Chaos and Strange Attractors in Discrete-time Dynamical Systems

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Abstract

In this project, we first introduce the essential concepts of discrete-time dynamical systems by comparing them to continuous-time dynamical systems. To analyse the asymptotic behaviors of a discrete system, we define the orbits and ω-limit sets of the system. Then, we introduce the Lyapunov exponents which measure the (in)stability and chaos of a dynamical system. We present the necessary definitions and theories for the analytical and numerical computation of the Lyapunov exponents. The project then focuses on chaotic and strange attractors and on three different types of dimensions of attractors. We explore the relationships between the dynamical systems and the attractors and their properties by different fractal dimensions. After that, we examine some not well-known dynamical systems using the above methods and theories. Finally, we introduce Ergodic theory which studies the statistical properties of dynamical systems. By Oseledets multiplicative ergodic theorem, we can prove some important results in the previous sections.

Contents

1 Introduction

In the realm of modern science, the concept of dynamical systems finds application in various fields, including physics, biology, data science, and economics. A dynamical system consists of a set of possible states and a rule that determines the current state. For continuous-time dynamical systems, this deterministic rule is given by a set of ordinary differential equations. Conversely, for discrete-time dynamical systems, the rule is given by difference equations.

The concept of a dynamical system, where the future state of a system is determined by a rule based on its current state, has been central to scientific thought for centuries. However, the formal study of dynamical systems in mathematics began in the late 19th and early 20th centuries with the work of Henri Poincaré, who made significant contributions to understanding deterministic chaos and the three-body problem in celestial mechanics.[1]

The study of discrete-time dynamical systems, where the state of the system evolves in discrete time steps, gained prominence in the mid-20th century. One of the most famous examples of a discrete-time dynamical system is the logistic map, introduced by biologist Robert May in the 1970s. This simple mathematical model of population dynamics was one of the first systems to demonstrate that simple deterministic rules could lead to complex and unpredictable behavior, a hallmark of chaos. This report will discuss this example in detail.[2]

The concept of chaos in dynamical systems, where long-term behavior is highly sensitive to initial conditions, was first recognized in the work of Edward Lorenz in the 1960s. Lorenz, a meteorologist, discovered chaotic behavior in a simplified model of atmospheric convection, now known as the Lorenz system. His work led to the realization that chaos is a widespread phenomenon in nature.[3]

The concept of a strange attractor, a set towards which a system tends to evolve regardless of the starting conditions, was introduced by David Ruelle and Floris Takens in the 1970s. One of the first examples of a strange attractor was found in the Lorenz system, where the system's state traces out a complex, fractal-like structure known as the Lorenz attractor.[4]

Another key development in the study of chaotic dynamical systems was the introduction of the Hénon map by French astronomer Michel Hénon in the 1970s. The Hénon map, a two-dimensional discrete-time dynamical system, was one of the first systems found to exhibit a strange attractor. This report will also provide a detailed explanation of this example.[5]

This project primarily focuses on discrete-time dynamical systems. We will begin by introducing the basic concepts of discrete-time dynamical systems, then delves into the phenomenon of chaos in these systems. Then we introduce Lyapunov exponents, a method of quantifying chaos, and provide detailed steps for their calculation. The concept of strange attractors is also discussed, which is a tool for describing the long-term behavior of chaotic systems. Furthermore, we introduce the concept of dimension, which is used for describing strange attractors. Detailed steps for calculating the Lyapunov exponents and dimension are provided in the corresponding sections, offering a practical approach to understanding these complex concepts. Each section thoroughly explains the related theories and concepts and we provide some examples to aid understanding.

2 Discrete-Time Dynamical Systems

2.1 Introduction

A dynamical system consists of a set of possible states, together with a rule that determines the present state from the past states.[6] For a **continuous-time dynamical system**, the deterministic rule is given by a system of ordinary differential equation. For example, we consider an open subset $D \subset \mathbb{R}^d$ and a locally Lipschitz continuous function $f : D \to \mathbb{R}^d$ of the differential equation

$$\dot{x} = f(x).$$

We can define $\varphi(t, x_0)$ as the flow of the autonomous differential equation for any initial value $x_0 \in D$. This flow indicates how the system evolves over time starting from the initial state x_0. Similarly, for a **discrete-time dynamical system**, the rule is given by a difference equation of the form

$$x_{n+1} = f(x_n)$$

where x_n is from the set of states X for all $n \in \mathbb{N}_0$ and $f : X \to X$ shows how states evolve in the system. We now define it formally.

Definition 2.1 (Discrete-Time Dynamical Systems)**.** *Consider a non-empty set X and a function $f : X \to X$, the tuple (X, f) fully characterizes a discrete-time dynamical system. X is called the **state space** and f is called the **evolution**.*

Notice that for any given initial state $x_0 \in X$ and for all $n \in \mathbb{N}$, the states $x_n \in X$ are evolved by $x_{n+1} := f(x_n)$. Write f^n as the n times composition of the https://www.overleaf.com/project/646f6b6aca9fdf65573aad3bfunction f. It is clear that $x_n = f^n(x_0)$ for all $n \in \mathbb{N}_0$.

2.2 Orbits and ω-Limit Sets

Similar to what we did in the Differential Equations course for continuous systems, we want to investigate how the discrete system evolved over infinite time by the orbits or trajectories. In general, for continuous systems, the orbits are curves; for discrete systems, the orbits are sequences of states.

Definition 2.2 (Orbits, [7])**.** *Let (X, f) be a discrete-time dynamical system. For all $x_0 \in X$, we call the set*

$$O(x_0) := \{f^n(x_0) : n \in \mathbb{N}_0\}$$

*the **orbit** through x_0.*

To understand the long-term behavior of a discrete system, additional structures on the state space are required, such as a metric function on X. Therefore, in the following sections, we assume X to be a subset of $\mathbb{R}^d$.

Definition 2.3 (ω-Limit Sets, [7])**.** *Let (X, f) be a discrete-time dynamical system. The **ω-limit set** is defined by*

$$\omega(x_0, f) = \bigcap_{n\in\mathbb{N}_0} \overline{O(f^n(x_0))}$$

for all $x_0 \in X$.

Notice that $\overline{O(f^n(x_0))}$ is the closure of the orbit $O(f^n(x_0))$ and the ω-limit set of x is the set of all limit points of the sequence $\{f^n(x_0)\}_{n\in\mathbb{N}_0}$.

2.3 Chaos in Discrete-Time Dynamical Systems

In a one-dimensional continuous-time dynamical system, all solutions are monotone. This means an orbit is either an equilibrium or a non-closed curve. In a two-dimensional continuous-time dynamical system, we have already seen the Poincaré–Bendixson theorem that classifies non-empty compact ω-limit sets containing only finitely many fixed points. However, for a discrete-time dynamical system, even in one or two dimensions, the analysis on the orbits and ω-limit sets can be quite complicated. Let us look at the following examples.

