Graduate Seminar in Arithmetic Geometry, Spring 2010

Ertl Veronika

May 3, 2010

Contents

1 The non-archimedean half-plane

1.1 The building of $\mathbf{PGL}_2(K)$

We start with the following situation: let K be a local non-archimedean field, $\mathcal{O}$ its ring of integers, π a uniformiser, $k = \mathcal{O}/\pi\,\mathcal{O}$ its residue field, $p = \mathrm{char}(k)$, $q = \#k$, C the completion of the algebraic closure of K, $|\cdot|$ the normalised norm on C by $|\pi| = \frac{1}{q}$, so the valuation v is $v(x) = -\log_q|x|$.

A lattice $M \subset K^2$ is a free $\mathcal{O}$-submodule of rank 2. Two lattices M and M' are homothetic, if there is $\lambda \in K^*$ such that $M' = \lambda M$. Denote by S the set of homothety classes and $[M]$ the class of M.

The building of $\mathbf{PGL}_2(K)$ is the following graph I: S is the set of vertices; two vertices s and s' are joint by an edge if $\pi M \subsetneq M' \subsetneq M$ for suitable representatives M and M'. So I is a tree where each vertex has $q+1$ neighbours: there is a bijection of the edges joining $s = [M]$ and the lines of $M/\pi M$, i.e. $\mathbb{P}^1(k)$. It is not obvious that this is indeed a tree.

For the geometric realisation $I_{\mathbb{R}}$ take the proportionality classes of the norms on K^2.

1. For a lattice M there is a norm $|\cdot|_M$ such that the unit disc is exactly M. For a basis $\{e_1, e_2\}$ of M and an element $v = a_1e_1 + a_2e_2$ it is defined by
$$|v|_M = \sup\{|a_1|, |a_2|\}.$$
A vertex $s = [M]$ corresponds to the class of $|\cdot|_M$.

2. For the edges: If s and s' are neighbours, by the definition, there is a basis $\{e_1, e_2\}$ of M such that $\{e_1, \pi e_2\}$ is a basis of M'. For $x = a_1e_1 + a_2e_2 = a_1e_1 + \frac{a_2}{\pi}\pi e_2$
$$\begin{aligned} |v|_M &= \sup\{|a_1|, |a_2|\}, \\ |v|_{M'} &= \sup\{|a_1|, q|a_2|\}. \end{aligned}$$
Therefore, a point between them, $x = (1-t)s + ts'$, $t \in (0,1)$, is represented by the norm defined by
$$|v|_t = \sup\{|a_1|, q^t|a_2|\}.$$
By definition, M is the unit disc for $|\cdot|_M$ and M' is the unit disc for $|\cdot|_{M'}$, so one could ask: how can we describe them as discs for $|\cdot|_t$? We see that
$$\begin{aligned} M &= \{v \in K^2 \,:\, |v|_t \le \lambda\} \text{ for } q^t \le \lambda < q, \\ M' &= \{v \in K^2 \,:\, |v|_t \le \lambda\} \text{ for } 1 \le \lambda < q^t. \end{aligned}$$
For the first case, the inclusion "$\subseteq$" is obvious: $m \in M \Rightarrow |m|_M \le 1 \Rightarrow |m|_t = \sup\{|a_1|, q^t|a_2|\} \le \sup\{1, q^t\} = q^t \le \lambda$. The other inclusion follows from the fact that the norm on K "is discrete": $v = a_1e_+a_2e_2 \in K^2$ with $|v|_t \le \lambda$, so $|a_1| \le \lambda < q$ meaning that $|a_1| \le 1$, on the other hand $q^t|a_2| \le \lambda < q$, meaning that $q^t|a_1| \le q^t$ or that $|a_1| \le 1$. The second case follows similarly.

3. Now we want to see that every norm on K^2 corresponds to a point on the tree $I_{\mathbb{R}}$. For $\lambda > 0$ we consider the disc $M_\lambda = \{v \in K^2 \,:\, |v| \le \lambda\}$, this is a lattice in K^2. Clearly, $M_{\lambda'} \subset M_\lambda$ if $\lambda' \le \lambda$ and $M_{q^{-1}\lambda} = \pi M_\lambda$. From this one can see that there are at most two homotety classes as λ varies. Indeed, choosing a norm on K^2 induced by the non-archimedean norm $|\cdot|$ on K is the same as to choose a ratio for the standard basis: $\frac{1}{\mu}$ corresponds to $|\cdot|_{K^2} = \sup\{|\cdot|, \mu|\cdot|\}$. If this ratio is a power of q, it defines a lattice, and therefore a homotety class od lattices. If the ratio is not a power of q it is squished between two unique powers of q and therefore, $[M_\lambda]$ can take two values.

In the first case, i.e. if $[M_\lambda] = s$ is constant, s corresponds to $|\cdot|$ so this norm represents a vertex of the graph. In the second case, the two resulting values of $[M_\lambda]$ represent two neighbouring vertices in the tree. WLOG, we can replace the norm by a proportional one, so that we get: $[M_\lambda] = s$ for $q^t \leq \lambda < q$ and $[M_\lambda] = s'$ for $1 \leq \lambda < q^t$, $t \in (0,1)$. In other words, $|\cdot|$ represents the point $(1-t)s + ts'$ on the joining edge.

1.2 The analytic rigid space Ω

Note $\Omega = \mathbb{P}^1_K(C) - \mathbb{P}^1_K(K)$, where we identify $\mathbb{P}^1_K(C)$ with the C^*-homothety classes of non-zero K-homomorphisms $K^2 \to C$. This means that $\mathbb{P}^1_K(K)$ are the classes of homomorphisms of rank 1 over K. And further, that Ω is the set of classes of inective homomorphisms.

We want to use what we said in the previous section to define a map $\lambda : \Omega \to I_{\mathbb{R}}$. Let $z : K^2 \to C$ be an injective homomorphism representing an element of Ω. Since z is injective, we get a well-defined norm on K^2 by composing with the norm on C:

$$|v|_z = |z(V)| \qquad \text{for} \qquad v \in K^2.$$

By what we already said, this defines the desired application by

$$\lambda(\text{ class of } z) = \text{ class of } |\cdot|_z.$$

The image of λ is $I_{\mathbb{Q}}$: Since $z \in \Omega$ is K-linear, there is a basis of K^2 and a basis of C, such that z is given by a matrix with K-coefficients. Further, without loss of generality, by an appropriate base change, we can assume that if $v = (a_1, a_2) \in K^2$ is given in this basis, $z(a_1, a_2) = a_1 + \xi a_2$ where $\xi \in K^2$. So for the norm:

$$|(a_1, a_2)|_z = |a_1 + \xi a_2| = \sup\{|a_1|, q^{-v(\xi)}|a_2|\}.$$

Let s and s' be neighbouring vertices of I and (e_1, e_2) as before. We can identify Ω and $C - K$: since $z \in \Omega$ is C^* homothetic, we can choose a representative such that $z(e_2) = 1$ and $z(e_1) = \zeta \in C - K$.

To study Ω, we examine the preimage of $I_{\mathbb{R}}$.

Lemma 1.1.

$$\begin{aligned}
\lambda^{-1}(s) &= \{\zeta \in C : |\zeta| \leq 1\} - \bigcup_{a \in \mathcal{O} \bmod \pi\, \mathcal{O}} \{\zeta \in C : |\zeta - a| < 1\}, \\
\lambda^{-1}(x) &= \{\zeta \in C : |\zeta| = q^t\} \textit{ for } x = (1-t)s + ts' \\
\lambda^{-1}(s') &= \{\zeta \in C : |\zeta| \leq q^{-1}\} - \bigcup_{b \in \pi\, \mathcal{O} \bmod \pi^2\, \mathcal{O}} \{\zeta \in C : |\zeta - b| < q^{-1}\}, \\
\lambda^{-1}([s, s']) &= \lambda^{-1}(s) \cup \lambda^{-1}(s').
\end{aligned}$$

Remark 1.2. Note that the preimages of vertices, respective of homothety classes of submodules are "discs", while the preimage of a point on the edge is just a circle. Further, the preimages of neighbouring vertices lie inside another. More precisely, we have closed discs of radius 1 (resp. of radius q^{-1}), where q open discs of radius 1 (resp. of radius q^{-1}) are cut out. We will see, why these are the q open discs of radius 1 (resp. q^{-1}) of K-rational centre that the respective "big" disc contains. Further, the preimage of the edge joining the two vertices is the open annulus centered at 0 between the circles of radius 1 and q^{-1}.

PROOF: An element of the tree I is given by a norm (more precisely by a proportionality class of norms) $|\cdot|_t = \sup\{|a_1|, q^t|a_2|\}$. A representative norm for an element $z \in \Omega$ is given by $|\cdot|_z = |\zeta a_1 + a_2|$, and ζ is the corresponding element in $C - K$. The two mentioned norms are proportional over K^2, if and only if $|\cdot|_z = q^{-t}|\cdot|_t$. Without loss of generality, we can by homothety assume that $a_1 = 1$, so that we get the necessary and sufficient condition

$$|\zeta + a_2| = \sup\{q^{-t}, |a_2|\} \text{ for all } a_2 \in K. \tag{1}$$

Now we consider three cases for t:

1. $t \in (0,1)$. By definition of an non-archimedean field, for any element of $a \in K$ the norm of a is of the form p^n for some $n \in \mathbb{Z}$. Thus, if t is not an integer, $|a_2| \neq q^{-t}$ for all $a_2 \in K$. Consequently we see that the equation (1) is equivalent to the condition that $|\zeta| = q^{-t}$.

2. $t = 0$. Since $0 \in K$ plugging in 0 in (1) gives, $|\zeta| = 1$ and for all a_2 of norm 1 $|\zeta + a_2| = 1$, and from this last one, $|\zeta| \leq \sup\{|\zeta|, |a_2|\} = |\zeta + a_2| = 1$. Further, for $a_2 = -a \in \mathcal{O}$, $|\zeta - a| = \sup\{1, |-a|\} \geq 1$. Plugging in $a_2 \in K - \mathcal{O}$ does not give new information.

3. $t = 1$. This is the same reasoning as in the previous case with 1 replaced by q^{-1}, and a replaced by $b \in \pi\,\mathcal{O}$, otherwise the equatioin would not give any more information.

Finally, we just take the union of the two sets to get the last expression. □

We have seen that the sets $\lambda^{-1}(s)$, $\lambda^{-1}(s')$ and $\lambda^{-1}([s,s'])$ are the complements in $\mathbb{P}^1_K$ of a finite number of open discs. This means that they are connected affinoid subsets of $\mathbb{P}^1_K$. As such they have naturally the structure of rigid analytic spaces over K. Furthermore, $\lambda^{-1}(s)$ and $\lambda^{-1}(s')$ are open subsets of $\lambda^{-1}([s,s'])$.

Now we can generalise our argument to the whole tree I. If T is a finite subtree of I, then $\lambda^{-1}(T)$ is an affinoid connected subset of $\mathbb{P}^1_K$, which is obtained by glueing the preimages of edges to the preimages of vertices, following the structure of T.

We can consider the union of the preimages of all finite subtrees T, which is $\Omega = \bigcup \lambda^{-1}(T)$, and therefore Ω has a natural structure as rigid analytic space (as union of affinoid rigid analytic spaces).

1.3 The formal scheme $\widehat{\Omega}$.

Let M be again a lattice in K^2. The generic fiber of the projective line $\mathbb{P}(M)$, which is the projective spectrum of the symmetric algebra of M, can be canonically identified with $\mathbb{P}^1_K$ over $\mathcal{O}$, since M is a $\mathcal{O}$-submodule of K^2. If M and M_1 are homothetic lattices, the homothety defines a unique $\mathcal{O}$-isomorphism between the corresponding projective lines, which gives the identity on the generic fibers. Therefore it is legitimate to denote the isomorphism class of the projective line corresponding to the homothety class $s = [M]$ corresponding to a vertex in I by $\mathbb{P}_s$ and identify the generic fibre with $\mathbb{P}^1_K$.

Looking at the previous section, we see that the points of $\lambda^{-1}(s)$ are exactly the points of $\mathbb{P}^1_K(C)$ which do not specialise to k-rational points in the special fibre of $\mathbb{P}_s$. The points of the set $\lambda^{-1}(s)$ satisfy $|\zeta - a| \geq 1$, which means, reducing $\operatorname{mod} \pi$ gives not a k-rational point.

To construct the formal scheme $\widehat{\Omega}$ we construct it for the vertices and edges of I as we did for the rigid analytic space and then glue. Let Ω_s be the open subset of $\mathbb{P}_s$, which is the complement of the points specialising to k-rational points. And $\widehat{\Omega}_s$ the formal scheme completed at the special fiber. We have canonical bijections

$$\mathbb{P}^1_K(C) = \mathbb{P}_s(\mathcal{O}_C) = \widehat{\mathbb{P}}_s(\mathcal{O}_C),$$

by the valuative criterion for properness and formal geometry since $\mathbb{P}^1$ is proper. This induces a bijection

$$\lambda^{-1}(s) = \widehat{\Omega}_s(\mathcal{O}_C).$$

In other words, the rigid analytic space $\lambda^{-1}(s)$ is the generic fiber of the formal scheme $\widehat{\Omega}_s$. Since it is an affine formal scheme, it makes sense to say that this means that the TATE algebra corresponding to $\lambda^{-1}(s)$, is $\Gamma(\widehat{\Omega}_s) \otimes_{\mathcal{O}} K$.

A neighbouring vertex s' of s defines a k-rational point in the special fiber of $\mathbb{P}_s$: for $s = [M]$ and $s' = [M']$ as before, the point is defined by the application

$$M \to M/M' \cong k.$$

If we blow up $\mathbb{P}_s$ in the point s' we get a $\mathcal{O}$-scheme $\mathbb{P}_{[s,s']}$ (which we can also obtain by blowing up $\mathbb{P}_{s'}$ in s). The generic fiber is again the same: $\mathbb{P}^1_K$. The blow-up of $\mathbb{P}_s$ in the k-rational point defined by s' is given globally by $\mathrm{Proj}(\bigoplus_n \mathfrak{I}^n)$, where $\mathfrak{I}$ is the ideal generated by π and a coordinate T. We have two open subsets: the formal spectra of $\mathcal{O}\{X,T\}/(X\pi - T)$ and $\mathcal{O}\{X,T\}/(XT - \pi)$. And it becomes clear, that the generic fiber is $\mathbb{P}^1_K$.

We consider again the complement of the k-rational points of the special fiber of $\mathbb{P}_{[s,s']}$ except the singular point and denote it by $\Omega_{[s,s']}$. Let $\widehat{\Omega}_{[s,s']}$ be the formal completion with respect to the special fiber. The identification of the generic fibers

$$\mathbb{P}^1_K(C) = \mathbb{P}_{[s,s']}(\mathcal{O}_C) = \widehat{\mathbb{P}}_{[s,s']}(\mathcal{O}_C),$$

induces a bijection

$$\lambda^{-1}([s,s']) = \widehat{\Omega}_{[s,s']}(\mathcal{O}_C). \tag{2}$$

We can see this bijection by looking at $\mathcal{O}\{X,T\}/(X\pi - T)$ and $\mathcal{O}\{X,T\}/(XT - \pi)$. The first one corresponds to taking away the points with $|\zeta - b| < q^{-1}$ $b \in \pi\Omega$, because $\frac{T}{\pi} = X$ has to be non-k-rational, so the norm of $|X - a| \geq 1$ and $|T - a\pi| \geq q^{-1}$ and the T will be the ζ (indeed, the natural coordinate in the generic fiber is $\frac{\zeta}{\pi}$ which corresponds to $X = \frac{T}{\pi}$. The second one is not as interesting, it doesn't give us new information, just the rest of the space. In particular, the inverse image of the open edge $]s,s'[$, the annulus $\lambda^{-1}(]s,s'[)$ specialise to the singular point of the special fiber of $\mathbb{P}_{[s,s']}$. Since the special fiber of $\mathcal{O}\{X,T\}/(TX - \pi)$ is $k[X,T]/(XT)$, the singular point on this open set (i.e. in the origin) correponds to the reduction of ζ being zero, and on the second one to the reduction of ζ being ∞. So it is justified that we kept the singular point.

Equation (2) tells us that $\lambda^{-1}([s,s'])$ is the generic fiber of the formal scheme $\widehat{\Omega}_{[s.s']}$. What is more, we get canonically open immersions of $\widehat{\Omega}_s$ and $\widehat{\Omega}_{s'}$ in $\widehat{\Omega}_{[s.s']}$, which corresponds on the generic fibers to the inclusions of $\lambda{-}1(s)$ and $\lambda{-}1(s')$ in $\lambda{-}1([s,s'])$.

Again we generalize this to finite subtrees T of I: by glueing together formal schemes corresponding to the vertices and edges of T following the indicated relations by T, we get a formal scheme $\widehat{\Omega}_T$ with the generic fiber canonically identified with $\lambda^{-1}(T)$.

Note that for an inclusion $T \subset T'$, we have open immersions $\widehat{\Omega}_T \subset \widehat{\Omega}_{T'}$ and induced inclusions on the generic fibers $\lambda^{-1}(T) \subset \lambda^{-1}(T')$. This means that we can take the inverse image to construct a formal scheme

$$\widehat{\Omega} = \varprojlim \widehat{\Omega}_T$$

with generic fiber Ω. (In particular $\widehat{\Omega}(\mathcal{O}_K) = \Omega$.) The special fiber of $\widehat{\Omega}$ is a tree, where the vertices are projective lines over k, intersecting in the k-rational points, which give the edges. It is the tree dual to I.

1.4 The functor $\widehat{\Omega}$ according to DELIGNE

Let $(\mathfrak{Compl})$ be the category of complete HAUSDORFF algebras for the π-adic topology. We follow an idea of DELIGNE to describe the functors on $(\mathfrak{Compl})$ represented by the formal schemes $\widehat{\Omega}_s$ and $\widehat{\Omega}_{[s,s']}$.

