



## COMPUTING PROJECT

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## MO-MILP Primal Heuristics

Improvement of Gravity Machine

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### 3 Definitions and Notations

We consider the problem as presented in [?]. The multi-objective 0-1 linear optimisation problems with  $p$  objectives ( $p$ -01LP) considered can be formulated as follows:

$$\begin{aligned} \min z(x) &= Cx \\ \text{subject to } Ax &\leqq b \\ x &\in \{0, 1\}^n \end{aligned}$$

where

- $x \in \{0, 1\}^n$ , the vector of  $n$  binary variables  $x_j, j = 1, \dots, n$ ;
- $A \in \mathbb{R}^{m \times n}$ , the  $m$  constraints  $A_i x \leqq b_i, i = 1, \dots, m$  and  $b \in \mathbb{R}^m$ ;
- $C \in \mathbb{R}^{p \times n}$ , the objective matrix where  $p \geq 2$ ;
- $X := \{x \in \{0, 1\}^n \mid Ax \leqq b\} \subseteq \mathbb{R}^n$ , the set of feasible solutions, with  $\mathbb{R}^n$  the decision space;
- $Y := \{Cx \mid x \in X\} \subseteq \mathbb{R}^p$ , the outcome set, with  $\mathbb{R}^p$  the objective space.

Thereafter,  $p$  is usually meant to be equal to 2 (e.g. we consider the bi-objective class of problems).

**Definition 1** ([?]). A feasible solution  $\hat{x} \in \mathcal{X}$  is called efficient or Pareto optimal if there is no other  $x \in \mathcal{X}$  such that  $f(x) \leq f(\hat{x})$ . If  $\hat{x}$  is efficient,  $f(\hat{x})$  is called nondominated point. If  $x^1, x^2 \in \mathcal{X}$  and  $f(x^1) \leq f(x^2)$  we say  $x^1$  dominates  $x^2$  and  $f(x^1)$  dominates  $f(x^2)$ . The set of all efficient solutions  $\hat{x} \in \mathcal{X}$  is denoted  $\mathcal{X}_E$  and called the efficient set. The set of all nondominated points  $\hat{y} = f(\hat{x}) \in \mathcal{Y}$ , where  $\hat{x} \in \mathcal{X}_E$ , is denoted  $\mathcal{Y}_N$  and called the nondominated set.

**Definition 2** ([?]). A feasible solution  $\hat{x} \in \mathcal{X}$  is called weakly efficient (weakly Pareto optimal) if there is no  $x \in \mathcal{X}$  such that  $f(x) < f(\hat{x})$ , i.e.  $f_k(x) < f_k(\hat{x})$  for all  $k = 1, \dots, p$ . The point  $\hat{y} = f(\hat{x})$  is then called weakly nondominated. A feasible solution  $\hat{x} \in \mathcal{X}$  is called strictly efficient (strictly Pareto optimal) if there is no  $x \in \mathcal{X}, x \neq \hat{x}$  such that  $f(x) \leq f(\hat{x})$ . The weakly (strictly) efficient and nondominated sets are denoted  $\mathcal{X}_{wE}(\mathcal{X}_{sE})$  and  $\mathcal{Y}_{wE}$ , respectively.

**Definition 3** (A lower bound set [?]). A lower bound set for  $Y'$  is a subset  $L \subseteq \mathbb{R}^p$

1. For each  $y \in Y'$  there is some  $l \in L$  such that  $l \leqq y$
2. There is no pair  $y \in Y', l \in L$  such that  $y$  dominates  $l$ .

**Definition 4** (An upper bound set [?]). An upper bound set for  $Y'$  is a subset  $U \subset \mathbb{R}^p$

1. For each  $y \in Y'$  there is some  $u \in U$  such that  $y \leqq u$
2. There is no pair  $y \in Y', u \in U$  such that  $u$  dominates  $y$ .

### 4 Backgrounds

Gravity Machine is an algorithm aiming to compute an upperbound set for a multi-objective linear optimization problem with binary variables.