Example 2.1. *Consider the difference equation on $\mathbb{R}$*

$$x_{n+1} = -\operatorname{sgn}(x_n)\sqrt{|x_n|}$$

We can define the discrete system (X, f) where $X = \mathbb{R}$ and $f(x) = -\operatorname{sgn}(x)\sqrt{|x|}$. Let us look at the following cases

1. *$x_0 = 0$. Then $f(x_0) = 0 = x_0$. This implies $O(0) = \omega(0, f) = \{0\}$. In this case, x_0 is called a **fixed** point.*

2. *$x_0 = \pm 1$. Then $f^n(x_0) = (-1)^n x_0$. This implies $O(\pm 1) = \omega(\pm 1, f) = \{-1, 1\}$. In this case, ± 1 is called a **period-2** point and the orbit of ± 1 is called a **period-2** orbit.*

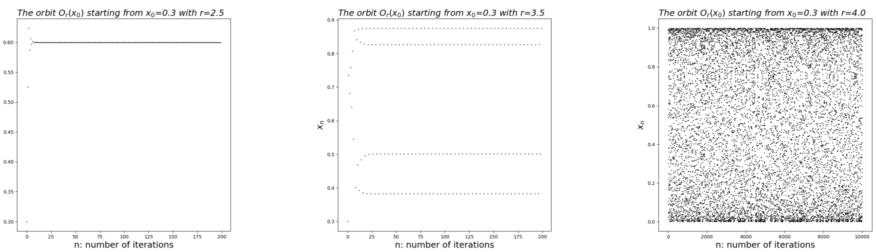
3. *$x_0 \in \mathbb{R} \setminus \{0, \pm 1\}$. Then $f^n(x_0) = (-1)^n \operatorname{sgn}(x_0)|x_0|^{2^{-n}}$. In this case, $O(x_0)$ is an infinite sequence and $\omega(x_0, f) = \{-1, 1\}$. For large enough n, $\{f^n(x_0), f^{n+1}(x_0)\}$ is close to the period-2 orbit $\{-1, 1\}$. This type of orbit is called an **asymptotically periodic** orbit.*

This example illustrates some regular asymptotic behaviors exhibited by the orbits from the discrete system. The next example will demonstrate a chaotic behavior of a discrete system.

Example 2.2 (Logistic Map)**.** *The logistic map is the evolution function of a discrete-time model of population growth.[8] Consider a recurrence relation on the interval $[0, 1]$ given by*

$$x_{n+1} = r x_n (1 - x_n)$$

where x_n represents the ratio of existing population to the maximum possible population and the parameter $r \in (0, 4]$ represents the biotic potential. For any $r \in (0, 4]$ and $x_0 \in [0, 1]$, define $O_r(x_0)$ as the orbit starting from x_0 following the evolution function $f_r(x) = rx(1-x)$. Through the numerical simulation, we can plot the orbit of any initial state over iterations. Figure 1 shows different long-time behaviors from different logistic models.



(a) (b) (c)

Figure 1: Three orbits over iterations of different logistic maps

*From Figure 1a we can see that $\omega(0.3, f_{2.5})$ is a singleton. Figure 1b shows that $\omega(0.3, f_{3.5})$ is a finite set with 4 points. This implies that $O_{3.5}(0.3)$ has a possibility of tending to a period-4 orbit after infinite time. However, Figure 1c shows a completely different result. Instead of exhibiting any asymptotic behaviors as mentioned in **Example 2.1**, there is no clear pattern that the system tends to follow. Moreover, if the orbit starts from the initial state $x_0 = 0.3000001$ which is relatively close to 0.3, the average distance between two orbits $d_n = \frac{1}{n}\sum_{i=1}^{n} |f^i(0.3) - f^i(0.3000001)|$ converges to a positive number (see Figure 2). The code that generated Figure 2 can be found in **Section 9**.*

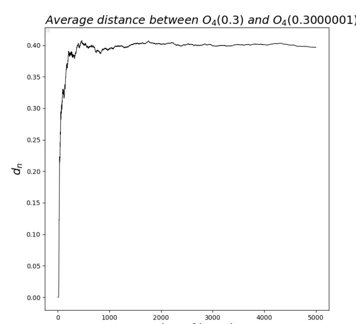


Figure 2

*This phenomenon is called **chaos**. The disorder patterns and the separation between two orbits are key features of a chaotic dynamical system.*

3 Lyapunov Exponents

For a given 2D dynamical system with a map f and initial condition x_0, we can encounter different types of behaviour of the points in the orbit. If our initial condition x_0 is itself a stable point, then all the points in the orbit will have a stable behaviour. Furthermore, even if our initial condition is unstable, the points in the orbit may have a stable behaviour in an infinite amount of time. We have seen such behaviour in continuous-time dynamical systems. However, under some specific conditions, we may experience instability in all points in the orbit. We may even have a dynamical system that has no stable points, and so all points of any orbit will be unstable. We can call such a phenomenon a **chaotic orbit**. Formally, a chaotic orbit is one that forever continues to experience the unstable behavior that an orbit exhibits near a source, but that is not itself fixed or periodic.[6] As the reader can imagine, this behaviour may be very difficult to grasp and so we would like to develop some quantitative theory to determine whether a dynamical system is chaotic or not. We will define Lyapunov exponents, Lyapunov spectrum and numerical methods to calculate them. In this section we will present the theory in a simpler form. The reader can go to **Section 7** where the theory is explained in more detail.

3.1 Introduction to Lyapunov exponents

Consider the 2D linear continuous-time dynamical system $\dot{x} = Ax$ where A is an arbitrary 2×2 matrix. We have seen in the Differential Equations course that we can define a Lyapunov exponent corresponding to the solution $\lambda(t) = e^{At}(x_0, y_0)^T$. The Lyapunov exponents show the growth or decay of e^{At}. For a non-linear discrete-time dynamical system we start at an initial point x_0. By **Definition 2.1** we have that $x_n = f^n(x_0)$. Since the dynamical system is non-linear we need to linearize it by taking the derivative of $f^n(x)$. This matrix $Df^n(x)$ is important for defining and calculating the Lyapunov exponents. By the chain rule

$$Df^n(x) = Df(f^n(x)) \cdot Df(f^{n-1}(x)) \cdot \cdots \cdot Df(x)$$

In the case of 2D dynamical systems, all terms on the right-hand side are 2×2 matrices, the Jacobian matrices of the corresponding map. For a given 2D discrete-time dynamical system with a map f and initial condition x_0, the Lyapunov exponents give a quantitative measure of chaos. The number of Lyapunov exponents is equal to the dimension of the system. We say that a **system is chaotic** if at least one of its Lyapunov exponents is positive. Another benefit of Lyapunov exponents is that they give the possibility to 'compare' the chaos of two or more dynamical systems.[9] For two chaotic systems, one is 'more chaotic' than the other if its positive Lyapunov exponent is bigger.

3.2 Definitions of Lyapunov exponents and Lyapunov spectrum

Below we will give some definitions of Lyapunov exponents for $n-$dimensional dynamical systems.

Definition 3.1 (Lyapunov Exponents, Geometrical Version, [6])**.** *Let f be a smooth map on $\mathbb{R}^n$. For $k = 1, \ldots, m$ and for all $n \in \mathbb{N}$, let r_k^n be the length of kth longest orthogonal axis of the ellipsoid $Df^n(x_0)\mathbb{S}$, where $\mathbb{S} = B_1(0)$ is the unit sphere, for an orbit with an initial point x_0. Then r_k^n measures the contraction or expansion near the orbit of x_0 during the first n iterations. The kth **Lyapunov exponent** of x_0 is defined by*

$$\lambda_k(x_0) = \lim_{n\to\infty} \frac{\log r_k^n}{n} \tag{1}$$

Definition 3.2 (Lyapunov Spectrum and Maximal Lyapunov Exponent)**.** *Let f be a smooth map on $\mathbb{R}^n$, with an initial condition x_0. Let $\lambda_1(x_0), \lambda_2(x_0), \ldots, \lambda_n(x_0)$ be the Lyapunov exponents defined as above (it follows that $\lambda_1(x_0) \geq \lambda_2(x_0) \geq \cdots \geq \lambda_n(x_0)$). We define the set $\{\lambda_1(x_0), \lambda_2(x_0), \ldots, \lambda_n(x_0)\}$ the **Lyapunov spectrum** of the dynamical system with map f and initial condition x_0 and we define $\lambda_1(x_0)$ to be the **maximal Lyapunov exponent** of the system.*

Definition 3.3 (Maximal Lyapunov Exponent, [10])**.** *Let f be a smooth map on $\mathbb{R}^n$, with an initial condition x_0. Then, the **maximal Lyapunov exponent** is equal to*

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x_0)\| \tag{2}$$

where $\|A\|$ is the matrix norm of A.

Remark 3.1. *By **Oseledets theorem**, the limit in (2) exists for almost all initial conditions x_0 and is equal to the maximal Lyapunov exponent $\lambda_1(x_0)$.[10] The reader can find more about the theorem in **Section 7** and the proof in [11].*

Now let us show a simple example that illustrates how **Definition 3.1** and **Definition 3.3** are related.