Definition 1.3. of the functor

$$F_s : (\mathfrak{Compl}) \to (\mathfrak{Set})\ ,\ R \mapsto \text{ class of } (\mathcal{L}, \alpha),$$

where $\mathcal{L}$ is a free R-module of rank one, and $\alpha : M \to \mathcal{L}$ a homomorphism of $\mathcal{O}$-modules (recall that $s = [M]$), with condition

$$\forall x \in \mathrm{Spec}(R/\pi R) \text{ the map } \alpha(x) : M/\pi M \to \mathcal{L} \otimes_R k(x) \text{ is injective.} \tag{3}$$

Proposition 1.4. *The functor F_s is represented by the formal scheme $\widehat{\Omega}_s$.*

PROOF: The condition (3) on α means that for a point x in the special fiber of $\mathrm{Spec}(R)$, the induced morphism of $k(x)$-modules, where $k(x)$ is the residue field at x, is injective. In other words, for $u \in M - \pi M$ $\alpha(u)$ is a generator of $\mathcal{L}$ as R module. In particular, tensoring with R gives an epimorphism $\alpha \otimes \mathrm{id} : M \otimes_{\mathcal{O}} R \to \mathcal{L}$. This shows that F_s is a subfunctor of the functor $\widehat{\mathbb{P}}_s$, as it is the formal projective line over derived by of the functor $\mathbb{P}_s$ defined by s. But we can do better: fix a basis $\{e_1, e_2\}$ which determines local coordinates $\{0, 1, \infty\}$ of $\widehat{\mathbb{P}}_s$. The paire $(\mathcal{L}, \alpha)$ is up to isomorphism defined by the ratio $\frac{\alpha(e_1)}{\alpha(e_2)} = \zeta \in R$. So F_s is a subfunctor of the affine formal line $\widehat{\mathbb{P}}_s - \{\infty\}$.

¿Why is this a formal functor?

The condition on α should give us the possibility to identify this functor as the object, we want it to be. Looking at the special fiber, this condition means that for the image of the ratio ζ modulo π, $a - \overline{\zeta}$ is never zero for any a and on any point in $\mathrm{Spec}(R/\pi R)$, more precisely, $\overline{\zeta} - a$ is invertible in $R/\pi R$. Looking at the Lemma 1.1 and the argumentation in the last section, this is the same as to say, F_s is represented by $\widehat{\Omega}_s$. □

Now we want to argue similar for the edges of our tree.

Definition 1.5. Let $F_{[s,s']} : (\mathfrak{Compl}) \to (\mathfrak{Set})$ be the functor, which sends R to the isomorphism classes of commutative diagrams of the form

$$\begin{tikzcd} \pi M \arrow[r, hook] \arrow[d, "\alpha/\pi"] & M' \arrow[r, hook] \arrow[d, "\alpha'"] & M \arrow[d, "\alpha"] \\ \mathcal{L} \arrow[r, "c"'] & \mathcal{L}' \arrow[r, "c'"'] & \mathcal{L}, \end{tikzcd}$$

where $\mathcal{L}$ and $\mathcal{L}'$ are free R-modules of rank 1, α and α' are $\mathcal{O}$-morphisms, c and c' are R-morphisms veryfying

$$\mathrm{Ker}(\alpha(x)) \subset M'/\pi M \ \text{ and } \ \mathrm{Ker}(\alpha'(x)) \subset \pi M/\pi M' \tag{4}$$

for all $x \in \mathrm{Spec}(R/\pi R)$.

Proposition 1.6. *The functor $F_{[s,s']}$ is represented by the formal scheme $\widehat{\Omega}_{[s,s']}$.*

PROOF: The idea is similar to the proof of the previous lemma. Let $\{e_1, e_2\}$ be a basis of M such that $\{e_1, \pi e_2\}$ is a basis of M'. The condition on α implies that $\alpha(e_2)$ generates M as a R-module, because, $e_2 \notin M'$, and $\alpha'(e_1)$ generates M', because although e_1 is in M' it is not in πM. We can even identifie the rank-1 R-modules M and M' with R by setting $\alpha(e_2) = 1$ and $\alpha'(e_1) = 1$ (this is possible as for our purpose only the ration counts). So let $\zeta = \alpha(e_1)$ and $\eta = \alpha'(e_2)$. Using the commutativity of the diagram gives:

$$\begin{aligned} c &= c \circ \alpha(e_2) = c \circ \frac{\alpha}{\pi}(\pi e_2) = \alpha'(\pi e_2) = \eta, \\ c' &= c' \circ \alpha'(e_1) = \alpha(e_1) = \zeta, \\ \zeta\eta &= c' \circ c = c' \circ c \circ \frac{\alpha}{\pi}(\pi e_2) = \alpha(\pi e_2) = \pi. \end{aligned}$$

These relations identify depending on the choice of $\{e_1, e_2\}$ the functor $F_{[s,s']}$ as a subfunctor of the formal scheme $\mathrm{Spf}\,(\mathcal{O}\{\zeta,\eta\}/(\zeta\eta-\pi))$. And as we have seen in the previous paragraph, this choice also identifies $\mathrm{Spf}\,(\mathcal{O}\{\zeta,\eta\}/(\zeta\eta-\pi))$ as an open subset of $\widehat{\mathbb{P}}_{[s,s']}$, the complement of the points at infinity $\overline{\zeta}=\infty$ and $\overline{\eta}=\infty$ in the special fiber.

To evaluate the condition given by (4) we look at the images of ζ and η in $R/\pi R$. It means that for all non-zero elements $a \in k$ $\overline{\zeta}-a$ and $\overline{\eta}-a$ are invertible in $R/\pi R$. Again we conclude that $F_{[s,s']}$ is the subfunctor of $\widehat{\mathbb{P}}_{[s,s']} - (\{\overline{\zeta}=\infty\}\cup\{\overline{\eta}=\infty\})$ that represents the complement of the k-rational points of the two components of the special fibre $\mathrm{Spec}(k[\overline{\zeta},\overline{\eta}]/\left(\overline{\zeta}\overline{\eta}\right))$ outside the singular point $\overline{\zeta}=\overline{\eta}=0$. And as we hav already seen, this is $\widehat{\Omega}_{[s,s']}$. □

The open immersion $\widehat{\Omega}_s \hookrightarrow \widehat{\Omega}_{[s,s']}$ corresponds to the restriction induced by the open immersion $\mathrm{Spf}(\mathcal{O}\{\zeta,\zeta^{-1}) \hookrightarrow \mathrm{Spf}\,(\mathcal{O}\,\zeta,\eta(\zeta\eta-\pi))$. We define functorially a map $F_s \to F_{[s,s']}$ by sending $\alpha : M \to \mathcal{L}$ to the commutative diagram

$$\begin{tikzcd} \pi M \arrow[r, hook] \arrow[d] & M' \arrow[r, hook] \arrow[d] & M \arrow[d, "\alpha"] \\ \mathcal{L} \arrow[r, "\pi"'] & \mathcal{L} \arrow[r, "\mathrm{id}"'] & \mathcal{L}, \end{tikzcd}$$

which makes F_s a subfunctor of $F_{[s,s']}$ consisting of the diagrams, where c' is invertible.

Similarly, we have functorially an identification of $F_{s'}$ as subfunctor of $F_{[s,s']}$ consisting this time of the diagrams, where c is invertible.

1.5 The functor $\widehat{\Omega}$ according to DRINFELD

In the previous approach, the functors F_s and $F_{[s,s']}$ can be seen as a modular description of the open affine subsets $\widehat{\Omega}_s$ and $\widehat{\Omega}_{[s,s']}$ composing to the whole formal scheme $\widehat{\Omega}$. The questioin arises, if we can describe $\widehat{\Omega}$ directly without decomposing it in terms of a unique functor F. And in fact, an approach suggested by DRINFELD provides such a functor on the category $(\mathfrak{Nilp})$ of $\mathcal{O}$-algebras, where the image of π is nilpotent.

Let B be an $\mathcal{O}$-algebra and $B[\Pi]$ be the quotient $B[X]/(X^2-\pi)$ of rank 2 generated by 1 and Π, the image of X. This algebra has a $\mathbb{Z}/2\,\mathbb{Z}$-graduation as usual. We define a functor F as desired as follows:

Definition 1.7. Let $b \in (\mathfrak{Nilp})$ and $S = \mathrm{Spec}(B)$. Then $F(B)$ is the isomorphism class of quadruples (η, T, u, r) where

1. η is a sheaf of flat $\mathcal{O}[\Pi]$-modules on S that is $\mathbb{Z}/2\,\mathbb{Z}$-graduated and constructible, with ZARISKI topology.
2. T is a sheaf of $\mathcal{O}_S[\Pi]$-modules, again $\mathbb{Z}/2\,\mathbb{Z}$-graduated, such that the homogenous components T_0 and T_1 are invertible sheafs on S.
3. u is a $\mathcal{O}[\Pi]$-homomorphism of degree 0 $\eta \to T$, such that tensoring with $\mathcal{O}_S$ makes it surjective.
4. r is a K-isomorphism of the constant sheaf $\underline{K}^2 \to \eta_0 \otimes_{\mathcal{O}} K$.

These data should further verify the conditions

C1 Let $S_i \subset S$ be the annihilator of the morphism $\Pi : T_i \to T_{i+1}$. Then the restriction $\eta_i|_{S_i}$ is a constant sheaf with fibres isomorphis to $\mathcal{O}^2$.

C2 For every geometric point $x \in S$, denote $T(x) = T \otimes_B k(x)$. Then the induced map $\eta_x/\Pi\eta_x \to T(x)/\Pi T(x)$ is injective.

C3 $(\bigwedge^2 \eta_i)|_{S_i} = \pi^{-i}\left(\bigwedge^2(\Pi^i r\underline{\mathcal{O}}^2)\right)|_{S_i}$ for $i = 0, 1$.

We want to prove that

Proposition 1.8. *The functor F is representable by the formal scheme $\widehat{\Omega}$.*

Proof (outline): define for each vertex s and each edge $[s, s']$ a morphism of functors $F_s \to F$ and $F_{[s,s']} \to F$ conpatible with the recollements, more precisely with open immersions $F_s \hookrightarrow F_{[s,s']}$ and $F_{s'} \hookrightarrow F_{[s,s']}$. This provides a morphism of functors, which is an isomorphism. □

1.6 The group action of $\mathbf{PGL}_2(K)$

¿Does this explain, why we call the tree "the building of $\mathbf{PGL}_2(K)$"?

The group $GL_2(K)$ act naturally on the tree via the quotient $\mathbf{PGL_2}(K)$. An element $g \in \mathbf{GL}_2(K)$ sends the vertex $[M] = s$ to the vertex $[gM]$ and the edge $[[M], [M']]$ to the edge $[[gM], [gM']]$. On the other side, there is an of $\mathbf{PGL}_2(K)$ on the set $\Omega = \mathbb{P}^1_K(C) - \mathbb{P}^1_K(K)$. By definition it is clear that the map $\lambda : \Omega \to I_{\mathbb{R}}$ is equivariant under the action of $\mathbf{PGL}_2(K)$. As this map help us to identify the rigid analytic structer of Ω, the group $\mathbf{PGL}_2(K)$ acts in fact by automorphisms of this structure. More precisely, it permutes the affino"id open subsets defined above. What is more, it is not difficult to see that all the different constructions leading to an understanding of $\widehat{\Omega}$ are equivariant and therefore the action of $\mathbf{PGL}_2(K)$ comes from an action on the formal scheme $\widehat{\Omega}$. Using Deligne's approach, we can describe this action as follows: for $s = [M]$ and $gs = [gM]$ there is a morphism of functors

$$g : F_s \to F_{gs},$$

given by

$$g : (\mathcal{L}, \alpha) \to (\mathcal{L}, \beta) = g \cdot (\mathcal{L}, \alpha),$$

where β is the composition

$$\beta : gM \xrightarrow{g^{-1}} [\cong] M \xrightarrow{\alpha} \mathcal{L}.$$

Further, for an edge $[s, s']$ the morphism $g : F_{[s,s']} \to F_{[gs,gs']}$ is given by

$$g \cdot (\mathcal{L}, \mathcal{L}', c, c', \alpha, \alpha') = (\mathcal{L}, \mathcal{L}', c, c', \alpha \circ g^{-1}, \alpha' \circ g^{-1}).$$

The next proposition gives a description of the action of $\mathbf{PGL}_2(K)$ for Drinfeld's functor.

Proposition 1.9. *An element $g \in \mathbf{GL}_2(K)$ acts on the functor F as follows:*

$$g \cdot (\eta, T, u, r) = (\eta[n], T[n], u[n], \Pi^n \circ r \circ g^{-1}),$$

where n is the valuation of $\det(g)$, and $[n]$ the shift of n modulo 2 of the graduation of (η, T, u).

2 The Theorem of Drinfeld

From now on, let K be of characteristic zero. (Recall: K is a local non-archimedean field, $\mathcal{O}$ its ring of integers, π a uniformiser, k the residue field of characteristic p and order q.

2.1 Cartier Theory of Formal $\mathcal{O}$-modules

To start recall the Witt vector functor:

Theorem 2.1. *If R is a perfect ring of characteristic p, then there exists a strict p ring $W(R)$ unique up to isomorphism, whose residue ring is R. It has the following universal property: If A is a p-ring of residue ring R', $\overline{\theta} : R \to R'$ a morphism of rings, $\widetilde{\theta} : R \to A$ a multiplicative map lifting $\overline{\theta}$ there is a unique morphism of rings $\theta : W(R) \to A$ such that $\theta([x]) = \widetilde{\theta}(x)$, where $[x]$ is the Teichmüller representative.*

The resulting ring is called the Witt ring and has the following properties:

1. If A is a strict p-ring of residue ring R, every element can be written uniquely as a sum $\sum_{i\in\mathbb{N}_0} p^i[x_i]$, where $[x_i]$ is the Teichmüller representative of some $x_i \in R$.
2. Addition and multiplication in these terms is given by certain polynomials.

We want to generalise this idea. There exists a unique functor $W_{\mathcal{O}} : (\mathcal{O}-\mathfrak{Alg}) \to (\mathcal{O}-\mathfrak{Alg})$ on the category of commutative $\mathcal{O}$-algebras which associates to each $\mathcal{O}$-algebra B the set $W_{\mathcal{O}}(B) = B^{\mathbb{N}}$ in a way such that for any $n \in \mathbb{N}_0$ the map $w_n : W_{\mathcal{O}}(B) \to B$ given by

$$w_n(a_0, a_1, \ldots) = a_0^{q^n} + \pi a_1^{q^{n-1}} + \cdots + \pi^n a_n$$

is a $\mathcal{O}$-algebra morphism, this means that for an element $a = (a_0, a_1, \ldots)$ the a_i can be seen as a sort of Teichmüller representatives and the w_n give the addition and multiplication law in $W_{\mathcal{O}}$. The images of an element under the w_n are called phantom components.

Further, we have functorially on $W_{\mathcal{O}}$ the shifting $\mathcal{O}$-endomorphism of

$$\tau(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$$

and an $\mathcal{O}$-algebra endomorphism σ such that the diagram

$$\begin{tikzcd} W_{\mathcal{O}}(B) \arrow[r, "\sigma"] \arrow[dr, "w_n"'] & W_{\mathcal{O}}(B) \arrow[d, "w_{n+1}"] \\ & B \end{tikzcd}$$

commutes, that is $w_n\sigma = w_{n+1}$. Looking at the diagram, it is not hard to verify that the introduced morphisms satisfy the relations

$$\begin{aligned} \sigma\tau &= \pi \\ ({}^{\tau}x).y &= {}^{\tau}(x.({}^{\sigma}y)), \quad \text{for} \quad x, y \in W_{\mathcal{O}}(B). \end{aligned}$$

For example

$$\begin{tikzcd} (a_0, a_1, \ldots) \arrow[r, "\tau"] \arrow[d, "w_n"'] & (0, a_0, a_1, \ldots) \arrow[r, "\sigma"] \arrow[d, "w_n"'] \arrow[dr, "w_{n+1}"] & \pi(a_0, a_1, \ldots) \arrow[d, "w_n"] \\ a_0^{q^n} + \pi a_1^{q^{n-1}} + \cdots + \pi^n a_n \arrow[r] & 0 + \pi a_0^{q^{n-1}} + \cdots + \pi^n a_{n-1} \arrow[r] & 0 + \pi a_0^{q^n} + \cdots + \pi^{n+1} a_{n+1}. \end{tikzcd}$$

For $a \in B$ we denote $[a] = (a, 0, \ldots)$. The structure of $W_{\mathcal{O}}$ tells us that

$$\begin{aligned} [ab] &= [a] \cdot [b] \\ {}^{\sigma}[a] &= [a^q] \end{aligned}$$

If B is a k-algebra, we even have ${}^{\sigma}(a_0, a_1, \ldots) = (a_0^q, a_1^q, \ldots)$ and $\tau\sigma = \sigma\tau = \pi$. For $\mathcal{O} = \mathbb{Z}_p$ we recognise this construction as the usual Witt functor.

Now we consider an additional structure: for any $\mathcal{O}$-algebra B let $W_{\mathcal{O}}(B)[F, V]$ be the non-commutative $\mathcal{O}$-algebra where F and V (Frobenius and Verschiebung) verify

$$\begin{aligned} Fx &= {}^{\sigma}xF \\ xV &= V^{\sigma}x \\ VxF &= {}^{\tau}x \\ FV &= \pi. \end{aligned}$$

Definition 2.2. The Cartier *ring* $E_{\mathcal{O}}(B)$ is the completion of this algebra with respect to V-adic topology (more precisely, with respect to the topology defined by the right ideals generated by the V^n).

This means, that every element in this ring can be written uniquely as sum

$$\sum_{m,n \in \mathbb{N}_0} V^m [a_{m,n}] F^n \,, \quad a_{m,n} \in B,$$

with the condition that $a_{m,n} = 0$ for n big enough depending on m. The diagonal map

$$(a_0, a_1, \ldots) \mapsto \sum_{n \in \mathbb{N}_0} V^n [a_n] F^n$$

identifies $W_{\mathcal{O}}(B)$ as a subalgebra of $E_{\mathcal{O}}(B)$ and this indicates, that each element of $E_{\mathcal{O}}(B)$ can be written uniquely as

$$\sum_{m \in \mathbb{N}} V^m x_m + x_0 + \sum_{n \in \mathbb{N}} y_n F^n \,, \quad x_m, y_n \in W_{\mathcal{O}}(B),$$

under condition that τ-adically $y_n \to 0$, since $VxF =^{\tau} x$.

By construction and definition $\mathcal{O}$ is embedded in B, $W_{\mathcal{O}}(B)$ and $E_{\mathcal{O}}(B)$ and B is alos embedded in $W_{\mathcal{O}}(B)$ and $E_{\mathcal{O}}(B)$. However, the structur of the embeddings imply, that we have to distinguish carefully the element $a \in \mathcal{O}$ and its multiplicative representative $[a]$ as an element $A.1 \in B$, because for $n \in \mathbb{N}_0$ we have $w_n(a) = a$ and $w_n([a]) = a^{q^n}$. In particular,

$$w_n(\pi - [\pi]) = \pi(1 - \pi^{q^n - 1}).$$

There is a unique element $\varepsilon \in W_{\mathcal{O}}(B)$ coming from the subalgebra $W_{\mathcal{O}}(\mathcal{O})$ such that

$$w_n(\varepsilon) = 1 - \pi^{q^{n+1} - 1},$$

meaning that

$$\pi - [\pi] =^{\tau} \varepsilon = V\varepsilon F.$$

However, the existence of this element is not a priori clear.

