sym}_{\mathcal{O}_{\overline{M}_J}}\mathcal{L}_j^{-1}\qquad\text{and}\qquad\overline{P}_J^j:=\operatorname{Proj}\operatorname{Sym}_{\mathcal{O}_{\overline{M}_J}}(\mathcal{O}_{\overline{M}_J}\oplus\mathcal{L}_j^{-1}).$$

We regard $\overline{V}_J^j$ as an open subscheme of $\overline{P}_J^j$ via the isomorphism

$$\begin{aligned}\operatorname{Sym}_{\mathcal{O}_{\overline{M}_J}}\mathcal{L}_j^{-1}&\xrightarrow{\sim}\operatorname{Sym}_{\mathcal{O}_{\overline{M}_J}}(\mathcal{O}_{\overline{M}_J}\oplus\mathcal{L}_j^{-1})[1_{\mathcal{O}_{\overline{M}_J}}^{-1}]_0\\ s&\mapsto 1_{\mathcal{O}_{\overline{M}_J}}^{-1}\otimes s\quad\text{for } s\in\mathcal{L}_j^{-1},\end{aligned}$$

where $1_{\mathcal{O}_{\overline{M}_J}}$ is considered as a degree-one element of $\operatorname{Sym}_{\mathcal{O}_{\overline{M}_J}}(\mathcal{O}_{\overline{M}_J}\oplus\mathcal{L}_j^{-1})$, and $\operatorname{Sym}_{\mathcal{O}_{\overline{M}_J}}(\mathcal{O}_{\overline{M}_J}\oplus\mathcal{L}_j^{-1})[1_{\mathcal{O}_{\overline{M}_J}}^{-1}]_0$ denotes the degree-zero part of $\operatorname{Sym}_{\mathcal{O}_{\overline{M}_J}}(\mathcal{O}_{\overline{M}_J}\oplus\mathcal{L}_j^{-1})[1_{\mathcal{O}_{\overline{M}_J}}^{-1}]$. For a subset $J'\subset J$, set

$$\overline{V}_J^{J'}:=\prod_{\overline{M}_J}(\overline{V}_J^j)_{j\in J'}\qquad\text{and}\qquad\overline{P}_J^{J'}:=\prod_{\overline{M}_J}(\overline{P}_J^j)_{j\in J'}.$$

We set $\overline{V}_J:=\overline{V}_J^J$ and $\overline{P}_J:=\overline{P}_J^J$. For $j\in J$, the pull-back of the divisor $\overline{P}_J^j\setminus\overline{V}_J^j$ on $\overline{P}_J^j$ is a divisor on $\overline{P}_J$ denoted by $N_{j,\infty}$. Let $N_{j,0}$ be the divisor on $\overline{P}_J$ which is the pull-back of the zero section divisor $\overline{M}_J\hookrightarrow\overline{V}_J^j\hookrightarrow\overline{P}_J^j$ on $\overline{P}_J^j$. Set $N_\infty:=\bigcup_{j\in J}N_{j,\infty}$ and $N_0:=\bigcup_{j\in J}N_{j,0}$. Let N' and D' be the divisors on $\overline{P}_J$ given by the pull-back of the divisors $\overline{M}_J\cap\bigcup_{i\in\Upsilon_{\overline{Y}}\setminus J}\overline{Y}_i$ and $\overline{M}_J\cap(\overline{Y}\setminus Y)$ on $\overline{M}_J$ respectively, via the structure morphism $\overline{P}_J\to\overline{M}_J$. We endow $\overline{P}_J$ with the log structure associated to the normal crossing divisor $N_\infty\cup N_0\cup N'\cup D'$. We consider $\overline{P}_J^{J'}$, and thus in particular $\overline{M}_J=\overline{P}_J^{\infty}$, as an exact closed log subscheme of $\overline{P}_J$ by identifying it with the intersection in $\overline{P}_J$ of all $N_{j,0}$ for $j\in J\setminus J'$. The global sections

s_i of $\mathscr{L}_i$ for $i\in\Upsilon_{\overline{Y}}\setminus J$ define a map $\eta\colon \bigotimes_{\mathcal{O}_{\overline{M}_J}}(\mathscr{L}_i^{-1})_{i\in\Upsilon_{\overline{Y}}\setminus J}\to\mathcal{O}_{\overline{M}_J}$, and hence a map

$$\mathcal{O}_{\overline{M}_J}\cong\bigotimes_{\mathcal{O}_{\overline{M}_J}}(\mathscr{L}_j^{-1})_{j\in J}\otimes\bigotimes_{\mathcal{O}_{\overline{M}_J}}(\mathscr{L}_i^{-1})_{i\in\Upsilon_{\overline{Y}}\setminus J}\xrightarrow{1\otimes\eta}\bigotimes_{\mathcal{O}_{\overline{M}_J}}(\mathscr{L}_j^{-1})_{j\in J}\to\mathrm{Sym}_{\mathcal{O}_{\overline{M}_J}}(\bigoplus_{j\in J}\mathscr{L}_j^{-1}).$$

We consider $\overline{V}_J$ as a log scheme over T by sending t to the image of $1_{\mathcal{O}_{\overline{M}_J}}$ under this map. Then $(\overline{V}_J^{J'},\overline{P}_J^{J'})$ is a T-log scheme with boundary. As in [22, § 2.4] (or Example 1.61), one can see that $(\overline{V}_J^{J'},\overline{P}_J^{J'})$ is $\mathcal{T}$-embeddable.

For a set Υ and an integer $m\geq 0$ we set

$$\widetilde{\Lambda}_m(\Upsilon):=\{\lambda=(J_0(\lambda),\dots,J_m(\lambda))\mid\varnothing\neq J_0(\lambda)\subset J_1(\lambda)\subset\cdots\subset J_m(\lambda)\subset\Upsilon\}.$$

Now we define a simplicial log scheme $\overline{M}_\bullet$ and a simplicial T-log scheme with boundary $(\overline{V}_\bullet,\overline{P}_\bullet)$ by

$$\overline{M}_m:=\coprod_{\lambda\in\widetilde{\Lambda}_m(\Upsilon_{\overline{Y}})}\overline{M}_{J_m(\lambda)}\qquad\text{and}\qquad(\overline{V}_m,\overline{P}_m):=\coprod_{\lambda\in\widetilde{\Lambda}_m(\Upsilon_{\overline{Y}})}(\overline{V}^{J_0(\lambda)}_{J_m(\lambda)},\overline{P}^{J_0(\lambda)}_{J_m(\lambda)}).$$

Remark 2.1. Here we use the set $\widetilde{\Lambda}_m(\Upsilon)$ instead of the set $\Lambda_m(\Upsilon)$ in [22, § 3.2], in order to ensure the functoriality of the simplicial construction. More precisely, a morphism $f\colon\overline{Y}\to\overline{Y}'$ in $\underline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ naturally induces a map $\rho_f\colon\widetilde{\Lambda}_m(\Upsilon_{\overline{Y}})\to\widetilde{\Lambda}_m(\Upsilon_{\overline{Y}'})$, but not necesarily a map $\Lambda_m(\Upsilon_{\overline{Y}})\to\Lambda_m(\Upsilon_{\overline{Y}'})$. If we let $\overline{M}'_\bullet$ and $(\overline{V}'_\bullet,\overline{P}'_\bullet)$ be the simplicial objects constructed from $\overline{Y}'$ as above, ρ_f induces natural morphisms $\overline{M}_\bullet\to\overline{M}'_\bullet$ and $\overline{V}_\bullet\to\overline{V}'_\bullet$. Note that this does not extend to a morphism $\overline{P}_\bullet\to\overline{P}'_\bullet$ in general. The following lemma implies that the log rigid complexes of $(\overline{V}_\bullet,\overline{P}_\bullet)$ given by $\widetilde{\Lambda}_m(\Upsilon_{\overline{Y}})$ and $\Lambda_m(\Upsilon)$ are quasi-isomorphic to each other.

Lemma 2.2. *Let X be a Grothendieck topological space, $\{U_j\}_{j\in\Upsilon}$ an admissible open covering of X, and $\mathcal{F}^\bullet$ a complex of sheaves of abelian groups on X. For $J\subset\Upsilon$ we set $U_J:=\bigcap_{j\in\Upsilon}U_j$. Then for the simplicial space $\widetilde{U}_\bullet$ given by $\widetilde{U}_m:=\coprod_{\lambda\in\widetilde{\Lambda}_m(\Upsilon)}U_{J_m(\lambda)}$ we have an isomorphism*

$$\mathrm{R}\Gamma(X,\mathcal{F}^\bullet)\cong\mathrm{R}\Gamma(U_\bullet,\mathcal{F}^\bullet|_{U_\bullet}).$$

Proof. We consider the set

$$\Lambda_m(\Upsilon):=\{\lambda=(J_0(\lambda),\dots,J_m(\lambda))\mid\varnothing\neq J_0(\lambda)\subsetneq J_1(\lambda)\subsetneq\cdots\subsetneq J_m(\lambda)\subset\Upsilon\}.$$

Then for a sheaf $\mathcal{F}$ of abelian groups on X we obtain two types of Cech complexes $C^\bullet(X,\mathcal{F})$ and $\widetilde{C}^\bullet(X,\mathcal{F})$ given by

$$C^m(X,\mathcal{F}):=\bigoplus_{\lambda\in\Lambda_m(\Upsilon)}\Gamma(U_{J_m(\lambda)},\mathcal{F}),\qquad\widetilde{C}^m(X,\mathcal{F}):=\bigoplus_{\lambda\in\widetilde{\Lambda}_m(\Upsilon)}\Gamma(U_{J_m(\lambda)},\mathcal{F}).$$

The natural direct decomposition $\widetilde{C}^m(X,\mathcal{F})=C^m(X,\mathcal{F})\oplus\bigoplus_{\lambda\in\widetilde{\Lambda}_m(\Upsilon)\setminus\Lambda_m(\Upsilon)}\Gamma(U_{J_m(\lambda)},\mathcal{F})$ induces an inclusion $i\colon C^\bullet(X,\mathcal{F})\to\widetilde{C}^\bullet(X,\mathcal{F})$ and a projection $p\colon\widetilde{C}^\bullet(X,\mathcal{F})\to C^\bullet(X,\mathcal{F})$ which are morphisms of complexes and satisfy $p\circ i=\mathrm{id}$. Moreover, by astraightforward calculation one can see that $i\circ p$ and id are homotopic with a homotopy h defined as follows. Fix $m\geq 0$ and an element $\lambda\in\widetilde{\Lambda}_m(\Upsilon)$, and take positive integers $k_1,\dots,k_n$ satisfying $J_{k_1+\dots+k_{j-1}}(\lambda)=\cdots=J_{k_1+\cdots+k_j-1}(\lambda)$ and $J_{k_1+\cdots+k_j-1}(\lambda)\neq J_{k_1+\cdots+k_j}(\lambda)$ for any j. Let r be the number of j's satisfying $k_j\geq 2$ if they exist, and otherwise we set $r=1$. For $1\leq j\leq n$, we set

$$s_j(\lambda):=(J_0(\lambda),\dots,J_{k_1+\cdots+k_j-2}(\lambda),J_{k_1+\cdots+k_j-1}(\lambda),J_{k_1+\cdots+k_j-1}(\lambda),J_{k_1+\cdots+k_j}(\lambda),\dots,J_m(\lambda)).$$

Then for an element $f=(f_\mu)_{\mu\in\widetilde{\Lambda}_{m+1}(\Upsilon)}\in\widetilde{C}^m(X,\mathcal{F})$, the λ-component of $h(f)$ is defined by

$$h(f)_\lambda:=\sum_{j=1}^n\frac{1}{r}(-1)^{k_1+\cdots k_j}f_{s_j(\lambda)}.$$

Hence we see that $C^\bullet(X,\mathcal{F})$ and $\widetilde{C}^\bullet(X,\mathcal{F})$ are naturally quasi-isomorphic. This result and [23, Lem. 3.5] imply the statement. ☐

Let $(Y,\overline{Y})$ be an object in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$, and $\overline{M}_\bullet$ and $(\overline{V}_\bullet,\overline{P}_\bullet)$ the associated simplicial objects defined above. We denote the compositions

$$\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}_\bullet)/\mathcal{T}) \xrightarrow{\Theta_b} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{V}_\bullet/\mathcal{T}) \xrightarrow{\Theta_0} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{V}_\bullet \times_T k^0/\mathscr{O}^0_F) \to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathscr{O}^0_F) \qquad \text{and}$$
$$\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}_\bullet)/\mathcal{T}) \xrightarrow{\Theta_b} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{V}_\bullet/\mathcal{T}) \xrightarrow{\Theta_\pi} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{V}_\bullet \times_T k^0/\mathscr{O}^\pi_K) \to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathscr{O}^\pi_K)$$

by $\overline{\xi}_0$ and $\overline{\xi}_\pi$ respectively. If we repeat the constructions for (Y,Y), we obtain simplicial objects $M_\bullet$, $(V_\bullet,P_\bullet)$, and morphisms

$$\xi_0\colon \mathrm{R}\Gamma_{\mathrm{rig}}((V_\bullet,P_\bullet)/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}^0_F),$$
$$\xi_\pi\colon \mathrm{R}\Gamma_{\mathrm{rig}}((V_\bullet,P_\bullet)/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}^\pi_K).$$

Taking into account the canonical morphisms induced by the inlcusions $Y \hookrightarrow \overline{Y}$, $(Y,Y) \hookrightarrow (Y,\overline{Y})$, $M_\bullet \hookrightarrow \overline{M}_\bullet$, and $(V_\bullet,P_\bullet) \hookrightarrow (\overline{V}_\bullet,\overline{P}_\bullet)$, they fit together into a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F) \arrow[r, "(b)"] & \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}^0_F) & \mathrm{R}\Gamma_{\mathrm{rig}}((V_\bullet,P_\bullet)/\mathcal{T}) \arrow[l, "\xi_0"'] \arrow[r, "\xi_\pi"] & \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}^\pi_K) & \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K) \arrow[l, "(b)"'] \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}^0_F) \arrow[u, "(a)"] \arrow[r, "(b)"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathscr{O}^0_F) \arrow[u] & \mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}_\bullet)/\mathcal{T}) \arrow[u] \arrow[l, "\overline{\xi}_0"'] \arrow[r, "\overline{\xi}_\pi"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathscr{O}^\pi_K) \arrow[u] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}^\pi_K) \arrow[u] \arrow[l, "(b)"']
\end{tikzcd} \tag{2.1}$$

Here, the morphisms (b) are quasi-isomorphisms by cohomological descent of admissible coverings of dagger spaces. Moreover ξ_0 and $\xi_\pi \otimes 1\colon \mathrm{R}\Gamma_{\mathrm{rig}}((V_\bullet,P_\bullet)/\mathcal{T}) \otimes_F K \to \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}^\pi_K)$ are quasi-isomorphisms by [22, Thm. 3.1].

Lemma 2.3. *Let $(Y,\overline{Y})$ be an object in $\overline{\mathsf{LS}}^{ss}_{k^0}$, and let $\overline{M}_\bullet$, $M_\bullet$, $(\overline{V}_\bullet,\overline{P}_\bullet)$, and $(V_\bullet,P_\bullet)$ be as above. The morphisms*

$$\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathscr{O}^0_F) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}^0_F), \tag{2.2}$$
$$\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathscr{O}^\pi_K) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}^\pi_K), \tag{2.3}$$
$$\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}_\bullet)/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}((V_\bullet,P_\bullet)/\mathcal{T}) \tag{2.4}$$

are quasi-isomorphisms.

Proof. By taking an affine open covering of $\overline{Y}$, we may assume that $\overline{Y}$ is affine and admits a global chart as in Definitin 1.25 (i). As in the proof of Lemma 1.28, one can take an exact closed immersion $(Y,\overline{Y}) \hookrightarrow (\mathscr{Z},\overline{\mathscr{Z}})$ into an object in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{\mathcal{T}}$ with $\overline{Y} \cong \overline{\mathscr{Z}} \times_{\mathcal{T}} k^0$ and $Y \cong \mathscr{Z} \times_{\mathcal{T}} k^0$. Set $\overline{\mathscr{Y}} := \overline{\mathscr{Z}} \times_{\mathcal{T}} \mathscr{O}^0_F$ and $\mathscr{D} := \overline{\mathscr{Z}} \setminus \mathscr{Z}$. Let Y' be the exact closed log subscheme of $\overline{\mathscr{Z}}$ whose underlying scheme is the special fiber of $\overline{\mathscr{Y}} \cup \mathscr{D}$. Then (Y',Y') is an object in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$, and there is a natural bijection $\Upsilon_{Y'} \cong \Upsilon_{\overline{Y}} \amalg \Upsilon_D$. Through this bijection we regard $\Upsilon_{\overline{Y}}$ and Υ_D as subsets of $\Upsilon_{Y'}$. Let $J \subset \Upsilon_{\overline{Y}}$ be a non-empty subset, and let $M'_J := \bigcap_{j\in J} Y'_j$ be the log scheme associated to (Y',Y'). Then we have isomorphisms

$$\overline{M}_J \cong M'_J \qquad \text{and} \qquad M_J \cong M'_J \setminus (M'_J \cap \bigcup_{i\in\Upsilon_D} Y'_i).$$

If we set $M'_{J,\heartsuit} := M'_J \setminus (M'_J \cap \bigcup_{i\in\Upsilon_{Y'}\setminus J} Y'_i)$, we can apply [22, Lem. 4.4] to obtain that the morphisms

$$\mathrm{R}\Gamma_{\mathrm{rig}}(M'_J/\mathscr{O}^0_F) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M'_{J,\heartsuit}/\mathscr{O}^0_F) \quad \text{and} \quad \mathrm{R}\Gamma_{\mathrm{rig}}((M'_J \setminus (M'_J \cap \bigcup_{i\in\Upsilon_D} Y'_i))/\mathscr{O}^0_F) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M'_{J,\heartsuit}/\mathscr{O}^0_F)$$

are quasi-isomorphisms. Hence we see that (2.2) is a quasi-isomorphism. The statements for (2.3) and (2.4) follow similarly. □

The commutativity of the diagram (2.1) implies the following statement.

Corollary 2.4. *In the setting of Lemma 2.3, the morphisms*

$$
\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_{\bullet},\overline{P}_{\bullet})/\mathcal{T}) \xrightarrow{\overline{\xi}_0} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_{\bullet}/\mathscr{O}_F^0),\\
&\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_{\bullet},\overline{P}_{\bullet})/\mathcal{T})\otimes_F K \xrightarrow{\overline{\xi}_\pi\otimes 1} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_{\bullet}/\mathscr{O}_K^\pi),\\
&\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_K^\pi) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_K^\pi),\\
&\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0) \to \mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0)
\end{aligned}
$$

are quasi-isomorphisms.

Let us point out that we did not have to assume that $\overline{Y}$ is proper to obtain the results in this corollary and the previous lemma. As a consequence, the lower horizontal arrow in (2.1) gives a morphism

$$\iota_\pi^{\mathrm{rig}}\colon \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0) \to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_K^\pi)$$

in $\mathscr{D}_F^+$, which induces a quasi-isomorphism after tensoring with K.

Proposition 2.5. *The morphism ι_π^{rig} is functorial on $(Y,\overline{Y})$.*

To prove this proposition, let $f\colon (Y,\overline{Y}) \to (Y',\overline{Y}')$ be a morphism in $\overline{\mathsf{LS}}_{k^0}^{\mathrm{ss}}$, and $\rho=\rho_f\colon \Upsilon_{\overline{Y}} \to \Upsilon_{\overline{Y}'}$ the map induced by f. For $I\subset J\subset \Upsilon_{\overline{Y}}$, we set $J' := \rho(J)\subset \Upsilon_{\overline{Y}'}$ and $I' := \rho(I)\subset I$. Denote by $\overline{M}_J$, $(\overline{V}_J^I,\overline{P}_J^I)$ and $\overline{M}'_{J'}$, $(\overline{V}'^{I'}_{J'},\overline{P}'^{I'}_{J'})$ the log schemes (with boundary) associated to $(Y,\overline{Y})$ and $(Y',\overline{Y}')$. Then f induces morphisms $\overline{M}_J \to \overline{M}'_{J'}$ and $\overline{V}_J^I \to \overline{V}'^{I'}_{J'}$, but not $\overline{P}_J^I \to \overline{P}'^{I'}_{J'}$ in general. Let $\widetilde{P}_J^I$ be the log schematic image of the diagonal map $\overline{V}_J^I \to \overline{P}_J^I \times_k \overline{P}'^{I'}_{J'}$.