Example 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x, y) = (\frac{1}{2}x, \frac{1}{2}y)$ for any $(x, y) \in \mathbb{R}^2$ be a map of a 2D discrete-time dynamical system $x_{n+1} = f(x_n)$ for all $n \geq 0$. Let $x_0 = (1, 1)$ be the initial condition.*
*First, let us calculate the Lyapunov exponents using **Definition 3.1**. In this case initially we have a unit circle with center $x_0 = (1, 1)$ and with every iteration the radius of this circle is halved. In n iterations, the radius of the circle will be $\frac{1}{2^n}$, so the length of both orthogonal axis (the x-axis and the y-axis) will be $\frac{1}{2^n}$. Therefore*

$$\lambda_1(x_0) = \lambda_2(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{2^n} = \lim_{n\to\infty} \frac{1}{n} n \log \frac{1}{2} = \log \frac{1}{2}$$

We can see that in this example the Lyapunov exponents do not depend on the initial condition.
Now, let us calculate the maximal Lyapunov exponent using ***Definition 3.3****.*
We can write

$$f(x,y) = A(x,y)^T$$

where

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

In this case $A = Df(x_0)$*. Since* A *is a constant matrix, we can see that*

$$Df^n(x_0) = A^n = \begin{pmatrix} \frac{1}{2^n} & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix}$$

By Oseledets theorem, we will obtain the same maximal Lyapunov exponent for any unitary u_0*, so we can choose* $u_0 = (1,0)^T$*. Then*

$$\lambda_1(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left\| \begin{pmatrix} \frac{1}{2^n} & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix} \right\| = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{2^n} = \log \frac{1}{2}$$

So indeed the maximal Lyapunov exponents from the two definitions are the same. However, we can see that **Definition 3.1** also gives us the second Lyapunov exponent. Is there a way deducing the second Lyapunov exponent using **Definition 3.3**? This might be useful if we want to calculate the Lyapunov exponents of a 2D map analytically. Fortunately, there is a very powerful result for discrete-time dynamical systems with dimension n which we state below.

Theorem 3.1. *For a discrete-time dynamical system, let* $\lambda_1(x_0), \lambda_2(x_0), \dots, \lambda_n(x_0)$ *be its Lyapunov exponents. Then*

$$\sum_{k=1}^{n} \lambda_k(x_0) = \lim_{n\to\infty} \frac{1}{n} ln|det(Df^n(x_0))|$$

For the proof of this theorem, we require some background on Ergodic measure and Birkhoff's Ergodic Theorem. The reader can find more about those and the proof of **Theorem 3.1** in **Section 7**.

3.3 Numerical calculation of Lyapunov exponents using QR decomposition

Developing a numerical method for calculating the Lyapunov spectrum of any discrete-time dynamical system will be a very powerful tool to determine whether a system is chaotic or not. One way to calculate the Lyapunov spectrum is by applying the so-called QR decomposition algorithm to $Df^n(x)$.
Mathematically, the logic of the algorithm is to apply it sequentially to the definition of $Df^n(x)$. Starting with $Q_0 = I$, we obtain a QR decomposition by the algorithm below.[12]

$$Df^n(x) = Df(f^n(x)) \cdot Df(f^{n-1}(x)) \cdot \dots \cdot Df(f^2(x)) \cdot (Df(x) \cdot Q_0) =$$

$$= Df(f^n(x)) \cdot Df(f^{n-1}(x)) \cdot \dots \cdot Df(f^2(x)) \cdot Q_1 \cdot R_1 =$$
$$= Df(f^n(x)) \cdot Df(f^{n-1}(x)) \cdot \dots \cdot (Df(f^2(x)) \cdot Q_1) \cdot R_1 =$$
$$= Df(f^n(x)) \cdot Df(f^{n-1}(x)) \cdot \dots \cdot Q_2 \cdot R_2 \cdot R_1 =$$
$$= \dots =$$
$$= Q_n \cdot (R_n \cdot R_{n-1} \cdot \dots \cdot R_1) =$$
$$:= Q_n \cdot R^{[n]} := Q \cdot R.$$

where Q is a $n \times n$ orthogonal matrix and $R = R_n \cdot R_{n-1} \cdot \dots \cdot R_1$ is a $n \times n$ right-triangular matrix with positive entries on the diagonal.

Theorem 3.2. *Now having $Df^n(x) = Q \cdot R$, the k^{th} Lyapunov exponent is equal to*

$$\lambda_k(x_0) = \lim_{n\to\infty} \frac{1}{n} \log[R_{kk}].$$

The reader can find the proof of this theorem in **Section 7**.
To implement the algorithm, we need to define n_iter - the number of iterations (n_iter and n in **Theorem 3.2** are equivalent). To obtain precise results, we will pick $n\_iter = 1,000,000$.
In words, the algorithm is as follows. [12]

1. Initialize Q to be the $n \times n$ Identity Matrix
2. Initialize LE to be a zero n-vector
3. for $i = 1$ to n_iter :
   $B = Df(f^i(x)) \cdot Q$
4. Compute the QR decomposition of B
5. $LE = LE + log(diag[R])$
   end
6. $LE = LE/n\_iter$

Step 5 and 6 are equivalent to the statement in **Theorem 3.2** because

$$\frac{1}{n} ln[R_{kk}] = \frac{1}{n} \log[R_{n_{kk}} \cdot R_{n-1_{kk}} \cdot \dots \cdot R_{1_{kk}}] = \frac{1}{n} \sum_{i=1}^{n} \log[R_{i_{kk}}]$$

since each of the diagonal elements of R is the product of the corresponding diagonal elements of R_i.
The reader can find the code that implements the above QR decomposition for $n = 2$ in **Section 9**.

3.4 Hénon map

Here, we will introduce the Hénon map - a very popular example of a 2D discrete-time dynamical system, which we will be using later as well.[9] The Hénon map is given by

$$x_{n+1} = 1 - ax_n^2 + y_n$$

$$y_{n+1} = bx_n$$

We will consider the Hénon map for $a = 1.4$ and $b = 0.3$ and initial condition $x_0 = (0, 0)$ - this is also known as the classical Hénon map. Now let us apply the QR algorithm and check that **Theorem 3.1** holds.
To apply **Theorem 3.1** we need to calculate $det(Df^n(x_0))$. We know that

$$Df(f^n(x_0)) = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{pmatrix} = \begin{pmatrix} -2ax_n & 1 \\ b & 0 \end{pmatrix}$$

So $det(Df(f^n(x_0))) = -b$ for any n. Also

$$Df^n(x) = Df(f^n(x)) \cdot Df(f^{n-1}(x)) \cdot \dots \cdot Df(x)$$

Therefore,

$$det(Df^n(x_0)) = \prod_{i=1}^{n} det(Df(f^i(x_0))) = \prod_{i=1}^{n} -b = (-b)^n$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log|det(Df^n(x_0))| = \lim_{n\to\infty} \frac{1}{n} \log|(-b)^n| = \lim_{n\to\infty} \frac{1}{n} n \log(|b|) = \log(|b|)$$

So, by **Theorem 3.1**

$$\lambda_1(x_0) + \lambda_2(x_0) = \log(|b|) \approx -1.204 \tag{3}$$

Applying the QR algorithm, we obtain

$$\lambda_1 \approx 0.42$$

and

$$\lambda_2 \approx -1.624$$

which agrees with (3). Since we have a positive Lyapunov exponent we know that this system is chaotic.

4 Chaotic and Strange Attractors

In dynamical systems, an attractor is a set of points towards which the system gravitates, regardless of varying initial conditions. Attractors can be chaotic, leading to diverging paths that make precise prediction challenging, especially when the system is disturbed by even the slight noise. This chaos, characterized by sensitivity to initial conditions, results in a system that is locally unstable but globally stable. In other words, while points on the attractor can diverge, they remain within the confines of the attractor. The term "strange" describes the attractor's geometric form, while "chaotic" refers to the dynamical behavior of orbits on the attractor. In the following, we will explore the properties of these chaotic and strange attractors[13].