A formal $\mathcal{O}$-module over an $\mathcal{O}$-algebra B is a smooth formal group over B with an action of $\mathcal{O}$ (a formal group scheme over B is a representable functor from the category of profinite B-algebras to the category of groups, it is locally of the form $\mathrm{Spf}(A)$ for a B-algebra A). In case it is connected, it is called a formal Lie group. The action of $\mathcal{O}$ is given by a morphism of rings $i : \mathcal{O} \to \mathrm{End}(X)$ such that the induced action on the tangent space $\mathfrak{Lie}(X)$ coincides with the action coming from the structure of $\mathfrak{Lie}(X)$ as B-module.

Definition 2.3. An $\mathcal{O}$-CARTIER module M over B is a left $E_{\mathcal{O}}(B)$-module such that

1. M/VM is a free B-module of finite rank.
2. V is injective on M.
3. M is HAUSDORFF and V-adically complete.

The main resutl of DIEUDONNÉ-CARTIER theory is the following

Theorem 2.4. *There is an equivalence of categories between formal* $\mathcal{O}$*-modules over* B *and* $\mathcal{O}$*-*CARTIER *modules over* B*. Furthermore, if* M *is the* CARTIER *modules associated to the formal module* X*,* $M/VM = \mathfrak{Lie}(X)$*.*

Base change is equivalent to tensoring: If B' is a B algebra, then the CARTIER module of the formal module $X_{B'}$ derived from X by base change is $M' = E_{\mathcal{O}}(B')\widehat{\otimes}_{E_{\mathcal{O}}(B)}M$, where the hat indicates V-adic completion.

Definition 2.5. If M is an $\mathcal{O}$-CARTIER module over B, we say elements $\gamma_1, \ldots, \gamma_d$ of M are a *V-basis* of M, if their images modulo V are a basis of the free B-module M/VM.

Thus every element in M can be written uniquely as a sum $\sum_{m\in\mathbb{N}_0}\sum_{i=1}^{d} V^m[c_{m,i}]\gamma_i$, $c_{m,i} \in B$. Consequently, by what we said about F and V, the choice of γ_i defines a family $c_{m,i,j} \in B$. such that

$$F(\gamma_j) = \sum_{m\in\mathbb{N}_0}\sum_{i=1}^{d} V^m[c_{m,i,j}]\gamma_i.$$

Conversely, if we are given a family of elements $c_{m,i,j} \in W_{\mathcal{O}}(B)$ there exists, up to isomorphism, a unique $\mathcal{O}$-Cartier module M, and a V-basis $\gamma_1, \ldots, \gamma_d$ of M satisfying

$$F(\gamma_j) = \sum_{m\in\mathbb{N}_0}\sum_{i=1}^{d} V^m c_{m,i,j}\gamma_i.$$

This module has the presentation

$$0 \to E_{\mathcal{O}}(B)^d \xrightarrow{\psi} E_{\mathcal{O}}(B)^d \xrightarrow{\varphi} M \to 0,$$

where φ and ψ are defined in terms of the canonical basis $\{\varepsilon_1, \ldots, \varepsilon_d\}$ of $E_{\mathcal{O}}(B)$ by

$$\begin{aligned} \varphi(\varepsilon_i) &= \gamma_i \\ \psi(\varepsilon_i) &= F(\varepsilon_i) - \sum_{m\in\mathbb{N}_0}\sum_{i=1}^{d} V^m c_{m,i,j}\varepsilon_i. \end{aligned}$$

Therefore the choice of a V-basis gives us a correspondence between CARTIER modules and structure constants. However, for the sake of simplicity we will use the following modification: the choice of a V-basis of M determines a family $d_{m,i,j} \in B$ such that

$$\pi\gamma_j = [\pi]\gamma_j + \sum_{m\in\mathbb{N}}\sum_{i=1}^{d} V^m[d_{m,i,j}]\gamma_i.$$

Conversely, for a family $d_{m,i,j} \in W_{\mathcal{O}}(B)$ there is, up to isomorphism, a unique $\mathcal{O}$-CARTIER module M and a V-basis $\{\gamma_1, \ldots, \gamma_d\}$ of M such that

$$\pi\gamma_j = [\pi]\gamma_j + \sum_{m\in\mathbb{N}}\sum_{i=1}^{d} V^m d_{m,i,j}\gamma_i.$$

How can we from this recover our original relations? Recall that $\pi - [\pi] = V\varepsilon F$ with a unit $\varepsilon \in W_{\mathcal{O}}(B)$ and thus

$$\begin{aligned} F\gamma_j &= \varepsilon^{-1}V^{-1}\pi\gamma_j - [\pi]\gamma_j \\ &= \varepsilon^{-1}V^{-1}\left(\sum_{m\in\mathbb{N}}\sum_{i=1}^{d} V^m d_{m,i,j}\gamma_i\right) \\ &= \sum_{m\in\mathbb{N}}\sum_{i=1}^{d} V^{m-1\sigma^{m-1}}\varepsilon^{-1}d_{m,i,j}\gamma_i, \end{aligned}$$

which makes sense since V is injective on M and the exponents of V in the sum are ≥ 1. This provides the relation with $c_{m,i,j} =^{\sigma^m} \varepsilon^{-1}d_{m+1,i,j}$.

2.2 Cartier Theory of Formal $\mathcal{O}_D$-modules

Let D be quaternion field over K and as usual $\mathcal{O}_D$ its ring of integers. Further, let K' be a unramified quadratic extension of K contained in D, $\mathcal{O}'$ its ring of integers, and σ the automorphism of conjugation of K' over K. Let Π be an element of $\mathcal{O}_D$ such that $\Pi^2 = \pi$ and $\Pi a =^\sigma a\Pi$ for all $a \in K'$.

Definition 2.6. formal $\mathcal{O}_D$-module over an $\mathcal{O}$-algebra B is a formal $\mathcal{O}$-module X over B with an action of $\mathcal{O}_D$, $i : \mathcal{O}_D \to \mathrm{End}(X)$ compatible with the action of $\mathcal{O}$. A formal $\mathcal{O}_D$-module is *special* if $\mathfrak{Lie}(X)$ is a free $B \otimes_{\mathcal{O}} \mathcal{O}'$-module of rank one via the action of $\mathcal{O}'$.

Let now B be an $\mathcal{O}'$-algebra an X a formal $\mathcal{O}_D$-module over B. Then the B-module $\mathfrak{Lie}(X)$ is $\mathbb{Z}/2\mathbb{Z}$-graduated via the action of $\mathcal{O}'$:

$$\begin{aligned} \mathfrak{Lie}(X)_0 &= \{m \in \mathfrak{Lie}(X) \,:\, i(a)m = am,\, a \in \mathcal{O}'\}, \\ \mathfrak{Lie}(X)_1 &= \{m \in \mathfrak{Lie}(X) \,:\, i(a)m =^\sigma am,\, a \in \mathcal{O}'\}. \end{aligned}$$

This means, that X is special if every component of $\mathfrak{Lie}(X)$ is a free B-module of rank 1.

Consider a $\mathbb{Z}/2\mathbb{Z}$-graduation of the Cartier ring $E_{\mathcal{O}}(B)$ given by

$$\begin{aligned} \deg V = \deg F &= 1, \\ \deg[B] &= 0 \quad \text{for} \quad b \in B. \end{aligned}$$

Recall the "diagonal" embedding of $W_{\mathcal{O}}(B)$ in $E_{\mathcal{O}}(B)$ which gives an element $(a_0, a_1, \ldots)$ uniquely as $\sum_{n\in\mathbb{N}_0} V^n[a_n]F^n$, $a_n \in B$, which has obviously degree 0, so that $W_{\mathcal{O}}(B)$ is contained in the degree-0 component of $E_{\mathcal{O}}(B)$.

Definition 2.7. A graduated $\mathcal{O}[\Pi]$-Cartier module over B is a $\mathcal{O}$-Cartier module which is $\mathbb{Z}/2\mathbb{Z}$-graduated, $M = M_1 \oplus M_2$ together with an $E_{\mathcal{O}}(B)$-endomorphism Π of degree 1 such that $\Pi^2 = \pi$.

The fact that $W_{\mathcal{O}}(B)$ is contained in the 0-component of $E_{\mathcal{O}}(B)$ implies that M_1 and M_0 are $W_{\mathcal{O}}(B)$-submodules of M.

Definition 2.8. M is called special if M_0/VM_1 and M_1/VM_0 are free B-modules of rank 1.

Similarly to the case of $\mathcal{O}$-modules, we have the following

Theorem 2.9. *If B is an $\mathcal{O}'$-algebra, there is an equivalence of categories of formal $\mathcal{O}_D$-modules over B and graduated $\mathcal{O}[\Pi]$-Cartier modules over B. Furthermore, a formal $\mathcal{O}_D$-module is special if and only if the corresponding $\mathcal{O}[\Pi]$-Cartier module is special.*

PROOF: Since B is not only an $\mathcal{O}$-algebra but an $\mathcal{O}'$-algebra, the $\mathcal{O}$-algebras $W_{\mathcal{O}}(B)$ and $E_{\mathcal{O}}(B)$ become also $\mathcal{O}'$-algebras. More precisely, $\mathcal{O}'$ is generated over $\mathcal{O}$ by the q^2-1^{th} roots of unity. For an element $\zeta \in \mathcal{O}'$ solving the corresponding equation $X^{q^2-1}-1=0$ consider the multiplicative (TEICHMÜLLER) representative $[\zeta] \in W_{\mathcal{O}}(B)$ of the image of ζ in B. The map $\zeta \mapsto [\zeta]$ gives an isomorphism of groups of the $(q^2-1)^{\text{th}}$ roots of unity of $\mathcal{O}'$ and those of $W_{\mathcal{O}}(B)$. This gives a unique homomorphism $j : \mathcal{O}' \to W_{\mathcal{O}}(B)$ induced by this isomorphism, i.e. $j(\zeta) = [\zeta]$.

Now we have to verify that the homomorphism of conjugation of $\mathcal{O}'$ sur $\mathcal{O}$ and the FROBENIUS endomorphism of $W_{\mathcal{O}}(B)$ both denoted by σ coincide. Indeed, j is compatible with σ. The conjugation has degree 2, in other words ${}^{\sigma}({}^{\sigma}\zeta) =^{\sigma} \zeta^2 = 1$ which means for a q^2-1^{th} root of unity $\zeta \in \mathcal{O}'$ that ${}^{\sigma}\zeta = \zeta^q$. On the other hand, we saw already, that ${}^{\sigma}[\zeta] = [\zeta^q]$ in $W_{\mathcal{O}}(B)$. Consequently, $j({}^{\sigma}a) =^{\sigma} j(a)$ for $s \in \mathcal{O}'$,

So any $E_{\mathcal{O}}(B)$-module, particularly any $\mathcal{O}$-CARTIER module over B has two natural structures as $\mathcal{O}'$-modules by j and $j\sigma$. For the sake of simplicity, we write simply am and ${}^{\sigma}am$, respectively $a \in \mathcal{O}'$, $m \in M$.

We have already seen that the category of formal $\mathcal{O}_D$-modules over B is equivalent to the category of $\mathcal{O}$-CARTIER modules M over B with an action $i : \mathcal{O}_D \to \mathrm{End}(M)$ compatible with the action of $\mathcal{O}$. Only the graduation needs some effort. In particular, there is an action of $\mathcal{O}' \subset \mathcal{O}_D$ on those modules via i, such that we have a decomposition $M = M_0 + M_1$ given by

$$\begin{aligned} M_0 &= \{m \in M \,:\, i(a)m = am,\, a \in \mathcal{O}'\}, \\ M_1 &= \{m \in M \,:\, i(a)m =^{\sigma} am,\, a \in \mathcal{O}'\}. \end{aligned}$$

The operators V, F and $[b], b \in B$ are homomorphisms of $\mathcal{O}$-modules of degrees $1,1,0$ respectively. This is clear by the earlier mentioned equalitites for $a \in \mathcal{O}, b \in B$

$$\begin{aligned} aV &= V^{\sigma}a \\ Fa &= {}^{\sigma}aF \\ a[b] &= [b]a \end{aligned}$$

Conversely, to give a $\mathbb{Z}/2\,\mathbb{Z}$-graduation of the $\mathcal{O}$-module M, such that $\deg V = \deg F = 1$ and $\deg[b] = 0$ for $b \in B$, comes to the same as to give an action of $\mathcal{O}'$ compatible with the action of $\mathcal{O}$.

Hereto comes the action of $\mathcal{O}_D$ which is the same as to give an action of Π. We denote by Π the action given by $i(\Pi)$. We have $\Pi^2 = \pi$. We see that this action has degree 1, since for all $a \in \mathcal{O}'$ $\Pi(a) =^{\sigma} a\Pi$.

Now for the second part of the theorem: the graduation on the LIE algebra of a formal $\mathcal{O}_D$-module X and on the corresponding CARTIER module M are both defined by the action of $\mathcal{O}'$ and they are compatible:

$$\begin{aligned} \mathfrak{Lie}(X)_0 &= M_0/VM_1, \\ \mathfrak{Lie}(X)_1 &= M_1/VM_0 \end{aligned}$$

as $\mathfrak{Lie}(X) = M/VM$ and V is of degree 1 and as the graduations are defined similarly. This shows, that M is special if and only if X is special. □

Now we want to give a unique way to write elements of our module. Let M be a graduated special $\mathcal{O}[\Pi]$-module over B and $\{\gamma_0 \in M_0, \gamma_1 \in M_1\}$ a homogenous V-basis of M. Every element in M can be written uniquely as

$$x = \sum_{m \in \mathbb{N}_0} \left(V^m[c_{m,o}]\gamma_0 + V^m[c_{m,1}]\gamma_1\right), \quad c_{mi} \in B.$$

Given that V is of degree 1 and $[c_{mi}]$ is of degree 0, a decompostion in homogenous components

$x_0 \in M_0$ and $x_1 \in M_1$ is

$$\begin{aligned} x_0 &= [c_{00}]\gamma_0 + \sum_{m\in\mathbb{N}} V^m[c_{m\overline{m}}\gamma_{\overline{m}} \\ x_1 &= [c_{01}]\gamma_1 \sum_{m\in\mathbb{N}} V^m[c_{m,\overline{m+1}}]\gamma_{\overline{m+1}}. \end{aligned}$$

If we choose a homogenous V-basis γ_0, γ_1 of M, this determines elements in B such that

$$\begin{aligned} \Pi\gamma_0 &= [a_{00}]\gamma_1 + \sum_{m\in\mathbb{N}} V^m[a_{m0}]\gamma_{\overline{m+1}} \\ \Pi\gamma_1 &= [a_{01}]\gamma_0 + \sum_{m\in\mathbb{N}} V^m[a_{m1}]\gamma_{\overline{m}}. \end{aligned}$$

Thus,

$$\Pi^2 = [a_{00}.a_{01}] \bmod VM.$$

This follows since $\Pi^2 = \pi$ and $\pi \cong [\pi] \bmod VM$, so

$$a_{00}.a_{01} = \pi.$$

Conversely, we have the following

Proposition 2.10. *Let B be an $\mathcal{O}'$-algebra. Given elements $a_{mi} \in B$ such that $a_{00}.a_{01} = \pi$, there exists a graduated special $\mathcal{O}[\Pi]$-*Cartier *module M over B, unique up to isomorphism, and a homogenous V-basis γ_0, γ_1 of M satisfying the relations*

$$\begin{aligned} \Pi\gamma_0 &= a_{00}\gamma_1 + \sum_{m\in\mathbb{N}} V^m a_{m0}\gamma_{\overline{m+1}} \\ \Pi\gamma_1 &= a_{01}\gamma_0 + \sum_{m\in\mathbb{N}} V^m a_{m1}\gamma_{\overline{m}}. \end{aligned}$$

Proof: We have formulas giving the action of Π which determine likewise the action of $\pi = \Pi^2$. To be precise we should mention, that Π as endomorphism of an $E_{\mathcal{O}}(B)$-module commutes with V and $[a_{mi}]$. The hypothesis that $a_{00}.a_{01}$ implies that one can find elements $d_{mi} \in B$ such that

$$\pi\gamma_i = [\pi]\gamma_i + \sum_{m\in\mathbb{N}} V^m[d_{mi}]\gamma_{\overline{m+i}}.$$

We already know that there is up to isomorphism a unique $\mathcal{O}$-Cartier module over B and a V-basis γ_0, γ_1 of M such that the relations are satisfied. Now for the graduation

$$M_i = \left\{ \sum_{m\in\mathbb{N}_0} V^m x_m \gamma_{\overline{m+i}} \,:\, x_m \in W_{\mathcal{O}}(B) \right\}.$$

So M_0 and M_1 are $W_{\mathcal{O}}(B)$-submodules of M such that $M = M_0 \oplus M_1$. The operators Π, V and $[b]$, $b \in B$ have by construction the degrees 110. It follows, that $\pi - [\pi] : M \to VM$ is of degree 1 and likewise F is of degree 1, since V is injective on M and $F = \varepsilon^{-1}V^{-1}(\pi - [\pi])$. This makes M a graduated $\mathcal{O}[\Pi]$-Cartier module. To see that it is special, it is enough to say, that M_0/VM_1 is free of basis γ_0 and M_1/VM_0 is free of basis γ_1. □

Let B' be a B-algebra and $M' = E_{\mathcal{O}}(B')\widetilde{\otimes}_{E_{\mathcal{O}}(B)} M$ the Cartier module over B' deduced from M by base change. Then M' is a graduated $\mathcal{O}[\Pi]$-Cartier module over B'. The image of γ_i in M' is a homogenous V-basis of M' verifying

$$\Pi\gamma_i' = [a_{0i}']\gamma_{i+1}' + \sum_{m\in\mathbb{N}} V^m[a_{mi}']\gamma_{\overline{m+1+i}}',$$

where the a_{mi}' are the images of the $a_{mi} \in B$ in B'.

2.3 Construction of (η_M, T_M, u_M)

2.4 The Homogenous Components of η_M

2.5 Formal special $\mathcal{O}_D$-modules over an algebraically closed field

In this section suppose $B = L$ is an algebraically closed field of characteristic p. Let $\mathcal{W} = W_{\mathcal{O}}(L)$ be the associated ring of Witt vectors and $\mathcal{K}$ the field of fractions of $\mathcal{W}$. Per definitionem, $\mathcal{W}$ is a complete discrete valuation ring with uniformiser π and residue field L. The FROBENIUS automorphism on $\mathcal{W}$ fixes $\mathcal{O}$.

For a formal smooth $\mathcal{O}$-module over L (a formal LIE group), the (by DIEUDENNÉ-CARTIER theory) associated CARITER module is in particular a free $\mathcal{W}$-module of finite rank. The rank of M over $\mathcal{W}$ is said to be the height of X.