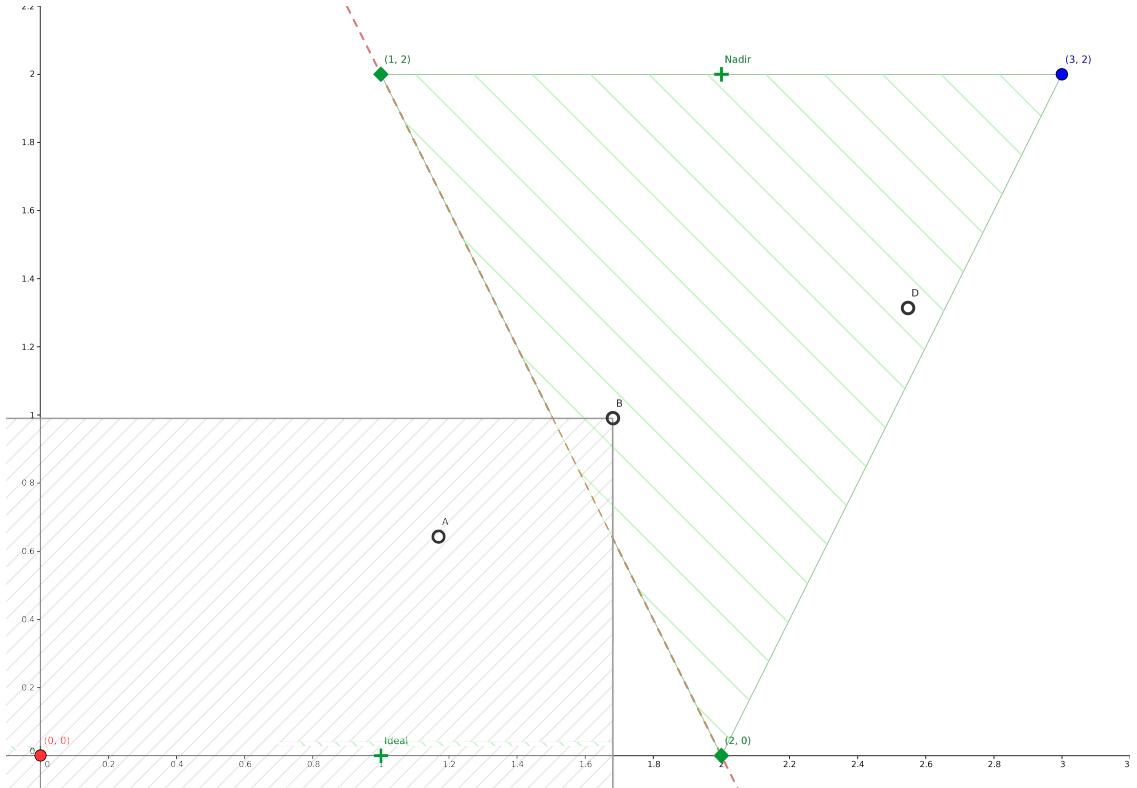


Figure 1: Projection of the solution space onto the cost space

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**Algorithm 1:** Outline of Gravity Machine

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**Data:**  $\mathcal{D}, n_L, \text{maxTrial}, \text{maxTime}$

**Result:**  $L, U$

```
1 begin
2   |    $L, F \leftarrow \text{COMPUTEGENERATORS}(\mathcal{D}, n_L);$ 
3   |   forall  $z(\bar{x}^k) \in L, \bar{x}^k \notin F$  do //each k-th unfeasible gen
4   |   |   timeStart  $\leftarrow \text{TIME}();$  trial  $\leftarrow 0;$ 
5   |   |   timeout  $\leftarrow \text{false};$  feasible  $\leftarrow \text{false};$ 
6   |   |    $\tilde{x}, H, \text{cycle} \leftarrow \text{ROUNDINGSOLUTION}(\bar{x}^k, H);$ 
7   |   |   while  $\neg \text{feasible} \wedge \neg \text{timeout}$  do
8   |   |   |   trial  $\leftarrow \text{trial} + 1;$ 
9   |   |   |    $\bar{x}, F, \text{feasible} \leftarrow \text{PROJECTSOLUTION}(\tilde{x});$ 
10  |   |   |   if  $\neg \text{feasible}$  then
11  |   |   |   |    $\tilde{x}, H, \text{cycle} \leftarrow \text{ROUNDINGSOLUTION}(\bar{x}, k, h);$ 
12  |   |   |   |   if  $\text{cycle}$  then
13  |   |   |   |   |    $\tilde{x} \leftarrow \text{PERTURBSOLUTION}(\tilde{x});$ 
14  |   |   |   |   end
15  |   |   |   end
16  |   |   |   timeout  $\leftarrow (\text{time}() - \text{timeStart} \geq \text{maxTime}) \vee (\text{trial} = \text{maxTrial});$ 
17  |   |   end
18  |   end
19  |    $U \leftarrow \text{EXTRACTNONDOMINATEDPOINT}(F);$ 
20 end
```

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## 5 Soft constraint on cones

A method inteneded to guide the search briefly mentioned in the seminal paper is here presented to a wider extent.

## 6 Improving generators

Generators are obtained by using the  $\epsilon$ -constraints method which is one of the well-known scalarization techniques [?] aimed at getting a sample of efficient solutions of a MOLP. Those generators form an efficient set for the relaxed problem. That is to say, the set of all generators is also the lower bound set  $L$ . Since GM starts the search based on each  $k$ -th generator, one would like to have a better initial solution in order to reach a feasible solution faster.

### 6.1 Bugs in Gravity Machine

#### 6.1.1 Generators not feasible

The previous version of Gravity Machine did use the GLPK solver to generate the lower bound set by the  $\epsilon$ -constraints method. However, closer to the line passing through the Nadir and the origin the more constrained the MILP representing the current generator. Since considered problems have lot of constraints and variables some numerical issues may arise [?]. Infeasibility hence may appear during the solving process for problems which has at least one (feasible) optimal solution. In addition, the *Julia* wrapper for GLPK seems to be known for returning the solution minimizing constraints violation<sup>4</sup>. Finally, as the termination status was not correctly handled, some generators turn out to be used whereas there were not "trully" feasible.