4.1 Basic Definitions

Building on the concepts introduced in **Section 3**, a system is classified as chaotic if it exhibits sensitivity to initial conditions, characterized by at least one positive Lyapunov exponent.

Definition 4.1 (Attractor, [6])**.** *Let (X, f) be a discrete-time dynamical system. An attractor is a ω-limit set which attracts a set of initial values that has non-zero measure (non-zero length, area, or volume, depending on whether the dimension of the map's domain is one, two, or higher). This set of initial conditions is called the* ***basin of attraction*** *of the attractor.*

An attractor is a set of system states that a system gravitates towards from various initial conditions. Once system values are near the attractor, they remain close despite small disturbances. Here are some common types of attractors.

1. **Attracting periodic point and attracting fixed point**: Let (X, f) be a time-discrete dynamical system. A point x is a periodic point of the map f if there exists a positive integer p such that $f^p(x) = x$ and $f^q(x) \neq x$ for all $0 < q < p$. If $p = 1$, then x is a fixed point of f. The basin of attraction of x is the set of points x' such that $\left|f^k(x') - f^k(x)\right| \to 0$ as $k \to \infty$[6]. All trajectories with initial conditions in the basin of attraction will converge to the attractor.

2. **Limit Cycle**: A limit cycle is a closed orbit $O(x_0)$ such that $O(x_0) \subset \omega(x_0, f)$, the ω-limit set, for some $x \notin O(x_0)$. A limit cycle is an attractor because a dynamical system with initial conditions in the limit cycle's basin of attraction will approach the limit cycle over time.[14]

3. **Limit Torus**: This is a higher-dimensional generalization of a limit cycle that can occur in discrete-time systems with three or more dimensions. The system cycles through a closed, donut-shaped sequence of points in its state space.[14]

In this report we will focus on chaotic and strange attractors.

Definition 4.2 (Chaotic Orbit, [6]). *Let* (X, f) *be a time-discrete dynamical system, and let* $O(x_0)$ *be an orbit of* f *for some* $x_0 \in X$. *The orbit is chaotic if*
1. $O(x_0)$ *is not asymptotically periodic, i.e. it does not converge to a periodic orbit* $\left\{x, f(x), f^2(x), ..., f^{p-1}(x)\right\}$ *as* $n \to \infty$.
2. at least one of its Lyapunov exponents λ *of* x_0 *is greater than zero.*

Definition 4.3 (Chaotic set, [6]). *Let* $O(x_0)$ *be a chaotic orbit. If* x_0 *is in* $\omega\left(x_0, f\right)$, *the* ω*-limit set, then* $\omega\left(x_0, f\right)$ *is called a* ***chaotic set****.*

Definition 4.4 (Chaotic Attractor, [6]). *A chaotic attractor is a chaotic set that is also an attractor.*

Definition 4.5 (Fractal, [15]). *A set of points is said to be fractal if one of its fractal dimensions is non-integer.*

Definition 4.6 (Strange Attractor, [13]). *An attractor is strange if it is fractal.*

We will discuss the definitions of three fractal dimensions and their algorithms thoroughly in **Section 5**.

4.2 Difference between Chaotic Attractor and Strange Attractor

It is possible for chaotic attractors not to be strange, just as it is possible for attractors to be strange but not chaotic. To illustrate this, we will provide two examples using the logistic map, which has been previously mentioned in **Example 2.2**.

Example 4.1 (Chaotic Attractor that is not Strange). *We consider the logistic map* $x_{n+1} = rx_n(1 - x_n), x_n \in [0, 1]$, *which is one-dimensional. The logistic map at* $r = 4$ *has a chaotic attractor with Lyapunov exponent* $h = ln(2) > 0$. *However, the attractor of the system under this condition is not strange. This is because for the logistic map with* $r = 4$, *all points within the interval* $[0, 1]$ *form part of its attractor, resulting in a one-dimensional attractor, not a fractal-dimensional strange attractor by definition (the dimension of the attractor is 1, which is an integer). In other words, although the behavior of the system is chaotic, the structure of its attractor is not complex, so we say it is not strange.[16]*

Example 4.2 (Strange Attractor that is not Chaotic). *The logistic map at* $r \approx 3.56995$ *(the accumulation point of the logistic map) is the onset of chaos. For this value of* r, *we no longer see oscillations of finite period for all initial conditions. Slight variations in the initial population yield dramatically different results over time. It has an attractor which has a Lyapunov exponent* $\lambda = 0$ *but is a Cantor set with Hausdorff dimension* $d \approx 0.538$.*[17] Hence it is a strange non-chaotic attractor.*

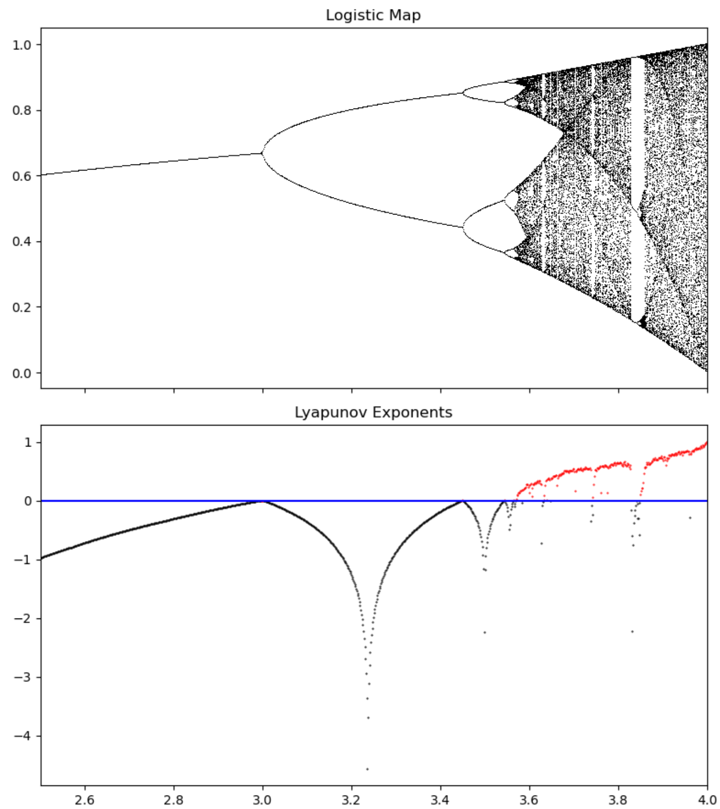


Figure 3: The horizontal axis is the parameter a. The upper figure is the bifurcation diagram of the logistic map. In the lower figure, black points indicate Lyapunov exponents less than 0, suggesting stability, whereas red points indicate Lyapunov exponents greater than 0, suggesting chaos. The exponent rises and hits zero at period doublings, a phenomenon where the period of the system doubles, and becomes positive in the chaotic region. It drops below 0 when there is a periodic attractor (a periodic orbit that is also an attractor). The code used to generate this figure can be found in **Section 9**.

4.3 Properties of Chaotic and Strange Attractors

Despite the two examples above, strangeness and chaos commonly occur together.