Remark 2.11. We saw in general, that there is a correspondence of categories between formal $\mathcal{O}_D$-modules over an $\mathcal{O}'$-algebra B and graded $\mathcal{O}[\Pi]$-CARTIER modules over B. We constructed for any such CARTIER module M a triple (η, T, u) where

1. η is a graded $\mathcal{O}[\Pi]$-module;
2. T is a graded $B[\Pi]$-module such that the homogenous components are of dimension 1;
3. $u : \eta \to T$ is a $\mathcal{O}[\Pi]$-morphism of degree 0.

Now we want, for the case when B is a algebrically closed field, to establish an equivalence of categories (in fact groupoids) of formal special $\mathcal{O}_D$-modules and such triples, with certain conditions on both categories which come up almost naturally as we will see.

Proposition 2.12. *If X is in addition a formal special $\mathcal{O}_D$-module, then its height is a multiple of* 4.

PROOF: Let M be the associated CARTIER module to X. Since X is special, $[M_i : VM_{i+1}] = 1$, that is they are free of rank one over L (and have length 1 over $\mathcal{W}$. Further, V is injective on M. Thus, M_0 and M_1 have the same rank let's say r over $\mathcal{W}$ and M as direct product has rank $2r$. We will see that r is even. As $\Pi^2 = \pi$, Π is injective as well and $M_0/\pi M_0$ is free of rank r over $\mathcal{W}$

$$\begin{aligned} r = [M_0 : \pi M_0] &= [M_0 : \Pi M_1] + [\Pi M_1 : \Pi^2 M_0] \\ &= [M_0 : \Pi M_1] + [M_1 : \Pi M_0]. \end{aligned}$$

Since $\Pi V M_0 \subset V M_1, \Pi M_1 \subset M_0$,

$$[M_0 : VM_1] + [VM_1 : \Pi V M_0] = [M_0 : \Pi M_1] + [\Pi M_1 : \Pi V M_0].$$

Or other

$$\begin{aligned} [\Pi M_1 : \Pi V M_0] &= [M_1 : VM_0] = [M_0 : VM_1] = 1 \\ [VM_1 : \Pi V M_0] &= [M_1 : \Pi M_0] \\ [M_1 : \Pi M_0] &= [M_0 : \Pi M_1] \end{aligned}$$

Thus, r is even. □

Proposition 2.13. *There is only one isogeny class of formal special $\mathcal{O}_D$-modules of height* 4.

PROOF: The isogeny class of a formal $\mathcal{O}$ module X is uniquely determined by the isocrystal $(M \otimes_{\mathcal{W}} \mathcal{K}, V)$, where M is the associated $\mathcal{O}$-CARTIER module. However, Π identifies (over the field $\mathcal{W}$ the homogenous components $M_0 \otimes \mathcal{K}$ and $M_1 \otimes \mathcal{K}$, which means that the isogeny class in question is already determined by the isocristal $(M_0 \otimes \mathcal{K}, V\Pi^{-1})$.

If i is a critical index, then we know, that $\Pi M_i \subseteq V M_i$. But in our case of height 4 $[M_{i-1} : \Pi M_i] = 1$ and in general $[M_{i-1} : V M_i] = 1$, too. So this induces that $\Pi M_i = V M_i$ and this is true after tensoring with $\mathcal{K}$ as well. However, over $\mathcal{K}$ Π is invertible, so the lattice M_i is stable under $V\Pi^{-1}$ in $M \otimes \mathcal{K}$ and the restriction to M_i is bijective. In this case we say it is a unit isocrystal. Since L is algebraically closed, $V\Pi^{-1}$ is diagonalisable over L, which means, we can find a basis $\{e_1, e_2\}$ of M_i even over $\mathcal{W}$ invariant under $V\Pi^{-1}$ and the isocristal is unique up to isomorphism. □

Remark 2.14. The formal $\mathcal{O}$-module X is isogen to a direct sum of two formal $\mathcal{O}$-modules of dimension 1 and height 2, meaning that the corresponding isocristal has dimension 2 and slope $\frac{1}{2}$.

Proposition 2.15. $\mathrm{End}^0_D(X) \cong M_2(K)$, *where* $\mathrm{End}^0_D(X) = \mathrm{End}_{\mathcal{O}_D}(X) \otimes \mathbb{Q}$.

PROOF: In the proof of the previous proposition, we established a correspondence between formal special $\mathcal{O}_D$ modules and isocristals of the form $X \mapsto (M \otimes \mathcal{K}) \mapsto (M_0 \otimes \mathcal{K}$. Now we see that this induces an isomorphism

$$\mathrm{End}^0_D(X) 0 \,\mathrm{End}_D(M \otimes_{\mathcal{W}} \mathcal{K}, V) = \mathrm{End}_K(M_0 \otimes \mathcal{K}, V\Pi^{-1}).$$

We have to see that this last ring is isomorphic to the 2×2-matrices over K. As mentioned, $(M_0 \otimes \mathcal{K}, V\Pi^{-1})$ is a unit isocristal. A $\mathcal{K}$-endomorphism on $M_0 \otimes \mathcal{K}$ has to commute with the σ^{-1}-linear map $V\Pi^{-1}$ and it does so if and only if the matrix in the basis stable under $V\Pi^{-1}$ has coefficients in $\mathcal{K}^\sigma = K$. □

From now on we assume that X is a formal special $\mathcal{O}_D$-module of height 4, in particular $[M_1 : \Pi M_0] = [M_0 : \Pi M_1] = 1$.

Proposition 2.16. *The homogenous components of* η *are free* $\mathcal{O}$*-modules of rank* 2 *and the application* $\eta \otimes_{\mathcal{O}} L \to T$ *is surjective.*

PROOF: We have seen that for a critical index $\eta_i \cong M_i^{V^{-1}\Pi}$ and $u_i : \eta_i \to T_i$ factors through $M_i^{V^{-1}\Pi} \hookrightarrow M_i \to M_i / V M_{i-1}$. Further from the previous proof, $V^{-1}\Pi$ is bijective on M_i, in other words, $(M_i, V^{-1}\Pi)$ is a unit cristal and $M_i^{V^{-1}\Pi}$ is a free $\mathcal{O}$-module of rank 2 and $M_i^{V^{-1}\Pi} \otimes_{\mathcal{O}} \mathcal{W} \to M_i$ is bijective. The composition of this ma with the projection to M_i/VM_{i-1} is surjective.

In case j is not critical, we have seen, that there is an isomorphism $\eta_j \to \eta_{j+1}$ induced by Π and as usual an isomorphism $T_j \to T_{j+1}$ and $j+1$ is a critical index. The diagram

$$\begin{tikzcd}
\eta_j \arrow[r, "\Pi"] \arrow[d, "u_j"'] & \eta_{j+1} \arrow[d, "u_{j+1}"'] \\
T_j \arrow[r, "\Pi"'] & T_{j+1}
\end{tikzcd}$$

shows then the claim. □

Proposition 2.17. *The map* $\eta/\Pi\eta \to T/\Pi T$ *induced by* u *is injective.*

PROOF: We show the claim for the homogenous components distinguishing wether one or both indices are critical.

1. *i and $i-1$ are critical*: Similarly to the previous proof we get a commutative diagram

$$\begin{tikzcd}
\eta_{i-1} \arrow[r, "\Pi"] \arrow[d] & \eta_i \arrow[d] \\
M_{i-1}^{V^{-1}\Pi}\Pi \arrow[r] \arrow[d] & M_i^{V^{-1}\Pi} \arrow[d] \\
T_{i-1} \arrow[r, "0"] & T_i.
\end{tikzcd}$$

Since Π is zero on T_{i-1}, $T_i/\Pi T_{i-1}$ is the same as T_i, and we have to verify, that the preimage of zero is in the image of Π in η_i. Let $x \in M_i^{V^{-1}\Pi}$ such that its image in $T_i = M_i/VM_{i-1}$ is zero, this means that there is $y \in M_{i-1}$ such that $x = Vy$. But we know already $\Pi x = Vx$ so since V is injective $\Pi y = Vy$ and $y \in M_{i-1}^{V^{-1}\Pi}$ and $x = \Pi y$.

2. *i is not critical but $i-1$ is critical*: Now we have a commutative diagram

$$\begin{tikzcd} \eta_{i-1} \arrow[r, "\Pi"] \arrow[d, "u_{i-1}"'] & \eta_i \arrow[r, "\Pi"] \arrow[d, "u_i"'] & \eta_{i-1} \arrow[d, "u_{i-1}"'] \\ T_{i-1} \arrow[r, "0"'] & T_i \arrow[r, "\Pi"'] & T_{i-1}. \end{tikzcd} \tag{5}$$

The second part of the diagram shows that $\Pi\eta_i = \eta_{i-1}$ so $\eta_{i-1}/\Pi\eta_i = 0$ and $\eta_{i-1}/\Pi\eta_i \to T_{i-1}/\Pi T_i$ is injective. To verify that $\eta_i/\Pi\eta_{i-1} \to T_i/\Pi T_{i-1} = T_i \cong T_{i-1}$ is injective, it suffices since $\Pi^2 = \pi$ to show that $\eta_{i-1}/\pi\eta_{i-1} \to T_{i-1}$ induced by u_{i-1}, which as $i-1$ is critical is identified with $M_{i-1}^{V^{-1}\Pi} \to M_{i-1}/VM_i$, is injective. Let therefore $x \in M_{i-1}^{V^{-1}\Pi}$ mapp to zero in the quotient M_{i-1}/VM_i, meaning that there is $y \in M_i$ such that $x = Vy$. As before $\Pi x = Vx$ induces $\Pi y = Vy$, in particular the image of y in M_i/VM_{i-1} is zero, because $\Pi\overline{y} = V\overline{y} = \overline{x} = 0 \in M_{i-1}/VM_i$ and $\Pi : M_i/VM_{i-1} \to M_{i-1}/VM_i$ is an isomorphism. Thus again, there exists $z \in M_{i-1}$ such that $y = Vz$, and again as before $\Pi z = Vz$ so $z \in M - i - 1^{V^{-1}\Pi}$ and $x = \pi z$.

□

Proposition 2.18. *The triple (η, T, u) determines X up to isomorphism.*

PROOF: By what we have seen, it is sufficient to show, that the triple in question determines M_0, M_1, V and Π (where M is the associated CARTIER module for X).

If i is a critical index and $\sigma = \mathrm{id} \otimes \sigma$ the induced automorphism on $\eta_i \otimes_{\mathcal{O}} \mathcal{W}$, which is a $\mathcal{W}$-module of rank 2 since η_i is a free $\mathcal{O}$-module of rank 2, then the inclusion $\eta_i = M_i^{V^{-1}\Pi} \subset M_i$ induces an isomorphism of isocristals

$$(\eta_i \otimes \mathcal{W}, \sigma) \to (M_i, \Pi V^{-1}).$$

This induces a surjection $\eta_i \otimes \mathcal{W} \xrightarrow{u_i \otimes \mathrm{id}} T_i = M_i/VM_{i-1}$ whose kernel $\mathcal{H}$ is identified with VM_{i-1} and further $\sigma\mathcal{H}$ with $\Pi V^{-1}VM_{i-1} = \Pi V_{i-1}$. This finally identifies the diagrams

$$M_{i-1} \underset{V}{\overset{\Pi}{\rightarrow}} M_i \underset{V}{\overset{\Pi}{\rightarrow}} M_{i-1}$$

and

$$\sigma(\mathcal{H}) \underset{\mathrm{incl}\circ\sigma^{-1}}{\xrightarrow{\mathrm{incl}}} \eta_i \otimes W \underset{\pi\circ\sigma^{-1}}{\xrightarrow{\pi}} \sigma(\mathcal{H}.$$

But since Π is an isomprphism $\mathbf{M}_{i-1}/VM_i \to M_i/VM_{i-1}$ we have all the data we want. □

Definition 2.19. A triple (η, T, u) is called admissible if it satisfies the conditions (2.16) and (2.17). An index $i \in \{0, 1\}$ is critical if $\Pi : T_i \to T_{i-1}$ is zero.

Lemma 2.20. *An admissible triple (η, T, u) is determined up to isomorphism by (η_i, T_i, u_i) for a critical index i.*

PROOF: Let $H = \operatorname{Ker} u_i$. We know $\pi\eta_i \subset H \subset \eta_i$ and $H \neq \eta_i$ by diagram (5).

If $i-1$ is not critical by Proposition (2.17) $\Pi T_{i-1} = T_i$ and $\Pi\eta_{i-1} = \eta_i$ and the diagram

$$\begin{tikzcd}
\eta_i \arrow[r, "\Pi"] \arrow[d, "u_i"'] & \eta_{i-1} \arrow[r, "\Pi"] \arrow[d, "u_{i-1}"'] & \eta_i \arrow[d, "u_i"'] \\
T_i \arrow[r, "\Pi=0"'] & T_{i-1} \arrow[r, "\Pi"'] & T_i
\end{tikzcd}$$

can be identified with

$$\begin{tikzcd}
\eta_i \arrow[r, "\pi"] \arrow[d, "u_i"'] & \eta_i \arrow[r, "\mathrm{id}"] \arrow[d, "u_i"'] & \eta_i \arrow[d, "u_i"'] \\
T_i \arrow[r, "0"'] & T_i \arrow[r, "\mathrm{id}"'] & T_i.
\end{tikzcd}$$

Further, $H = \pi\eta_i$.

If $i-1$ is critical, the diagram (5) implies that $\Pi\eta_{i-1} \neq \eta_i$ (otherwise it would contradict (2.16). And by hypothesis $\Pi\eta_i \neq \eta_{i-1}$. So, by (2.17) u_i and $u_{i-1}\Pi^{-1}$ induce isomorphisms

$$\eta_i/\Pi\eta_{i-1} \otimes_k L \to T_i \quad \text{and} \quad \Pi\eta_{i-1}/\pi\eta_i \otimes_k L \to T_{i-1}.$$

In this case $H = \Pi\eta_{i-1} \neq \pi\eta_i$ and the diagram

$$\begin{tikzcd}
\eta_i \arrow[r, "\Pi"] \arrow[d, "u_{i-}"'] & \eta_{i-1} \arrow[r, "\Pi"] \arrow[d, "u_{i-1}"'] & \eta_i \arrow[d, "u_i"'] \\
T_i \arrow[r, "0"'] & T_{i-1} \arrow[r, "0"'] & T_i
\end{tikzcd}$$

is identified with the diagram

$$\begin{tikzcd}
\eta_i \arrow[r, "\pi"] \arrow[d] & H \arrow[r, "\mathrm{incl}"] \arrow[d] & \eta_i \arrow[d] \\
\eta_i/H \otimes_k L \arrow[r, "0"] & H/\pi\eta_i \otimes_k L \arrow[r, "0"] & \eta_i/H \otimes_k L,
\end{tikzcd}$$

where the vertical maps are given canonically. Note that you can tell wether $i-1$ is critical or not by $H \neq \pi\eta_i$ or $H = \pi\eta_i$, which gives us finally all data we need to determine (η, T, u). □

Conversely we have the following

Proposition 2.21. *Any admissible triple* (η, T, u) *is isomorphic to the triple associated to a formal special* $\mathcal{O}_D$*-module of height* 4.

Proof: For any given admissible triple, we have to cinstruct an isomorphic triple coming from a special Cartier module. We again look at the critical index i. Let σ again be the automorphism $\mathrm{id}\otimes\sigma$ on $\eta_i \otimes_{\mathcal{O}} \mathcal{W}$ and $\mathcal{H}$ the kernel of $u_i \otimes \mathrm{id} : \eta_i \otimes \mathcal{W} \to T_i$. By condition (2.16) $\eta_i \otimes L \to T_i$ is surjective. This and the proof of the previous lemma imply $\pi(\eta_i \otimes_{\mathcal{O}} \mathcal{W} \subsetneq \mathcal{H} \subsetneq \eta_i \otimes \mathcal{W}$ and similarly for $\sigma(\mathcal{H})$. We can define the diagram

$$M_{i-1} \xrightarrow[V]{\Pi} M_i \xrightarrow[V]{\Pi} M_{i-1}$$

as the equivalent of

$$\sigma(\mathcal{H} \xrightarrow[\mathrm{incl}\circ\sigma^{-1}]{\mathrm{incl}} \eta_i \otimes \mathcal{W} \xrightarrow[\pi\circ\sigma^{-1}]{\pi} \sigma\mathcal{H}.$$

So V is σ^{-1}-linear, Π is linear and $\Pi^2 = \pi$, and

$$[M_i : VM_{i-1}] = [M_{i-1} : VM_i] = [M_i : \Pi M_{i-1}] = [M_{i-1} : \Pi M_i] = 1,$$

and we constructed the CARTIE module (M, Π, V) corresponding to a formal special $\mathcal{O}_D$-module of height 4.

Moreover, the index i is also critical for M, the homogenous component of i of the triple associated to M is

$$(M_i^{V\Pi^{-1}}, M_i/VM_{i-1}, \mathrm{can}) \cong (\eta_i, \eta_i \otimes \mathcal{W}/\mathcal{H}, \mathrm{can}) \cong (\eta_i, T_i, u_i),$$

so the constructed triple is isomorphis to the original one. □

All together, we have proven the following

Theorem 2.22. *Over an algebraically closed field of characteristic p, the correspondence $X \mapsto (\eta, T, u)$ gives an equivalents of groupoids between the formall special $\mathcal{O}_D$-modules of height 4 and the admissible triples with their repective isomorphisms.*

Lemma 2.23. *Let (η, T, u) be an adimissible triple associated to a formal special $\mathcal{O}_D$-module of height 4. Then the isoctistals $(M_i \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1})$ are canonically isomorphic to $(\eta_i \otimes_{\mathcal{O}} \mathcal{K}, \sigma^{-1})$.*

PROOF: For a critical index $(M_i, V\Pi^{-1})$ is a unit cristal, meaning that $V\Pi^{-1}$ is a σ^{-1}-linear bijection of M_i. Further, η_i is identified with $M_i^{V\Pi^{-1}}$. By the structure theorem of unit cristals, $(M_i, V\Pi^{-1})$ can be identified with $(\eta_i \otimes \mathcal{W}, \sigma^{-1})$ Tensoring with $\mathcal{K}$ gives the statement.

For i non-critical, the statement can be derived from the first case since Π induces an isomorphism of isocristals:

$$\begin{array}{rcl} (M_i \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1}) & \to & (M_{i+1} \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1}), \\ (\eta_i \otimes_{\mathcal{O}} \mathcal{K}, \sigma^{-1}) & \to & (\eta_{i+1} \otimes_{\mathcal{O}} \mathcal{K}, \sigma^{-1}); \end{array}$$

which are compatible with the mentioned isomorphism in case both indices are critical. □

Definition 2.24. Let X and X' be two formal special $\mathcal{O}_D$-modules of height 4 over L. A quasi-isogeny between X and X' is an element in $\mathrm{Hom}_{\mathcal{O}_D}(X, X') \otimes_{\mathcal{O}} K$ which is invertible in $\mathrm{Hom}_{\mathcal{O}_D}(X', X) \otimes_{\mathcal{O}} K$, i.e. for an quasi-isogeny α there is $n \in \mathbb{N}_0$ such that $\pi^n \alpha$ is an $\mathcal{O}_D$-idogeny. α is of height 0 if $h(\pi^n \alpha) = h(\pi^n)$.