Lemma 2.6. *Let $f\colon (Y,\overline{Y}) \to (Y',\overline{Y}')$ be a morphism in $\overline{\mathsf{LS}}_{k^0}^{ss}$ and let $I\subset J\subset \Upsilon_{\overline{Y}}$. With the notation as above, $(\overline{V}_J^I,\widetilde{P}_J^I)$ is a locally $\mathcal{T}$-embeddable T-log scheme with boundary.*

Proof. Without loss of generality we assume there are rigid Hyodo-Kato data $((\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$ for $(Y,\overline{Y})$ and $((\mathcal{Z}',\overline{\mathcal{Z}}'),i',\phi')$ for $(Y',\overline{Y}')$. We define an object A in $\mathsf{SET}^0_{\mathrm{HK}}(Y,\overline{Y})$ by

$$A := \{(\mathrm{id}_{(Y,\overline{Y})},((\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)),(f,((\mathcal{Z}',\overline{\mathcal{Z}}'),i',\phi'))\}.$$

Let $(\mathcal{Z}_A,\overline{\mathcal{Z}}_A)$ be the rigid Hyodo-Kato datum for $(Y,\overline{Y})$ constructed as in § 1.2. As in the proof of Proposition 1.39 there is a commutative diagram

$$
\begin{tikzcd}
\overline{Y} \arrow[r] \arrow[d] & \overline{\mathcal{Z}}_A \arrow[r] \arrow[d] & \mathrm{Spwf}\,\dfrac{\mathscr{O}_F[t_\alpha,s_\beta,u_\gamma^{\pm1},v_\delta^{\pm1}]^\dagger_{\alpha\in\Upsilon_{\overline{Y}},\beta\in\Upsilon_D,\gamma\in\Upsilon_{\overline{Y}'},\delta\in\Upsilon_{D'}}}{(\prod_{\gamma\in\Upsilon_{\overline{Y}'}}u_\gamma-1)} \arrow[d] \\
\overline{Y}' \arrow[r] & \overline{\mathcal{Z}}' \arrow[r] & \mathrm{Spwf}\,\mathscr{O}_F[t'_\gamma,s'_\delta]^\dagger_{\gamma\in\Upsilon_{\overline{Y}'},\delta\in\Upsilon_{D'}}.
\end{tikzcd}
$$

Here the right horizontal arrows are smooth. The vertical arrow on the right is defined by

$$t'_\gamma \mapsto u_\gamma \prod_{\alpha\in\Upsilon_{\overline{Y}}^\gamma} t_\alpha \qquad\text{and}\qquad s'_\delta \mapsto v_\delta \prod_{\beta\in\Upsilon_D} s_\beta^{m_{\delta,\beta}},$$

where $\Upsilon_{\overline{Y}}^\gamma = \rho_f^{-1}(\{\gamma\})\subset\Upsilon_{\overline{Y}}$ and $m_{\delta,\beta}$ is the multiplicity of f^*D' at D_β. Consider the diagram

$$
\begin{tikzcd}
\mathcal{O}_{\overline{\mathcal{Z}}_A} & \dfrac{\mathscr{O}_F[t_\alpha,s_\beta,u_\gamma^{\pm1},v_\delta^{\pm1}]^\dagger_{\alpha\in\Upsilon_{\overline{Y}},\beta\in\Upsilon_D,\gamma\in\Upsilon_{\overline{Y}'},\delta\in\Upsilon_{D'}}}{(\prod_{\gamma\in\Upsilon_{\overline{Y}'}}u_\gamma-1)} \arrow[l] & \mathbb{N}^{\Upsilon_{\overline{Y}}}\oplus\mathbb{N}^{\Upsilon_D}\oplus\mathbb{Z}^{\Upsilon_{\overline{Y}'}}\oplus\mathbb{Z}^{\Upsilon_{D'}} =: Q_A \arrow[l, "\kappa"'] \\
\mathcal{O}_{\overline{\mathcal{Z}}'} \arrow[u] & \mathscr{O}_F[t'_\gamma,s'_\delta]^\dagger_{\gamma\in\Upsilon_{\overline{Y}'},\delta\in\Upsilon_{D'}} \arrow[l] \arrow[u] & \mathbb{N}^{\Upsilon_{\overline{Y}'}}\oplus\mathbb{N}^{\Upsilon_{D'}} =: Q', \arrow[l, "\kappa'"'] \arrow[u, "c"']
\end{tikzcd}
$$

where κ is defined by

$$1_\alpha \mapsto t_\alpha \quad (\alpha \in \Upsilon_{\overline{Y}}), \qquad\qquad 1_\beta \mapsto s_\beta \quad (\beta \in \Upsilon_D),$$
$$1_\gamma \mapsto u_\gamma \quad (\gamma \in \Upsilon_{\overline{Y}'}), \qquad\qquad 1_\delta \mapsto v_\delta \quad (\delta \in \Upsilon_{D'}),$$

κ' is defined by

$$1_\gamma \mapsto t'_\gamma \quad (\gamma \in \Upsilon_{\overline{Y}'}), \qquad\qquad 1_\delta \mapsto s'_\delta \quad (\delta \in \Upsilon_{D'}),$$

c is defined by

$$1_\gamma \mapsto 1_\gamma + \sum_{\alpha \in \Upsilon^\gamma_{\overline{Y}}} 1_\alpha \quad (\gamma \in \Upsilon_{\overline{Y}'}), \qquad\qquad 1_\delta \mapsto 1_\delta + \sum_{\beta \in \Upsilon_D} m_{\delta,\beta} 1_\beta \quad (\delta \in \Upsilon_{D'}).$$

Here 1_α denotes 1 in the α-component of $\mathbb{N}^{\Upsilon_{\overline{Y}}}$. We used a similar notation for 1_β, 1_γ, and 1_δ. The above diagram gives a chart of $\overline{\mathscr{Z}}_A \to \overline{\mathscr{Z}}'$. Set

$$\overline{\mathscr{M}}_J := \bigcap_{j \in J} \overline{\mathscr{Y}}_{A,j}, \qquad \overline{\mathscr{V}}_J := \overline{\mathscr{M}}_J \times \mathscr{T}^J, \qquad \overline{\mathscr{P}}_J := \overline{\mathscr{M}}_J \times \overline{\mathscr{T}}^J,$$
$$\overline{\mathscr{M}}'_{J'} := \bigcap_{j \in J'} \overline{\mathscr{Y}}'_j, \qquad \overline{\mathscr{V}}'_{J'} := \overline{\mathscr{M}}'_{J'} \times \mathscr{T}^{J'}, \qquad \overline{\mathscr{P}}_{J'} := \overline{\mathscr{M}}'_{J'} \times \overline{\mathscr{T}}^{J'}.$$

For $\alpha \in J$ (resp. $\gamma \in J'$), we denote the coordinate of the α-component (resp. γ-component) of $\mathscr{T}^I$ (resp. $\mathscr{T}^{I'}$) by x_α (resp. y_γ). We regard $\overline{\mathscr{V}}_J$ and $\overline{\mathscr{V}}'_{J'}$ as log schemes over $\mathscr{T}$ via the maps

$$t \mapsto \prod_{\alpha \in \Upsilon_{\overline{Y}} \setminus J} t_\alpha \cdot \prod_{\alpha' \in J} x_{\alpha'} \qquad \text{and} \qquad t \mapsto \prod_{\gamma \in \Upsilon_{\overline{Y}'} \setminus J'} t'_\gamma \cdot \prod_{\gamma' \in J'} y_{\gamma'}$$

respectively. For $\gamma \in J'$

$$y_\gamma \mapsto u_\gamma \cdot \prod_{\alpha \in \Upsilon^\gamma_{\overline{Y}} \setminus J} t_\alpha \cdot \prod_{\alpha' \in \Upsilon^\gamma_{\overline{Y}} \cap J} x_{\alpha'}$$

defines a morphism $\overline{\mathscr{V}}_J \to \overline{\mathscr{V}}'_{J'}$ which lifts $\overline{V}_J \to \overline{V}'_{J'}$. Let $\widetilde{\mathscr{P}}_J$ be the (weak formal) log schematic image of the diagonal embedding $\overline{\mathscr{V}}_J \hookrightarrow \overline{\mathscr{P}}_J \times \overline{\mathscr{P}}'_{J'}$. Note that $\overline{\mathscr{P}}_J \times \overline{\mathscr{P}}'_{J'}$ is covered by

$$\overline{\mathscr{U}}_\varepsilon := \overline{\mathscr{M}}_J \times \overline{\mathscr{M}}'_{J'} \times \operatorname{Spwf} \mathscr{O}_F[x^{\varepsilon_\alpha}_\alpha, y^{\varepsilon_\gamma}_\gamma]^\dagger_{\alpha \in J, \gamma \in J'} \subset \overline{\mathscr{P}}_J \times \overline{\mathscr{P}}'_{J'}$$

for all $\varepsilon = ((\varepsilon_\alpha)_{\alpha \in J}, (\varepsilon_\gamma)_{\gamma \in J'}) \in \{\pm 1\}^J \times \{\pm 1\}^{J'}$. Let

$$\widetilde{\mathscr{U}}_\varepsilon := \overline{\mathscr{U}}_\varepsilon \cap \widetilde{\mathscr{P}}_J \qquad \text{and} \qquad \mathscr{U}_\varepsilon := \widetilde{\mathscr{U}}_\varepsilon \cap \overline{\mathscr{V}}_J,$$

and endow them with the log structure which is the pull-back of the log structure on $\widetilde{\mathscr{P}}_J$. We define a map $\psi_J \colon Q_A \to Q^{\mathrm{gp}}_A$ by

$$1_\alpha \mapsto \begin{cases} 1_\alpha & (\alpha \in \Upsilon_{\overline{Y}} \setminus J, \text{ or } \alpha \in J \text{ and } \varepsilon_\alpha = 1), \\ -1_\alpha & (\alpha \in J \text{ and } \varepsilon_\alpha = -1), \end{cases} \qquad 1_\beta \mapsto 1_\beta \quad (\beta \in \Upsilon_D),$$
$$1_\gamma \mapsto 1_\gamma \quad (\gamma \in \Upsilon_{\overline{Y}'}), \qquad\qquad 1_\delta \mapsto 1_\delta \quad (\delta \in \Upsilon_{D'}),$$

and a second map $\psi'_{J'} \colon Q' \to Q'^{\mathrm{gp}}$ by

$$1_\gamma \mapsto \begin{cases} 1_\gamma & (\gamma \in \Upsilon_{\overline{Y}'} \setminus J', \text{ or } \gamma \in J' \text{ and } \varepsilon_\gamma = 1), \\ -1_\gamma & (\gamma \in J' \text{ and } \varepsilon_\gamma = -1), \end{cases} \qquad 1_\delta \mapsto 1_\delta \quad (\delta \in \Upsilon_{D'}).$$

Let Q be the image of

$$(\psi_J, c^{\mathrm{gp}} \circ \psi'_{J'}) \colon Q_A \oplus Q' \to Q^{\mathrm{gp}}_A.$$

Then $(\mathcal{U}_\varepsilon, \widetilde{\mathcal{U}}_\varepsilon)$ has a chart $(\xi\colon Q_{\widetilde{\mathcal{U}}_\varepsilon} \to \mathcal{M}_{\widetilde{\mathcal{U}}_\varepsilon}, \zeta\colon \mathbb{N} \to Q^{\mathrm{gp}})$, where ξ is induced by the map $Q^{\mathrm{gp}}_{A,\overline{\mathcal{V}}_J} \to \mathcal{M}^{\mathrm{gp}}_{\overline{\mathcal{V}}_J}$ which sends

$$
\begin{array}{lll}
1_\alpha \mapsto t_\alpha \ (\alpha \in \Upsilon_{\overline{Y}} \setminus J), & 1_{\alpha'} \mapsto x_{\alpha'} \ (\alpha' \in J), & 1_\beta \mapsto s_\beta \ (\beta \in \Upsilon_D), \\
1_\gamma \mapsto u_\gamma \ (\gamma \in \Upsilon_{\overline{Y}'}), & 1_\delta \mapsto v_\delta \ (\delta \in \Upsilon_{D'}), &
\end{array}
$$

and ζ sends $1 \mapsto \sum_{\alpha \in \Upsilon_{\overline{Y}} \setminus J} 1_\alpha + \sum_{\alpha' \in J} 1_{\alpha'}$. Set $a := -\sum_{\alpha \in J,\ \varepsilon_\alpha = -1} 1_\alpha \in Q$ and $b := \sum \alpha \in \Upsilon_{\overline{Y}} \setminus J 1_\alpha + \sum_{\alpha' \in J,\ \varepsilon_{\alpha'} = 1} 1_{\alpha'} \in Q$. Then the chart (ξ, ζ) and $a, b \in Q$ satisfy the conditions in Definition 1.17. Since we have boundary immersions $(\overline{V}^I_J, \widetilde{P}^I_J) \hookrightarrow (\overline{V}^J_J, \widetilde{P}^J_J) \hookrightarrow (\overline{\mathcal{V}}_J, \widetilde{\mathcal{P}}_J)$, the lemma holds. ☐

Lemma 2.7. *The morphism* $\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}^I_J, \overline{P}^I_J)/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}^I_J, \widetilde{P}^I_J)/\mathcal{T})$ *induced by the natural projection* $(\overline{V}^I_J, \widetilde{P}^I_J) \to (\overline{V}^I_J, \overline{P}^I_J)$ *is a quasi-isomorphism.*

Proof. Again we can work locally on $\overline{Y}$ and $\overline{Y}'$. Thus assume that $\overline{Y}$ and $\overline{Y}'$ are affine. Let $(\overline{\mathcal{V}}_J, \widetilde{\mathcal{P}}_J)$ be as in the proof of Lemma 2.6, and $\pi\colon \widetilde{\mathcal{P}}_{J,\mathbf{Q}} \to \overline{\mathcal{P}}_{J,\mathbf{Q}}$ be the morphism of dagger spaces induced by the natural projection. By the computation in the proof of Lemma 2.6, we see that for any $i \geq 0$ we have $\omega^i_{(\overline{\mathcal{V}}_J, \widetilde{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}} \cong \pi^* \omega^i_{(\overline{\mathcal{V}}_J, \overline{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}}$ and it is free over $\mathcal{O}_{\widetilde{\mathcal{P}}_{J,\mathbf{Q}}}$. Now we claim that

$$
\mathrm{R}\pi_* \mathcal{O}_{\widetilde{\mathcal{P}}_{J,\mathbf{Q}}} \cong \mathcal{O}_{\overline{\mathcal{P}}_{J,\mathbf{Q}}}. \tag{2.5}
$$

If this claim holds, we have $\mathrm{R}\pi_* \omega^i_{(\overline{\mathcal{V}}_J, \widetilde{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}} \cong \omega^i_{(\overline{\mathcal{V}}_J, \overline{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}}$. Since $]\widetilde{P}^I_J[_{\widetilde{\mathcal{P}}_J} = \pi^{-1}(]\overline{P}^I_J[_{\overline{\mathcal{P}}_J})$, we obtain

$$
H^j(]\widetilde{P}^I_J[_{\widetilde{\mathcal{P}}_J}, \omega^i_{(\overline{\mathcal{V}}_J, \widetilde{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}}) \cong H^j(]\overline{P}^I_J[_{\overline{\mathcal{P}}_J}, \omega^i_{(\overline{\mathcal{V}}_J, \overline{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}})
$$

for any $i, j \geq 0$. Thus we have an isomorphism between the spectral sequences

$$
\begin{aligned}
E_1^{i,j} &= H^j(]\widetilde{P}^I_J[_{\widetilde{\mathcal{P}}_J}, \omega^i_{(\overline{\mathcal{V}}_J, \widetilde{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}}) \Rightarrow H^{i+j}(]\widetilde{P}^I_J[_{\widetilde{\mathcal{P}}_J}, \omega^\bullet_{(\overline{\mathcal{V}}_J, \widetilde{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}}) = H^{i+j}_{\mathrm{rig}}((\overline{V}^I_J, \widetilde{P}^I_J)/\mathcal{T}) \qquad \text{and} \\
E_1^{i,j} &= H^j(]\overline{P}^I_J[_{\overline{\mathcal{P}}_J}, \omega^i_{(\overline{\mathcal{V}}_J, \overline{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}}) \Rightarrow H^{i+j}(]\overline{P}^I_J[_{\overline{\mathcal{P}}_J}, \omega^\bullet_{(\overline{\mathcal{V}}_J, \overline{\mathcal{P}}_J)/\mathcal{T},\mathbf{Q}}) = H^{i+j}_{\mathrm{rig}}((\overline{V}^I_J, \overline{P}^I_J)/\mathcal{T}),
\end{aligned}
$$

and this implies the lemma. It remains to prove the claim (2.5). For any subset $I \subset J$ and $L \subset J'$ let

$$
\begin{array}{ll}
\overline{\mathcal{V}}^I_J := \overline{\mathcal{M}}_J \times \mathcal{T}^I, & \overline{\mathcal{P}}^I_J := \overline{\mathcal{M}}_J \times \overline{\mathcal{T}}^I, \\
\overline{\mathcal{V}}'^L_{J'} := \overline{\mathcal{M}}'_{J'} \times \mathcal{T}^L, & \overline{\mathcal{P}}'^L_{J'} := \overline{\mathcal{M}}'_{J'} \times \overline{\mathcal{T}}^L.
\end{array}
$$

Consider the composition of natural morphisms $\overline{\mathcal{V}}_J \to \overline{\mathcal{V}}'_{J'} \to \overline{\mathcal{V}}'^L_{J'} \to \overline{\mathcal{P}}'^L_{J'}$, where the second map is induced by the natural projection $\mathcal{T}^{J'} \to \mathcal{T}^L$. Let $\widetilde{\mathcal{P}}_{J,L}$ be the (weak formal) log schematic image of the diagonal immersion $\overline{\mathcal{V}}_J \to \overline{\mathcal{P}}_J \times \overline{\mathcal{P}}'^L_{J'}$. Then for all subsets $L_1, L_2 \subset J'$ with $L_1 \cap L_2 = \emptyset$, we have $\widetilde{\mathcal{P}}_{J,L_1 \cup L_2} = \widetilde{\mathcal{P}}_{J,L_1} \times_{\overline{\mathcal{P}}_J} \widetilde{\mathcal{P}}_{J,L_2}$. Let $\pi^L\colon \widetilde{\mathcal{P}}_{J,L,\mathbf{Q}} \to \overline{\mathcal{P}}_{J,\mathbf{Q}}$ be the morphism induced by the natural projection. Note that $\widetilde{\mathcal{P}}_{J,J'} = \widetilde{\mathcal{P}}_J$ and $\pi^{J'} = \pi$. Recall that locally rigid Hyodo-Kato data can be constructed as weak completion of a usual scheme (see the proof of Lemma 1.28), hence one can apply all constructions for schemes, and obtain a morphism of schemes $\widetilde{Z}_{J,L} \to \overline{Z}_J$ whose weak completion is $\widetilde{\mathcal{P}}_{J,L} \to \overline{\mathcal{P}}_J$. Let $\pi^L_s\colon \widetilde{Z}_{J,L,\mathbf{Q}} \to \overline{Z}_{J,\mathbf{Q}}$ be the induced morphism. By [31, Thm. A], it suffices to show that

$$
\mathrm{R}\pi^{J'}_{s,*} \mathcal{O}_{\widetilde{Z}_{J,J',\mathbf{Q}}} \cong \mathcal{O}_{\overline{Z}_{J,\mathbf{Q}}}.
$$