Example 4.3. *[5] As mentioned in **Section 3.4**, the Hénon map* ($a = 1.4, b = 0.3$)

$$x_{n+1} = a - x_n^2 + by_n \tag{4}$$

$$y_{n+1} = x_n \tag{5}$$

*has a positive Lyapunov exponent ($\lambda \approx 0.42$) so we know that this system is chaotic. And as calculated in **Example 5.4** later, the attractor of the Hénon map is strange as it has a non-integer dimension (its Lyapunov dimension $dim_L \approx 1.25812$). Hence this is a chaotic and strange attractor.*

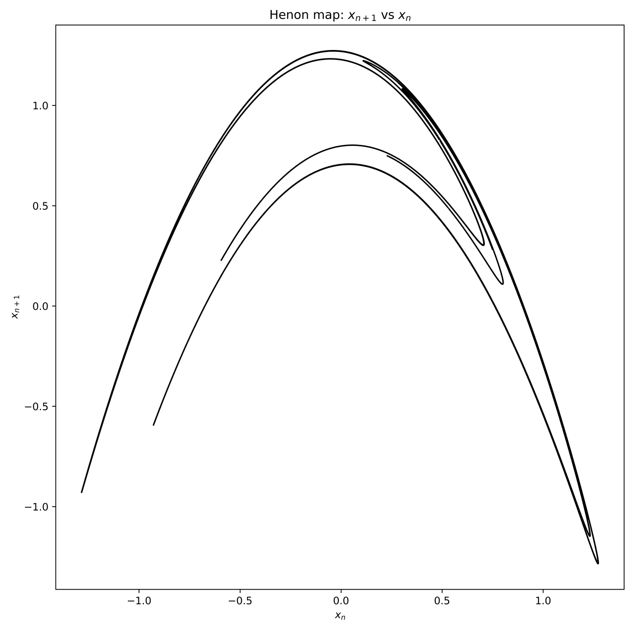


Figure 4: This figure shows the result of plotting 10,000 successive points obtained by iterating (4) and (5) with parameters $a = 1.4$ and $b = 0.3$ and initial condition $(x_0, y_0) = (0, 0)$. The figure x_{n+1} against x_n displays the evolution of the dynamical system.[6] The reader can find the code generating Figure 4 in **Section 9**.

Here are the properties of chaotic and strange attractors, and these properties can be shown through the Hénon map example.

1. **Non-periodic Trajectory**: Non-periodicity in Hénon map ($a = 1.4, b = 0.3$) means that the system's evolution does not repeat in a regular cycle, but instead exhibits complex and unpredictable patterns. This can be observed in Figure 4 - there are no integers n and p such that $(x_n, y_n) = (x_{n+p}, y_{n+p})$.
   The non-periodic nature of the Hénon map arises from its non-linear dynamics, which generate a complex attractor, known as a strange attractor. The existence of this strange attractor allows for continuous evolution

of the system without returning to a repeating cycle, resulting in non-periodic behavior.[18]

2. **Sensitivity to initial conditions**: Let (X, f) be a dynamical system. A point x_0 has sensitive dependence on initial conditions if there is a non-zero distance d such that some points arbitrarily near to x_0 are eventually mapped at least d units from the corresponding image of x_0. More precisely, there exists $d > 0$ such that any neighborhood N of x_0 contains a point x such that $\left|f^k(x) - f^k\left(x_0\right)\right| \geq d$ for some non-negative integer k[6].

3. **Non-predictability**: The Hénon map, like many chaotic systems, exhibits sensitivity to initial conditions, which contributes to its unpredictability. Consider two initial points $(0, 0)$ and $(0.00001, 0)$, which are slightly apart. As time evolves, these points will diverge. With an initial difference of δ, the difference at time n can be approximated as $\delta \cdot e^{n\lambda}$, where λ is the Lyapunov exponent (approximately 0.42 for the Hénon map).
   In reality, exact initial conditions are unobtainable due to measurement constraints. Because of the system's sensitivity to initial conditions, even the smallest measurement error in the initial state can lead to significant differences in the long-term behavior of the Hénon map, making it unpredictable. This unpredictability is shown in Figure 5, demonstrating the difference between two trajectories over time.
   Nevertheless, the system's evolution is confined within the strange attractor, ensuring the divergence doesn't become infinite. Therefore, although the exact trajectory becomes unpredictable over time, it remains within the boundaries of the attractor.[18]

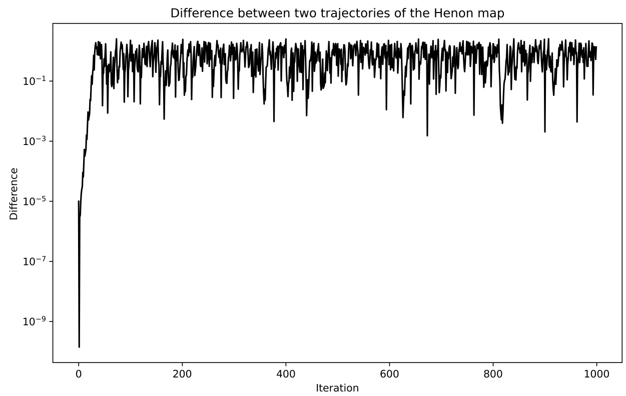


Figure 5: This figure illustrates the difference between two trajectories of the same Hénon map ($a = 1.4, b = 0.3$) with initial conditions $(x_0, y_0) = (0, 0)$ and $(x_0, y_0) = (0.00001, 0)$. The reader can find the code generating Figure 5 in **Section 9**.

5 Dimensions

We would like to learn more about the characteristics of different attractors in dynamical systems in order to distinguish them, and one approach to do this is to measure their fractal dimensions. We will look into the Lyapunov dimension, the Box-Counting dimension and the Hausdorff dimension.[19]
An attractor is difficult to describe precisely, but since we can consider it as a fractal-like set, we can approximate it using fractal dimensions. We can also check the correctness of the approximation by using other properties of the dynamical system, such as the Lyapunov exponents.[20]
In this section we will first give some fundamental notations and definitions for the various dimensions and then attempt to make them clearer by interpreting them geometrically. Secondly, we will provide some examples of dynamical systems to help the reader understand attractors better. Finally, we will attempt to connect the Lyapunov exponents and the dimension theory.

5.1 Basic Notations

Definition 5.1 (Lyapunov Dimension, [20]). *The Lyapunov dimension is defined as*

$$\dim_L(Z) = \begin{cases} 0 & \text{if } \lambda_1 \leq 0 \\ k + \frac{\lambda_1+\lambda_2+\ldots+\lambda_k}{|\lambda_{k+1}|} & \text{otherwise} \end{cases}$$

where λ_i is the i^{th} greatest Lyapunov exponent ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$), k is the largest value such that $\sum_{i=1}^{k} \lambda_i > 0$, and Z is the attractor associated with the system. Notice the Lyapunov dimension is determined only by the Lyapunov exponents associated with the system.

Definition 5.2 (Lower and Upper Box-Counting Dimensions, [21]). *Let X be a subset of a smooth manifold, the lower and upper Box-Counting dimensions of $Z \subset X$ are defined, respectively, by*

$$\underline{\dim}_B Z = \liminf_{\varepsilon \to 0} \frac{\log N(Z,\varepsilon)}{-\log \varepsilon}$$

and

$$\overline{\dim}_B Z = \limsup_{\varepsilon \to 0} \frac{\log N(Z,\varepsilon)}{-\log \varepsilon}$$

where (N, ε) denotes the least number of balls of radius ε that are needed to cover the set Z. If they are equal, we define this common value as the Box-Counting dimension of Z

$$\dim_B Z = \lim_{\varepsilon \to 0} \frac{\log N(Z,\varepsilon)}{-\log \varepsilon}$$

Note that here we assume ε is sufficiently small to make sure that $-\log \varepsilon$ is positive. However, the Box-Counting dimension may not always exist, If the upper and lower Box-Counting dimensions are not equal, then it is not well defined. Hausdorff dimension can solve this problem.