Proposition 2.25. *Let (η, T, u) and (η', T', u') the admissible triples corresponding to X and X'. Then there is a canonical isomorphism*

$$\text{Quasi-isog}(X, X') \cong \mathrm{Isom}_K(\eta_0 \otimes_{\mathcal{O}} K, \eta_0' \otimes_{\mathcal{O}} K).$$

PROOF: Let M and M' be the respective corresponding CARTIER modules. We know already, that we have a canonical isomorphism

$$\text{Quasi-isog}(X, X') = \mathrm{Isom}((M \otimes_{\mathcal{W}} \mathcal{K}, V), (M' \otimes_{\mathcal{W}} \mathcal{K}, V)),$$

where the isomorphism of isocristal have to respect the gradation and the action of Π, meaning, that they are already determined by their action on the homogenous component of degree 0 or in other words

$$\mathrm{Isom}((M \otimes_{\mathcal{W}} \mathcal{K}, V), (M' \otimes_{\mathcal{W}} \mathcal{K}, V)) = \mathrm{Isom}((M_0 \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1}), (M_0' \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1})).$$

But $(M_0 \otimes_{\mathcal{W}} \mathcal{K}, V\Pi^{-1})$ is by Lemma (2.23) canonically isomorphic to $\eta_0 \otimes_{\mathcal{O}} \mathcal{K}, \sigma^{-1})$ and the assertion follows. □

Proposition 2.26. *Let* 0 *be a critical index for* X*. Then in the previous isomorphism the quasi-isogenies of height* 0 *correspond to isomorphisms* $r : \eta_0 \otimes_{\mathcal{O}} K \to \eta_0' \otimes_{\mathcal{O}} K$ *with*

$$\begin{aligned}[\eta_0' : r(\eta_0)] &= 0 \quad \textit{if } 0 \textit{ is critical for } X', \\ [\eta_1' : \Pi r(\eta_0)] &= 1 \quad \textit{if } 1 \textit{ is critical for } X';\end{aligned}$$

in other words $\bigwedge^2 \eta_i' = \pi^{-i} \bigwedge^2 \Pi^i r \eta_0$ *if* i *is critical.*

PROOF: A quasi-isogeny is of height 0 if and only if the corresponding isomorphism $\alpha : M \otimes_{\mathcal{W}} \mathcal{K} \to M' \otimes_{\mathcal{W}} \mathcal{K}$ is such that $[M' : \alpha(\pi^n M)] = [M : \pi^n M]$ for n such that $\alpha(\pi^n M) \subset M'$ or in other words $[M' : \alpha M] = 0$. Taking into account that $[M_1 : \Pi M_0] = [M_0 : \Pi M_1]$ and likewise for M', this is the same as to ask that $[M_0' : \alpha M_0] = 0$ or $[M_1' : \Pi\alpha M_0] = 1$.

Since 0 is critical for X by assumption, M_0 can be identified with $\eta_0 \otimes_{\mathcal{O}} \mathcal{W}$ as we have seen in the proof of Lemma (2.23). If 0 is critical for X', the similar statement holdes for M_0'. Thus in this case $[M_0' : \alpha M_0] = [\eta_0' : r\eta_0] = 0$. On the other hand, if 1 is critical for X' M_1' is identified with$\eta_1' \otimes_{\mathcal{O}} \mathcal{W}$ and $[M_1' : \Pi\alpha M_0] = [\eta_1' : \Pi r \eta_0] = 1$. □

Now consider an algebraic closure $\overline{k}$ of the residue field $k = \mathcal{O}/\pi\,\mathcal{O}$ and choose a formal special $\mathcal{O}_D$-modules Φ of height 4 over $\overline{k}$ such that 0 is critical for Φ, which we already know is unique up to isogeny) and fix a basis of the associated $\mathcal{O}$-module $\eta_{0,\Phi}$, that is an isomorphism $\mathcal{O}^2 \cong \eta_{0,\Phi}$.

Definition 2.27. A formal special $\mathcal{O}_D$-module X of height 4 over an extension field L of $\overline{k}$ is called rigidified if it disposes of a quasi-isogeny $\rho : \Phi_L \to X$.

An admissible triple (η, T, u) over L is rigidified if it disposes of an isomorphism $r : K^2 \to \eta_0 \otimes_{\mathcal{O}} K$ such that $[\eta_i : \Pi^i r\,\mathcal{O}^2] = i$ if i is critical for η.

Therefore, we have seen, that if (η, T, u) corresponds to X, a rigidification of X corresponds to a rigidification of (η, T, u), more precisely

Theorem 2.28. *Let* L *be an algebraically closed extension of* $\overline{k}$*. The map* $(X, \rho) \mapsto (\eta, T, u, r)$ *is a bijection between the set of isomorphism classes of rigidified formal special* $\mathcal{O}_D$*-modules of height* 4 *over* L *and the set of isomorphism classes of rigidified admissible triples over* L*.*

2.6 Filtration of $N(M)$ and η_M

In this section assume that B is a $\mathcal{O}'$-algebra such that $\pi B = 0$ and M a special graded $\mathcal{O}[\Pi]$-CARTIER module over B. Further assume that i is a critical index, i.e. $\Pi M_i \subset V M_i$. Denote $N(M)$ by N, φ_M by φ, η_M by η.

Consider the filtrations of N_i and N_{i-1} given by the $\mathcal{O}$-submodules

$$\begin{aligned} V^{2n} M_i = ((0, V^{2n} M_i)) \subset N_i \\ V^{2n-1} M_i = ((0, V^{2n-1} M_i)) \subset N_{i-1}\end{aligned}$$

for $n \in \mathbb{N}$. Denote $N_{i,n} = N_i / V^{2n} M_i$ and $N_{i-1,n} = N_{i-1}/V^{2n-1} M_i$. For $j \in \{0, 1\}$ Let $\varepsilon = 0$ if $j = i$ and $\varepsilon = 1$ if $j \equiv i - 1$, such that $N_{j,n} = N_j / V^{2n-\varepsilon} M_i$.

Lemma 2.29. *For all* $r > 0$*,* $\varphi(V^r M_i) \subset V^r M_i$*.*

PROOF: Let $m \in M_i$. Then

$$\varphi((0, V^r m)) = ((V^r m, 0)) = ((0, \Pi V^{r-1} m)),$$

since we mod out by $(Vm, -\Pi m)$. However, i is critical and thus $\Pi m \in \Pi M_i \subset V M_i$, so $\Pi V^{r-1} m \in V^r M_i$. □

So φ induces an $\mathcal{O}$-linear endomorphism of $N_{j,n}$, and we denote $\eta_{j,n} = \{z \in N_{j,n} \quad | \quad \varphi(z) = z\}$.

Lemma 2.30. *$N_j = \lim_{\leftarrow} N_{j,n}$ and $\eta_j = \lim_{\leftarrow} \eta_{j,n}$.*

PROOF: As V is injective, we have an exact sequence of $\mathcal{O}$-modules

$$0 \to M_{j-1} \xrightarrow{\alpha} M_j \oplus M_j \to N_j \to 0, \tag{6}$$

with $\alpha(m) = (Vm, -\Pi m)$. Since V is injective, $\alpha(M_{j-1}) \cap (0, V^{2n-\varepsilon} M_i) = 0$ for any n, which provides a projective system of exact sequences

$$0 \to M_{j-1} \to M_j \oplus M_j / V^{2n-\varepsilon} M_i \to N_{j,n} \to 0,$$

taking projective limits

$$0 \to M_{j-1} \to M_j \oplus \lim_{\leftarrow} M_j / V^{2n-\varepsilon} M_i \to \lim_{\leftarrow} N_{j,n} \to 0.$$

But M is by definition complete with respect to the V-adic topology and the assertion follows.
The corresponding assertion for η_j follows by taking the kernel of $1 - \varphi$. □

We will consider M_j, N_j, η_j, $N_{j,n}$, $\eta_{j,n}$ as functors on the category of B-algebras as follows: for a B-algebra B', set $M(B') = M \widehat{\otimes}_{E_{\mathcal{O}}(B)} E_{\mathcal{O}}(B')$, which is a graded special $\mathcal{O}[\Pi]$-CARTIER module over B' and $N_j(B')$ etc. are obtained from $M(B')$ as described, and since φ is compatible with base change, the morphisms are clear.

Lemma 2.31. *The functor $N_{j,n}$ is representable by an affine scheme of $\mathcal{O}$-modules over B, where the underlying scheme is the affine space of dimension $2n+1-\varepsilon$ over B.*

PROOF: Let γ_0, γ_1 be a homogenous V-basis of M then

$$M_{j,(0)}(B') = \{[a]\gamma_j \quad : \quad a \in B'\}$$

defines a subfunctor of M_j. The exact sequence (6) induces a natural bijective map

$$M_{j,(0)} \times M_j / V^{2n-\varepsilon} M_i \to N_{j,n}.$$

To prove surjectivity, let $m, m' \in M_j$, there is $m_0 \in M_{j,(0)}$ and $m_1 \in M_{j-1}$ such that $m = m_0 + V m_1$, therefore $((m, m')) = ((m_0, m' + \Pi m_1))$ in N_j. To prove injectivity, for m_0 and $\ell \in M_{j,(0)}$, $m', \ell' \in M_j$ such that $((m_0, m'))$ and $((\ell_0, \ell'))$ have the same image in $N_{j,n}$ there exists $m_1 \in M_{j-1}$ such that

$$\begin{aligned} m_0 + V m_1 &= \ell_0 \\ m' - \Pi m_1 &\equiv \ell' \bmod V^{2n-\varepsilon} M_i. \end{aligned}$$

From this follows that $m_1 = 0$ (since γ_i is a V-basis).
Hence we only have to see that the maps

$$B' \to M_{j,(0)}(B'), \quad a \mapsto [a]\gamma_j$$

and

$$B'^{2n-\varepsilon} \to M_j / V^{2n-\varepsilon} M_i(B'), \quad (a_k) \mapsto \sum_{0 \le k < 2n-\varepsilon} V^k [a_k] \gamma_{j-k}$$

are funcrorial and bijective. This provides a functorial bijection between $\mathbb{A}^{2n+1-\varepsilon}$ and $N_{j,n}$. □

Lemma 2.32. *The functor $\eta_{j,n}$ is representable by an affine étale scheme of $\mathcal{O}$-modules of finite presentation over B.*

PROOF: By definition, $\eta_{j,n}$ is the kernel of $1-\varphi$ which is the inverse image of the zero section of $N_{j,n}$ by $1-\varphi$ and by the revious discussion, this is isomorphic to the zero section of $\mathbb{A}^{2n+1-\varepsilon}$. Therefore, it is a closed immersion of finite presentation.

It remains to show, that $\eta_{j,n}$ is étale over B. Let $B' \to B''$ be a surjection of B-algebras, defined by an ideal of square zero. We have to see, that $\eta_{j,n}(B') \to \eta_{j,n}(B'')$ is bijective. We have the following commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & V^{2n-\varepsilon}M_i(B') \arrow[r] \arrow[d, "\alpha"] & N_j(B') \arrow[r] \arrow[d, "\beta"] & N_{j,n}(B') \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & V^{2n-\varepsilon}M_i(B'') \arrow[r] & N_j(B'') \arrow[r] & N_{j,n}(B'') \arrow[r] & 0, \end{tikzcd}$$

and since we know allready that α and β are surjective γ is surjective, and since φ is nilpotent on the kernel of β it is nilpotent on the kernel of γ. Since $\eta_{j,n}$ is the kernel of $1-\Phi$, this shows the desired bijection. $\square$

This means, that $\eta_{j,n}$ is a constructible sheaf for étale topology over B, and this is compatible with base change. Let $S = \operatorname{Spec}(B)$ and S_{i-1} the closed subset, where $i-1$ is critical.

Proposition 2.33. *1. $\eta_{i,n}$ is a smooth sheaf of free $\mathcal{O}/\pi^n$-modules of rank 2 over S.*

2. $\eta_{i-1,n}$ is constructible over S and $\Pi : \eta_{i-1,n} \to \eta_{i,n}$ is injective.

3. $\eta_{i-1,n}|_{S-S_{i-1}}$ is smooth and $\Pi|_{S-S_{i-1}}$ is an isomorphism.

4. $\eta_{i-1,n}|_{S_{i-1}}$ is smooth and $(\eta_{i,n}/\Pi\eta_{i-1,n})|_{S_{i-1}}$ is a smooth sheaf of $\mathcal{O}/\pi$-vectorspaces of rank 1.

PROOF: We have already seen constructibility. To show smoothness, it suffices to verify, that the fibers over the geometric points of S have the same cardinality. So wlog we can assume that $B = L$ is an algebraically closed field of characteristic p.

1. We have an isomorphism
$$N_{i,n} \cong M_i/VM_{i-1} \oplus M_i/V^{2n}M_i$$
such that φ is given by
$$(\overline{m}, \overline{m}'') \mapsto (\overline{m}''V^{-1}\Pi\overline{m}'')$$
so we get an isomorphism
$$\eta_{i,n} \cong (M_i/V^{2n}M_i)^{V^{-1}\Pi}.$$
Identifying $(M_i, V^{-1}\Pi)$ with $(\eta_i \otimes \mathcal{W}, \sigma)$, V^2 is identified with $\pi\sigma^{-2}$ and $V^{2n}M_i$ with $\pi^n\eta_i\otimes\mathcal{W}$. Thus
$$\eta_{i,n} \cong \eta_i/\pi^n\eta_i \cong (\mathcal{O}/\pi^n)^2.$$

2. The map $\Pi : N_{i-1} \to N_i$ is injective and $\Pi N_{i-1} \cap V^{2n}M_i = V^{2n}M_i = \Pi V^{2n-1}M_i$ so $\Pi : N_{i-1,n} \to N_{i,n}$ is injective. Consequently, the induced application $\Pi : \eta_{i-1,n} \to \eta_{i,n}$ is injective.

3. If $i-1$ is not critical, then we have an isomorphism
$$N_{i-1,n} \cong M_i/V^{2n}M_i$$
such that φ corresponds to $V^{-1}\Pi$ which induces an isomorphism
$$\eta_{i-1,n} \cong (M_i, V^{2n}M_i)^{V^{-1}\Pi} \cong \eta_{i,n}.$$

4. If $i-1$ is critical, then as in (1) we can use the fact that $\eta_{i-1,n} \cong (M_{i-1}/V^{2n-1}M_i)^{V^{-1}\Pi}$. The following diagrams correspond to each other
$$M_{i-1} \xrightarrow{\Pi} [V]M_i \xrightarrow{\Pi} [V]M_{i-1}$$
and
$$\eta_{i-1} \otimes_{\mathcal{O}} \mathcal{W} \xrightarrow{\pi} [\pi \circ \sigma^{-1}]\eta_i \otimes \mathcal{W} \xrightarrow{\pi} [\pi \circ \sigma^{-1}]\eta_{i-1} \otimes \mathcal{W}.$$
This means that the inclusions $\Pi V^{2n-1}M_i = V^{2n}M_i \subset \Pi M_{i-1} \subset M_i$ are σ-invariant coming from $\pi^n \eta_i \subset \Pi\eta_{i-1} \subset \eta_i$ by tensoring with $\mathcal{W}$. Thus
$$\eta_{i-1,n} \cong \Pi\eta_i/\pi^n\eta_i.$$
Moreover, $[\eta_i : \Pi\eta_{i-1}] = 1$, giving a (non-canonical) isomorphism
$$\eta_{i,n}/\Pi\eta_{i-1,n} \cong \mathcal{O}/\pi$$
.

□

Remark 2.34. The calculatioin show in particular that for $j = 0, 1$ and $m \geq n$, the canonical maps $\eta_{j,m} \otimes_{\mathcal{O}} \mathcal{O}/\pi^n \to \eta_{j,n}$ are isomorphisms. This means, that the projective system of $\eta_{j,n}$s defines a π-adic sheaf η_j. And we can rewrite

Proposition 2.35. *1. η_i is a π-adic smooth sheaf of free $\mathcal{O}$-modules of rank 2 over S.*

2. η_{i-1} is a π adic and constructible sheaf of free $\mathcal{O}$-modules of rank 2 over S and $\Pi : \eta_{i-1} \to \eta_i$ is injective.

3. $\eta_{i-1}|_{S-s_{i-1}}$ is smooth and $\Pi|_{S-S_{i-1}}$ is an isomorphism.

4. $\eta_{i-1}|_{S_{i-1}}$ is smooth and $(\eta_{i,n}/\Pi\eta_{i-1})|_{S_{i-1}}$ is a smooth sheaf of $\mathcal{O}/\pi$-vectorspaces of rank 1.

2.7 Rigidification

2.8 Drinfeld's Theorem

Let again $\overline{k}$ be an algebraic closure of k, Φ a formal special $\mathcal{O}_D$-module of height 4 over $\overline{k}$ such that wlog 0 is a critical index for Φ and we have an isomorphism $\mathcal{O}^2 \cong \eta_{\Phi,0}$. Let $\mathcal{O}^{nr}$ be the maximal unramified extension of $\mathcal{O}$ which is the strict henselisation with residue field $\overline{k}$.

Definition 2.36. Let $\overline{\mathfrak{Nilp}}$ be the category of $\mathcal{O}^{nr}$-algebras where the image of π is nilpotent. Define a functor $\overline{G}$ on this category in the following way: for $B \in \overline{\mathfrak{Nilp}}$ let $\overline{G}(B)$ be the set of isomorphism classes of pairs (X, ρ) such that

1. X is a formal special $\mathcal{O}_D$-module of hight 4 over B,
2. $\rho : \Phi_{B/\pi B} \to X_{B/\pi B}$ is a quasi-isogeny of hight zero.

It is convenient to take more general formal $\mathcal{O}_D$-modules: we only want $\mathfrak{Lie}(X)$ to be a projective B-module. Locally for the Zariski topology nothing changes. The main result of Drinfeld can be written as

Theorem 2.37. *The functor $\overline{G}$ is representable by the formal $\widehat{\mathcal{O}}^{nr}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}$.*

The original formulation of Drinfeld was slightly different.