We will prove that $\mathrm{R}\pi^L_{s,*} \mathcal{O}_{\widetilde{Z}_{J,L,\mathbf{Q}}} \cong \mathcal{O}_{\overline{Z}_{J,\mathbf{Q}}}$ for any L by induction on $|L|$. Note that $\pi^L_{s,*} \mathcal{O}_{\widetilde{Z}_{J,L,\mathbf{Q}}} = \mathcal{O}_{\overline{Z}_{J,\mathbf{Q}}}$ since $\overline{Z}_{J,\mathbf{Q}}$ is normal. If $|L| = 1$, we have $\mathrm{R}\pi^L_{s,*} \mathcal{O}_{\widetilde{Z}_{J,L,\mathbf{Q}}} = 0$ by [24, Cor. 11.2]. Let $g\colon X \to \widetilde{Z}_{J,L,\mathbf{Q}}$ be a resolution of singularities. Since X is smooth, $\mathrm{R}(\pi^L_s \circ g)_* \mathcal{O}_X = \mathcal{O}_{\overline{Z}_{J,\mathbf{Q}}}$ holds automatically. Hence via the Grothendieck–Leray spectral sequence

$$
E_2^{p,q} = \mathrm{R}^p \pi^L_{s,*} \mathrm{R}^q g_* \mathcal{O}_X \Rightarrow \mathrm{R}^{p+q}(\pi^L_s \circ g)_* \mathcal{O}_X,
$$

we obtain $E_2^{1,0} = \mathrm{R}^1\pi_{s,*}^L \mathcal{O}_{\widetilde{Z}_{J,L,\mathbf{Q}}} = 0$. Assume $|L| \geq 2$, fix an element $\ell \in L$, and set $L' := L \setminus \{\ell\}$. Since $\pi_s^{\{\ell\}}$ is flat, we have

$$\mathrm{R}\pi_{s,*}^{L,\{\ell\}}\mathcal{O}_{\widetilde{Z}_{J,L,\mathbf{Q}}} = \mathrm{R}\pi_{s,*}^{L,\{\ell\}}\pi_s^{L,L',*}\mathcal{O}_{\widetilde{Z}_{J,L',\mathbf{Q}}} = \pi_s^{\{\ell\},*}\mathrm{R}\pi_{s,*}^{L'}\mathcal{O}_{\widetilde{Z}_{J,L',\mathbf{Q}}} = \mathcal{O}_{\widetilde{Z}_{J,\{\ell\},\mathbf{Q}}}, \tag{2.6}$$

where $\pi_s^{L,L'}\colon \widetilde{Z}_{J,L,\mathbf{Q}} \to \widetilde{Z}_{J,L',\mathbf{Q}}$ and $\pi_s^{L,\{\ell\}}\colon \widetilde{Z}_{J,L,\mathbf{Q}} \to \widetilde{Z}_{J,\{\ell\},\mathbf{Q}}$ are natural projections, and the third equality is given by the induction hypothesis. Now we also have $\mathrm{R}\pi_{s,*}^{\ell}\mathcal{O}_{\widetilde{Z}_{J,\{\ell\},\mathbf{Q}}} = \mathcal{O}_{\overline{Z}_{J,\mathbf{Q}}}$. Hence by combining this with (2.6) we obtain $\mathrm{R}\pi_{s,*}^L\mathcal{O}_{\widetilde{Z}_{J,L,\mathbf{Q}}} = \mathcal{O}_{\overline{Z}_{J,\mathbf{Q}}}$ as desired. □

Consequently, for a morphism $(Y,\overline{Y}) \to (Y',\overline{Y}')$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathcal{O}_F^0) & \mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}_\bullet)/\mathcal{T}) \arrow[l, "\overline{\xi_0}"'] \arrow[r, "\overline{\xi_\pi}"] \arrow[d, "\cong"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathcal{O}_K^\pi) & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_K^\pi) \arrow[l] \\
 & & \mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\widetilde{P}_\bullet)/\mathcal{T}) & & \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}'/\mathcal{O}_F^0) \arrow[r] \arrow[uu] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}'_\bullet/\mathcal{O}_F^0) \arrow[uu] & \mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}'_\bullet,\overline{P}'_\bullet)/\mathcal{T}) \arrow[l, "\overline{\xi_0}"'] \arrow[r, "\overline{\xi_\pi}"] \arrow[u] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}'_\bullet/\mathcal{O}_K^\pi) \arrow[uu] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}'/\mathcal{O}_K^\pi). \arrow[l] \arrow[uu]
\end{tikzcd}$$

This gives the functoriality of the rigid Hyodo-Kato map and finishes the proof of Proposition 2.5.

2.2 Frobenius and monodromy on the rigid Hyodo–Kato complex

Keeping in mind that we want to compare log rigid syntomic and log crystalline syntomic cohomology, we will now show that the locally explicit Frobenius and monodromy operators on the rigid Hyodo–Kato complex are compatible with ones of a more classical form.

Let us start with the Frobenius. We take $(Y,\overline{Y}) \in \overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$. On the one hand, Definition 1.30 induces a Frobenius endomorphism φ_{HK} on $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ coming from local data. On the other hand, in (1.17) we obtained a Frobenius endomorphism φ on $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0)$ by base change and functoriality. We will now see, that they are compatible with the comparison quasi-isomorphism between $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ and $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0)$.

Proposition 2.8. *For an object $(Y,\overline{Y}) \in \overline{\mathsf{LS}}^{ss}_{k^0}$, the isomorphism $\Theta_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0) \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ is compatible with the Frobenius endomorphism. In other words, there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0) \arrow[r, "\varphi"] \arrow[d, "\Theta_{\mathrm{HK}}"', "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0) \arrow[d, "\Theta_{\mathrm{HK}}"', "\sim"] \\
\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) \arrow[r, "\varphi_{\mathrm{HK}}"] & \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}).
\end{tikzcd}$$

Proof. As in Remark 1.50 we can take a simplicial object $((Y_\bullet,\overline{Y}_\bullet),(\mathcal{Z}_\bullet,\overline{\mathcal{Z}}_\bullet),i_\bullet,\phi_\bullet)$ in $\mathsf{RQ}_{\mathrm{HK}}$ such that $\{(Y_\bullet,\overline{Y}_\bullet)\}$ is a Zariski hyper covering of $(Y,\overline{Y})$. Let $(\overline{Y}_\bullet\overline{\mathcal{Y}}_\bullet,i_\bullet)$ be the simplicial object in $\mathsf{RT}_{\mathcal{O}_F^0}$ given by $\overline{\mathcal{Y}}_\bullet := \overline{\mathcal{Z}}_\bullet \times_{\mathcal{T}} \mathcal{O}_F^0$. Then we have a commutative diagram of quasi-isomorphisms

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{O}_F^0) \arrow[r, "\sim"] \arrow[d, "\Theta_{\mathrm{HK}}"] & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}_\bullet/\mathcal{O}_F^0) \arrow[d, "\widehat{\theta}_{\mathrm{HK}}"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}_\bullet/\mathcal{O}_F^0)_{\overline{\mathcal{Y}}_\bullet} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}_\bullet/\mathcal{O}_F^0)_{\overline{\mathcal{Y}}_\bullet} \arrow[d, "\theta_{\mathrm{HK}}"] \\
\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) \arrow[r, "\sim"] & \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet,\overline{Y}_\bullet) & \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet,\overline{Y}_\bullet)_{(\mathcal{Z}_\bullet,\overline{\mathcal{Z}}_\bullet)} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet,\overline{Y}_\bullet)_{(\mathcal{Z}_\bullet,\overline{\mathcal{Z}}_\bullet)}.
\end{tikzcd}$$

We denote by ρ_{rig} and $\rho^{\mathrm{rig}}_{\mathrm{HK}}$ the compositions of the upper respectively lower horizontal arrows in the above diagram. Since $\rho^{\mathrm{rig}}_{\mathrm{HK}}$ is a morphism in $\mathscr{D}^+_F(\varphi,N)$, the Frobenius endomorphisms φ_{HK} on $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ and $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet,\overline{Y}_\bullet)_{(\mathcal{Z}_\bullet,\overline{\mathcal{Z}}_\bullet)}$ induced locally by Definition 1.30 are compatible with each other via $\rho^{\mathrm{rig}}_{\mathrm{HK}}$.

On the other hand, since the Frobenius $\phi_\bullet$ of the simplicial rigid Hyodo–Kato datum induces a lift of the absolute Frobenius on $\overline{\mathcal{Y}}_\bullet$, we obtain a lift $F_{\overline{\mathcal{Y}}_\bullet}\colon \overline{\mathcal{Y}}_\bullet \to \overline{\mathcal{Y}}^\sigma_\bullet$ over $\mathcal{T}$ of the relative Frobenius endomorphism $f_{\overline{Y}_\bullet}\colon \overline{Y}_\bullet \to \overline{Y}^\sigma_\bullet$. Now we have a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0) \arrow[r, "\sim"] \arrow[d, "\Theta_\sigma"] & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}_\bullet/\mathscr{O}_F^0) \arrow[d, "\widehat{\theta}_\sigma"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}_\bullet/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}_\bullet} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}_\sigma"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}_\bullet/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}_\bullet} \arrow[d, "\theta_\sigma"] \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}^\sigma/\mathscr{O}_F^0) \arrow[r, "\sim"] \arrow[d, "f^*_{\overline{Y}}"] & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}^\sigma_\bullet/\mathscr{O}_F^0) \arrow[d, "f^*_{\overline{Y}_\bullet}"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}^\sigma_\bullet/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}^\sigma_\bullet} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "(f_{\overline{Y}_\bullet}, F_{\overline{\mathcal{Y}}_\bullet})^*"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}^\sigma_\bullet/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}^\sigma_\bullet} \arrow[d, "(f_{\overline{Y}_\bullet}, F_{\overline{\mathcal{Y}}_\bullet})^*"] \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0) \arrow[r, "\sim"] & \widehat{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}_\bullet/\mathscr{O}_F^0) & \widetilde{\mathrm{R}\Gamma}_{\mathrm{rig}}(\overline{Y}_\bullet/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}_\bullet} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}_\bullet/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}_\bullet},
\end{tikzcd}$$

and accordingly the Frobenius φ on $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)$ is given by

$$\varphi := f^*_{\overline{Y}} \circ \Theta_\sigma = \rho_{\mathrm{rig}}^{-1} \circ (f_{\overline{Y}_\bullet}, F_{\overline{\mathcal{Y}}_\bullet})^* \circ \theta_\sigma \circ \rho_{\mathrm{rig}}.$$

Moreover, the morphism $(f_{\overline{Y}_\bullet}, F_{\overline{\mathcal{Y}}_\bullet})^* \circ \theta_\sigma$ on $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}_\bullet/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}_\bullet}$ is induced by the action of $\phi_\bullet$ on $\overline{\mathcal{Y}}_\bullet$, and hence compatible the Frobenius φ_{HK} on $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet, \overline{Y}_\bullet)_{(\mathcal{Z}_\bullet, \overline{\mathcal{Z}}_\bullet)}$, so that we have

$$\varphi_{\mathrm{HK}} \circ \theta_{\mathrm{HK}} = \theta_{\mathrm{HK}} \circ (f_{\overline{Y}_\bullet}, F_{\overline{\mathcal{Y}}_\bullet})^* \circ \theta_\sigma \colon \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}_\bullet/\mathscr{O}_F^0)_{\overline{\mathcal{Y}}_\bullet} \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet, \overline{Y}_\bullet)_{(\mathcal{Z}_\bullet, \overline{\mathcal{Z}}_\bullet)}.$$

This shows the desired compatibility. □

Let us now discuss the monodromy and show that it can be interpreted as boundary morphism of a certain short exact sequence. Since this is a discussion on the level of cohomology, we will identify the (canonical) rigid Hyodo–Kato complexes which are induced by the quasi-isomorphic complexes in Proposition 1.35. If $((Y, \overline{Y}), (\mathcal{Z}, \overline{\mathcal{Z}}), i, \phi)$ is a rigid Hyodo–Kato quadruple and $\overline{\mathcal{Y}} := \overline{\mathcal{Z}} \times_{\mathcal{T}} \mathscr{O}_F^0$ the fibre of $\overline{\mathcal{Z}}$ over $t = 0$, then $(\overline{Y}, \overline{\mathcal{Y}}, i)$, where we denote $\overline{Y} \to \overline{\mathcal{Y}}$ again by i, is a log rigid triple. The short exact sequence (1.2) induces a short exact sequence

$$0 \to \omega^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}}[-1] \xrightarrow{d\log t} \widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to \omega^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to 0 \tag{2.7}$$

of sheaves on the generic fibre $\overline{\mathcal{Y}}_{\mathbf{Q}}$ of $\overline{\mathcal{Y}}$. Note that despite our notation the use of the immersion $(Y, \overline{Y}) \to (\mathcal{Z}, \overline{\mathcal{Z}})$ for the construction of the complexes $\widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ and the existence of the short exact sequence is essential. Nevertheless the definition

$$\mathrm{R}\Gamma^{\sim}_{\mathrm{rig}}((Y, \overline{Y})/\mathscr{O}_F^0)_{(\mathcal{Z}, \overline{\mathcal{Z}})} := \mathrm{R}\Gamma(]\overline{Y}[^{\log}_{\overline{\mathcal{Y}}}, \widetilde{\omega}^\bullet_{\overline{\mathcal{Y}}_{\mathbf{Q}}})$$

makes sense. Out of this complex, we can define canonical rigid complexes as we have done before. For $(Y, \overline{Y}) \in \overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$, and $A \in \mathsf{SET}^0_{\mathrm{HK}}(Y, \overline{Y})$ we consider $(\mathcal{Z}_A, \overline{\mathcal{Z}}_A)$ as in § 1.2 after Definition 1.38. Note that $\overline{\mathcal{Y}}_A := \overline{\mathcal{Z}}_A \times_{\mathcal{T}} \mathscr{O}_F^0$ together with the diagonal embedding $i_A \colon \overline{Y} \hookrightarrow \overline{\mathcal{Y}}_A$ is a log-rigid datum for $\overline{Y}$. We define the log rigid complexes

$$\widehat{\mathrm{R}\Gamma}^{\sim}_{\mathrm{rig}}((Y, \overline{Y})/\mathscr{O}_F^0) := \varinjlim_{A \in \mathsf{SET}^0_{\mathrm{HK}}(Y, \overline{Y})} \mathrm{R}\Gamma^{\sim}_{\mathrm{rig}}((Y, \overline{Y})/\mathscr{O}_F^0)_{(\mathcal{Z}_A, \overline{\mathcal{Z}}_A)}. \tag{2.8}$$

This construction is again functorial in $\mathsf{RQ}_{\mathrm{HK}}$ and by a similar intermediate construction as in Proposition 1.45 we see that for a rigid Hyodo–Kato quadruple $((Y, \overline{Y}), (\mathcal{Z}, \overline{\mathcal{Z}}), i, \phi)$ and $\overline{\mathcal{Y}}$ as above the complexes $\mathrm{R}\Gamma^{\sim}_{\mathrm{rig}}((Y, \overline{Y})/\mathscr{O}_F^0)_{(\mathcal{Z}, \overline{\mathcal{Z}})}$ and $\widehat{\mathrm{R}\Gamma}^{\sim}_{\mathrm{rig}}((Y, \overline{Y})/\mathscr{O}_F^0)$ are quasi-isomorphic.

Let $\mathfrak{C}^{\sim}_{\mathrm{rig}}$ be the sheafification of the presheaf $(Y, \overline{Y}) \mapsto \widehat{\mathrm{R}\Gamma}^{\sim}_{\mathrm{rig}}((Y, \overline{Y})/\mathscr{O}_F^0)$ on $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$ with respect to the Zariski topology. By Proposition 1.2 and Lemma 1.46, we may extend them to sheaves on $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$. For any object $(Y, \overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$, we define a canonical rigid complex

$$\mathrm{R}\Gamma^{\sim}_{\mathrm{rig}}((Y, \overline{Y})/\mathscr{O}_F^0) := \mathrm{R}\Gamma(\overline{Y}_{\mathrm{Zar}}, \mathfrak{C}^{\sim}_{\mathrm{rig}}).$$

By the same argument used in Definition 1.48 for the rigid Hyodo–Kato complex, this definition extends to simplicial objects in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$. As in Proposition 1.49, we see that if $(Y,\overline{Y})$ is an object in $\overline{\mathsf{ELS}}^{\mathrm{ss}}_{k^0}$ the natural morphism

$$\widehat{\mathrm{R}\Gamma}^{\sim}_{\mathrm{rig}}((Y,\overline{Y})/\mathscr{O}_F^0) \to \mathrm{R}\Gamma^{\sim}_{\mathrm{rig}}((Y,\overline{Y})/\mathscr{O}_F^0)$$

is a quasi-isomorphism.

For $(Y,\overline{Y})$ in $\overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ let now $N:\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ be the morphism locally induced by the boundary morphism of the short exact sequence (2.7). In particular, there is a distinguished triangle of the form

$$\mathrm{R}\Gamma^{\sim}_{\mathrm{rig}}((Y,\overline{Y})/\mathscr{O}_F^0) \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) \xrightarrow{N} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}). \tag{2.9}$$

On the other denote here by $N_{\mathrm{HK}}:\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) \to \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})$ the morphism wich is locally induced by the monodromy operator in Definition 1.30.

Proposition 2.9. *For a strictly semistable log scheme with boundary over k^0, i.e. $(Y,\overline{Y}) \in \overline{\mathsf{LS}}^{ss}_{k^0}$, the monodromy operators N and N_{HK} coincide on the level of cohomology.*

Proof. Let $((Y_\bullet,\overline{Y}_\bullet),(\mathcal{Z}_\bullet,\overline{\mathcal{Z}}_\bullet),i_\bullet,\phi_\bullet)$ and $(\overline{Y}_\bullet,\overline{\mathcal{Y}}_\bullet,i_\bullet)$ be the simplicial objects from the previous proposition. We have quasi-isomorphisms

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) \xrightarrow{\sim} \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet,\overline{Y}_\bullet) \xleftarrow{\sim} \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet,\overline{Y}_\bullet)_{(\mathcal{Z}_\bullet,\overline{\mathcal{Z}}_\bullet)} \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y_\bullet,\overline{Y}_\bullet)_{(\mathcal{Z}_\bullet,\overline{\mathcal{Z}}_\bullet)}. \tag{2.10}$$

Since this zigzag of maps is compatible with N and N_{HK} respectively, we can compare N and N_{HK} on $\overline{Y}_\bullet$, and we may therefore assume without loss of generality that $(Y,\overline{Y})$ is HK-embeddable. Thus let $((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi)$ be a rigid Hyodo–Kato quadruple.

For $\overline{\mathcal{Y}}_{\mathbf{Q}}$ as above we define a double complex $B^{\bullet,\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ by setting

$$\begin{aligned}
&B^{ij}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} := A^{i-1,j}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \oplus A^{ij}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}\\
d':\quad &B^{ij}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to B^{i+1,j}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}, \quad d'(\omega_1,\omega_2) = ((-1)^j d\omega_1, (-1)^{j+1} d\omega_2)\\
d'':\quad &B^{ij}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to B^{i,j+1}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}, \quad d''(\omega_1,\omega_2) = ((-1)^i d\log t \wedge \omega_1 + \nu(\omega_2), (-1)^i d\log t \wedge \omega_2)
\end{aligned}$$

Let $B^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ be the total complex of $B^{\bullet,\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$. There is a morphism $\lambda:\widetilde{\omega}^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \to B^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ induced by $\omega \mapsto (\omega, \omega\wedge d\log t)$ such that there exists a morphism of short exact sequences (see [34, (11.8.1)])

$$\begin{tikzcd}
0 \arrow[r] & \omega^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}[-1] \arrow[r, "\wedge d\log t"] \arrow[d, "\sim"', "\wedge d\log t"] & \widetilde{\omega}^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \arrow[r] \arrow[d, "\lambda"] & \omega^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \arrow[r] \arrow[d, "\sim"', "\wedge d\log t"] & 0\\
0 \arrow[r] & A^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}[-1] \arrow[r] & B^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \arrow[r] & A^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}} \arrow[r] & 0
\end{tikzcd}$$

where the left and right vertical map is the quasi-isomorphism from Lemma 1.29. Note that in contrast to [34] we don't consider the Frobenius action here. By definition of $B^{\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$ the boundary morphism of the lower exact sequence coincides with the morphism on $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z},\overline{\mathcal{Z}})}$ induced by the endomorphism N_{HK} on the double complex $A^{\bullet,\bullet}_{\overline{\mathcal{Y}}_{\mathbf{Q}}}$, while the boundary morphism of the upper exact sequence is the monodromy N, and the statement follows. ☐

3 Syntomic cohomology

In this section we recall the definition of crystalline syntomic cohomology from [35] and give a new definition of rigid syntomic cohomology in the case of a strictly semistable log scheme with boundary. For this we use the canonical rigid complexes introduced in the previous section. Both of these cohomology theories can be understood in the context of p-adic Hodge complexes. For the convenience of the reader we give a short overview over the situation.