Definition 5.3 (Hausdorff Dimension, [21])**.** *Let X be a n-dimensional Euclidean space, and let d be the distance in X. We define the diameter of a collection A of subsets of X by*

$$\operatorname{diam} A = \sup\{\operatorname{diam} U : U \in A\}$$

where

$$\operatorname{diam} U = \sup\{d(x,y) : x, y \in U\}$$

is the diameter of the set U. Given $Z \subset X$ and $\alpha \subset R$, we define the α-dimensional Hausdorff measure of Z by

$$m(Z,\alpha) = \lim_{\varepsilon \to 0} \inf_{A} \sum_{U \in A} (\operatorname{diam} U)^{\alpha}$$

where the infimum is taken over all finite or countable covers A of the set Z with diam $A \leq \varepsilon$. One can easily verify that the function $\alpha \to m(Z,\alpha)$ jumps from ∞ to 0 at a single point. We define this number to be the Hausdorff dimension of Z.
Thus the Hausdorff dimension of a set $Z \subset X$ is defined by

$$\begin{aligned} \dim_H Z &= \inf\{\alpha \in \mathbb{R} : m(Z,\alpha) = 0\} \\ &= \sup\{\alpha \in \mathbb{R} : m(Z,\alpha) = +\infty\} \end{aligned}$$

The Hausdorff dimension $\dim_H Z$ is defined for all set $Z \subset \mathbb{R}^n$, but normally it is hard to compute. Only the Julia sets (the reader can find more about this in [22]) and self-similar sets can be computed exactly. Box-Counting dimensions are typically simpler to compute.
Note that the values of the dimensions do not have to be integers. We now explore more about the relationship between the Box-Counting dimension and the Hausdorff dimension.
When we calculate the Hausdorff dimension, we assign different values to the diameter of different U, whereas for the Box-Counting dimensions we assign the same value ε for each covering set Z. In contrast to the Hausdorff dimension, which involves coverings by sets of small but potentially widely varied size, Box dimensions are thought of as demonstrating the efficiency with which a set may be covered by small sets of equal size, this is also part of the reasons why it is easier to compute.

Proposition 5.1. *By the Kaplan-Yorke Conjecture [20], we have*

$$\dim_B Z \geq \dim_L Z$$

So the Lyapunov dimension is a lower bound of the Box-Counting dimension, this means that we expect the Box-Counting dimension to be positive if we have one positive Lyapunov exponent, which also indicates that the dynamical system is chaotic.

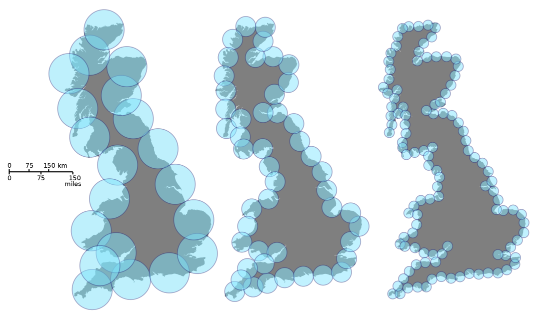
Proposition 5.2. *We also have*

$$\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z$$

The reader can find the proof of the above in [23]. We can see that the Box-Counting dimension is an upper bound of the Hausdorff dimension, we do not generally obtain the equality.

5.2 Geometric Interpretation

Example 5.1 (The Coast of Great Britain)**.**



[24]

We aim to measure the fractal dimensions of the coast line. The radius of the balls is the same in one graph, and as they go smaller, we are getting closer to the limit, which is the value of the Box-Counting dimension. Also it can be seen in the image where the least number of balls required to cover the coast line are drawn. The approximate results for Box-Counting and Hausdorff dimensions are 1.18 *and* 1.25 *[25], respectively.*

Example 5.2 (Vicsek Fractal)**.**

[26]

This is a self-similar fractal, instead of considering number of balls to cover these figures, we can directly think of number of boxes needed to cover them. When the size of the boxes gets smaller, again we approach the limit, this time the fractal dimensions have exact and consistent values.
It has an exact Box-Counting dimension

$$\dim_B = \lim_{n\to\infty} \frac{\log 5^n}{-\log(1/3)^n} = \frac{\log 5}{-\log(1/3)} = \frac{\log 5}{\log 3} \approx 1.465$$

where we assume the side length of the first square is 1*, and* n *is the number of iterations.*

It has an exact Hausdorff dimension

$$\dim_H = \frac{\log 5}{\log 3} \approx 1.465$$

5.3 Examples in Dynamical Systems

Example 5.3 (1D Logistic Map). *In Figure 3 we obtained the bifurcation diagram of the Logistic map. It is self-similar - the pattern is repeated if we zoom in. The graph also shows an exponential divergence, which indicates the existence of chaos. This is an example of the connection between chaos and fractals, it has a Hausdorff dimension of about* 0.538 *for* $r \approx 3.5699456$*.[17] The set of points produced by repeated iterations of the logistic function for this r is known as the Feigenbaum attractor.*

Example 5.4 (2D Hénon Map). *In Figure 4 we obtained the attractor of the Hénon map. It is also a self-similar diagram. For the classical map (*$a = 1.4$ *and* $b = 0.3$*), an initial point of the plane will either diverge to infinity or approach a set of points known as the Hénon strange attractor which is a fractal.*
For the Lyapunov dimension, we have approximately
$\lambda_1 = 0.4189, \lambda_2 = -1.6229, k = 1$*, so*

$$\dim_L = 1 + \frac{0.4189}{|-1.6229|} \approx 1.25812$$

We calculate the Box-Counting dimension using the code in ***Section 9****. We get* $\dim_B \approx 1.25167$*.*
For the Hausdorff dimension, $\dim_H \approx 1.261$*.[27]*

The results seem to disagree with what we expected - the Lyapunov dimension should be smaller than the Box-Counting dimension by **Proposition 5.1**. After we tried other examples, the computed values for the Box-Counting dimension seem to be less accurate than the actual values, whereas the computed values for the Lyapunov dimension appear more reliable. The computation method for calculating the Box-Counting dimension really depends on the size of the balls used to cover the set. If the size is not small enough, the result may be inaccurate.
After accounting for some numerical bound errors, we can see all these values seem to agree with the propositions we mentioned earlier. Therefore, we are more certain of the accuracy of our approximations.

6 Bedhead attractor

In this section we will apply the theory we have learnt in the above sections to another not so popular 2D discrete-time dynamical system - the Bedhead attractor. We will analyse its Lyapunov exponents, whether the system is chaotic or not and type of attractors it has. The map is given by

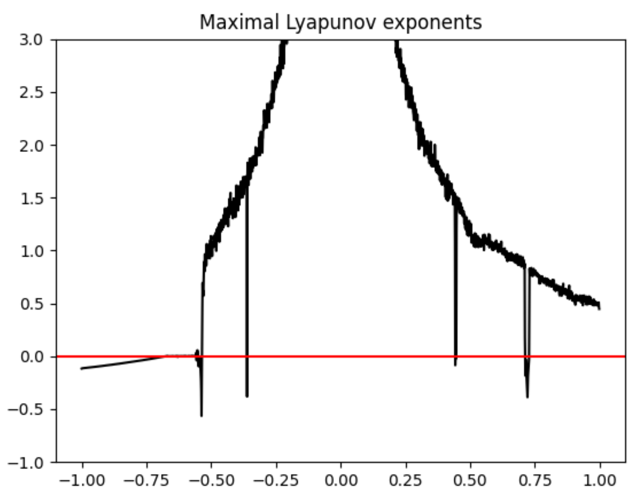
$$x_{n+1} = sin\left(\frac{x_n y_n}{b}\right) y_n + cos(ax_n - y_n)$$

$$y_{n+1} = x_n + \frac{sin(y_n)}{b}$$

with initial condition $(x_0, y_0) = 1$ and with parameters a and b both between -1 and 1 and $b \neq 0$.
We will explore the behaviour of this map when we keep a fixed.

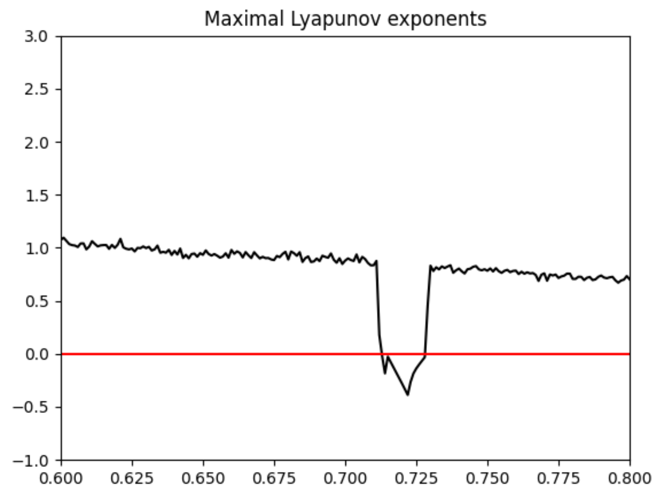
6.1 Bedhead attractor with $a = 0.65343$

We would like to start our analysis with determining the Lyapunov exponents for all values of b between -1 and 1 and therefore deciding when the system is chaotic.
Applying the QR decomposition algorithm we saw in **Section 3**, we can obtain the following plot for the maximal Lyapunov exponent when b is between -1 and 1. The reader can find the code generating the figure below in **Section 9**.