Definition 2.38. Let $\mathfrak{Nilp}$ be the category of $\mathcal{O}$-algebras, where π is nilpotent. Define a functor G on this category by associating to B the pairs $(\psi, (X, \rho))$ where

1. $\psi : \overline{k} \to B/\pi B$ is a k-homomorphism,
2. (X, ρ) is the isomorphism class with
   (a) X is a special formal $\mathcal{O}_D$. module of height 4 over B,
   (b) $\rho : \psi_* \Phi \to X_{B/\pi B}$ is a quasi-isogeny of height zero

If $B \to B'$ is a morphisme in $\mathfrak{Nilp}$, we define a map $G(B) \to G(B')$ by sending ψ to its composition with $B/\pi B \to B'/\pi B'$, and sending the pair (X, ρ) to the pair $(X_{B'}, \rho_{B'/\pi B'})$ induced by extension of the scalars.

It is obvious that G can be derived from $\overline{G}$ by retricting the scalars from $\mathcal{O}^{nr}$ to $\mathcal{O}$. Thus the theorem of Drinfeld takes the form

Theorem 2.39. *The functor G is representable by the formal $\mathcal{O}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}^{nr}}$.*

The goal is to show, that this functor is isomorphic to another functor we encountered earlier.

Recall the definition of the functor F on $\mathfrak{Nilp}$ in (1.7). We have seen, that F is representable by the formal $\mathcal{O}$-scheme $\widehat{\Omega}$. Now denote by $\overline{H}$ the restriction of F to the category of $\overline{\mathfrak{Nilp}}$. It follows, that $\overline{H}$ is representable by $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}^{nr}}$.

For $B \in \overline{\mathfrak{Nilp}}$ we define a map $\overline{\xi}_B : \overline{G}(B) \to \overline{H}(B)$ by sending the pair (X, ρ) to that quadruple $(\eta_X, T_X, u_x, r_{(X,\rho)})$ where we have as discussed

1. $\eta_X = \eta_M$ as sheaf over $\operatorname{Spec} B$: if B' is a B-algebra and $M' = M_{B'}$ we have $\eta_X(B') = \eta_{M'}$;
2. $T_X = \mathfrak{Lie}(X) = M/VM$;
3. $u_X : \eta_X \to T_X$ is the sheaf homomorphism such that $u_X(B') = u_{M'} : \eta_{\mathbf{M}'} \hookrightarrow N(M') \to M'/VM' = (M/VM) \otimes_B B'$;
4. $r_{(X,\rho)} : \underline{K}^2 \xrightarrow{\sim} \eta_{X,0}$ is the isomorphism associated to the rigidification ρ of X.

Our earlier discussion showed that the quadruple $(\eta_X, T_X, u_x, r_{(X,\rho)})$ satisfies the conditions of the definition in (1.7) and so this mao is well defined. Furthermore, we have the following functoriality: if $B \to B'$ is a morphism of the category $\overline{\mathfrak{Nilp}}$ then the diagram

$$\begin{tikzcd}
\overline{G}(B) \arrow[r, "\overline{\xi}_B"] \arrow[d] & \overline{H}(B) \arrow[d] \\
\overline{G}(B') \arrow[r, "\overline{\xi}'_B"'] & \overline{H}(B')
\end{tikzcd}$$

is commutative. This is follows, because η and r commute by construction with base change. So indeed, we have constructed a natural transformation $\overline{\xi} : \overline{G} \to \overline{H}$. In particular this means, that it suffices to show that this natural transformation is an isomorphism in order to show Theorem 2.37.

In order to derive from here the second formulation of Drinfeld's theorem, let H be the functor on $\mathfrak{Nilp}$ deduced from $\overline{H}$ by restricting the scalars from $\mathcal{O}^{nr}$ to $\mathcal{O}$. This means, for $B \in \mathfrak{Nilp}$, an element of $H(B)$ is the data of a k-homomorphism $\psi : \overline{k} \to B/\pi B$ and of an element of $\overline{H}(B_\psi) = F(B)$. By what we have seen, it is clear that H is representable by the formal $\mathcal{O}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}$.

By restriction of scalars, the natural transformation defined above induces a natural transformation $\xi : G \to H, (\psi, a) \mapsto (\psi, \overline{\xi}_{B_\xi}(a))$, where $a \in \overline{G}(B_\psi)$. Hence, if we proof the above statement it follows also the following

Theorem 2.40. *The natural transformation $\xi : G \to H$ is an isomorphism of functors.*

Remark 2.41. Note that the map $\overline{\xi}_{B_\psi} : \overline{G}(B_\psi) \to \overline{H}(B_\psi) = F(B)$ depends on the $\mathcal{O}^{nr}$-algebra structure of B_ψ, in particular the $\mathbb{Z}/2\mathbb{Z}$ graduation of η and T associated to X depend on its $\mathcal{O}'$-algebra structure.

2.9 Group actions of $\mathbf{GL}_2(K)$ and D^*

2.10 Deformationtheory

This and the following two sections are meant to prove the Theorem 2.40. However, as we have seen, it is sufficient to work in the category $\overline{\mathfrak{Nilp}}$ of $\mathcal{O}^{nr}$-algebras, where π is nilpotent. We call infinitesimal extension a surjective homomorphism $B' \to B$ with nilpotent kernel.

Proposition 2.42. *Let $B' \to B$ be an infinitesimal extension, $B'_0 = B'/\pi B'$ and $B_0 = B/\pi B$. Let X' be a special formal $\mathcal{O}_D$-module of height 4 over B' and $X = X'_B$. Suppose there is a rigidification $\rho : \Phi_{B_0} \to X_{B_0}$ of X. Then*

1. *X' is π-divisible (i.e. $\pi : X' \to X'$ is an isogeny);*
2. *ρ can be lifted uniquely to a rigidification $\rho' : \Phi_{B'_0} \to X_{B'_0}$ of X'.*

PROOF:

1. Since ρ is a quasi-isogenie, there is $n \in \mathbb{N}$ such that $\alpha = \pi^n \rho$ is an isogeny. We know, that Φ_{B_0} is π-divisible, meaning that $\pi : \Phi_{B_0} \to \Phi_{B_0}$ is an isogeny. Thus, $\alpha \circ \pi \Phi_{B_0} \to X_{B_0}$ is also an isogeny. But ρ is π-equivariant, so $\alpha \circ \pi = \pi \circ \alpha$, which implies, that $\pi : X_{B_0} \to X_{B_0}$ is alos an isogeny. BY deformation theory, this result can be extended to $\pi : X' \to X'$.

2. By hypothesis $B' \to B$ is infinitesimal, meaning that $I = \mathrm{Ker}(B' \to B)$ is nilpotent. Wlog we can assume that already $I^2 = 0$, and then do it by induction (have a property for B/I want to show the property for B/I^n, show it first for B/I^2, then keep going). Further we can assume that $\pi I = 0$ because π and I are both nilpotent. We look at the associated CARTIER modules to lift ρ. The associated isomorphism of sheaves r, can be lifted by lifting the structure constants. However, it has to commute with Π. Although $\pi^n \rho = \alpha$ is an isogeny it might not a priori commute with Π, but this can be fixed by multiplying it with π, as this can be lifted such that it respects the structure of the modules induced by Π. So we obtain an isogeny $\beta' : \Phi_{B'_0} \to X_{B'_0}$ and set $\rho' = \beta'/\pi^{n+1}$.

   The *Lemma on Strictness for p-Divisible Groups* implies uniqueness of the lift, as it emphasises a certain rigidity.

   Lemma 2.43. *Let B be a ring, $I \subset B$ an ideal with $I^2 = 0$ and $pI = 0$, and let Γ_1 and Γ_2 be p-divisible groups over B. If the homomorphism $\varphi : \Gamma_1 \to \Gamma_2$ induce the zero homomorphism by the module I, then it is already the zero momomorphism.*

   PROOF: We have to verify that $p\phi = 0$. Then, since p is surjective on Γ_1, ϕ must be 0. □
   Now assume that the lift of ρ is not unique, then take the difference and apply the lemma. □

This means in particular, that we don't have to worry about rigidifications when we study deformation theory. Moreover, we don't have infinitesimal automorphism.

Proposition 2.44. *Let $B' \to B$ and $B'' \to B$ be two infinitesimal extensions. Then the canonical map*

$$\overline{G}(B' \times_B B'') \to \overline{G}(B') \times_{\overline{G}(B)} \overline{G}(B'')$$

is bijective.

PROOF: Mutatis mutandis, this is the proof in the usual case, of p-divisible groups. Let X', X'' be deformations over B', B'' respectively of X over B. Then there exists a unique deformation $\widetilde{X}$ of X' and X'' simultanously over $B'\times_B B''$. Its graded $\mathcal{O}[\Pi]$-CARTIER module is $M_{\widetilde{X}} = M_{X'}\times_{M_X} M_{X''}$. By definition of $\overline{G}$ this provides all the required data. □

Let $x \in \overline{G}(B)$, then x is a class represented by some (X, ρ). And let $C \to B$ be infinitesimal. Denote by $\overline{G}_x(C)$ the inverse image of x in $\overline{G}(C)$ and $\overline{H}_x(C)$ the inverse image of $\bar{\xi}(x)$ in $\overline{H}(C)$.

Corollary 2.45. *Let $B' \to B$ be an infinitesimal extension where the kernel I satisfies $I^2 = 0$ and let $B[I]$ be the B-algebra $B \oplus I$. Then*

1. *$\overline{G}_x(B[I])$ is an abelian group;*
2. *if $\overline{G}_x(B')$ is non-empty, then it's a homogenous principal $\overline{G}_x(B[I])$-set.*

These structures are canonical.

PROOF: This follows from classical results. The group structure on $\overline{G}_x(B[I])$ is by the previous result given by

$$B[I] \times_B B[I] \to B[I]\,, \quad (b+i) \times_b (b+j) \mapsto b+i+j.$$

The group action is induced by the isomorphism

$$B' \times_B B' \xrightarrow{\sim} B' \times_B B[I]\,, \quad a \times_b c \mapsto (a, b+(c-a))$$

which gives a bijection

$$\overline{G}_x(B') \times \overline{G}_x(B') \xrightarrow{\sim} \overline{G}_x(B') \times \overline{G}_x(B[I]),$$

and this is trivially free and transitiv. □

Since the functor $\overline{H}$ is representable, it commutes with fibre productes, and the corollary also holds for $\overline{H}$. What is more, the natural transformation ξ between $\overline{H}$ and $\overline{G}$ is compatible with this, since all the structurs are canonical.

Proposition 2.46. *If $\overline{H}_x(B') \neq \emptyset$, then $\overline{G}_x(B') \neq \emptyset$.*

PROOF: Let (X, ρ) representing $x \in \overline{G}(B)$, M the associated CARITER module over B. Taking the preimage of x means lifting X and ρ over B'. The CARTIER module is defined by the srtucture constants, so we only have to lift them. Note that they have to verify $a'_{0,0} \cdot a'_{0,1} = \pi$.

$\bar{\xi}$ associates to (X, ρ) among other things $T = M/VM$ with Π. The image of the homogenous bases of M in T is still homogenous and satisfies $\Pi\overline{\gamma}_i = a_{0,i}\overline{\gamma}_{i+1}$ If $\overline{H}_x(B')$ is non-empty, there exists a pair (T', Π) over B' lifting (T, Π). Thus there exist $a'_{0,i}$ lifting $a_{0,i}$ such that the product is π. That the lift of th other constants exist is trivial. □

The principal result of this section is:

Proposition 2.47. *That $\bar{\xi} : \overline{G} \to \overline{H}$ is an isomorphism it suffices that is is one restricted to $\bar{k}$-algebras.*

PROOF: This follows by induction. Let $\overline{\mathfrak{Nilp}}_n$ be the (full) subcategory of $\mathcal{O}^{nr}$-algebras such that the image of π^n is zero. Then $\overline{\mathfrak{Nilp}}_1$ is the category of $\bar{k}$-algebras. If $\bar{\xi}_{|\overline{\mathfrak{Nilp}}_n}$ be an isomorphism. Let $B' \in \overline{\mathfrak{Nilp}}_{n+1}$, $B = B'/\pi^n B'$, $I = \pi^n B'$. Then B and $B[I]$ are in $\overline{\mathfrak{Nilp}}_n$ and the corresponding ξ's are bijective. But $\overline{G}$ and $\overline{H}$ commute with fiber products, so $\xi_{B'}$ is bijective. □

2.11 Tangent Spaces

2.12 Completing the Proof

Our goal is to show that $G : \mathfrak{Nilp} \to \mathfrak{Sets}\,,\ B \mapsto (X,\rho)$ is representable by the formal $\mathcal{O}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}$. To this point we have the following results:

1. We know that $H : \mathfrak{Nilp} \to \mathfrak{Sets}$ and $\overline{H} : \overline{\mathfrak{Nilp}} \to \mathfrak{Sets}$ induced by Drinfeld's functor F is representable by the formal $\mathcal{O}$-scheme (resp. $\mathcal{O}^{nr}$-scheme) $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}$.
2. We defined natural transformations $\overline{\xi} : \overline{G} \to \overline{H}$ and $\xi : G \to H$.
3. We saw that ξ is an isomorphism if $\overline{\xi}$ is an isomorphism.
4. It is sufficient to show that $\overline{\xi}$ is an isomorphism restricted to the category of $\overline{k}$-algebras (i.e. $\pi = 0$).
5. $\overline{\xi}(\overline{k} : \overline{G}(\overline{k}) \to \overline{H}(\overline{k})$ is an isomorphism.
6. The tangent maps $t_{\overline{\xi}}(x) : t_{\overline{G}}(x) \to t_{\overline{H}}(\overline{\xi}(x))$ in the geometric points $x \in \overline{G}(\overline{k})$ is bijective.

Now the idea is to show that for $\overline{k}$-algebras $\overline{G}$ is the union of functors which are representable by projective $\overline{k}$-schemes. Using information about the tangent maps, we conclude that $\overline{\xi}$ restricted to these subfunctors is étale in the neighbourhood of the preimage ofa geometric point. But since these are open immersions, we can find neighbourhoods of the geometric points of $\overline{H}$ over which $\overline{\xi}$ is an isomorphism.

Definition 2.48. For $n, m \in \mathbb{N}_0$, we define a subfunctor $G_{nm}\overline{k}\text{-}\mathfrak{Alg} \to \mathfrak{Sets}$ of $\overline{G}$ associating to a $\overline{k}$-algebra B the isomorphism classes of paires (X,ρ) where

1. $\pi^n\rho : \Phi_B \to X$ is an isogeny.
2. $\mathrm{Ker}(\pi^n\rho) \subset \Phi_B(\pi^{n+m})$, where $\Phi_B(\pi^{n+m})$ is the kernel of π^{n+m} in Φ_B. This is equivalent to say that there exist an isogeny $\beta : X \to \Phi_B$ such that $\beta\pi^n\rho = \pi^{n+m}$.

With this it is clear, that G_{nm} is a subfunctor of $G_{n'm'}$ if $n \le n'$ and $m \le m'$.

Proposition 2.49. *The functors G_{nm} are representable by a projective $\overline{k}$-scheme.*

Proof: Denote by $\mathcal{A}$ the algebra ofthe group scheme $\Phi(\pi^{n+m})$ over $\overline{k}$. Instead of giving a pair (X,ρ) over a $\overline{k}$-algebra B we could give the kernel Z of $\pi^n\rho$. The associated algebra to this group scheme is a locally free B-algebra of rank q^{4n}, in fact it's a quotient of $\mathcal{A}_B$. So G_{nm} parametrises such algebras and consequently it is a subfunctor of the Hilbert scheme $\mathrm{Hilb}(\mathcal{A}, q^{4n})$ which is a projective $\overline{k}$-scheme.

Moreover, if we show that the inclusion of $G_{nm} \subset \mathrm{Hilb}(\mathcal{A}, q^{4n})$ is represented by a closed immersion, we know, that G_{nm} is representable as well. Let Z be as defined above representing a point in $G_{nm}(B)$. The condition on Z is that it is a closed subscheme in $\Phi_B(\pi^{n+m})$. Thus it has to be closed under multiplication, it has to contain a neutral element and inverses and be stable under the $\mathcal{O}_D$-action. These conditions define a closed subscheme of $\mathrm{Hilb}(\mathcal{A}_B, q^{4n}$. We verify this contravariantly on the associated algebras using the universal object $\mathcal{B}$ of $\mathrm{Hilb}(\mathcal{A}, q^{4n})$.

Let $\mathfrak{I}_H$ be the kernel of the quotient map $\mathcal{A} \to \mathcal{B}$. Since $\mathcal{A}$ is closed under mulitplication there is a homomorphism $\mu^* : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}_H} \mathcal{A}$ and we want to deduce a similar homomorphism for $\mathcal{B}$. In other words, we need to find a homomorphism, such that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A} \otimes \mathcal{A} \arrow[d] \\ \mathcal{B} \arrow[r, dashed] & \mathcal{B} \otimes \mathcal{B} \end{tikzcd}$$

commutes. But this is the case if the homomorphism $\mathfrak{J}_H \subset \mathcal{A} \to \mathcal{A}\otimes\mathcal{A} \twoheadrightarrow \mathcal{B}\otimes\mathcal{B}$ is the zeromorphism. Since the involved $\mathcal{O}_H$-modules are locally free, this condition gives rise to euqations in $\mathcal{O}_H$ which defines a closed subscheme $H_1 \subset \mathrm{Hilb}(\mathcal{A}, q^{4n})$.

Similarly, since $\mathcal{A}$ has a unit element, we want to carry the homomorphism $\varepsilon^* : \mathcal{A} \to \mathcal{O}_H$ on to $\mathcal{B}$, therefore, the diagram

$$\begin{tikzcd}
\mathfrak{J}_H \arrow[r, hook] & \mathcal{A} \arrow[r] \arrow[d, two heads] & \mathcal{O}_H \arrow[d, two heads] \\
& \mathcal{B} \arrow[r, dashed] & \mathcal{O}_H
\end{tikzcd}$$

has to commute. And as before, this is the case, when the induced homomorphism on $\mathfrak{J}_H$ is zero. This gives a second closed subscheme H_2.

For inverses, we have the diagram

$$\begin{tikzcd}
\mathfrak{J}_H \arrow[r, hook] & \mathcal{A} \arrow[r] \arrow[d, two heads] & \mathcal{A} \arrow[d, two heads] \\
& \mathcal{B} \arrow[r, dashed] & \mathcal{B}
\end{tikzcd}$$

which has to commute coming from the homomorphism $\mathrm{inv}^* : \mathcal{A} \to \mathcal{A}$. As before this gives us a third closed subscheme H_3.