**Patch:** The termination status is now considered and GM properly fails if a generator is not feasible. Tolerance attribute was hand-tuned to allow the solver to find a feasible solution. . It is worth noticing that no difference in the performance has been observed. The robustness is the matter here. In addition, GLPK was replaced by HiGHS [?]. Indeed, the latter shows better performances than GLPK [?] and is still open-source.

[EM] ça serait bien que ce ne soit pas "hand-tuned"  
<https://discourse.julialang.org/t/access-infeasible-optimization-results-glpk-with-semicontinuous-variable/59972>

#### 6.1.2 Soft conic constraints $\lambda_1$ and $\lambda_2$

Both vectors  $\lambda_1$  and  $\lambda_2$  were accidentally swaped in their definition such that the "guidance mechanism" was useless and a fortiori misleading.

### 6.2 Empirical findings

Empirical evidences tend to show that very few variables are set to non-integral value for each generator.

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<sup>4</sup><https://discourse.julialang.org/t/access-infeasible-optimization-results-glpk-with-semicontinuous-variable/59972>

### 6.3 Setting some variables binary

At first sight, non-solved to integrality variables should be set binary and the problem re-optimized. The latter allows GM to find initial solution Then, a generator closer to  $Y$  should result from a the re-optimization process. However, several issues arises:

#### 6.3.1 Communicating vessels effect

Since not all constraints are set binary, some variables found to be zero or one might become fractional. A number of variables found fractional after the second stage solving greater than the number of variables found fractional after the first stage solving represents the worst case. As far as our experimental results show us, this situation is very rare.

[EM] à reformuler

#### 6.3.2 Same generators lead to the same solution

Ideally, one would that each generator  $\tilde{y}^k$  converges to a different  $\tilde{y}^{k,t}$  for some iterations  $t \in \mathbb{N}$ . Below, we consider  $J_0(x)$  (and  $J_1(x)$ ) the set containing the index of variables set to zero (respectively to one) into the solution  $x$ . Some well known modelization techniques are briefly presented and their pertinence for GM is studied. For the major part of them, they are inspired by canonical cut constraints introduced by [?] and used for the same purpose by [?].

**Canonical cut constraint :**

**Proposition 1.** *Let  $x^0$  be a vector in  $\{0,1\}^n$ . The following inequality*

$$\sum_{j \in J_1(x^0)} x_j + \sum_{j \in J_1(x^0)} 1 - x_j \geq 1 \quad (1)$$

*cutts off solution  $x^0$  without cutting off any other solution in  $\{0,1\}^n$ .*

*Proof.* Let's suppose that  $x^0$  is not cutted off by the constraint. Then  $x = x^0$  implies  $0 \geq 1$  from (1). The cutting off is proved.

Finally, a solution  $x^1 \neq x^0$  is assumed to be cutted off by the constraint. That is to say

$$\sum_{j \in J_1(x^0)} x_j^1 + \sum_{j \in J_1(x^0)} 1 - x_j^1 = 0 \quad (2)$$

$$\iff \|x^0 - x^1\| = 0 \quad (3)$$

In other words  $x^0 = x^1$  and  $x^0 \neq x^1$  which is impossible. Then, ?? holds.  $\square$

**Proposition 2.** *Gravity Machine does not longer guarantee the rectitude of  $L$  if  $x^0 \in [0,1]^n$*

*Proof.* We exhibit a counter-example. Let be a 1-01LP of the form:

$\square$

### Corollary 6.0.1.

Provided a feasible solution  $s$  (known to be binary), the following constraint allows the solving process to avoid returning  $s$  **only** as a solution [?]:

$$\sum_{j \in \{i | s_i = 0\}} x_j + \sum_{j \in \{i | s_i = 1\}} 1 - x_j \geq 1$$

It is **The lower bound set is no longer guaranteed**

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### Algorithm 2: Improving generators

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**Data:**  $L, \lambda_1, \lambda_2, \tau, \alpha$   
**Result:**  $L_I, U$

```
1 begin
2   |    $L, F \leftarrow \text{COMPUTEGENERATORS}(\mathcal{D}, n_L);$ 
3   |   forall  $z(\bar{x}^k) \in L, \bar{x}^k \notin F$  do //each k-th unfeasible gen
4   |   |
5 end
```

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### Cones

#### 6.3.3 Too much binary variables increase the solving time

As shown by our empirical data, less than 20

## 7 Improving projection

By the same token as the generators improvement, the projection used during GM can be improved by adding some integrity constraints on the non

## 8 Numerical experiments

Gravity Machine is implemented in Julia 1.8.5 [?]

### 8.1 Numerical instances

### 8.2 Quality measure

### 8.3 Numerical results

Experiments have been performed on-top of the following configuration:

- Laptop: Dell - XPS 13 9300
- OS: Linux Mint 21.1 Cinnamon - v5.6.8

- Kernel version: 5.19.0-38-generic
- Intel® Core™ i7-1065G7 CPU @ 1.30GHz × 4
- L1: 320 KiB, L2: 2 MiB, L3: 8 MiB
- RAM: 15.2 Go LDDR4
- SSD: 1024.2 Go

### 8.3.1

## 9 Conclusion and discussion