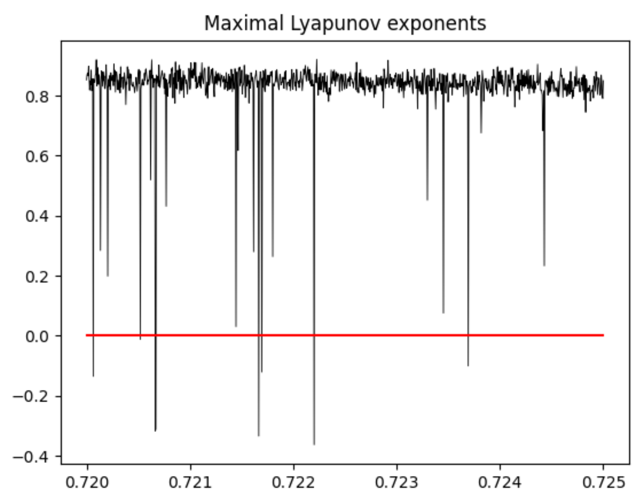


As we can see, the plot for positive and negative values of b is pretty smooth apart from some dips, where the attractor suddenly becomes non-chaotic. Moreover, as b gets close to 0, the maximal Lyapunov exponent explodes. This is due to the fact that the determinant of the derivative of the map gets large because there are terms with denominator b. Let us examine the behaviour of the attractor around one of the dips. Let us fix b to be between 0.6 and 0.8.

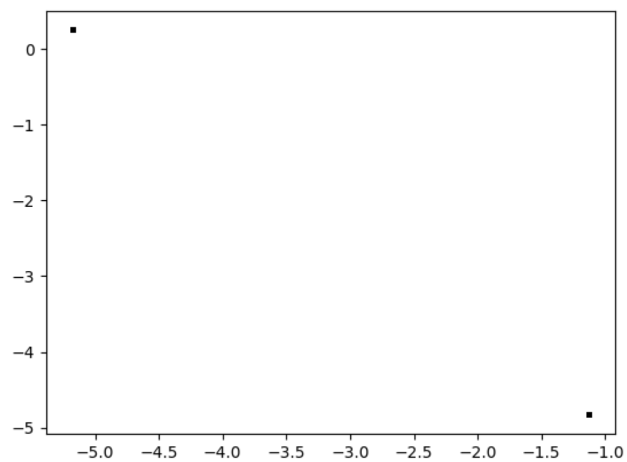
The zoomed in plot looks like this.



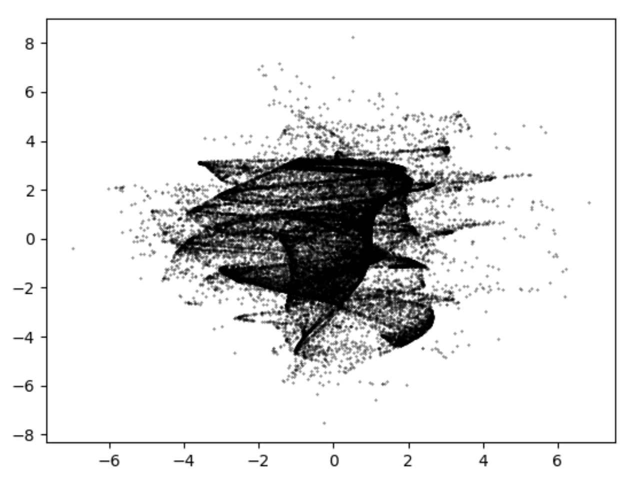
As we can see, the dip occurs in a small interval around $b = 0.7225$. Let us zoom in further - we will plot the maximal Lyapunov exponents for 1000 points in the interval $[0.72, 0.725]$.



Let us examine the first dip that occurs. This happens when $b = 0.7200650650650651$. We have a negative maximal Lyapunov exponent, so we are expecting a non-chaotic behaviour. For this value, the attractor looks like this



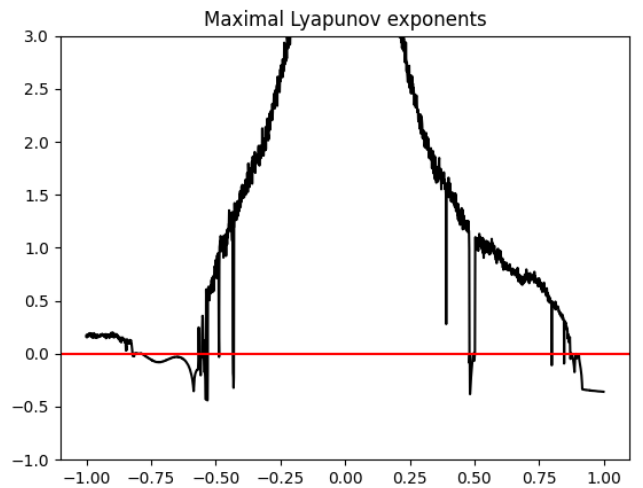
We can see that the attractor is only 2 points! However, it turns out when we plot the orbit, these two points contain a lot of different points that are infinitely close to them. The orbit is asymptotically periodic to the two points in the above figure. Therefore the attractor is indeed non-chaotic.
Now let us plot the attractor for $b = 0.7201$ to demonstrate the difference. For $b = 0.7201$ the maximal Lyapunov exponent is positive, so we are expecting a chaotic behaviour.



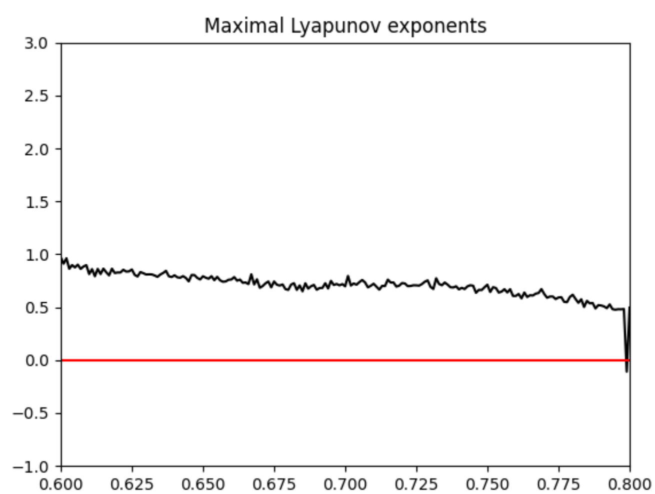
We can see that the attractor is indeed chaotic and strange. This shows that a very small change in one of the parameters in a dynamical system can have a dramatic effect on the behaviour of the attractor. The reader can find the code generating the plots of the attractors in **Section 9**.

6.2 Bedhead attractor with $a = -0.81$

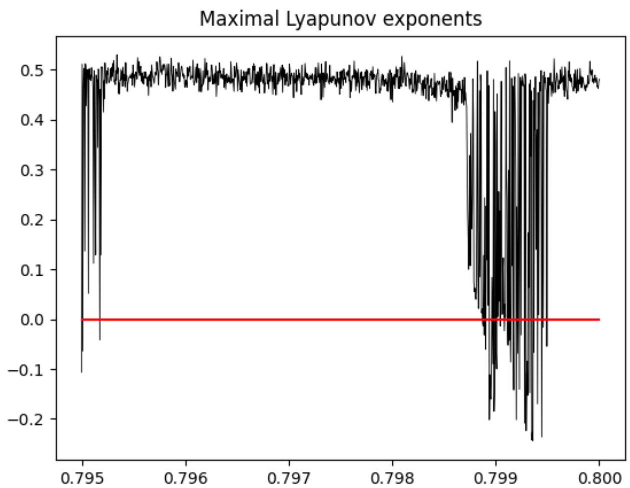
Again, we start our analysis with determining the Lyapunov exponents for all values of b between -1 and 1 and therefore deciding when the system is chaotic. Applying the QR decomposition algorithm, we obtain the following plot for the maximal Lyapunov exponent when b is between -1 and 1.