3.1 p-adic Hodge complexes

The philosophy behind p-adic Hodge cohomology is in analogy with Beilinson's interpretation of Deligne–Beilinson cohomology as absolute Hodge cohomology. Denote by

$$\mathrm{MF}_K^{\mathrm{ad}}(\varphi) \subset \mathrm{MF}_K^{\mathrm{ad}}(\varphi, N) \subset \mathrm{MF}_K^{\mathrm{ad}}(\phi, N, G_K)$$

the categories of (weak) admissible filtered φ-modules, (φ, N)-modules and (φ, N, G_K)-modules as defined by Fontaine. They are equivalent to crystalline, semistable, and potentially semistable Galois representations. The objects we want to consider lie in the derived category of admissible filtered φ-, (φ, N)- and (φ, N, G_K)-modules respectively, depending on the scheme under consideration. These derived categories correspond to the various categories of p-adic Hodge complexes.

Recall that a φ-module over F is a pair (M, φ), where M is an F-vector space and φ is a σ-semilinear endomorphism of M. Likewise a (φ, N)-module over F is a triple (M, φ, N), such that (M, φ) is a φ-module over F and N is an F-linear endomorphism of M such that $N\varphi = p\varphi N$. Finally, a (φ, N, G_K)-module is a quadruple (M, φ, N, ρ), where M is a F^{nr}-vector space, φ is a σ-semilinear endomorphism of M, N is an F^{nr}-linear endomorphism of M such that $N\varphi = p\varphi N$, and ρ is an F^{nr}-semilinear action of G_K on M_0 factoring through a quotient of the inertia group and commuting with φ and N. Usually φ is called Frobenius and N is called monodromy operator.

Definition 3.1. (i) A filtered φ-module over K is a triple $(M_0, \varphi, \mathrm{Fil}^\bullet)$, where (M_0, φ) is a φ-module over F, and $\mathrm{Fil}^\bullet$ is a decreasing, separated, exhaustive filtration on $M = M_0 \otimes_F K$ called the Hodge filtration.

(ii) A filtered (φ, N)-module $(M_0, \varphi, N, \mathrm{Fil}^\bullet)$ over K consists of a filtered φ-module $(M_0, \varphi, \mathrm{Fil}^\bullet)$ together with an F-linear operator N on M_0 satisfying the relation $N\varphi = p\varphi N$.

(iii) A filtered (φ, N, G_K)-module is a quintuple $(M_0, \varphi, N, \rho, \mathrm{Fil}^\bullet)$ where (M_0, φ, N, ρ) is a (φ, N, G_K)-module, and $\mathrm{Fil}^\bullet$ is a decreasing, separate, exhaustive filtration of $M = (M_0 \otimes_{F^{\mathrm{nr}}} \overline{K})^{G_K}$.

A filtered φ-, (φ, N), or (φ, N, G_K)-module is called (weakly) admissible if M_0 is finite dimensional (over F in the first two cases, over F^{nr} in the third case) and if for any filtered φ-, (φ, N), or (φ, N, G_K)-submodule the Hodge and Newton number coincide.

The categories of admissible filtered φ-, (φ, N), and (φ, N, G_K)-modules are known to be Tannakian. Thus it makes sense to consider their respective differential graded bounded derived categories denoted by $\mathscr{D}^\sharp(\mathrm{MF}_K^{\mathrm{ad}}(\varphi))$, $\mathscr{D}^\sharp(\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N))$ and $\mathscr{D}^\sharp(\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N, G_K))$ respectively.

On the other hand there are p-adic Hodge complexes. Depending on the situation to be studied, this means different objects.
Consider the following categories:

- For a field $L \in \{F, F^{\mathrm{nr}}, K, \overline{K}\}$, let V_L be the category of L-vector spaces .
- For a field $L \in \{F, F^{\mathrm{nr}}, K, \overline{K}\}$, let V_L^{dR} be the category of L-vector spaces together with a separated, exhaustive, descending filtration.
- Let $V_{\overline{K}}^G$ be the category of $\overline{K}$-vector spaces together with a smooth $\overline{K}$-linear G_K-action.
- Let $M_K(\varphi)$, $M_K(\varphi, N)$ and $M_K(\varphi, N, G_K)$ be the categories of φ-, (φ, N)- and (φ, N, G_K)-modules (defined as above without the filtration).

Depending on the situation, denote by F_{dR} either of the functors

$$F_{\mathrm{dR}} : V_K^{\mathrm{dR}} \to V_K, \quad (E, \mathrm{Fil}^\bullet) \mapsto E;$$
$$F_{\mathrm{dR}} : V_K^{\mathrm{dR}} \to V_{\overline{K}}^G, \quad (E, \mathrm{Fil}^\bullet) \mapsto E \otimes \overline{K}.$$

Similarly, depending on the situation, denote by F_0 either of the functors

$$F_0 : M_K(\varphi) \to V_K, \qquad (M_0, \varphi) \mapsto M_0 \otimes K;$$
$$F_0 : M_K(\varphi, N) \to V_K, \qquad (M_0, \varphi, N) \mapsto M_0 \otimes K;$$
$$F_0 : M_K(\varphi, N, G_K) \to V_{\overline{K}}^G, \quad (M_0, \varphi, N, \rho) \mapsto (M_0 \otimes \overline{K}, \rho).$$

These functors are used to glue the differential graded bounded derived categories of the categories just introduced in order to obtain the differential graded categories of p-adic Hodge complexes.

Definition 3.2. Let $\mathscr{D}_{pH}^{\mathrm{pst}}$ be the differential graded category given by the homotopy limit

$$\mathscr{D}_{pH}^{\mathrm{pst}} := \operatorname{holim}\left(\mathscr{D}^\sharp(M_K(\varphi, N, G_K)) \xrightarrow{F_0} \mathscr{D}^\sharp(V_{\overline{K}}^G) \xleftarrow{F_{\mathrm{dR}}} \mathscr{D}^\sharp(V_K^{\mathrm{dR}})\right).$$

Let $\mathscr{D}_{pH}^{\mathrm{st}}$ be the differential graded category given by the homotopy limit

$$\mathscr{D}_{pH}^{\mathrm{st}} := \operatorname{holim}\left(\mathscr{D}^\sharp(M_K(\phi, N)) \xrightarrow{F_0} \mathscr{D}^\sharp(V_K^G) \xleftarrow{F_{\mathrm{dR}}} \mathscr{D}^\sharp(V_K^{\mathrm{dR}})\right).$$

Let $\mathscr{D}_{pH}^{\mathrm{cr}}$ be the differential graded category given by the homotopy limit

$$\mathscr{D}_{pH}^{\mathrm{cr}} := \operatorname{holim}\left(\mathscr{D}^\sharp(M_K(\varphi)) \xrightarrow{F_0} \mathscr{D}^\sharp(V_K^G) \xleftarrow{F_{\mathrm{dR}}} \mathscr{D}^\sharp(V_K^{\mathrm{dR}})\right).$$

Concretely, an object of $\mathscr{D}_{pH}^{\mathrm{pst}}$ is given via a quasi-pushout construction by a tuple of the form $((M_0, \varphi, N, \rho), (M_K, \mathrm{Fil}^\bullet), \alpha : M_0 \otimes \overline{K} \xrightarrow{\sim} M_K \otimes \overline{K})$, where (M_0, φ, N, ρ) is a complex of (φ, N, G_K)-modules, $(M_K, \mathrm{Fil}^\bullet)$ is a complex of filtered K-vector spaces, and α is a comparison quasi-isomorphism between them over $\overline{K}$. Similar presentations can be specified in the semistable and crystalline case.

For a more detailed explanation of this gluing process see [14, Sec. 2.2]. For more concrete descriptions of this construction see also [1, Section 2] and [12, Section 2]. Déglise and Nizioł point out that one consequence of this gluing process is that p-adic Hodge complexes have a canonical t-structure. Moreover, there is an obvious functor of differential graded categories

$$\Theta\colon \mathscr{D}^\sharp(\mathrm{MF}_K(\phi, N, G_K)) \to \mathscr{D}_{pH.}^{\mathrm{pst}}$$

It restricts to functors

$$\begin{aligned} \mathscr{D}^\sharp(\mathrm{MF}_K(\phi, N)) &\to \mathscr{D}_{pH}^{\mathrm{st}} \\ \mathscr{D}^\sharp(\mathrm{MF}_K(\phi)) &\to \mathscr{D}_{pH}^{\mathrm{cr}} \end{aligned}$$

and is compatible with the t-structures in all three cases. Moreover, the restriction to the hearts with respect to the t-structures is fully faithful (cf. [14, Lem. 3.6]). This fact leads to the following definition.

Definition 3.3. A strict p-adic Hodge complex $M \in \mathscr{D}_{pH}^{\mathrm{pst}}$ (or $\mathscr{D}_{\mathrm{pH}}^{\mathrm{st}}$ or $\mathscr{D}_{pH}^{\mathrm{cr}}$) is called admissible, if its cohomology objects with respect to the t-structure are contained in $\mathrm{MF}_K^{\mathrm{ad}}(\phi, N, G_K)$ (or $\mathrm{MF}_K^{\mathrm{ad}}(\phi, N)$ or $\mathrm{MF}_K^{\mathrm{ad}}(\phi)$). Denote by $\mathscr{D}_{pH,\mathrm{ad}}^{\mathrm{pst}} \subset \mathscr{D}_{pH}^{\mathrm{pst}}$ (and $\mathscr{D}_{pH,\mathrm{ad}}^{\mathrm{st}} \subset \mathscr{D}_{pH}^{\mathrm{st}}$ and $\mathscr{D}_{pH,\mathrm{ad}}^{\mathrm{cr}} \subset \mathscr{D}_{pH}^{\mathrm{cr}}$ respectively) the full subcategory of admissible p-adic Hodge complexes.

Then the functor Θ identifies $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N, G_K)$ (or $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ or $\mathrm{MF}_K^{\mathrm{ad}}(\varphi)$) with the heart of the induced t-structures.

Theorem 3.4 (K.Bannai, F. Déglise and W. Nizioł)**.** *The functor Θ induces equivalences of triangulated differential graded categories*

$$\begin{aligned} D^\sharp(\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N, G_K) &\xrightarrow{\sim} \mathscr{D}_{pH,\mathrm{ad}}^{pst} \\ D^\sharp(\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N) &\xrightarrow{\sim} \mathscr{D}_{pH,\mathrm{ad}}^{st} \\ D^\sharp(\mathrm{MF}_K^{\mathrm{ad}}(\varphi) &\xrightarrow{\sim} \mathscr{D}_{pH,\mathrm{ad}}^{\mathrm{cr}}. \end{aligned}$$

Proof. This is the summary of [14, Theo. 2.17] and [1, Theo. 3.2]. □

3.2 Review of crystalline syntomic cohomology

In this section, we recall the definition of crystalline syntomic cohomology. For crystalline cohomology together with Frobenius, monodromy and Hyodo–Kato morphism is an essential part of the data, we start with a brief review thereof. We refer to [25] and [35, Sec. 3.1] for more details.

Crystalline cohomology is defined as the cohomology on the (log) crystalline topos. In the context of this paper we are mostly concerned with the cohomology of the crystalline structure sheaf. Let $\mathscr{O}_F\langle t_l\rangle$ be the divided power polynomial algebra generated by elements t_l for $l \in \mathfrak{m}_K/\mathfrak{m}_K^2\backslash 0$ with the relation $t_{al} = [\overline{a}]t_l$ for $a \in \mathscr{O}_K^*$. Let $\mathscr{T}_{PD}$ be the p-adic completion of the subalgebra of $\mathscr{O}_F\langle t_l\rangle$ generated by t_l

and $\frac{t_l^{ie}}{i!}$. If we fix l and set $t = t_l$ it can be seen as an $\mathscr{O}_F$-subalgebra of $F[[t]]$. As before, we extend the Frobenius by setting $\sigma(t_l) = t_l^p$. By abuse of notation we denote by $\mathcal{T}_{PD}$ also the scheme $\operatorname{Spec} \mathcal{T}_{PD}$ with the log structure generated by the t_l's. There are exact closed embeddings

$$\mathscr{O}_F^0 \xrightarrow{i_0} \mathcal{T}_{PD} \xleftarrow{i_\pi} \mathscr{O}_K^\pi$$

via $t_l \mapsto 0$ and $t_l \mapsto [\overline{\frac{l}{\pi}}]\pi$, For a fine proper log smooth log scheme of Cartier type X over $\mathscr{O}_K^\pi$ we consider the log crystalline complexes

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathscr{O}_K^\pi) &:= \operatorname{holim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_n/\mathscr{O}_{K,n}^\pi), \\ \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{T}_{PD}) &:= \operatorname{holim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_n/\mathcal{T}_{PD,n}), \\ \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X) &:= \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}_F^0) := \operatorname{holim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}_{F,n}^0). \end{aligned}$$

As indicated by the notation which we borrowed from [35, Sec. 3.1], the latter one is often called the Hyodo–Kato complex.

The objects $\mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{T}_{PD})$ and $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X)$ have a monodromy operator and an action of Frobenius. The Frobenius action φ is in both cases given by the relative Frobenius and base change by σ, similarly to the rigid case described in Proposition 1.79, that is

$$\varphi : \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{T}_{PD}) \xrightarrow{\Theta_\sigma} \mathrm{R}\Gamma_{\mathrm{cr}}(X^\sigma/\mathcal{T}_{PD}) \xrightarrow{f_X^*} \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{T}_{PD})$$
$$\varphi : \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X) \xrightarrow{\Theta_\sigma} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X^\sigma) \xrightarrow{f_{X_0}^*} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X)$$

where Θ_σ is the base change by σ on $\mathcal{T}_{PD}$ or $\mathscr{O}_F$, and f_X and f_{X_0} are the relative Frobenii of X over $\mathcal{T}_{PD}$ and of X_0 over $\mathscr{O}_F$ respectively. The Frobenius action is invertible on $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X)_{\mathbf{Q}}$. As usual we denote by φ_r the Frobenius divided by p^r.

We explain how to obtain the monodromy operator on $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X)$ as the boundary morphism of a short exact sequence. Let $\mathscr{O}_F\langle t\rangle$ be the divided power polynomial algebra in one variable t over $\mathscr{O}_F$ Denote by $\mathcal{T}^{\mathrm{cr}}$ the log scheme with underlying scheme $\operatorname{Spf} \mathscr{O}_F\langle t\rangle$ and log structure associated to the map $1 \mapsto t$ and let $i_0 \colon \mathscr{O}_F^0 \to \mathcal{T}^{\mathrm{cr}}$ be the exact closed immersion induced by the map $t \mapsto 0$. Choose an embedding system $\theta : U_\bullet \to X_0$ for X_0 over $\operatorname{Spec}(\mathscr{O}_{F,n}[t], 1 \mapsto t)$ as explained in [25, (3.6)]. We also denote by $\theta : (U_\bullet)_{\text{ét}}^{\mathrm{tot}} \to X_{0,\text{ét}}$ the induced morphism of étale sites. Then we obtain crystalline complexes $C_{X_0/\mathscr{O}_{F,n}}$ and $C_{X_0/\mathscr{O}_{F,n}\langle t\rangle}$ on $(U_\bullet)_{\text{ét}}^{\mathrm{tot}}$ related by the short exact sequence

$$0 \to C_{X_0/\mathcal{T}_n^{\mathrm{cr}}}[-1] \xrightarrow{\wedge d\log t} C_{X_0/\mathscr{O}_{F,n}^\varnothing} \xrightarrow{\mathrm{can}} C_{X_0/\mathcal{T}_n^{\mathrm{cr}}} \to 0.$$

Taking the pull-back along i_0 we obtain a short exact sequence on $(U_\bullet)_{\text{ét}}^{\mathrm{tot}}$

$$0 \to C_{X_0/\mathscr{O}_{F,n}^0}[-1] \xrightarrow{\wedge d\log t} i_0^* C_{X_0/\mathscr{O}_{F,n}^\varnothing} \xrightarrow{\mathrm{can}} C_{X_0/\mathscr{O}_{F,n}^0} \to 0, \tag{3.1}$$

where $C_{X_0/\mathscr{O}_{F,n}^0}$ computes Hyodo–Kato cohomology with finite coefficients. The monodromy N is now the connecting homomorphism of the induced long exact sequence of cohomology. One can now take the homotopy limit and invert p to obtain the desired map

$$N \colon \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X)_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X)_{\mathbf{Q}}.$$

The homotopy limit of $R\theta_* C_{X_0/\mathscr{O}_{F,n}^0}$ is commonly denoted by $W\omega_{X_0}^\bullet$ and called the logarithmic de Rham–Witt complex. Similarly, we use the notation

$$W\tilde{\omega}_{X_0}^\bullet := \operatorname{holim} \mathrm{R}\theta_* i_0^* C_{X_0/\mathscr{O}_{F,n}^\varnothing}. \tag{3.2}$$

On the rational complexes $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X)_{\mathbf{Q}}$ and $\mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{T}_{PD})_{\mathbf{Q}}$ one replaces the usual monodromy operators by the normalised ones $e^{-1}N$ to make them compatible with base change.

The morphisms of log schemes i_0 and i_π induce morphisms on cohomology

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(X) \xleftarrow{i_0^*} \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{T}_{PD}) \xrightarrow{i_\pi^*} \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathscr{O}_K^\pi). \tag{3.3}$$

The map $\mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{I}_{PD}) \to \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X)$ from the above diagram has in the derived category a unique functorial $\mathcal{O}_F$-linear section $s_\pi : \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X)_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathcal{I}_{PD})_{\mathbf{Q}}$ which commutes with the Frobenius and the normalised monodromy and whose $\mathcal{I}_{PD}$-linear extension is a quasi-isomorphism. We set

$$\iota^{\mathrm{cr}}_\pi = i^*_\pi \circ s_\pi : \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X)_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{O}^\pi_K)_{\mathbf{Q}}.$$

It induces a K-linear functorial quasi-isomorphism $\iota_\pi : \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \otimes_{W(k)} K \to \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{O}^\pi_K)_{\mathbf{Q}}$.

Moreover, by the crystalline Poincaré Lemma there exists a canonical quasi-isomorphism

$$\gamma\colon \mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{O}^\pi_K)_{\mathbf{Q}},$$

where we endow X_K with the pull-back log structure of X and the left hand side is the log de Rham cohomology of X_K with the Hodge filtration given by the stupid filtration of the log de Rham complex. The composition in the derived category

$$\iota^\pi_{\mathrm{dR}} := \gamma^{-1} \circ i^*_\pi \circ s_\pi : \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X)_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)$$

is called the Hyodo–Kato morphism.