Stability under the action of $\mathcal{O}_D$ is a little more complicated. Of course, the diagram

$$\begin{tikzcd}
\mathfrak{J}_H \arrow[r, hook] & \mathcal{A} \arrow[r] \arrow[d, two heads] & \mathcal{O}_D \otimes \mathcal{A} \arrow[d, two heads] \\
& \mathcal{B} \arrow[r, dashed] & \mathcal{O}_D \otimes \mathcal{B}
\end{tikzcd}$$

induces a closed subscheme as before, but we have to check that this is compatible with the group structure. Let's check this for multiplication. We want the following diagram to commute:

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[dd, two heads] & \mathcal{O}_D \otimes \mathcal{A} \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \mathcal{O}_D \otimes (\mathcal{A} \otimes \mathcal{A}) \arrow[dd, two heads] \\
& & (\mathcal{O}_D \otimes \mathcal{A}) \otimes (\mathcal{O}_D \otimes \mathcal{A}) \arrow[ur] \arrow[dd, two heads] & \\
\mathcal{B} \arrow[r] & \mathcal{O}_D \otimes \mathcal{B} \arrow[rr] \arrow[dr] & & \mathcal{O}_D \otimes (\mathcal{B} \otimes \mathcal{B}) \\
& & (\mathcal{O}_D \otimes \mathcal{B}) \otimes (\mathcal{O}_D \otimes \mathcal{B}) \arrow[ur] &
\end{tikzcd}$$

But this is clear, since all of the squares commute. This provides the fourth closed subscheme H_4.

Finally taking the intersection H_0 of these subschemes give the desired one. Now the claim follows by base change. □

Lemma 2.50. *For any $\overline{k}$-algebra B $\overline{G}(B) = \bigcup_{n,m} G_{nm}(B)$.*

PROOF: The inclusion "$\supseteq$" is evident. So let (X, ρ) represent an element in $\overline{G}(B)$. Since ρ is a quasi-isogeny there is $n \in \mathbb{N}_0$ such that $\pi^n\rho$ is an isogeny. For this isogeny there is an integer m and an isogeny $\beta : X \to \Phi_B$ such that $\beta\pi^n\rho = \pi^{n+m} : X \to \Phi_B \to \Phi_B \to X$, which is the second condition of the definition of G_{nm}. □

Lemma 2.51. *For $x \in G_{nm}(\overline{k})$ the induced tangent map $t_{G_{n'm'}}(x) \to t_{\overline{G}}(x)$ is bijective whenever $n' > n$ and $m' > m$.*

PROOF: Since $G_{n'm'}$ is a subfunctor of $\overline{G}$, it is clear that the tangent map is injective. So we only need to show surjectivity.

Recall that $t_{\overline{G}}(x) = \{x' \in \overline{G}(\overline{k}[\varepsilon]) \quad \text{over} \quad x\}$ where $\varepsilon^2 = 0$. Let (X, ρ) represent x and (X', ρ') be a deformation over $\overline{k}[\varepsilon]$, which would represent an element of $t_{\overline{G}}(x)$. By choice of x $\pi^n \rho : \Phi_{\overline{k}} \to X$ is an isogeny. Hence we can lift $\pi^{n+1}\rho$ to an isogeny $\Phi_{\overline{k}[\varepsilon]} \to X'$ and by rigidity of π-divisible groups this must be $\pi^{n+1}\rho'$.

By choice of x, $\mathrm{Ker}(\pi^n \rho) \subset \Phi_{\overline{k}}(\pi^{n+m})$, or by the variant, there is an isogeny $\beta : X \to \Phi_{\overline{k}}$ such that $\beta \pi^n \rho = \pi^{n+m}$. Then $\pi\beta$ can be lifted to $\beta' : X' \to \Phi_{\overline{k}[\varepsilon]}$ such that $\beta' \pi^{n+1} \rho' = \pi^{(n+1)+m+1}$, the additional π's coming from β' and $\pi^{n+1}\rho'$ respectively. Thus the chosen deformation (X', ρ') of (X, ρ) represents indeed an element of $G_{n'm'}(\overline{k}[\varepsilon])$ for some $n' \geq n+1$ and $m' \geq m+1$. □

Now we have everything in hand to conclude. Let ξ_{nm} be the morphism $G_{nm} \hookrightarrow \overline{G} \xrightarrow{\overline{\xi}} \overline{H}$. This morphism is of finite type since G_{nm} is of finite type over $\overline{k}$.

Since $\overline{H}(\overline{k})$ and $\overline{G}(\overline{k})$ are bijective, for $y \in \overline{H}(\overline{k})$ there is a unique $x \in \overline{G}(\overline{k})$ such that $y = \overline{\xi}(x)$ and we have seen in the last but one lemma that there is $n_y, m_y \in \mathbb{N}_0$ such that $x \in G_{n_y m_y}(\overline{k})$. For $n = n_y + 1$ and $m = m_y + 1$ the tangent map at x of ξ_{nm} is bijective. This means, that there is a neighbourhood of x where ξ_{nm} is étale. What is more, as ξ_{nm} is injective on the geometric points, it is an open imersion on this neighbourhood of x. But this is no more as to say, there exist a neighbourhood $\mathcal{V}_y$ of y in $\overline{H}$, where ξ_{nm} is an isomorphism.

This is alos true for $n' > n$ and $m' > m$ as we can restrict $\xi_{n'm'}$ to $\mathcal{V}_y$ and get an isomorphism. This shows the compatibility $G_{mn | \mathcal{V}_y} = G_{m'n' | \mathcal{V}_y}$ and by the last bat one lemma $\overline{G}_{| \mathcal{V}_y} = G_{nm | \mathcal{V}_y}$ as well. Consequently, the morphisms $\overline{\xi}$ and ξ_{nm} coincides over $\mathcal{V}_y$ which is an isomorphism.

Arguing similar for every point in $\overline{H}$ shos that $\overline{\xi}$ is an isomorphism.

2.13 Construction of a system of coverings

We proved that the functor G is representable by the formal scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}$ which implies that it has a universal object X, a formal $\mathcal{O}_D$-module, which is π-divisible. Hence π^n is an isogeny for all $n \in \mathbb{N}$ and it is well known that in this case the kernel X_n of π^n is representable by a finite flat formal group scheme of rank q^{4n} over $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}$. Since $\mathcal{O}_D$ acts on X, we have naturally an action of $\mathcal{O}_D / \pi^n \mathcal{O}_D$ on X_n. Furthermore, the module of relative KÄHLER differentials is annihilated by π^n. Indeed, note that the by π^n induced tangent map is again multiplication by π^n (i.e. it acts via the structure map on the tangent space at zero $\mathfrak{Lie}(X)$,), consequently it also acts as multiplication by π^n on the KÄHLER differentials. Since π^n acts as 0 on X_n, we have a commutative diagram

$$\begin{tikzcd}
X_n \arrow[rr, "\pi^n"] \arrow[dr] & & X_n \\
& \widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr} \arrow[ur] &
\end{tikzcd}$$

which induces a commutative diagram on the modules of differentials

$$\begin{tikzcd}
\Omega^1_{X_n/\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}} & & \Omega^1_{X_n/\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}} \arrow[ll, "\pi^n"'] \arrow[dl] \\
& \Omega^1_{\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}/\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}} \arrow[ul] &
\end{tikzcd} .$$

But for any scheme $\Omega^1_{Y/Y} = 0$, so $\pi^n = 0$ on $\Omega_{X_n/\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}}$.

Let $\mathcal{X}_n$ be the generic fiber of X_n seen as the associated rigid space. By scalar extension we know from the previous paragraph that it is flat over $\Omega \otimes_K \widehat{K}^{nr}$ and unramified since the Kähler differentials vanish by multiplication with π^n. Thus it is a finite étale covering fibered in $\mathcal{O}_D / \pi^n \mathcal{O}_D$-modules. $\mathcal{X}_{n-1}$ is the subspace of $\mathcal{X}_n$ of elements annihilated by π^{n-1}. The intermediate spaces annihilated by Π^{2n-1} are denoted by $\mathcal{X}_{n-\frac{1}{2}}$.

It is known that the cardinality of the fibers of $\mathcal{X}_n$ is q^{4n}, which is exactly the cardinality of $\mathcal{O}_D / \pi^n \mathcal{O}_D$, so that it is immediate that the fibers are free modules of rank 1. Basis elements are the elements exactly killed by π^n, $\Sigma_n := \mathcal{X}_n \backslash \mathcal{X}_{n-\frac{1}{2}}$. Similarly to $\mathcal{X}_n$, Σ_n corresponding to the invertible elements in $\mathcal{O}_D / \pi^n \mathcal{O}_D$, is also an étale covering of $\Omega \otimes_K \widehat{K}^{nr}$. What is more, since $(\mathcal{O}_D / \pi^n \mathcal{O}_D)^*$ acts free transitive and discontinuous on the fibers, this is even an étale covering with Galois group $(\mathcal{O}_D / \pi^n \mathcal{O}_D)^*$. Via π, the Σ_n form a projective system as do the X_n, $\mathcal{X}_n$, with Galois group the profinite completion $\widehat{\mathcal{O}}_D^*$ of $\mathcal{O}_D^*$.

Finally note, that the constructed covering spaces are equivariant with respect to the action of $\mathbf{GL}_2(K)$, which we defined earlier, since this action only effects the rigidifications.

3 The theorem of Čerednik-Drinfeld

3.1 A moduli problem over $\mathbb{C}$

3.2 Tate-Honda-theory

3.3 The moduli problme over $\mathbb{Z}_p$

3.4 Polarisations

3.5 The theorem of Čerednik-Drinfeld

Let $U \subset \Delta^*(\mathbb{A}_f)$ be an open compact subgroup of the form $U_p^0 U^p$ where U_p^0 is the unique maximal compact subgroup of $\Delta^*(\mathbb{Q}_p)$ and $U^p \subset \Delta^*(\mathbb{A}_f^p)$ any compact open subgroup. Let $\overline{\Delta}$ be the quaternion algebra deduced from Δ be changing the invariants at p and at ∞ such that $\overline{Delta}$ is defined and unramified at p and $\overline{\Delta}_\ell \cong \Delta_\ell$ for $\ell \neq p, \infty$. We fix an isomorphism of groups

$$\Delta^*(\mathbb{A}_f^p) = (\Delta \otimes \mathbb{A}_f^p)^* \text{ and } \overline{\delta}^*(\mathbb{A}_f^p) = (\overline{\Delta} \otimes \mathbb{A}_f^p)^*$$

induced by an anti-isomorphism of the algebras $\Delta \otimes \mathbb{A}_f^p$ and $\overline{\Delta} \otimes \mathbb{A}_f^p$. Hence, U^p can be seen as a subgroup of $\overline{\delta}^*(\mathbb{A}_f^p)$. We also fix an isomorphism

$$\overline{\delta}^*(\mathbb{Q}_p) \equiv \mathbf{GL}_2(\mathbb{Q}_p)$$

coming from an isomorphism $\overline{\Delta} \otimes \mathbb{Q}_p \cong \mathbf{M}_2(\mathbb{Q}_p)$ Consider the following double cosets

$$Z_U = U^p \backslash \overline{Delta}^*(\mathbb{A}_f) / \overline{\Delta}^*(\mathbb{Q}).$$

The group $\overline{\Delta}^*(\mathbb{Q}_p)$ acts on it from left and the quotient is finite. Indeed, each orbit contains the double class of an element whose p-component is trivial. Thus the stabiliser of x is given by

$$\Gamma_x = \overline{\Delta}^*(\mathbb{Q}) \cap x^{-1} U^p x,$$

where the intersection is in $\overline{\Delta}^*(\mathbb{A}_f^p)$. These are discrete and co-compact subgroups of $\overline{\Delta}^*(\mathbb{Q}_p) \cong \mathbf{GL}_2(\mathbb{Q}_p)$, containing a positive power of the matrix $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$. Taking U^p smaller and smaller, we obtain a projective systems of the Z_U with an action of $\overline{\Delta}^*(\mathbb{A}_f^p)$.

Theorem 3.1. *For each small enough compact open subgroup $U^p \subset \Delta^*(\mathbb{A}_f^p)$ there is an isomorphism of formal $\mathbb{Z}_p$-schemes*

$$\widehat{S}_U \cong \mathbf{GL}_2(\mathbb{Q}_p) \backslash [(\widehat{\Omega} \widehat{\otimes} \widehat{\mathbb{Z}}_p^{nr}) \times Z_U]$$

where $\overline{S}_U$ is the formal completion of S_U along the special fiber.

Addendum 3.2. *The isomorphisms are, as U^p varies, compatible with the projection operations. The isomorphism of projective systems is compatible with the action over the two members of the group $\Delta^*(\mathbb{A}_f^p) \cong \overline{\Delta}^*(\mathbb{A}_f^p)$. Thus one can lift these isomorphisms to isomorphisms between the formal special $\mathcal{O}_{\Delta_p}$-modules naturally supported by the two members.*

3.6 Proof of the Theorem

In this section we will proof the theorem of Čerednik-Drinfeld.we will do this in several steps.

1. Give a description of Z_U in terms of algebraisaitons $\mathfrak{Alg}_U(\Phi)$.

2. Construct a morphism of special fibers $\theta : [(\widehat{\Omega}\widehat{\otimes}\widehat{\mathbb{Z}}_p^{nr}) \otimes \mathbb{F}_p] \times Z_u = (\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U \to S_U \otimes \mathbb{F}_p$.

3. Show that it factorises over the quotient $\mathbf{GL}_2(\mathbb{Q}_p)\backslash[(\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U]$.

4. Show that $\overline{\theta}$ is an isomorphism between the special fibers.

5. Extend $\overline{\theta}$ to the formal schemes.

Fix an abelian variety A_0 over $\overline{\mathbb{F}}_p$ of dimension 2 with a "special" action of the ring $\mathcal{O}_\Delta$. It is clear that such a variety exists: take the product of two supersingular elliptic curves and define the action of $u \in \mathcal{O}_{\Delta_p} \cong \mathrm{End}(\mathcal{E})$ by u on the first factor and by tut^{-1} on the second factor, where $t \in \Delta^*$ is a uniformiser of Δ_p (of negative square in $\mathbb{Q}$). Let $\Phi := \widehat{A}_0$ be the associated formal group, which is consequently a formal special $\mathcal{O}_{\Delta_p}$-module. By HONDA-TATE theory we can choose an identification of the quaternion algebra $\overline{\Delta}$ with the algebra $\mathrm{End}_{\mathcal{O}_\Delta}(A) \otimes \mathbb{Q} = \mathrm{End}^0_\Delta(A_0)$, what is more $\overline{\Delta}^*(\mathbb{Q}) = \overline{\Delta}^* = \mathrm{Aut}^0_\Delta(A_0)$.

This induces identifications

$$\overline{\Delta}_p = \overline{\Delta} \otimes \mathbb{Q}_p = \mathbf{M}_2(\mathbb{Q}_p) = \mathrm{End}^0_{\Delta_p}(\Phi),$$

and thus $\overline{\Delta}^*(\mathbb{Q}_p) = \overline{\Delta}_p^* = \mathbf{GL}_2(\mathbb{Q}_p) = \mathrm{Aut}^0_{\Delta_p}(\Phi)$. Finally fix isomorphisms for $\ell \neq p$

$$\nu_{0,\ell} : V_\ell(A_0) \to V_\ell$$

which are compatible with the fixed isomorphism

$$\Delta \otimes \mathbb{A}_f^p \cong (\overline{\Delta} \otimes \mathbb{A}_f^p)^{opp}.$$

Remark 3.3. For an abelian variety A of dimension 2 with an action of $\mathcal{O}_\Delta$, $V_\ell(A)$ is as Δ_ℓ module isomorphic to V_ℓ with an Δ_ℓ-linear action of $\mathrm{End}^0_\Delta(A)$. $\mathrm{End}_{\Delta_\ell}(V_\ell)$ is naturally identified with the opposite algebra Δ_ℓ^{opp} acting by rightmulitplication. Hence, the choice of $V_\ell(A) \cong V_\ell$ and $\overline{\Delta} \cong \mathrm{Aut}^0_\Delta(A)$ determines an isomorphism $\overline{\Delta}_\ell \cong \Delta_\ell^{opp}$ and therefore an isomorphism

$$\overline{\Delta} \otimes \mathbb{A}_f^p \cong \Delta^{opp} \otimes \mathbb{A}_f^p\,.$$

Via $\nu_{0,\ell}$ the action of $\overline{\Delta} = \mathrm{End}^0_\Delta(A_0)$ on V_ℓ is given by the composition

$$\overline{\Delta} \hookrightarrow \overline{\Delta}_\ell \cong \Delta_\ell^{opp} \to \mathrm{End}_{\Delta_\ell}(V_\ell)$$

by rightmultiplication.

In the following, let S be a $\mathbb{Z}_p$-scheme, where the image of p is nilpotent and X a formal special $\mathcal{O}_{\Delta_p}$-module over S.

Definition 3.4. An algebraisation of X is the data of a pair (A, ε) where

1. A is an abelian scheme over S with an action of $\mathcal{O}_\Delta$

2. $\varepsilon : \widehat{A} \to X$ is an $\mathcal{O}_\Delta$-equivariant isomorphism.

If A has a niveau structur U, we say the same of the algebrisation.

Further let $\mathfrak{Alg}_U(\Phi)$ be the set of isomorphism classes of algebraisations of U with niveau structure U. A representant of such a class is a triple $(A, \varepsilon, \overline{\nu})$, where

1. A is an abelian $\mathcal{O}_\Delta$-variety over $\overline{\mathbb{F}}_p$,

2. $\varepsilon : \widehat{A} \to \Phi$ is an equivariant isomorphism,

3. $\overline{\nu}$ is a class of $\mathcal{O}_\Delta$-isomorphism modulo U^p
$$\nu : \prod_{\ell \neq p} T_\ell(A) \to \prod_{\ell \neq p} W_\ell,$$
where $W = \mathcal{O}_\Delta$ as left $\mathcal{O}_\Delta$-module.

One can see, that it is the same to consider triples $(A, \varepsilon, \overline{\nu})$, where

1. A is an abelian variety with an action of Δ up to isogeny,
2. $\varepsilon : \widehat{A} \to \Phi$ is an equivariant quasi-isogeny,
3. $\overline{\nu}$ is a class of Δ-isomorphisms mod U^p
$$\nu : \prod_{\ell \neq p} V_\ell(A) \to \prod_{\ell \neq p} V_\ell.$$

Taking the U^p smaller and smaller, we can take the projective limit $\mathfrak{Alg}_\infty(\Phi)$ over the $\mathfrak{Alg}_U(\Phi)$ to get triple (A, ε, ν), where in fact the only element, whic is influenced by taking this limit is ν. We have a left-action of $\overline{\Delta}^*(\mathbb{A}_f) = \overline{\Delta}^*(\mathbb{Q}_p) \times \overline{\Delta}^*(\mathbb{A}_f^p)$ in the following way: $\overline{\Delta}^*(\mathbb{Q}_p)$ act by composition on ε, which is well defined, since A has up to isogeny a Δ-action and ε is Δ-equivariant and Φ is defined as $\mathcal{O}_{\Delta_p}$-module; $\overline{\Delta}^*(\mathbb{A}_f^p)$ act by composition on ν.

By Honda-Tate-theory we know, that there is a unique isogeny class of A. By definition of Φ as formal $\mathcal{O}_{\Delta_p}$-module, it is then clear, that two ε's differ by multiplication by an element in $\overline{\Delta}^*(\mathbb{Q}_p)$ and moreover, that each ε can be derived from one in this way. The same holds for ν which is even more obvious, as ν is anvectorspace isomorphism. This shows, that the action of $\overline{\Delta}^*(\mathbb{A}_f)$ is transitive.

By what we have just seen, it makes sense to choose one triple. Consider the element in $\mathfrak{Alg}_\infty(\Phi)$ given by
$$(A_0\,,\, \varepsilon_0 = \mathrm{id} : \widehat{A}_0 \to \Phi\,,\, \nu = \prod_{\ell \neq p} \nu_{0,\ell}).$$

We've already seen, that $\overline{\Delta}^*(\mathbb{Q}) = \mathrm{Aut}^0_\Delta(A_0)$, so the stabiliser of this element has to be a subgroup of $\overline{\Delta}^*(\mathbb{Q})$. Let $\gamma \in \overline{\Delta}^*(\mathbb{Q})$, γ_ℓ its image in $\overline{\Delta}^*(\mathbb{Q}_\ell)$, γ_p its image in $\overline{\Delta}^*(\mathbb{Q}_p)$, $\widehat{\gamma}$ the induced map on $\widehat{A}_0$ and $V_\ell(\gamma)$ the induced map on $V_\ell(A_0)$. Then we have the following commutative diagrams:

$$\begin{tikzcd} A_0 \arrow[d, "\gamma"'] \\ A_0 \end{tikzcd} \qquad \begin{tikzcd} \widehat{A}_0 \arrow[r] \arrow[d, "\widehat{\gamma}"'] & \Phi \arrow[d, "\gamma_p"] \\ \widehat{A}_0 \arrow[r] & \Phi \end{tikzcd} \qquad \begin{tikzcd} V_\ell(A_0) \arrow[r, "\nu_{0,\ell}"'] \arrow[d, "V_\ell(\gamma)"'] & V_\ell \arrow[d, "\gamma_\ell"] \\ V_\ell(A_0) \arrow[r, "\nu_{0,\ell}"'] & V_\ell \end{tikzcd}$$

where the commutativity in both squares is clear. In the first square it is also clear, that the resulting map is in the same isogeny class. In the second square, obviously, if the isomorphism should remain, we can still mulriply them by scalers in $\mathbb{Q}$. This shows indeed, that $\overline{\Delta}^*(\mathbb{Q}) \subset \overline{\Delta}^*(\mathbb{A}_f)$ is the stabiliser of the mentioned element.

Since the action is transitive and there, this means, that there is an bijection between $\mathfrak{Alg}_\infty(\Phi)$ and the homogenous space $\overline{\Delta}^*(\mathbb{A}_f)/\overline{\Delta}^*(\mathbb{Q})$. But since we derived $\mathfrak{Alg}_\infty(\Phi)$ from $\mathfrak{Alg}_U(\Phi)$ by taking the projective limit over varying U^p, we end up with an bijection
$$\mathfrak{Alg}_U(\Phi) = U^p \backslash \mathfrak{Alg}_\infty(\Phi) \cong U^p \backslash \overline{\Delta}^*(\mathbb{A}_f)/\overline{\Delta}^*(\mathbb{Q}) = Z_U.$$

Now we will define a morphism of special fibers: $\theta : [(\widehat{\Omega}\widehat{\otimes}\widehat{\mathbb{Z}}_p^{nr}) \otimes \mathbb{F}_p] \times Z_u = (\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U \to S_U \otimes \mathbb{F}_p$.

Let S be a scheme of characteristic p.

Definition 3.5. We define a p-quasi-isogeny between two abelian schemes over S to be a quasi-isogeny $g : A_1 \to A_2$ such that the product of g with a big enough power of p is an isogeny of p-power order. For $\ell \neq p$ this induces an isomorphism $T_\ell(A_1) \cong T_\ell(A_2)$.

Lemma 3.6. *Let X_1 and X_2 formal special $\mathcal{O}_{\Delta_p}$-modules over S and $f : X_1 \to X_2$ a quasi-isogeny. Let (A_1, ε_1) be an algebraisation of X_1. There is an algebraisation (A_2, ε_2) of X_2 and a p-quasi-isogeny $h : A_1 \to A_2$ such that the following diagram is commutative*

$$\begin{array}{ccc} \widehat{A}_1 & \xrightarrow{\varepsilon_1} & X_1 \\ {\scriptstyle \widehat{h}}\downarrow & & \downarrow{\scriptstyle f} \\ \widehat{A}_2 & \xrightarrow{\varepsilon_2} & X_2. \end{array}$$

The triple (A_2, ε_2, h) is uniquely determined by this property. If A_1 has a niveau-structur U, A_2 has a niveau-structur via h.

Proof: First consider the case when f is an isogeny. Then take A_2 to be the quotient $A_1/\varepsilon_1^{-1}(\operatorname{Ker} f)$. Then $\varepsilon_1 : \widehat{A}_1 \to X_1$ induces an isomorphism $\varepsilon_2 : \widehat{A}_2 \to X_2$ which is $\mathcal{O}_{\Delta_p}$-equivariant as well and the quotient map $h : A_1 \to A_2$ is obviously an isogeny.

In general, if f is a quasi-isogeny, there exist a natural number such that $p^n f$ is an isogeny. Then do the same as before to get A_2', h' and ε_2'.

□

Now we will use Drinfels's theorem and the previous result. We have seen, that $Z_U \cong \mathfrak{Alg}_U(\Phi)$, and by Drinfeld's theorem, the functor G is representable by the formal $\mathcal{O}$-scheme $\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}}\widehat{\mathcal{O}}^{nr}$. This means, that a section of $(\widehat{\Omega}^2 \otimes \overline{\mathbb{F}}_p) \times Z_U$ over a connected scheme $S = \operatorname{Spec} B$ of characteristic p is determined by the following data:

1. A homomorphism $\psi : \overline{\mathbb{F}}_p \to B$.
2. An algebraisation $(A, \varepsilon, \overline{\nu})$ of Φ with niveau-structure U.
3. An isomorphism class of pairs (X, ρ) where
   (a) X is a formal special $\mathcal{O}_{\Delta_p}$-module over S
   (b) $\rho : \psi_*\Phi \to X$ is a quasi-isogeny of height 0.

So it is given by a tuple $(\psi, X, \rho, A, \varepsilon, \overline{\nu})$. To this we want to associate an element of $S_U(B)$. Let in the notations of the above lemma $X_1 = \psi_*\Phi$, $X_2 = X$, $f = \rho$, $A_1 = \psi_*A$, $\varepsilon_1 = \psi_*\varepsilon$. Hence, we find an algebraisation of X with niveau-structure U. But this is an element in $M_U(S)$ and we have seen, that the functor M_U is represented by S_U. So we get indeed an point in $S_U(B)$, and this defines a morphism of functors , hence the following morphism of $\mathbb{F}_p$-schemes

$$\Theta : (\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U \to S_U \otimes \mathbb{F}_p\,.$$

But since we want a morphism from the quotient $\mathbf{GL}_2(\mathbb{Q}_p)\backslash[(\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U]$ to $S_U \otimes \mathbb{F}_p$, **we should check, that Θ is invariant under the left action of $\mathbf{GL}_2(\mathbb{Q}_p)$.**

Recall that the action of an element of this group on the functor G is given by

$$g(\psi, X, \rho) = (\psi \circ \mathsf{Frob}^{-n}, X, \rho \circ \psi_*(g^{-1}\,\mathsf{Frob}^n)),$$

where $n = v(\det g)$.

To see the action on Z_U we use the lemma: since $g : \Phi \to \Phi$ is a quasi-isogeny, the image of $(A, \varepsilon, \overline{\nu}) \in Z_U$ under $g \in \mathbf{GL}_2(\mathbb{Q}_p)$ is characterised by the of a p-quasi-isogeny making the following diagram commutative:

$$\begin{tikzcd} \widehat{A} \arrow[r, "\varepsilon", "\sim"'] \arrow[d, "\widehat{h}_g"'] & \Phi \arrow[d, "g"] \\ \widehat{A}_1 \arrow[r, "\varepsilon_1", "\sim"'] & \Phi. \end{tikzcd}$$

Now we have to check that this is compatible with the action on $S_U \otimes \mathbb{F}_p$. Let $(\psi, X, \rho, A, \varepsilon, \overline{\nu})$ as before and (A_2, ε_2) the associated algebraisation of X. So we have a commutative diagram

$$\begin{tikzcd} \psi_* \widehat{A} \arrow[r, "\psi_* \varepsilon", "\sim"'] \arrow[d, "\widehat{h}"'] & \psi_* \Phi \arrow[d, "\rho"] \\ \widehat{A}_2 \arrow[r, "\varepsilon_2", "\sim"'] & X. \end{tikzcd}$$

Now we look at the image of this point under g

$$(\psi_= \psi \circ \mathsf{Frob}^{-n}, X, \rho_1 = \rho \circ \psi_*(g^{-1}\, \mathsf{Frob}^n), A_1, \varepsilon_1, \overline{\nu}_1),$$

where $(A_1, \varepsilon_1, \overline{\nu}_1)$ was defined before. Since X is not changed under g, we can look at the algebraisation of X associated to this point defined by the diagram

$$\begin{tikzcd} \psi_1 \widehat{A}_1 \arrow[r, "\psi_{1*} \varepsilon_1"] \arrow[d, "\widehat{h}'"'] & \psi_1 \Phi \arrow[d, "\rho_1"] \\ \widehat{A}'_2 \arrow[r, "\varepsilon'_2", "\sim"'] & X. \end{tikzcd}$$

But this factors in the following way:

$$\begin{tikzcd} \psi_{1*} \widehat{A}_1 \arrow[r, "\psi_{1*} \varepsilon_1", "\sim"'] \arrow[d, "\psi_* \mathsf{Frob}^n"'] & \psi_{1*} \Phi \arrow[d, "\psi_* \mathsf{Frob}^n"'] \\ \psi_* \widehat{A}_1 \arrow[r, "\psi_{1*} \varepsilon_1", "\sim"'] \arrow[d, "\psi_* \widehat{h}_g^{-1}"'] & \psi_* \Phi \arrow[d, "\psi_* g^{-1}"'] \\ \psi_* \widehat{A} \arrow[r, "\psi_* \varepsilon", "\sim"'] \arrow[d, "\widehat{h}"'] & \psi_* \Phi \arrow[d, "\rho"'] \\ \widehat{A}_2 \arrow[r, "\varepsilon_2", "\sim"'] & X. \end{tikzcd}$$

This shows, that we get indeed the same algebraisation. And Θ factorises to a morphism of $\mathbb{F}_p$-schemes

$$\overline{\Theta} : \mathbf{GL}_2(\mathbb{Q}_p) \backslash [(\widehat{\Omega} \otimes \overline{\mathbb{F}}_p) \times Z_U] \to S_U \otimes \mathbb{F}_p\,.$$

The next step is to show that $\overline{\Theta}$ is an isomorphism.

The quotient in this morphism is exactly the special fiber of the quotient appearing in the theorem of ČEREDNIK-DRINFELD. First we note that it is convenient to extend scalers of the morphism to $\overline{\mathbb{F}}_p$ by linearity and we denote the extended morphism by $\overline{\Theta}_{\overline{\mathbb{F}}_p}$. Further, the quotient class can be identified with

$$\mathbf{GL}_2'(\mathbb{Q}_p)\backslash(\widehat{\Omega}_{\overline{\mathbb{F}}_p} \times Z_U),$$

where $\mathbf{GL}_2'(\mathbb{Q}_p)$ is the subgroup of $\mathbf{GL}_2(\mathbb{Q}_p)$ containing the elements g with $v(\det g) = 0$. $\widehat{\Omega}_{\overline{\mathbb{F}}_p}$ represents the functor $\overline{G}$ which classifies the pairs (X, ρ) and $\mathbf{GL}'$ acts by composition with ρ.

$\overline{\Theta}_{\overline{\mathbb{F}}_p}$ comes from a morphism of $\overline{\mathbb{F}}_p$ schemes

$$\Theta_1 : \widehat{\Omega}_{\overline{\mathbb{F}}_p} \times Z_U \to S_U \otimes \overline{\mathbb{F}}_p,$$

which associates to a point $(X, \rho, A, \varepsilon, \overline{\nu})$ defined over a $\overline{\mathbb{F}}_p$-algebra B the algebraisation of X obtained by applying the lemma with $X_1 = \Phi_B$, $X_2 = X$, $f = \rho$, $A_1 = A_B$, $\varepsilon_1 = \varepsilon_B$.

Now we will show, that $\overline{\Theta}_{\overline{\mathbb{F}}_p}$ induces a bijection between the sets of $\overline{\mathbb{F}}_p$-points of the two schemes.

Injectivity: Let $(X, \rho, A, \varepsilon, \overline{\nu})$ and $(X', \rho', A', \varepsilon', \overline{\nu}')$ be two $\overline{\mathbb{F}}_p$-points having the same image A_2 – a special abelian $\mathcal{O}_\Delta$ variety with a niveau-structure – under Θ_1. Since A_2 is an algebraisation of X and X', we have isomorphisms $X \cong \widehat{A}_2 \cong X'$. Since we have the same Φ and the map is $\overline{\mathbb{F}}_p$-linear, ρ and ρ' can only differ by an element $g \in \mathbf{GL}_2'(\mathbb{Q}_p)$. Since we are moding out, we can assume $\rho = \rho'$. Moreover, $(A, \varepsilon, \overline{\nu})$ is an algebraisation of Φ which can be obtained by applying the lemma to the algebraisation A_2 of X via ρ^{-1} and the same holds for $(A', \varepsilon', \overline{\nu}')$, so they are equal.

This shows, that the two points are related via the action of $\mathbf{GL}_2'(\mathbb{Q}_p)$.

Surjectivity: Let A_2 be an abelian S-scheme with relative dimension 2 given with an action of $\mathcal{O}_{\Delta_p}$ and a niveau-structure. Then obviously, we define $X := \widehat{A}_2$ its formal completion. Since all the formal special $\mathcal{O}_{\Delta_p}$-modules of height 4 over $\overline{\mathbb{F}}_p$ are isogenous, there exist a quasi-isogeny $\rho : \Phi \to X$. What is more, one can assume, that ρ is of height 0 by composition with an appropriate endomorphism of Φ.

Now apply the lemma to the algebraisation A_2 of X and ρ^{-1} to get an algebraisation $(A, \varepsilon, \overline{\nu})$ of Φ with a niveau-structure. It is clear, that the image of $(X, \rho, A, \varepsilon, \overline{\nu})$ by Θ_1 is A_2.

Now it remains to show, that Θ_1 is étale and therefore $\overline{\Theta}_{\overline{\mathbb{F}}_p}$ is étale . Then it follows that it is an isomorphism on th $\overline{\mathbb{F}}_p$-points.

Let B an $\overline{\mathbb{F}}_p$-algebra and $B' \to B$ surjective where the kernel is an ideal of square 0 (an "infinitesimal extension". Let $x = (X, \rho, A, \varepsilon, \overline{\nu})$ be a point with values in B of $\widehat{\Omega}_{\overline{\mathbb{F}}_p} \times Z_U$ and $y = A_2$ the associated algebraisation of X by Θ_1. To deform x to a B'-point x' means to deform the formal $\mathcal{O}_{\Delta_p}$-module X. As we already have seen earlier, the quasi-isogeny can be lifted uniquely by the rigidity of p-divisible groups. Furthermore, Z_U is constant. A deformation of y is an deformation of $\widehat{A}_2$. Thus the map Θ_1 sends an deformation of x to a deformation of y. The inverse image of a deformation of A_2 would be the underlying deformation of the formal group $\widehat{A}_2 \cong X$.

This finally shows that $\overline{\Theta}$ is an isomorphism. This means, we constructed an isomorphism between special fibers. By construction, it can be lifted to a morphism of the formal $\mathcal{O}_{\Delta_p}$-modules supported by the two elements. One has just to verify that these isomorphisms are compatible as U^p varies, and the system of these isomorphisms is $\Delta^*(\mathbb{A}_f^p)$-equivariant.

Lastly, we use the theorem of SERRE-TATE to show that $\overline{\Theta}$ can be extended to the two formal schemes.

Let B a $\mathbb{Z}_p$-algebra, where p is nilpotent, and $B_0 = B/pB$. Giving a B_0-point x_0 of $\mathbf{GL}_2(\mathbb{Q}_p)\backslash[(\widehat{\Omega}\widehat{\otimes}\widehat{\mathbb{Z}}_p^{nr}) \times Z_U]$ provides B_0 with a formal special $\mathcal{O}_{\Delta_p}$-module X_0. Similar as before, to deform this point to B means to deform X_0. SInce this turns out to be a local question, we are to solve it for the functor G represented by the formal $\mathbb{Z}_p$-scheme $\widehat{\Omega}\widehat{\otimes}\widehat{\mathbb{Z}}_p^{nr}$.

To give a B-point of the quotient scheme $\mathbf{GL}_2(\mathbb{Q}_p)\backslash[(\widehat{\Omega}\widehat{\otimes}\widehat{\mathbb{Z}}_p^{nr}) \times Z_U]$ is the same as to give its restriction x_0 to B_0 and a deformation X of X_0 over B. On the other hand, let y be a B-point of

$\widehat{S}_U$. Then it is the same to consider its restriction y_0 to B_0 and a deformation A of A_0 the special abelian $\mathcal{O}_\Delta$-scheme over B_0 defined by y_0.

If x_0 and y_0 are correspondent via $\overline{\Theta}$, $X_0 \cong \widehat{A}_0$. By the theorem of SERRE-TATE the deformations of A_0 correspond to the deformations of $\widehat{A}_0 = X_0$. This defines indeed a natural isomorphism of formal schemes $\mathbf{GL}_2(\mathbb{Q}_p)\backslash[(\widehat{\Omega}\widehat{\otimes}\widehat{\mathbb{Z}}_p^{nr}) \times Z_U] \to \widehat{S}_U$ which extends $\overline{\Theta}$. As we mentioned above, it has the desired properties.

This concludes the proof of the theorem of ČEREDNIK and DRINFELD.

We cite the theorem of SERRE and TATE:

Theorem 3.7. *Let B be a ring, $\mathfrak{I} \subset B$ a nilpotent ideal, p a prime number, and let some power of p annihilate $\mathfrak{I}$. Let A_0 be an abelian scheme over $B/\mathfrak{I}$, and Γ_0 its p-divisible group. Then the natural map*

$$\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{lifts of } A_0 \text{ to an}\\ \text{abelian scheme over } B\end{array}\right\} \to \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{lifts of } \Gamma_0 \text{ to}\\ p\text{-divisible groups over } B\end{array}\right\}$$

is bijective.