Definition 3.5. Let X be a fine proper log smooth log scheme of Cartier type over $\mathcal{O}^\pi_K$. For $r \geqslant 0$ and a choice of uniformizer π, we define the crystalline syntomic cohomology as the homotopy limit

$$\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{syn}}(X, r, \pi) := \left[\begin{tikzcd} & \mathrm{Fil}^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \arrow[d, "{(0,\gamma)}"] \\ \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \arrow[r, "{(1-\varphi_r, \iota^{\mathrm{cr}}_\pi)}"] \arrow[d, "N"] & \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \oplus \mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathcal{O}^\pi_K) \arrow[d, "{(N,0)}"] \\ \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \arrow[r, "1-\varphi_{r-1}"] & \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \end{tikzcd}\right]. \tag{3.4}$$

Note that we used the normalized monodromy operator here. Since γ is invertible in the derived category, there is in fact a quasi-isomorphism

$$\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{syn}}(X, r, \pi) \cong \left[\begin{tikzcd} \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \arrow[r, "{(1-\varphi_r, \iota^\pi_{\mathrm{dR}})}"] \arrow[d, "N"] & \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \oplus \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/\mathrm{Fil}^r \arrow[d, "{(N,0)}"] \\ \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \arrow[r, "1-\varphi_{r-1}"] & \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(X) \end{tikzcd}\right].$$

In [35, Sec. 3.3] Nekovář and Nizioł use h-sheafification to extend syntomic cohomology to K-varieties. For a K-variety Z and $r \in \mathbf{Z}$ we denote it by $\mathrm{R}\Gamma^{NN}_{\mathrm{syn}}(Z, r)$. To accomplish this they use appropriate bases for the h-topology following [4, Sec. 2], which uses de Jong's alteration theorem. However, the fields of definition of semi-stable models in the bases for h-topology change. This is a problem if one wants to represent syntomic cohomology as a homotopy limit similar to Definition 3.5 because the usual Hyodo–Kato map depends on a choice of uniformiser. If it is changed, one has to take into account transition functions involving exponentials of the monodromy. But the usual monodromy as defined above is only homotopically nilpotent. They address this problem by using a different representation of the rational Hyodo–Kato complex with nilpotent monodromy introduced by Beilinson [4, 1.16]. We call it the Beilinson–Hyodo–Kato complex and denote it by $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X)$ for $X/\mathcal{O}^\pi_K$ as above. What is more, it admits a Hyodo–Kato morphism

$$\iota^B_{\mathrm{dR}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)$$

which is independent of the choice of a uniformiser. Since everything in sight now h-sheafifies well, we have by [35, Prop. 3.20] for a K-variety Z and $r \geqslant 0$ the identification

$$\mathrm{R}\Gamma^{NN}_{\mathrm{syn}}(Z, r) \cong \left[\begin{tikzcd} \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_h) \arrow[r, "{1-\varphi_r, \iota^B_{\mathrm{dR}}}"] \arrow[d, "N"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_h) \oplus \mathrm{R}\Gamma^D_{\mathrm{dR}}(Z)/\mathrm{Fil}^r \arrow[d, "{(N,0)}"] \\ \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_h) \arrow[r, "1-\varphi_{r-1}"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_h) \end{tikzcd}\right], \tag{3.5}$$

where $\mathrm{R}\Gamma^D_{\mathrm{dR}}(Z)$ is Deligne's de Rham cohomology [16]. To relate (3.4) and (3.5) consider an open embedding $Z \hookrightarrow X$ of a K-variety into a regular proper flat $\mathscr{O}_K$-scheme such that $X \setminus Z$ is a normal crossing divisor and X_0 is reduced. The pair (Z, X) is an ss-pair over K in the sense of [3, 2.2]. We endow X with the log structure associated to the divisor $X \setminus Z$. The following lemma due to Nekovář and Nizioł is basically h-descent for syntomic cohomology.

Lemma 3.6. *For any $r \geq 0$ there is a canonical isomorphism in $\mathscr{D}^+_{\mathbf{Q}_p}$*

$$\mathrm{R}\Gamma^{NN}_{\mathrm{syn}}(Z, r) \cong \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{syn}}(X, r, \pi) \tag{3.6}$$

depending on the choice of a uniformiser π.

Proof. This is the combination of [35, Prop. 3.18] and [35, Prop. 3.8]. □

Déglise–Nizioł in [14, Sec. 2.3] place the above theory in the framework of p-adic Hodge complexes.

Definition 3.7. Let Z be a variety over K. The geometric p-adic Hodge cohomology of Z is given by the complex

$$\mathrm{R}\Gamma_{pH}(Z_{\overline{K}}, 0) := (\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_{\overline{K},h}), \mathrm{Fil}^0(\mathrm{R}\Gamma_{\mathrm{dR}}(Z)), \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_{\overline{K},h}) \xrightarrow{\iota^B_{\mathrm{dR}}} \mathrm{R}\Gamma^D_{\mathrm{dR}}(Z_{\overline{K}})) \in \mathscr{D}^{\mathrm{pst}}_{pH,\mathrm{ad}},$$

where $\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_{\overline{K},h})$ is the h-sheafification of Beilinson's Hyodo–Kato cohomology [4], [35, Sec. 3.3], and ι^B_{dR} is the Beilinson–Hyodo–Kato quasi-isomorphism,

Denote by $\mathrm{R}\Gamma_{pH}(Z_{\overline{K}}, r) = \mathrm{R}\Gamma_{pH}(Z_{\overline{K}}, 0)(r)$ the r^{th} Tate twist, defined by tensoring with the Tate object. The geometric p-adic Hodge cohomology of a K-variety Z is a differential graded $\mathbf{Q}_p$-algebra, and as the Beilinson–Hyodo–Kato map is a map of differential graded F^{nr}-algebras, the assignment

$$Z \mapsto \mathrm{R}\Gamma_{pH}(Z_{\overline{K}}, r)$$

is a presheaf of differential graded $\mathbf{Q}_p$-algebras on Var_K and there is also an external product $\mathrm{R}\Gamma_{pH}(Z_{\overline{K}}, r) \otimes \mathrm{R}\Gamma_{pH}(Z_{\overline{K}}, s)$ in $\mathscr{D}^{\mathrm{pst}}_{pH,\mathrm{ad}}$, which satisfies the Künneth formula [14, Lem. 2.21].

Definition 3.8. The absolute p-adic Hodge cohomology of Z is defined as

$$\mathrm{R}\Gamma_{pH}(Z, r) := \mathrm{Hom}_{\mathscr{D}^{\mathrm{pst}}_{pH}}(K(0), \mathrm{R}\Gamma_{pH}(Z_{\overline{K}}, r)).$$

The equivalence of categories of Theorem 3.4 induces a quasi-isomorphism

$$\mathrm{R}\Gamma_{pH}(Z, r) \cong \mathrm{Hom}_{D^\sharp(\mathrm{MF}_K(\phi, N, G_K))}(K(0), \Theta^{-1}\mathrm{R}\Gamma_{pH}(Z_{\overline{K}}, r)).$$

More importantly, in [14, Thm. 2.26] they prove the following theorem.

Theorem 3.9 (F. Déglise, W. Nizioł). *For a K-variety Z as above, let $\mathrm{R}\Gamma^{NN}_{syn}(Z, r)$ be the syntomic cohomology defined by Nekovář and Nizioł in [35, Section 3.3]. Then there exists a natural quasi-isomorphism*

$$\mathrm{R}\Gamma^{NN}_{syn}(Z, r) \xrightarrow{\sim} \mathrm{R}\Gamma_{pH}(Z, r).$$

3.3 Syntomic cohomology for strictly semistable schemes

Although it is possible to place log rigid syntomic cohomology in the context of p-adic Hodge complexes, it is more suitable for our purposes to a definition of log rigid syntomic cohomology for strictly semistable log schemes with boundary over $\mathscr{O}^\pi_K$ as a homotopy limit analogous to [35].

Definition 3.10. A *strictly semistable log scheme with boundary* over $\mathscr{O}^\pi_K$ is an $\mathscr{O}^\pi_K$-log scheme with boundary $(X, \overline{X})$, such that Zariski locally on $\overline{X}$ there exists a chart $(\alpha\colon P_{\overline{X}} \to n_{\overline{X}},\ \beta\colon \mathbf{N} \to P^{\mathrm{gp}})$ which extends $c_{\mathscr{O}^\pi_K}$ of the following form:

- The monoid P equals $\mathbf{N}^m \oplus \mathbf{N}^n$ for some integers $m \geq 1$ and $n \geq 0$, and β is given by the composition of the diagonal map $\mathbf{N} \to \mathbf{N}^m$ and the canonical injection $\mathbf{N}^m \to \mathbf{Z}^m \oplus \mathbf{Z}^n$. In particular the morphism of X extends to a morphism $\overline{X} \to \mathscr{O}^\pi_K$ with a chart $\beta'\colon \mathbf{N} \to \mathbf{N}^m \oplus \mathbf{N}^n = P$.

- The morphism of schemes

$$\overline{X} \to \operatorname{Spec} \mathscr{O}_K \times_{\operatorname{Spec} \mathscr{O}_K[\mathbf{N}]} \operatorname{Spec} \mathscr{O}_K[\mathbf{N}^m \oplus \mathbf{N}^n] = \operatorname{Spec} \mathscr{O}_K[t_1, \ldots, t_m, s_1, \ldots, s_n]/(t_1 \cdots t_m - \pi)$$

  induced by β' is smooth, and makes the diagram

$$\begin{tikzcd} \overline{X} \arrow[r] & \operatorname{Spec} \mathscr{O}_K[t_1, \ldots, t_m, s_1, \ldots, s_n]/(t_1 \cdots t_m - \pi) \\ X \arrow[r] \arrow[u] & \operatorname{Spec} \mathscr{O}_K[t_1, \ldots, t_m, s_1^{\pm 1}, \ldots, s_n^{\pm 1}]/(t_1 \cdots t_m - \pi) \arrow[u] \end{tikzcd}$$

  Cartesian.

A strictly semistable log scheme with boundary $(X, \overline{X})$ over $\mathscr{O}_K^\pi$ is called proper if $\overline{X}$ is a proper scheme. This definition is independent of the choice of π.

A strictly semistable log scheme over $\mathscr{O}_K^\pi$ is a fine log scheme X over $\mathscr{O}_K$ such that (X, X) is a strictly semistable log scheme with boundary over $\mathscr{O}_K^\pi$. Equivalently, Zariski locally on X there exists an integer $m \geq 1$ and a chart $(\mathbb{N}_X^m \to \mathcal{N}_X, \mathbb{N} \xrightarrow{\text{diag}} \mathbb{N}^m)$ extending $c_{\mathscr{O}_K^\pi}$ such that the induced morphism

$$X \to \operatorname{Spec} \mathscr{O}_K \times_{\operatorname{Spec} \mathscr{O}_K[\mathbb{N}]} \operatorname{Spec} \mathscr{O}_K[\mathbb{N}^m]$$

of schemes is smooth.

We observe that a proper strictly semi-stable log scheme with boundary $(X, \overline{X})$ over $\mathscr{O}_K^\pi$ gives on the one hand rise to a strictly semistable pair $(Z, \overline{X})$ over K in the sense of Beilinson [3, 2.2] with $Z = X_K$. Recall that this means that $\overline{X}$ is regular and proper over V, $Z \to \overline{X}$ is a K-variety with dense image in $\overline{X}$, the subset $D := \overline{X} \setminus Z$ is a divisor with normal crossings, the special fibre $\overline{X}_0$ of $\overline{X}$ is reduced, and the irreducible components of the the special fibre are regular. On the other hand $(X, \overline{X})$ gives also rise to a strictly semi-stable log scheme with boundary $(Y, \overline{Y}) = (X_0, \overline{X}_0)$ over k^0 such that $\overline{Y}$ is proper.

We start first with a scheme $X/\mathscr{O}_K^\pi$ of strictly semistable reduction endowed with the canonical log structure without making any assumptions about a compactification. As we made clear in the definition, we can think about it as the strictly semistable log scheme with boundary (X, X) over $\mathscr{O}_K^\pi$. Instead of (Beilinson–)Hyodo–Kato cohomology we use the canonical rigid complex, which we have discussed in the previous two sections as our first building block. The second building block is the same as in [35], Deligne's de Rham complex. More precisely, we consider on the one hand the rigid Hyodo–Kato complex $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0)$ of the special fiber of X endowed with the pull-back log structure of X with monodromy operator N and Frobenius morphism φ, and on the other hand the de Rham complex $\mathrm{R}\Gamma_{\mathrm{dR}}^D(X_K)$ of the generic fibre with Deligne's Hodge filtration.

To connect these pieces of information, one needs an analogue of the Hyodo–Kato map. We laid the base for this in § 2.1 where we showed that Große-Klönne's Hyodo–Kato map which on the level of complexes consists of a zigzag is a functorial morphism in $\mathscr{D}_F^+$

$$\iota_\pi^{\mathrm{rig}} : \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0) \to \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi).$$

We combine this with the zigzag Θ_{HK} from Corollary 1.73, which makes up the comparison quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0) \cong \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0)$, and thus obtain a zigzag

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0) \to \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi), \tag{3.7}$$

which by abuse of notation we also denote by ι_π^{rig}. We now have to link the log rigid cohomology of the special fibre $\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi)$ and the de Rham cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}^D(X_K)$ of its generic fibre. This is done via a zig-zag of morphisms.

It is necessary to consider Deligne's de Rham complex in order to obtain the Hodge filtration. However, this complex is in the derived category of K-vector spaces quasi-isomorphic to the complex of Kähler differentials [16, Prop. 3.1.8].

Let $w : X_{K,\mathrm{an}} \to X_{K,\mathrm{Zar}}$ be the canonical morphism from the analytic to the Zariski site of X_K. Furthermore, we denote by X_K^{an} the rigid analytic spaces associated to the generic fiber of X. Finally, let $\mathcal{X}$ be a weak completion of X and $u : \mathcal{X}_{\mathbf{Q}} \to X_K^{\mathrm{an}}$ the canonical inclusion of its generic fibre in the analytification of X_K. Note that $(X_0, \mathcal{X}, i)$ with $i : X_0 \to X \to \mathcal{X}$ forms a log rigid triple over $\mathscr{O}_K^\pi$ as introduced in Definition 1.51. For the construction of the desired zig-zag, we closely follow [12], hence we use generalised Godement resolutions for Zariski and rigid analytic points which we indicate by $\mathrm{Gd}_{\mathrm{Zar}}$ and $\mathrm{Gd}_{\mathrm{an}}$.

Similarly to [12, Prop. 4.9] we consider the diagram of sites

$$\begin{tikzcd} Pt_{\mathrm{an}}(X_K) \arrow[r] \arrow[d, "u_1"] & Pt_{\mathrm{an+Zar}}(X_K) \arrow[d, "u_2"] & Pt_{\mathrm{Zar}}(X_K) \arrow[l] \arrow[d, "u_3"] \\ X_{K,\mathrm{an}} \arrow[r, "w"] & X_{K,\mathrm{Zar}} & X_{K,\mathrm{Zar}} \arrow[l, "\mathrm{id}"'] \end{tikzcd} \tag{3.8}$$

where $Pt_{\mathrm{an}}(X_K)$ denotes the discrete site of rigid points on $X_{K,\mathrm{an}}$, $Pt_{\mathrm{Zar}}(X_K)$ the discrete site of Zariski points of X_K, and $Pt_{\mathrm{an+Zar}}(X_K)$ their direct sum in the category of sites. Note that u_2 is induced by u_3 on the Zariski points of X_K and by $w \circ u_1$ on the rigid points of $X_{K,\mathrm{an}}$.

By [12, Lem. 3.2], the left square with the map $\Omega^\bullet_{X_K} \to w_*\Omega^\bullet_{X_K^{\mathrm{an}}}$ induces $\mathrm{Gd}_{\mathrm{an+Zar}}\,\Omega^\bullet_{X_K} \to w_*\,\mathrm{Gd}_{\mathrm{an}}\,\Omega^\bullet_{X_K^{\mathrm{an}}}$, and the right square induces $\mathrm{Gd}_{\mathrm{an+Zar}}\,\Omega^\bullet_{X_K} \to \mathrm{Gd}_{\mathrm{Zar}}\,\Omega^\bullet_{X_K}$. Hence we obtain canonical morphisms

$$\Gamma(X_K^{\mathrm{an}}, \mathrm{Gd}_{\mathrm{an}}\,\Omega^\bullet_{X_K^{\mathrm{an}}}) \leftarrow \Gamma(X_K, \mathrm{Gd}_{\mathrm{an+Zar}}\,\Omega^\bullet_{X_K}) \to \Gamma(X_K, \mathrm{Gd}_{\mathrm{Zar}}\,\Omega^\bullet_{X_K}). \tag{3.9}$$

We point out that we don't considere a filtration here. The complex $\mathrm{Gd}_{\mathrm{Zar}}\,\Omega^\bullet_{X_K}$ is by definition quasi-isomorphic to the complex of Kähler differentials. By what we said above it can thus be identified with Deligne's de Rham complex in the derived category.

Lemma 3.11. *Both arrows in diagram (3.9) are quasi-isomorphisms.*

Proof. For the first arrow, this is proved in [21, § 1.8 (b)].

For the second arrow, this follows essentially from the fact that u_1, u_3, and hence u_2 are exact and conservative. It is well known, that u_3^* is exact and conservative. In other words that the Zariski site of X_K has enough points. The same is true for u_1^* by [41, Sec. 4]. Note that u_2 is induced by u_3 on the Zariski points of X_K and by $w \circ u_1$ on the rigid points of $X_{K,\mathrm{an}}$. Recall that $w : X_{K,\mathrm{an}} \to X'_{K,\mathrm{Zar}}$ maps $X_{K,\mathrm{an}}$ bijectively to the closed points of X_K. It now follows from the fact that u_1^* and u_3^* are exact and conservative that the same is true for u_2^*.

By the definition of the Godement resolution we obtain canonical resolutions $\mathrm{Gd}_{\mathrm{an+Zar}}\,\Omega^\bullet_{X_K}$ and $\mathrm{Gd}_{\mathrm{Zar}}\,\Omega^\bullet_{X_K}$ of $\Omega^\bullet_{X_K}$ which are therefore quasi-isomorphic. □

To link this to rigid cohomology, we observe first, that $u : \mathscr{X}_{\mathbf{Q}} \to X_K^{\mathrm{an}}$ induces a canonical map of complexes

$$\Gamma(X_K^{\mathrm{an}}, \mathrm{Gd}_{\mathrm{an}}\,w^*\Omega^\bullet_{X_K}) \cong \Gamma(X_K^{\mathrm{an}}, \mathrm{Gd}_{\mathrm{an}}\,\Omega^\bullet_{X_K^{\mathrm{an}}}) \to \Gamma(\mathscr{X}_{\mathbf{Q}}, \mathrm{Gd}_{\mathrm{an}}\,\Omega^\bullet_{\mathscr{X}_K}). \tag{3.10}$$

By comparing the Čech spectral sequences in [22, Sec. 3.6] for the generic fibre $\mathscr{X}_K$ of the weak completion $\mathscr{X}$ and the special fibre X_0 of X, we obtain a quasi-isomorphism $\Gamma(\mathscr{X}_K, \mathrm{Gd}_{\mathrm{an}}\,\Omega^\bullet_{\mathscr{X}_K}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi)_{\mathscr{X}}$, where the right hand side is Große-Klönne's log-rigid complex (compare Definition 1.54). According to Proposition 1.57 it is quasi-isomorphic to the canonical complex $\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi)$ and hence combining this with the above map leads to

$$\Gamma(X_K^{\mathrm{an}}, \mathrm{Gd}_{\mathrm{an}}\,w^*\Omega^\bullet_{X_K}) \cong \Gamma(X_K^{\mathrm{an}}, \mathrm{Gd}_{\mathrm{an}}\,\Omega^\bullet_{X_K^{\mathrm{an}}}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi)$$

We can finally put everything together to obtain a zigzag of morphisms on the level of complexes

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi) \leftarrow \Gamma(X_K^{\mathrm{an}}, \mathrm{Gd}_{\mathrm{an}}\,w^*\Omega^\bullet_{X_K}) \leftarrow \Gamma(X_K, \mathrm{Gd}_{\mathrm{an+Zar}}\,\Omega^\bullet_{X_K}) \xrightarrow{\sim} \Gamma(X_K, \mathrm{Gd}_{\mathrm{Zar}}\,\Omega^\bullet_{X_K}) \tag{3.11}$$

It will turn out that in the situation we consider in our comparison theorem all morphisms are quasi-isomorphisms.

(HK) If all the maps in the zigzag (deRham-rig) are quasi-isomorphisms, we say that X satisfies the Hyodo–Kato condition.

In general it induces a morphism $\mathrm{sp} : \mathrm{R}\Gamma^D_{\mathrm{dR}}(X_K) \to \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi)\ \mathscr{D}_K^+$. We compose this now with the zigzag (3.13) to obtain a pair of morphisms in the derived category

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X_0, X_0) \xrightarrow{\iota_\pi^{\mathrm{rig}}} \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi) \xleftarrow{\mathrm{sp}} \mathrm{R}\Gamma^D_{\mathrm{dR}}(X_K), \tag{3.12}$$

where both ι_π^{rig} and ι_{rig} are given by zigzags on the level of complexes as described above. We think of it as the analogue of the Hyodo–Kato map in the crystalline case. If X satisfies (HK) it becomes an actual morphism in the derived category.

Remark 3.12. On the level of complexes, one can convert the zigzag (3.12) into one of the form

$$\begin{tikzcd}[column sep=small]
& M_1^\bullet & & \cdots & R\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi) & \cdots & & M_n^\bullet & \\
\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0) \arrow[ur] & & M_2^\bullet \arrow[ul] \arrow[ur] & & & & M_{n-1}^\bullet \arrow[ul] \arrow[ur] & & \mathrm{R}\Gamma_{\mathrm{dR}}^{D}(X_K) \arrow[ul]
\end{tikzcd}$$

if one inserts the identity map at appropriate places. By applying a quasi-pushout construction several times to this zigzag of complexes, we obtain a diagram of the form

$$\begin{tikzcd}
& R\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi) & \\
\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0) \arrow[ur] & & \mathrm{R}\Gamma_{\mathrm{dR}}^{D}(X_K) \arrow[ul]
\end{tikzcd}$$

It represents an object in the category of p-adic Hodge complexes. For more details we refer to [12, - 5.6 (Construction)]

For the definition of rigid syntomic cohmology as homotopy limit we use as in the crystalline case the normalised monodromy, that is we devide the monodromy coming from Definition 1.32 by the absolute ramification index e.

Definition 3.13. Let $X/\mathscr{O}_K^\pi$ be strictly semi-stable and denote by N and φ the monodromy operator and Frobenius on $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0)$ induced by the operators from Definition 1.32. For $r \geqslant 0$ and a choice of uniformiser π the logarithmic rigid syntomic cohomology of X is given by

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathrm{rig}}(X, r, \pi) := \left[\begin{tikzcd}
& \mathrm{Fil}^r \, \mathrm{R}\Gamma_{\mathrm{dR}}^{D}(X_K) \arrow[d, "{(0,\mathrm{sp})}"'] \\
\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0) \arrow[r, "{(1-\varphi_r, \iota_\pi^{\mathrm{rig}})}"] \arrow[d, "N"'] & \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0) \oplus \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi) \arrow[d, "{(N,0)}"'] \\
\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0) \arrow[r, "1-\varphi_{r-1}"] & \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0)
\end{tikzcd}\right],$$

where again φ_r denotes as usual the Frobenius divided by p^r. It is functorial in X.

Note that the diagram in the definition hides the fact that the homotopy limit is taken along zigzags of maps, as ι_π^{rig} and ι_{rig} are not necessarily morphisms on the level of complexes.

Assume now that X has a "nice" boundary, in other words, there exists a (not necessarily proper) scheme $\overline{X}/\mathscr{O}_K^\pi$, such that $(X, \overline{X})$ is a strictly semistable log scheme with boundary over $\mathscr{O}_K^\pi$ as in Definition 3.10. We have seen in Corollary 2.4 that the canonical morphisms

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0) &\to \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, X_0), \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0) &\to \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0), \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi) &\to \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi)
\end{aligned}$$

are quasi-isomorphisms. With this in mind, we repeat now the steps from the case of a trivial boundary. Explicitely, the lower horizontal zigzag in the diagram (2.1) induces a functorial morphism in the derived category

$$\iota_\pi^{\mathrm{rig}} : \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0) \to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi).$$

We combine this again with the zigzag Θ_{HK} from Corollary 1.73, which makes up the comparison quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0) \cong \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0)$, and thus obtain a zigzag

$$\iota_\pi^{\mathrm{rig}} : \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0) \to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi). \tag{3.13}$$

Furthermore, we simply compose the zigzag $\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi) \xleftarrow{\mathrm{sp}} \Gamma(X_K, \mathrm{Gd}_{\mathrm{Zar}}\,\Omega^\bullet_{X_K})$ from above with the quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi)$ and call the resulting zigzag again sp. In the derived category, we therefore obtain a pair of morphisms

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0) \xrightarrow{\iota_\pi^{\mathrm{rig}}} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi) \xleftarrow{\iota_{\mathrm{rig}}} \mathrm{R}\Gamma_{\mathrm{dR}}^D(X_K).$$

As a consequence, the following definition makes sense:

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathrm{rig}}((X,\overline{X}),r,\pi) := \left[\begin{array}{ccc} & & \mathrm{R}\Gamma_{\mathrm{dR}}^D(X_K)/\,\mathrm{Fil}^r \\ & & \downarrow{\scriptstyle (0,\mathrm{sp})} \\ \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0) & \xrightarrow{(1-\varphi_r,\iota_\pi^{\mathrm{rig}})} & \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0)\oplus \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle (N,0)} \\ \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0) \end{array}\right].$$

Here again we used the normalised monodromy. It is functorial in $(X,\overline{X})$. Again π indicates that it depends on the choice of a uniformiser. We emphasise again that on the level of complexes, sp and ι_π^{rig} are given by zigzags, which has to be taken into account when computing the above homotopy limit.

We note that rigid syntomic cohomology for X and $(X,\overline{X})$ are isomorphic.

Proposition 3.14. *Let $(X,\overline{X})$ be a strictly semistable log scheme over $\mathscr{O}_K^\pi$. Then the canonical morphism in $\mathscr{D}^+_{\mathbf{Q}_p}$*

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathrm{rig}}((X,\overline{X}),r,\pi) \to \mathrm{R}\Gamma_{\mathrm{syn}}^{\mathrm{rig}}(X,r,\pi)$$

is an isomorphism.

Proof. It is enough to showthat $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0) \to \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0)$ is an isomorphism. This follows from Corollary 2.4, since we have $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0) \cong \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0)$ and $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,X_0) \cong \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)$. □

To conclude this section, we show that rigid syntomic cohomology admits a cup product. Let $(X,\overline{X})$ be a strictly semistable log scheme with boundary over $\mathscr{O}_K^\pi$, and let

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0) \xrightarrow{a} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi) \xleftarrow{b} \mathrm{R}\Gamma_{\mathrm{dR}}^D(X_K)$$

be the associated p-adic Hodge complex. Each of $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0,\overline{X}_0)$, $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi)$, and $\mathrm{R}\Gamma_{\mathrm{dR}}^D(X_K)$ has a cup product induced by the wedge product of differential forms. We want to define a cup product

$$H_{\mathrm{syn}}^{\mathrm{rig},i}((X,\overline{X}),r,\pi)\times H_{\mathrm{syn}}^{\mathrm{rig},j}((X,\overline{X}),s,\pi) \to H_{\mathrm{syn}}^{\mathrm{rig},i+j}((X,\overline{X}),r+s,\pi)$$

on rigid syntomic cohomology. Thus let $\eta \in H_{\mathrm{syn}}^{\mathrm{rig},i}((X,\overline{X}),r,\pi)$ and $\eta' \in H_{\mathrm{syn}}^{\mathrm{rig},j}((X,\overline{X}),s,\pi)$. The class of η is represented by a sextuple (u,v,w,x,y,z) with

$$\begin{aligned} &u \in \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig},i}(X_0,\overline{X}_0), && v \in \mathrm{Fil}^r\mathrm{R}\Gamma_{\mathrm{dR}}^{D,i}(X_K), \\ &w,x \in \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig},i-1}(X_0,\overline{X}_0), && y \in \mathrm{R}\Gamma_{\mathrm{rig}}^{i-1}(\overline{X}_0/\mathscr{O}_K^\pi), \\ &z \in \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig},i-2}(X_0,\overline{X}_0), \end{aligned}$$

such that

$$\begin{aligned} &du = 0, && dv = 0, \\ &dw = (1-\frac{\varphi}{p^r})(u), && dx = N(u), && dy = a(u)-b(v), \\ &dz = N(w) - (1-\frac{\varphi}{p^{r-1}})(x). \end{aligned}$$

Let (u',v',w',x',y',z') be another sextuple representing the class of η'. We define a sextuple $(u'',v'',w'',x'',y'',z'')$ by

$$\begin{aligned}
u'' &:= u\cup u',\\
v'' &:= v\cup v',\\
w'' &:= w\cup \frac{\varphi(u')}{p^s}+(-1)^i(u\cup w'),\\
x'' &:= x\cup u'+(-1)^i(u\cup x'),\\
y'' &:= y\cup a(u')+(-1)^i(b(v)\cup y'),\\
z'' &:= z\cup \frac{\varphi(u')}{p^s}-(-1)^i(w\cup \frac{\varphi(x')}{p^{s-1}})+(-1)^i(x\cup w')-u\cup z'.
\end{aligned}$$

The class of $H^{\mathrm{rig},i+j}_{\mathrm{syn}}((X,\overline{X}),r+s,\pi)$ it represents is the cup product of η and η' and denoted by $\eta\cup\eta'$.

4 Comparison in the compactifiable case

In this section we compare the syntomic cohomologies introduced in the previous section in the case of a strictly semistable log scheme which has a normal crossings compactification. Our main result is the following.

Theorem 4.1. *Let $(X,\overline{X})$ be a proper strictly semistable log scheme with boundary over $\mathscr{O}_K^\pi$. Then for $r\geqslant 0$ there exists a canonical quasi-isomorphism*

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}((X,\overline{X}),r,\pi)\cong \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{syn}}(\overline{X},r,\pi).$$

Combined with Lemma 3.6 and Proposition 3.14, this implies the following.

Corollary 4.2. *Let $(X,\overline{X})$ be a proper strictly semistable log scheme with boundary over $\mathscr{O}_K^\pi$. Then for $r\geqslant 0$ there exists a canonical quasi-isomorphism*

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}(X,r,\pi)\cong \mathrm{R}\Gamma^{NN}_{\mathrm{syn}}(X_K,r).$$

which is compatible with cup products.

Recall that the homotopy limits in the theorem are given by

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}((X,\overline{X}),r,\pi)=\left[\begin{array}{ccc}
 & & \mathrm{Fil}^r\,\mathrm{R}\Gamma^D_{\mathrm{dR}}(X_K)\\
 & & \downarrow{\scriptstyle (0,\mathrm{sp})}\\
\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X_0,\overline{X}_0) & \xrightarrow{(1-\varphi_r,\iota^{\mathrm{rig}}_\pi)} & \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X_0,\overline{X}_0)\oplus\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^\pi)\\
{\scriptstyle N}\downarrow & & \downarrow{\scriptstyle (N,0)}\\
\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X_0,\overline{X}_0) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X_0,\overline{X}_0)
\end{array}\right],$$

and

$$\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{syn}}(\overline{X},r,\pi)=\left[\begin{array}{ccc}
 & & \mathrm{Fil}^r\,\mathrm{R}\Gamma_{\mathrm{dR}}(\overline{X}_K)\\
 & & \downarrow{\scriptstyle (0,\gamma)}\\
\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(\overline{X}) & \xrightarrow{(1-\varphi_r,\iota^{\mathrm{cr}}_\pi)} & \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(\overline{X})\oplus\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathscr{O}_K^\pi)\\
{\scriptstyle N}\downarrow & & \downarrow{\scriptstyle (N,0)}\\
\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(\overline{X}) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(\overline{X})
\end{array}\right].$$

Note that in this case Deligne's de Rham cohomology is computed by the compactification $X_K\hookrightarrow\overline{X}_K$, namely we have a natural filtered quasi-isomorphism $\mathrm{R}\Gamma^D_{\mathrm{dR}}(X_K)\cong\mathrm{R}\Gamma_{\mathrm{dR}}(\overline{X}_K)$. The first step to link them is to obtain a canonical quasi-isomorphism between $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X_0,\overline{X}_0)$ and $\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(\overline{X})$ which is naturally compatible with Frobenius φ and the (normalized) monodromy N. The next step is to show that the crystalline and rigid Hyodo–Kato morphisms are compatible. In particular, we need to show the comptibility of ι^{rig}_π and ι^{cr}_π. This is not obvious from the construction. However, it suffices to see the compatibility on Frobenius eigenspaces where it follows from formal arguments.

4.1 Logarithmic rigid and crystalline cohomology

In this section we compare for a strictly semi-stable scheme with boundary over k^0 log rigid cohomology with log crystalline cohomology passing through Shiho's log convergent cohomology.

In [22, 1.5] and the proof of [22, Thm. 5.3 (ii)], Große-Klönne mentions that his constructions can also be carried out for rigid spaces instead of dagger spaces which results in a different contruction of Shiho's logarithmic convergent cohomology introduced in [38]. Since our contructions are based on Große-Klönne's work the same is true for the canonical complexes we introduced in the previous two sections. We denote the different resulting canonical convergent complexes by replacing the index rig by conv.

More precisely, if we denote by $\mathsf{CQ}_{\mathrm{HK}}$ the category of convergent Hyodo–Kato quadruples, defined analogously to the category $\mathsf{RQ}_{\mathrm{HK}}$ of rigid Hyodo–Kato quadruples, by replacing the overconvergent setting (weak formal schemes and dagger spaces) by the convergent setting (formal schemes and rigid spaces) and follow the same procedures we obtain the following analogue of Proposition 1.45

Proposition 4.3. *There are convergent Hyodo–Kato complexes which define contravariant functors*

$$\begin{aligned}
\overline{\mathit{ELS}}_{k^0}^{ss} &\to \mathscr{C}_F^+(\varphi,N), && (Y,\overline{Y}) \mapsto \widehat{\mathrm{R}\Gamma}_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y}),\\
\mathit{CQ}_{\mathrm{HK}} &\to \mathscr{C}_F^+(\varphi,N), && ((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi) \mapsto \widetilde{\mathrm{R}\Gamma}_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})},\\
\mathit{CQ}_{\mathrm{HK}} &\to \mathscr{C}_F^+(\varphi,N), && ((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi) \mapsto \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})}.
\end{aligned}$$

If $((Y,\overline{Y}),(\mathscr{Z},\overline{\mathscr{Z}}),i,\phi)$ is a convergent Hyodo–Kato quadruple, we have quasi-isomorphisms

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y}) \leftarrow \widetilde{\mathrm{R}\Gamma}_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})} \to \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y})_{(\mathscr{Z},\overline{\mathscr{Z}})},$$

which are functorial on $\mathit{CQ}_{\mathrm{HK}}$. Here we regard $\widehat{\mathrm{R}\Gamma}_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y})$ as a functor from $\mathit{CQ}_{\mathrm{HK}}$ through the forgetful functor $\mathit{CQ}_{\mathrm{HK}} \to \overline{\mathit{ELS}}_{k^0}^{ss}$.

As in the rigid case, we can sheafify the first of these functors $(Y,\overline{Y}) \mapsto \widehat{\mathrm{R}\Gamma}_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y})$ with respect to the Zariski topology to obtain the canonical convergent Hyodo–Kato complex on strictly semistable log schemes with boundary over k^0 denoted by $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{conv}}(Y,\overline{Y})$.

Similarly, we obtain canonical log convergent complexes over $\mathscr{O}_F^0$, $\mathscr{O}_K^\pi$ and $\widehat{\mathscr{T}}$ for fine log schemes. Let $S \hookrightarrow \mathcal{S}$ be one of $k^0 \hookrightarrow \mathscr{O}_F^0$, $k^0 \hookrightarrow \mathscr{O}_K^\pi$, or $T \hookrightarrow \widehat{\mathscr{T}}$, denote by $\mathsf{CT}_{\mathcal{S}}$ the category of log convergent triples in anlogy with $\mathsf{RT}_{\mathcal{S}}$, and by $\mathcal{S}$-ELS_S the category of $\mathcal{S}$-embeddable objects of LS_S as before. We have the following convergent analogue of Proposition 1.57

Proposition 4.4. *There are log convergent complexes which define contravariant functors*

$$\begin{aligned}
\mathcal{S}\text{-}\mathsf{ELS}_S &\to \mathscr{C}_{\mathscr{O}_{\mathbf{Q}}}^+, && Y \mapsto \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathcal{S}),\\
\mathsf{CT}_{\mathcal{S}} &\to \mathscr{C}_{\mathscr{O}_{\mathbf{Q}}}^+, && (Y,\mathcal{Y},i) \mapsto \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathcal{S})_{\mathcal{Y}},\\
\mathsf{CT}_{\mathcal{S}} &\to \mathscr{C}_{\mathscr{O}_{\mathbf{Q}}}^+, && (Y,\mathcal{Y},i) \mapsto \mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathcal{S})_{\mathcal{Y}},
\end{aligned}$$

where $\mathscr{O}_{\mathbf{Q}}$ is the coordinate ring of $\mathcal{S}$ tensored with $\mathbf{Q}$. (So $\mathscr{O}_{\mathbf{Q}} = F, K,$ or $\widehat{F[t]}$.) If $(Y,\mathcal{Y},i)$ is a log rigid triple on S over $\mathcal{S}$, we have quasi-isomorphisms

$$\widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathcal{S}) \xleftarrow{\sim} \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathcal{S})_{\mathcal{Y}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathcal{S})_{\mathcal{Y}},$$

which are functorial in $\mathsf{CT}_{\mathcal{S}}$.

These complexes are a priori different from the log convergent cohomology in the sense of [39], but by [39, Corollary2.3.9] we may identify them.

Again we sheafify with respect to the Zariski topology and obtain for $Y \in \mathsf{LS}_S$, the canonical log convergent complex $\mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathcal{S})$.

For a strictly semistable log scheme with boundary $(Y,\overline{Y})$ over k^0, we can construct zigzags of quasi-isomorphisms between the convergent Hyodo–Kato and the log convergent complexes which give us the following analogon of Proposition 1.72 and Corollary 1.73

Proposition 4.5. *There are canonical and functorial isomorphisms in* $\mathscr{D}^+_F$

$$\begin{aligned}
&\widehat{\theta}_{\mathrm{HK}}\colon \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F) \xrightarrow{\sim} \widehat{\mathrm{R}\Gamma}^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y}) && \text{for } (Y,\overline{Y}) \in \overline{\mathit{ELS}}^{ss}_{k^0},\\
&\widetilde{\theta}_{\mathrm{HK}}\colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F)_{\overline{\mathcal{Y}}} \xrightarrow{\sim} \widetilde{\mathrm{R}\Gamma}^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z},\overline{\mathcal{Z}})} && \text{for } ((Y,\overline{Y}),(\mathcal{Z},\overline{\mathcal{Z}}),i,\phi) \in \mathit{CQ}_{\mathrm{HK}} \text{ and } \overline{\mathcal{Y}} = \overline{\mathcal{Z}} \times_{\mathcal{T}} \mathscr{O}^0_F,\\
&\Theta_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F) \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y}) && \text{for } (Y,\overline{Y}) \in \overline{\mathit{LS}}^{ss}_{k^0},
\end{aligned}$$

such that the diagrams

$$\begin{tikzcd}
\widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F) \arrow[d, "\widehat{\theta}_{\mathrm{HK}}"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F)_{\overline{\mathcal{Y}}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{\theta}_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F)_{\overline{\mathcal{Y}}} \arrow[d, "\theta_{\mathrm{HK}}"] \\
\widehat{\mathrm{R}\Gamma}^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y}) & \widetilde{\mathrm{R}\Gamma}^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z},\overline{\mathcal{Z}})} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y})_{(\mathcal{Z},\overline{\mathcal{Z}})}
\end{tikzcd}$$

and in case $(Y,\overline{Y}) \in \overline{\mathit{LS}}^{ss}_{k^0}$ *is embeddable*

$$\begin{tikzcd}
\widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F) \arrow[r, "\sim"] \arrow[d, "\widehat{\theta}_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F) \arrow[d, "\Theta_{\mathrm{HK}}"] \\
\widehat{\mathrm{R}\Gamma}^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y}) \arrow[r, "\sim"] & \mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y}),
\end{tikzcd}$$

commute.

As in the rigid case we obtain base change maps via the functors

$$\begin{aligned}
&\mathsf{LS}_T \to \mathsf{LS}_{k^0},\ Z \mapsto Y := Z \times_T k^0,\\
&\mathsf{LS}_{\widehat{\mathcal{T}}} \to \mathsf{LS}_{\mathscr{O}^0_F},\ \mathcal{Z} \mapsto \mathcal{Y} := \mathcal{Z} \times_{\mathcal{T}} \mathscr{O}^0_F,\\
&\mathsf{LS}_{\widehat{\mathcal{T}}} \to \mathsf{LS}_{\mathscr{O}^\times_K},\ \mathcal{Z} \mapsto \mathcal{X} := \mathcal{Z} \times_{\mathcal{T}} \mathscr{O}^\pi_K
\end{aligned}$$

Proposition 4.6. *There are canonical and functorial morphisms in* $\mathscr{A}^+_{\widehat{F[t]}}$

$$\begin{aligned}
&\Theta_0\colon \mathrm{R}\Gamma_{\mathrm{conv}}(Z/\widehat{\mathcal{T}}) \to \mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathscr{O}^0_F), && \Theta_\pi\colon \mathrm{R}\Gamma_{\mathrm{conv}}(Z/\widehat{\mathcal{T}}) \to \mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathscr{O}^\pi_K) && \text{for } Z \in \mathsf{LS}_T,\\
&\widehat{\theta}_0\colon \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Z/\widehat{\mathcal{T}}) \to \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathscr{O}^0_F), && \widehat{\theta}_\pi\colon \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Z/\widehat{\mathcal{T}}) \to \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathscr{O}^\pi_K) && \text{for } Z \in \widehat{\mathcal{T}}\text{-}\mathsf{ELS}_T,\\
&\widetilde{\theta}_0\colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Z/\widehat{\mathcal{T}})_{\mathcal{Z}} \to \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathscr{O}^0_F)_{\mathcal{Y}}, && \widetilde{\theta}_\pi\colon \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Z/\widehat{\mathcal{T}})_{\mathcal{Z}} \to \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathscr{O}^\pi_K)_{\mathcal{X}} && \text{for } (Z,\mathcal{Z},i) \in \mathsf{CT}_{\widehat{\mathcal{T}}},
\end{aligned}$$

such that the diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathscr{O}^0_F) & \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathscr{O}^0_F) \arrow[l, "\sim"'] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathscr{O}^0_F)_{\mathcal{Y}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathscr{O}^0_F)_{\mathcal{Y}} \\
\mathrm{R}\Gamma_{\mathrm{conv}}(Z/\widehat{\mathcal{T}}) \arrow[u, "\Theta_0"] \arrow[d, "\Theta_\pi"] & \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Z/\widehat{\mathcal{T}}) \arrow[l, "\sim"'] \arrow[u, "\widehat{\theta}_0"] \arrow[d, "\widehat{\theta}_\pi"] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Z/\widehat{\mathcal{T}})_{\mathcal{Z}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\widetilde{\theta}_0"] \arrow[d, "\widetilde{\theta}_\pi"] & \mathrm{R}\Gamma_{\mathrm{conv}}(Z/\widehat{\mathcal{T}})_{\mathcal{Z}} \arrow[u, "\theta_0"] \arrow[d, "\theta_\pi"] \\
\mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathscr{O}^\pi_K) & \widehat{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathscr{O}^\pi_K) \arrow[l, "\sim"'] & \widetilde{\mathrm{R}\Gamma}_{\mathrm{conv}}(Y/\mathscr{O}^\pi_K)_{\mathcal{X}} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{conv}}(Y/\mathscr{O}^\pi_K)_{\mathcal{X}}.
\end{tikzcd}$$

commutes for $(Z,\mathcal{Z},i) \in \mathsf{CT}_{\widehat{\mathcal{T}}}$. *Here we consider the log convergent complexes over* $\mathscr{O}^0_F$ *and* $\mathscr{O}^\pi_K$ *as objects in* $\mathscr{A}^+_{\widehat{F[t]}}$ *through the maps* $\widehat{F[t]} \to F,\ t \mapsto 0$ *and* $\widehat{F[t]} \to K,\ t \mapsto \pi$.

Let $(Y,\overline{Y}) \in \overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$ be a strictly semistable log scheme with boundary over k^0. The canonical convergent complexes are related to their rigid counterparts via canonical morphisms

$$\begin{array}{rclcl}
\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,Y) & \leftarrow & \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y}) & \to & \mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y})\\
\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^0_F) & \leftarrow & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}^0_F) & \to & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^0_F)\\
\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}^\pi_K) & \leftarrow & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}^\pi_K) & \to & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\mathscr{O}^\pi_K)\\
\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathcal{T}) & \leftarrow & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{T}) & \to & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\widehat{\mathcal{T}})
\end{array} \tag{4.1}$$

where the strictly semistable log scheme Y is identified with the strictly semistable log scheme with boundary (Y, Y) and the first map in each line is induced by the inclusion $(Y, Y) \to (Y, \overline{Y})$ of strictly semistable log schemes with boundary by functoriality and the second map in each line is the change of convergence map, that is, it is induced by the functor which sends a dagger space to its associated rigid space. We have seen in Corollary 2.4 that the second and third map on the left are quasi-isomorphisms, even when $\overline{Y}$ is not proper. Thus the same is true for the first one. We will see now that the upper three maps on the right are quasi-isomorphisms if $\overline{Y}$ is proper.

The proof we give provides at the same time another proof of the fact that the morphisms in the first two lines above are quasi-isomorphisms in case $\overline{Y}$ is proper. (Hence it is less general than Corollary 2.4). It is based on the existence of a weight spectral sequence which follows in the rigid and convergent case from the construction in terms of Steenbrink double complexes. Let $(Y, \overline{Y})$ be a strictly semistable log scheme with boundary over k^0, i.e. $(Y, \overline{Y}) \in \overline{\mathsf{LS}}^{\mathrm{ss}}_{k^0}$. We denote by $\overline{Y}^{\infty}$ the log scheme whose underlying scheme is $\overline{Y}$ and the log structure is associated to the horizontal divisor $D := \overline{Y} \backslash Y$. For each subscheme U of $\overline{Y}$, we write U^{∞} for the log scheme endowed with the pull-back log structure of $\overline{Y}^{\infty}$ whose underlying scheme is U. Note that with this convention the log structure of Y^{∞} is trivial, i.e. $Y^{\infty} = Y^{\varnothing}$.

Recall that we denote by $\Upsilon_{\overline{Y}}$, Υ_Y and Υ_D the set of irreducible components of $\overline{Y}$, Y and D and that for $J \subset \Upsilon_{\overline{Y}}$ we denote $\overline{Y}_J := \bigcap_{\alpha \in J} \overline{Y}_{\alpha}$ and similarly for $J \subset \Upsilon_Y$.

Lemma 4.7. *Let $(Y, \overline{Y})$ be a strictly semistable log scheme with boundary over k^0. The weight filtration induces spectral sequences*

$$E_1^{p,q} = \bigoplus_{\substack{j \geq 0 \\ j \geq p}} \bigoplus_{\substack{J \subset \Upsilon_{\overline{Y}} \\ |J| = 2j - p + 1}} H^{2p+q-2j}_{\mathrm{rig}}(\overline{Y}^{\infty}_J / \mathscr{O}_F^{\varnothing}) \Rightarrow H^{p+q}_{\mathrm{rig}}(\overline{Y} / \mathscr{O}_F^0),$$

and

$$E_1^{p,q} = \bigoplus_{\substack{j \geq 0 \\ j \geq p}} \bigoplus_{\substack{J \subset \Upsilon_{\overline{Y}} \\ |J| = 2j - p + 1}} H^{2p+q-2j}_{\mathrm{conv}}(\overline{Y}^{\infty}_J / \mathscr{O}_F^{\varnothing}) \Rightarrow H^{p+q}_{\mathrm{conv}}(\overline{Y} / \mathscr{O}_F^0).$$

Proof. As in Remark 1.50 we can take a simplicial object $((Y_{\bullet}, \overline{Y}_{\bullet}), (\mathscr{Z}_{\bullet}, \overline{\mathscr{Z}}_{\bullet}), i_{\bullet}, \phi_{\bullet})$ in $\mathsf{RQ}_{\mathrm{HK}}$ such that $\{(Y_{\bullet}, \overline{Y}_{\bullet})\}$ is a Zariski hyper covering of $(Y, \overline{Y})$. We have a zigzag of quasi-isomorphisms

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y, \overline{Y}) \xrightarrow{\sim} \widehat{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y_{\bullet}, \overline{Y}_{\bullet}) \xleftarrow{\sim} \widetilde{\mathrm{R}\Gamma}^{\mathrm{rig}}_{\mathrm{HK}}(Y_{\bullet}, \overline{Y}_{\bullet})_{(\mathscr{Z}_{\bullet}, \overline{\mathscr{Z}}_{\bullet})} \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y_{\bullet}, \overline{Y}_{\bullet})_{(\mathscr{Z}_{\bullet}, \overline{\mathscr{Z}}_{\bullet})},$$

and thus we may without loss of generality assume that $(Y, \overline{Y})$ is HK-embeddable. In this case, (1.3) induces a spectral sequence like the desired one for the cohomology groups of the rigid Hyodo–Kato complex.

Similarly, Remark 1.50 is valid mutatis mutandis in the convergent case, so that we can take a simplicial object $((Y_{\bullet}, \overline{Y}_{\bullet}), (\mathscr{Z}_{\bullet}, \overline{\mathscr{Z}}_{\bullet}), i_{\bullet}, \phi_{\bullet})$ in $\mathsf{CQ}_{\mathrm{HK}}$ such that $\{(Y_{\bullet}, \overline{Y}_{\bullet})\}$ is a Zariski covering of $(Y, \overline{Y})$. Again, we have a zigzag of quasi-isomorphisms

$$\mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{HK}}(Y, \overline{Y}) \xrightarrow{\sim} \widehat{\mathrm{R}\Gamma}^{\mathrm{conv}}_{\mathrm{HK}}(Y_{\bullet}, \overline{Y}_{\bullet}) \xleftarrow{\sim} \widetilde{\mathrm{R}\Gamma}^{\mathrm{conv}}_{\mathrm{HK}}(Y_{\bullet}, \overline{Y}_{\bullet})_{(\mathscr{Z}_{\bullet}, \overline{\mathscr{Z}}_{\bullet})} \xrightarrow{\sim} \mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{HK}}(Y_{\bullet}, \overline{Y}_{\bullet})_{(\mathscr{Z}_{\bullet}, \overline{\mathscr{Z}}_{\bullet})},$$

and may assume that $(Y, \overline{Y})$ is HK-embeddable. We obtain a spectral sequence for the cohomology groups of the convergent Hyodo–Kato complex via the convergent analogue of (1.3)

The statement then follows from the quasi-isomorphisms of Corollary 1.73 and Proposition 4.5 which allows us to identify the cohomology groups of the canonical rigid (resp. convergent) and rigid Hyodo–Kato (resp. convergent Hyodo–Kato) complex. □

We will use the weight spectral sequences for the strictly semistable log schemes with boundary (Y, Y) and $(Y, \overline{Y})$ over k^0 with $\overline{Y}$ proper.

Lemma 4.8. *Let $(Y, \overline{Y})$ be a strictly semistable log scheme with boundary over k^0 and assume that $\overline{Y}$ is proper. Then for any $J \subset \Upsilon_{\overline{Y}}$ the canonical morphism $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}^{\infty}_J / \mathscr{O}_F^{\varnothing}) \to \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}^{\infty}_J / \mathscr{O}_F^{\varnothing})$ is a quasi-isomorphism.*

Proof. We show first that there exist spectral sequences for the rigid and the convergent cohomology of the form

$$
\begin{aligned}
E_1^{p,q} &= \bigoplus\nolimits_{\substack{i\geq 0\\ i\geq p}} \bigoplus\nolimits_{\substack{I\subset \Upsilon_D\\ |I|=2i-p+1}} H_{\mathrm{rig}}^{2p+q-2i}(D^{\varnothing}_{J,I}/\mathscr{O}_F^{\varnothing}) \Rightarrow H_{\mathrm{rig}}^{p+q}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing}), \qquad (4.2)\\
E_1^{p,q} &= \bigoplus\nolimits_{\substack{i\geq 0\\ i\geq p}} \bigoplus\nolimits_{\substack{I\subset \Upsilon_D\\ |I|=2i-p+1}} H_{\mathrm{conv}}^{2p+q-2i}(D^{\varnothing}_{J,I}/\mathscr{O}_F^{\varnothing}) \Rightarrow H_{\mathrm{conv}}^{p+q}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing}),
\end{aligned}
$$

where $D^{\varnothing}_{J,I}$ is the scheme $\overline{Y}_J\cap\bigcap_{\beta\in I}D_\beta$ with the trivial log structure.

As in the proof of Lemma 4.7 it is enough to argue locally by Remark 1.50. In particular we might assume that $(Y,\overline{Y})$ has a rigid Hyodo–Kato datum $(\mathcal{Z},\overline{\mathcal{Z}},i,\phi)$. Consider the weak formal log scheme $\overline{\mathcal{Z}}^{\infty}$ whose underlying scheme is $\overline{\mathcal{Z}}$ with log strucure associated to $\mathcal{D}=\overline{\mathcal{Z}}\backslash\mathcal{Z}$. Let $\overline{\mathcal{Y}}^{\infty}:=\overline{\mathcal{Z}}^{\infty}\times_{\mathrm{Spwf}\,\mathscr{O}_F[t]^\dagger}\mathrm{Spwf}\,\mathscr{O}_F$ be the exact closed weak formal log subscheme defined by $t=0$. To obtain a spectral sequence for the log scheme $\overline{\mathcal{Y}}^{\infty}$ similar to above we define the filtration

$$
P_i^{\infty}\omega^k_{\overline{\mathcal{Y}}_J^{\infty}} := \mathrm{Im}(\omega^i_{\overline{\mathcal{Y}}_J^{\infty}}\otimes\omega^{k-i}_{\overline{\mathcal{Y}}_J^{\varnothing}}\to\omega^k_{\overline{\mathcal{Y}}_J^{\infty}}),
$$

for $i,k\geqslant 0$, where as before $\overline{\mathcal{Y}}_J^{\varnothing}$ is endowed with the trivial log structure. For a subset $I\subset\Upsilon_D$, taking the residue along $\mathcal{D}_{J,I}:=\overline{\mathcal{Y}}_J\cap\bigcap_{\beta\in I}\mathcal{D}_\beta$ induces an isomorphism

$$
\mathrm{Gr}_i^{P^{\infty}}\omega^{\bullet}_{\overline{\mathcal{Y}}^{\infty}_{J,\mathbf{Q}}}\cong\bigoplus_{\substack{I\subset\Upsilon_D\\ |I|=i}}\Omega^{\bullet}_{\mathcal{D}_{J,I,\mathbf{Q}}}[-i]. \qquad (4.3)
$$

Hence we obtain the desired spectral sequence for rigid cohomology.

If we repeat the argument with formal schemes instead of weak formal schemes, we see the analogous spectral sequence for convergent cohomology.

We use these spectral sequences now in the case that $(Y,\overline{Y})$ is a strictly semistable log scheme over k^0 with boundary where $\overline{Y}$ is proper. Since in this case $D=\overline{Y}\backslash Y$ is proper as well we have canonical isomorphisms $H^k_{\mathrm{rig}}(D^{\varnothing}_{J,I}/\mathscr{O}_F^{\varnothing})\cong H^k_{\mathrm{conv}}(D^{\varnothing}_{J,I}/\mathscr{O}_F^{\varnothing})$ between the left hand sides of the spectral sequences (4.2). Hence by the general theory of spectral squences this implies $H^k_{\mathrm{rig}}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing})\cong H^k_{\mathrm{conv}}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing})$ as desired. ☐

We are now able to relate the canonical logarithmic rigid and convergent complexes over $\mathscr{O}_F^0$.

Proposition 4.9. *Let $(Y,\overline{Y})$ be a strictly semistable log scheme with boundary over k^0 and assume that $\overline{Y}$ is proper. Then the canonical morphisms*

$$
\begin{aligned}
\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,Y)\leftarrow\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y,\overline{Y})\to\mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{HK}}(Y,\overline{Y}) \qquad \textit{and}\\
\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0)\leftarrow\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)\to\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\mathscr{O}_F^0,
\end{aligned}
$$

are quasi-isomorphisms.

Proof. According to Lemma 4.7 there are weight spectral sequences for log rigid cohomology with respect to $(Y,\overline{Y})$ and $Y=(Y,Y)$, and for log convergent cohomology with respect to $(Y,\overline{Y})$

$$
\begin{aligned}
E_1^{p.q} &= \bigoplus_{\substack{j\geq 0\\ j\geq p}}\ \bigoplus_{\substack{J\subset\Upsilon_Y\\ |J|=2j-p+1}} H^{2p+q-2j}_{\mathrm{rig}}(Y_J^{\infty}/\mathscr{O}_F^{\varnothing}) \;\Rightarrow\; H^{p+q}_{\mathrm{rig}}(Y/\mathscr{O}_F^0)\\
E_1^{p.q} &= \bigoplus_{\substack{j\geq 0\\ j\geq p}}\ \bigoplus_{\substack{J\subset\Upsilon_{\overline{Y}}\\ |J|=2j-p+1}} H^{2p+q-2j}_{\mathrm{rig}}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing}) \;\Rightarrow\; H^{p+q}_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0)\\
E_1^{p.q} &= \bigoplus_{\substack{j\geq 0\\ j\geq p}}\ \bigoplus_{\substack{J\subset\Upsilon_{\overline{Y}}\\ |J|=2j-p+1}} H^{2p+q-2j}_{\mathrm{conv}}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing}) \;\Rightarrow\; H^{p+q}_{\mathrm{conv}}(\overline{Y}/\mathscr{O}_F^0).
\end{aligned}
$$

By functoriality the second line of (4.1) induces canonical morphisms between them, which on the right hand side of the spectral sequences correspond exactly the morphisms from the statement. It thus suffices to show that the canonical maps

$$
H^k_{\mathrm{rig}}(Y_J^{\infty}/\mathscr{O}_F^{\varnothing})\leftarrow H^k_{\mathrm{rig}}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing})\to H^k_{\mathrm{conv}}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing})
$$

are isomorphisms. Now these morphisms fit into a commutative diagram

$$\begin{tikzcd}
& H^k_{\mathrm{rig}}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing}) \arrow[dl] \arrow[dr] & \\
H^k_{\mathrm{rig}}(Y_J^{\infty}/\mathscr{O}_F^{\varnothing}) \arrow[dr] & & H^k_{\mathrm{conv}}(\overline{Y}_J^{\infty}/\mathscr{O}_F^{\varnothing}) \arrow[dl] \\
& H^k_{\mathrm{rig}}(Y_J/\mathscr{O}_F) &
\end{tikzcd}$$

where $H^k_{\mathrm{rig}}(Y_J/\mathscr{O}_F)$ is the (non-logarithmic) rigid cohomology defined by Berthelot [6] in terms of formal schemes and rigid spaces. By [19, Thm. 5.1] it can also be computed in terms of dagger spaces. In particular it is isomorphic to the log rigid cohomology group of $Y_J^{\varnothing} = Y_J^{\infty}$ over $\mathscr{O}_F^{\varnothing}$ where Y_J is considered with trivial log structure. This means that the left lower map is an isomorphism. We have already seen in Lemma 4.8 that the upper right map is an isomorphism. The lower right map is the isomorphism from [39, Cor. 2.4.13]. Consequently the upper left arrow is also an isomorphism. □

The comparison between log rigid and log crystalline cohomology over $\mathscr{O}_F^0$ follows immediately from the comparison between log rigid and log convergent cohomology.

Corollary 4.10. *Let $(Y, \overline{Y})$ be a strictly semistable log scheme with boundary over k^0 and assume that $\overline{Y}$ is proper. Then there is a canonical quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0) \cong \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\mathscr{O}_F^0)_{\mathbf{Q}}$$

between log rigid cohomology and log crystalline cohomology as defined by Hyodo and Kato in [25].

Proof. As $\overline{Y}$ is in particular a fine log-scheme over k^0, there is a canonical quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\mathscr{O}_F^0) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\mathscr{O}_F^0)_{\mathbf{Q}}$$

by [39, Thm. 3.1.1]. Together with the maps from (4.1) we obtain a zig-zag of quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^0) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{Y}/\mathscr{O}_F^0) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\mathscr{O}_F^0)_{\mathbf{Q}},$$

as stated. □

Remark 4.11. Consider the case that $(Y, \overline{Y}) = (X_0, \overline{X}_0)$ comes from a proper strictly semi-stable log scheme with boundary $(X, \overline{X})$ over $\mathscr{O}_K^{\pi}$ as introduced in Definition 3.10. In this case the crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}_0/\mathscr{O}_F^0)$ which appears here is just the Hyodo–Kato cohomology $\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(\overline{X}_0)$ and we have a canonical quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0) \cong \mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(\overline{X})_{\mathbf{Q}}$$

Lemma 4.12. *Let $(X, \overline{X})$ be a proper strictly semistable log scheme with boundary over $\mathscr{O}_K^{\pi}$. Then*

$$\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^{\pi}) \to \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_K^{\pi})$$

is a quasi-isomorphism.

Proof. The statement follows from [19, Thm. 3.2], since $\overline{X}_0$ has a global lifting $\overline{\mathcal{X}}$, the weak completion of $\overline{X}$, which is partially proper. □

Corollary 4.13. *Let $(X, \overline{X})$ be a proper strictly semistable log scheme with boundary over $\mathscr{O}_K^{\pi}$. Then there is a canonical quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^{\pi}) \cong \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathscr{O}_K^{\pi})_{\mathbf{Q}}.$$

Proof. The functor defined in [38, § 5.3] induces also a map

$$\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_K^{\pi}) \to \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}_0/\mathscr{O}_K^{\pi})_{\mathbf{Q}}.$$

But since the natural morphism $\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}_0/\mathscr{O}_K^{\pi}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathscr{O}_K^{\pi})$ is in fact a quasi-isomorphism, and we obtain $\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_K^{\pi}) \to \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathscr{O}_K^{\pi})_{\mathbf{Q}}$. By the crystaline and convergent Poincaré Lemmas [39, Sec. 2.3] there is a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_K^{\pi}) \arrow[dd] & \\
& \mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \arrow[ul, "\mathrm{sp}"'] \arrow[dl, "\gamma"] \\
\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathscr{O}_K^{\pi})_{\mathbf{Q}} &
\end{tikzcd}$$

where γ and sp are quasi-isomorphisms. It follows that the vertical morphism is a quasi-isomorphism as well.

Together with the third morphism from Corollary 2.4 and the morphism of Lemma 4.12 we obtain a zig-zag of quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^{\pi}) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^{\pi}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}/\mathscr{O}_K^{\pi}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathscr{O}_K^{\pi})_{\mathbf{Q}}$$

as desired. □

4.2 Compatibility of structures

As we have established that for a proper strictly semistable pair with boundary $(X, \overline{X})$ over $\mathscr{O}_K$ as in Theorem 4.1 rational Hyodo–Kato and log rigid cohomology coincide, we turn our attention to the structure given by Frobenius, monodromy and Hyodo–Kato morphism.

To begin with we show the compatibility of Frobenius on log rigid and log crystalline cohomology via the comparison map obtained § 4.1.

Proposition 4.14. *Let $(X, \overline{X})$ be a proper strictly semistable log scheme with boundary over $\mathscr{O}_K^{\pi}$. Then the Frobenius morphisms on $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0)$ and $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}}$ are compatible with each other.*

Proof. While the Frobenius on $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0)$ is given by local data via Definition 1.30, we have seen in § 1.4 that it is compatible via the comparison quasi-isomorphism Θ_{HK} with the Frobenius on $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0)$ given by

$$\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0) \xrightarrow{\Theta_\sigma} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0^{\sigma}/\mathscr{O}_F^0) \xrightarrow{f_{\overline{X}_0}^*} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0),$$

where Θ_σ is the base change by σ, $\overline{X}_0^{\sigma} = \overline{X}_0 \otimes_\sigma \mathscr{O}_F$, and $f_{\overline{X}_0}^*$ is induced by the relative Frobenius $f_{\overline{X}_0} : \overline{X}_0 \to \overline{X}_0^{\sigma}$ as described in § 1.4. However, they are compatible with the comparison quasi-isomorphism Θ_{HK} as we have seen in § 2.2. Thus it suffices to show that φ on $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0)$ is compatible with the Frobenius on crystalline cohomology.

As convergent and crystalline cohomology are functorial with respect to base change, the Frobenius operator on them can be obtained in the same way as just described for rigid cohomology (c.f. [25, (3.2)]). But the comparison morphisms between convergent and rigid cohomology as well as between convergent and crystalline cohomology are functorial, so that we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0) \arrow[r, "\Theta_\sigma"] \arrow[d, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0^{\sigma}\mathscr{O}_F^0) \arrow[r, "f_{\overline{X}_0}^*"] \arrow[d, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0) \arrow[d, "\sim"] \\
\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_F^0) \arrow[r, "\Theta_\sigma"] \arrow[d, "\sim"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0^{\sigma}/\mathscr{O}_F^0) \arrow[r, "f_{\overline{X}_0}^*"] \arrow[d, "\sim"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_F^0) \arrow[d, "\sim"] \\
\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}} \arrow[r, "\Theta_\sigma"] & \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}} \arrow[r, "f_{\overline{X}}^*"] & \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}}.
\end{tikzcd}$$

and this shows the compatibility. □

Similarly, we check the compatibility of the monodromy operators by passing through convergent cohomology.

Proposition 4.15. *Let $(X, \overline{X})$ be a proper strictly semistable log scheme with boundary over $\mathscr{O}_K^\pi$. Then the monodromy operators on $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0)$ and $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}}$ are compatible.*

Proof. By construction the monodromy operators on the rigid Hyodo–Kato complexe $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0)$ and the convergent Hyodo–Kato complex $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{conv}}(X_0, \overline{X}_0)$ are compatible via the canonical morphisms from the first line of (4.1) as can easily be checked locally. Thus we have a commutative diagram

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0) \arrow[r, "N"] \arrow[d, "\sim"'] & \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0) \arrow[d, "\sim"'] \\ \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{conv}}(X_0, \overline{X}_0) \arrow[r, "N"] & \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{conv}}(X_0, \overline{X}_0) \end{tikzcd}$$

and it remains to show the compatibility between the monodromy on convergent and the monodromy on crystalline Hyodo–Kato cohomology. For this we use the alternative construction from [29], which we used as a guide for Definition 1.32.

Let $W\tilde{\omega}^\bullet[u]_{\overline{X}_0}$ be the commutative differentially graded algebra obtained from $W\tilde{\omega}^\bullet_{\overline{X}_0}$ defined in (3.2) by adjoining the divided powers $u^{[i]}$ of a variable u in degree zero, i.e. they satisfy the relations $du^{[i]} = \frac{dt}{t}u^{[i-1]}$ and $u^{[0]} = 1$. The monodromy operator on $W\tilde{\omega}^\bullet_{\overline{X}_0}[u]$ is defined to be the $W\tilde{\omega}^\bullet_{\overline{X}_0}$-linear morphism that maps $u^{[i]}$ to $u^{[i-1]}$. By [29, Lem. 6] and [29, Lem. 7], the natural morphism $W\tilde{\omega}^\bullet_{\overline{X}_0}[u] \to W\omega^\bullet_{\overline{X}_0}$ is a quasi-isomorphism, compatible with Frobenius and monodromy. Hence $W\tilde{\omega}^\bullet_{\overline{X}_0}[u]$ also computes the crystalline Hyodo–Kato cohomology and its structures.

To compare this to the construction of the monodromy on the convergent Hyodo–Kato complex, we note that by Lemma 1.28 and Remark 1.50 (or rather their convergent analogue) we can choose a simplicial object $((U_\bullet, \overline{U}_\bullet), (\mathscr{Z}_\bullet, \overline{\mathscr{Z}}_\bullet), i_\bullet, \phi_\bullet)$ in the category of convergent Hyodo–Kato quadruples $\mathsf{CQ}_{\mathrm{HK}}$ such that $\{(U_\bullet, \overline{U}_\bullet)\}$ is a Zariski hyper covering of $(X_0, \overline{X}_0)$, and for each $n \geqslant 0$ the reduction modulo p^n of the pair $(\overline{U}_\bullet, \overline{\mathscr{Z}}_\bullet)$ is an embedding system for $\overline{X}_0 \to \mathrm{Spec}(\mathscr{O}_{F,n}[t], 1 \mapsto t)$ in the Zariski topology in the sense of [25, 2.18]. Note that we can obtain the fromer from the latter by p-adic completion. Hence, without loss of generality we assume that $(X_0, \overline{X}_0)$ is HK-embeddable. Let $((X_0, \overline{X}_0), (\mathscr{Z}, \overline{\mathscr{Z}}), i, \phi)$ be an appropriate convergent Hyodo–Kato quadruple and set $\overline{\mathscr{X}}_0 := \overline{\mathscr{Z}} \times_{\widehat{\mathscr{T}}} \mathscr{O}_F^0$. Then $(\overline{X}_0, \overline{\mathscr{X}}_0, i)$ is a log convergent triple

Compare the definition of $W\tilde{\omega}^\bullet_{\overline{X}_0}$ in (3.2) and $\tilde{\omega}^\bullet_{\overline{\mathscr{X}}_{0,\mathbf{Q}}}$ in (1.1). In fact, the complex $i_0^*C_{X_0/\mathscr{O}^\varnothing_{F,n}}$ is given by $\mathscr{O}_{F,n} \otimes_{\mathscr{T}_n^{\mathrm{cr}}} \omega^\bullet_{\overline{\mathscr{Z}}_n}$. Hence the functor from [38, § 5.3] induces a map between the cohomology groups of $\tilde{\omega}^\bullet_{\overline{\mathscr{X}}_{0,\mathbf{Q}}}$ and $W\tilde{\omega}^\bullet_{X_0,\mathbf{Q}}$. If we furthermore send the variable u to u, we obtain a morphism $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{conv}}(X_0, \overline{X}_0)_{(\mathscr{Z},\overline{\mathscr{Z}})} \to \mathrm{R}\Gamma(X_0, W\tilde{\omega}^\bullet_{\overline{X}_0}[u])_{\mathbf{Q}}$ which fits into a commutative diagram

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{conv}}(X_0, \overline{X}_0)_{(\mathscr{Z},\overline{\mathscr{Z}})} \arrow[r] \arrow[d, "\sim"'] & \mathrm{R}\Gamma(\overline{X}_0, W\tilde{\omega}^\bullet_{\overline{X}_0}[u])_{\mathbf{Q}} \arrow[d, "\sim"'] \\ \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_F^0)_{\overline{\mathscr{X}}_0} \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}_0/\mathscr{O}_F^0)_{\mathbf{Q}} \end{tikzcd}$$

where the left vertical morphism is a quasi-isomorphism by Lemma 1.34 and the right vertical morphism is a quasi-isomorphism by [29, Lem. 6], while the horizontal map at the bottom is again the quasi-isomorphism from [?, Thm. 3.1.1]. Consequently, the upper horizontal map is a quasi-isomorphism as well. Since the monodromy operator in both cases is defined by sending $u^{[i]}$ to $u^{[i-1]}$, the desired compatibility follows immediately. □

Lastly we turn our attention to the Hyodo–Kato morphisms.

Proposition 4.16. *Let $(X, \overline{X})$ be a proper strictly semistable log scheme with boundary over $\mathscr{O}_K^\pi$. The Hyodo–Kato morphisms on $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(X_0, \overline{X}_0)$ and $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}}$ are compatible on Frobenius eigenspaces.*

Proof. To compare the crystalline and rigid Hyodo–Kato morphisms we once again pass through log convergent cohomology. Analogous to (3.3) we consider the exact closed immersions

$$\mathscr{O}_F^0 \xrightarrow{i_0} \widehat{\mathcal{T}} \xleftarrow{i_\pi} \mathscr{O}_K^\pi$$

given by $t \mapsto 0$ and $t \mapsto \pi$, which induce the base change morphisms

$$\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_F^0) \xleftarrow{i_0^*} \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\widehat{\mathcal{T}}) \xrightarrow{i_\pi^*} \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_K^\pi). \tag{4.4}$$

The functor between the log convergent and log crystalline site defined in [38, § 5.3] together with the crystalline and convergent Poincaré Lemma induces a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_F^0) \arrow[d, "\sim"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\widehat{\mathcal{T}}) \arrow[l, "i_0^*"'] \arrow[r, "i_\pi^*"] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathscr{O}_K^\pi) \arrow[d, "\sim"] & \\
\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}} & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}_0/\mathcal{T}_{PD})_{\mathbf{Q}} \arrow[l, "i_0^*"', "\sim_\varphi"] \arrow[r, "i_\pi^*"] & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}_0/\mathscr{O}_K^\pi)_{\mathbf{Q}} & \mathrm{R}\Gamma_{\mathrm{dR}}(\overline{X}_K) \arrow[ul, "\mathrm{sp}"', "\sim"] \arrow[dl, "\gamma", "\sim"'] \\
& \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathcal{T}_{PD})_{\mathbf{Q}} \arrow[ul, "i_0^*", "\sim_\varphi"'] \arrow[u, "\sim_\varphi"'] \arrow[r, "i_\pi^*"] & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathscr{O}_K^\pi)_{\mathbf{Q}} \arrow[u, "\sim"'] &
\end{tikzcd} \tag{4.5}$$

where $\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}_0/\mathcal{T}_{PD})_{\mathbf{Q}} \xrightarrow{i_0^*} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}}$ and $\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathcal{T}_{PD})_{\mathbf{Q}} \xrightarrow{i_0^*} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{cr}}(\overline{X})_{\mathbf{Q}}$ are quasi-isomorphisms on Frobenius eigenspaces, i.e. on the homotopy limits of $1-\varphi_r$, by [35, Proof of Prop. 3.8].

In the case of rigid cohomology, the closed immersions of weak formal log schemes,

$$\mathscr{O}_F^0 \xrightarrow{i_0} \mathcal{T} \xleftarrow{i_\pi} \mathscr{O}_K^\pi$$

induce canonical morphisms on cohomology

$$\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0) \xleftarrow{i_0^*} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathcal{T}) \xrightarrow{i_\pi^*} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi), \tag{4.6}$$

and similarly for X_0. By construction they are compatible with the convergent analogue (4.4) via the comparison morphisms from the previous section.

However it is unclear how to obtain a section of the left morphism. Instead, we use the construction in diagram (2.1) to obtain ι_π^{rig}. In what follows we use the same notation as in § 2.1 with $(X_0, \overline{X}_0)$ instead of $(Y, \overline{Y})$. Note that in (2.1) $\overline{\xi}_0$ is induced by $t \mapsto 0$ while $\overline{\xi}_\pi$ is induced by $t \mapsto \pi$.

Let $\overline{M}_\bullet$ be the simplicial log scheme over k^0 corresponding to $(\overline{V}_\bullet, \overline{P}_\bullet)$ as in § 2.1. The augmentation $\overline{M}_\bullet \to \overline{Y}$ induces quasi-isomorphisms for $\mathcal{S} = \mathscr{O}_F^0, \mathscr{O}_K^\pi, \mathcal{T}$

$$\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathcal{S}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathcal{S}).$$

Together with the canonical morphism $\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet, \overline{P}_\bullet)/\mathcal{T}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathcal{T})$ they allow us to fit (4.6) and (2.1) into one diagram

$$\begin{tikzcd}
& \mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet, \overline{P}_\bullet)/\mathcal{T}) \arrow[dl, "\sim"'] \arrow[d] \arrow[dr] & \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathscr{O}_F^0) & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathcal{T}) \arrow[l] \arrow[r] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathscr{O}_K^\pi) \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_F^0) \arrow[u, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/\mathcal{T}) \arrow[l] \arrow[r] \arrow[u, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathscr{O}_K^\pi) \arrow[u, "\sim"]
\end{tikzcd}$$

which obviously commutes since the all maps which point left are induced by $t \mapsto 0$ and all maps which point right are induced by $t \mapsto \pi$.

To see that ι_π^{rig} and ι_π^{cr} are compatible, we have to invert the morphism i_0^* in (3.3). This is possible on Frobenius eigenspaces. To consider the Frobenius on $\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet, \overline{P}_\bullet)/\mathcal{T})$, we need the following lemma.

Lemma 4.17. *The morphism $\Theta^\sharp\colon \mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}_\bullet)/\mathcal{T})\to \mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}^\sharp_\bullet)/\mathcal{T})$ of Proposition ?? is a quasi-isomorphism.*

Proof. It suffices to show that $\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}^I_J,\overline{P}^I_J)/\mathcal{T})\cong \mathrm{R}\Gamma^\sharp((\overline{V}^I_J,\overline{P}^{I,\sharp}_J)/\mathcal{T})$ for any $I\subset J\subset \Upsilon_{\overline{X}_0}$. Recall that $(\overline{V}^I_J,\overline{P}^{I,\sharp}_J):=(\overline{V}^I_J,\overline{P}^I_J)\times_T(T,\overline{T})$. This is just the construction $(\overline{V}^I_J,\widetilde{P}^I_J)$ in § 2.1 with respect to the structure morphism $\overline{X}_0\to k^0$. Thus we have $\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}^I_J,\overline{P}^I_J)/\mathcal{T})\cong \mathrm{R}\Gamma((\overline{V}^I_J,\overline{P}^{I,\sharp}_J)/\mathcal{T})$ by Lemma ??. By Definition ?? the right hand side is quasi-isomorphic to $\mathrm{R}\Gamma^\sharp_{\mathrm{rig}}((\overline{V}^I_J,\overline{P}^{I,\sharp}_J)/\mathcal{T})$. □

Consequently, one can define the Frobenius endomorphism on $\mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}_\bullet)/\mathcal{T})$ as at the end of § 1.4, which is compatible with the Frobenius on $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{M}_\bullet/\mathcal{T})$ and hence with the Frobenius on $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathcal{T})$ and by Proposition 1.79 with the Frobenius on $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathcal{O}_F^0)$.

Putting everything that we discussed together we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathrm{R}\Gamma_{\mathrm{rig}}((\overline{V}_\bullet,\overline{P}_\bullet)/\mathcal{T})^{\varphi=p^r} \arrow[dl, "\sim"'] \arrow[d] \arrow[dr] & & \\
\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathcal{O}_F^0)^{\varphi=p^r} \arrow[d, "\sim"'] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathcal{T})^{\varphi=p^r} \arrow[l] \arrow[r] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathcal{O}_K^\pi) \arrow[d, "\sim"'] & \\
\mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathcal{O}_F^0)^{\varphi=p^r} \arrow[d, "\sim"'] & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\widehat{\mathcal{T}})^{\varphi=p^r} \arrow[l] \arrow[r] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{conv}}(\overline{X}_0/\mathcal{O}_K^\pi) \arrow[d, "\sim"'] & \mathrm{R}\Gamma_{\mathrm{dR}}(\overline{X}_K) \arrow[ul, "\mathrm{sp}"'] \arrow[l, "\mathrm{sp}", "\sim"'] \arrow[dl, "\gamma", "\sim"'] \\
\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{HK}}(\overline{X})^{\varphi=p^r}_{\mathbf{Q}} & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathcal{T}_{PD})^{\varphi=p^r}_{\mathbf{Q}} \arrow[l, "\sim"'] \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}/\mathcal{O}_K^\pi)_{\mathbf{Q}} &
\end{tikzcd}$$

where we restricted the cohomology theories in the left and middle vertical row to Frobenius eigenspaces. From the commutativity of the triangles on the right hand side of the diagram we conclude that the canonical morphism $\mathrm{sp}\colon \mathrm{R}\Gamma_{\mathrm{dR}}(\overline{X}_K)\to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/\mathcal{O}_K^\pi)$ and hence $\mathrm{sp}\colon \mathrm{R}\Gamma^D_{\mathrm{dR}}(X_K)\to \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathcal{O}_K^\pi)$ constructed in § 3.3 are quasi-isomorphisms.

This shows as desired that the rigid and crystalline Hyodo–Kato morphisms are compatible on Frobenius eigenspaces. □

As a consequence, we obtain a quasi-isomorphism between the diagrams which define $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}((X,\overline{X}),r,\pi)$ and $\mathrm{R}\Gamma^{\mathrm{cr}}_{\mathrm{syn}}(\overline{X},r,\pi)$. Since our construction of the cup product on $\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}((X_0,\overline{X}_0),r,\pi)$ follows the construction of [11, § 2.4], this quasi-isomorphism is compatible with cup products.

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