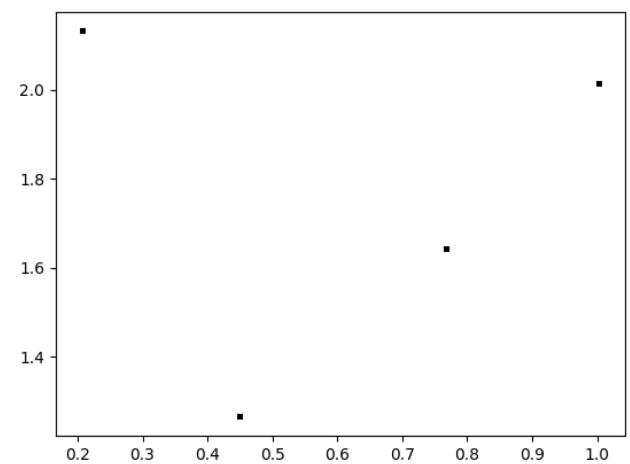
Similarly to before, for positive and negative values of b the plot is relatively smooth apart from some dips, where the attractor suddenly becomes non-chaotic. As before, when b is close to 0, the maximal Lyapunov exponent explodes. Again, let us examine the behaviour of the attractor around one of the dips. Fixing b to be between 0.6 and 0.8 and zooming in we get



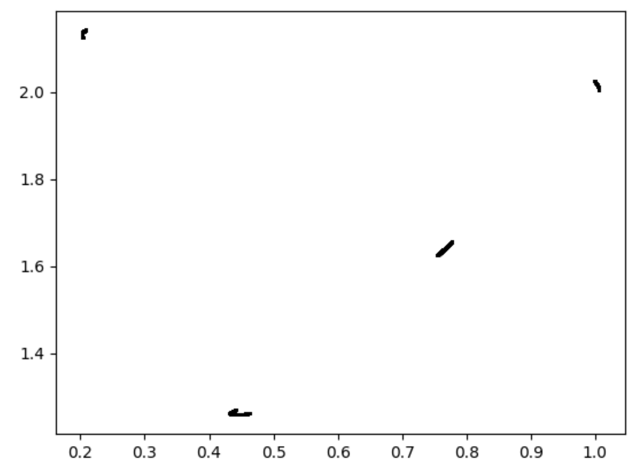
As we can see, the dip occurs in a small interval around $b = 0.8$. Let us zoom in further - we will plot the maximal Lyapunov exponents for 1000 points in the interval $[0.795, 0.8]$.



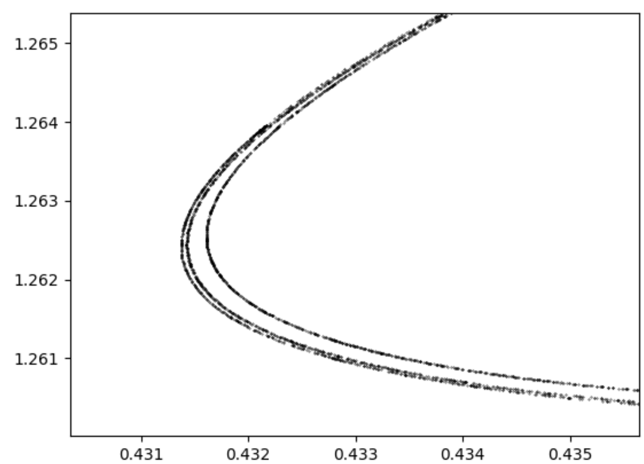
One of the dips occurs when $b = 0.799304304304304$. We have a negative maximal Lyapunov exponent, so we are expecting a non-chaotic behaviour. For this value, the attractor looks like this



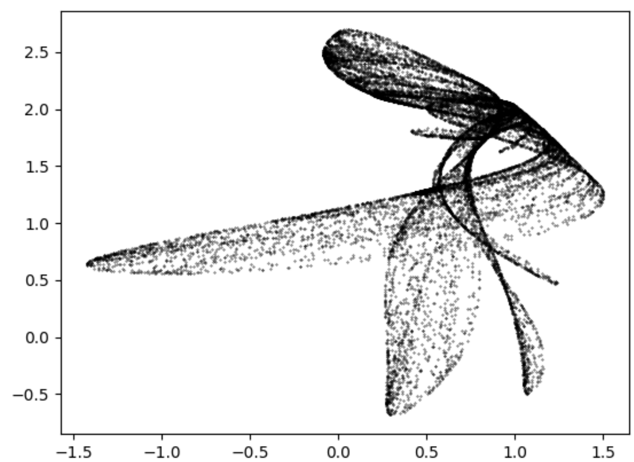
We can see the attractor is a set of 4 points which is a period-4 orbit. Now let us see plot the attractor for a smaller $b = 0.798734$. We have a positive maximal Lyapunov exponent, so we are expecting a chaotic behaviour.



As we can see, there are four separated parts in the figure. When we further zoom in the botttom one, we get.



This a strange attractor with a similar pattern to the strange attractor of the Hénon map. Now let us plot the attractor for $b = 0.798733$ which is very close to 0.798734 to demonstrate the difference in the behaviour of the attactors.



We can see that the attractor is indeed chaotic. Again, this shows that a very small change in one of the parameters in a dynamical system can have a dramatic effect on the behaviour of the attractor.

7 Ergodic Theory

7.1 Introduction and Basic Notions

Ergodic theory studies statistical properties of dynamical systems. The statistical properties are always expressed through the behavior of time averages of various random variables (i.e., measurable functions) along the orbits of dynamical systems. Invariant measures or measure-preserving transformations play a fundamental role in the description of dynamical systems. We introduce invariant measures here.

Definition 7.1 (Invariant Measures, [28])**.** *Let $(X, \mathcal{F})$ be a measurable space. A probability measure $\mu : \mathcal{F} \to [0, \infty]$ is called an* ***invariant measure*** *with respect to a measurable mapping $f : X \to X$ if*

$$\mu(f^{-1}(A)) = \mu(A)$$

for all $A \in \mathcal{F}$. The measurable mapping f is called the ***measure-preserving transformation*** *of μ.*

To analyze a statistical property of a dynamical system, we seek a connection between the invariant measure and the behaviors of typical orbits. In particular, for any $A \in \mathcal{F}$, we aim to attain $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_A(f^i(x)) = \mu(A)$ for μ-almost $x \in X$. To achieve this, the invariant measures should be restricted for invariant sets, which motivates the definition of ergodic measures.

Definition 7.2 (Ergodic Invariant Measures, [28])**.** *Let $(X, \mathcal{F}, \mu)$ be a probability space and $f : X \to X$ be a measurable function such that μ is invariant with respect to f. The invariant probability measure μ is called* ***ergodic*** *if any $A \in \mathcal{F}$ with $f^{-1}(A) = A$ has either measure zero or full measure, i.e., $\mu(A) \in \{0, 1\}$.*

After ergodic measures are defined, the following important theorem gives a sufficient condition for the existence of the time averages and a relation between the time averages and the space averages.

Theorem 7.1 (Birkhoff's Ergodic Theorem, [28])**.** *Let $(X, \mathcal{F}, \mu)$ be a probability space and $f : X \to X$ be a measurable function such that μ is invariant with respect to f. Let $g : X \to \mathbb{R}$ be an integrable random variable. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x))$$

exists for μ-almost all $x \in X$ and defines a μ-integrable random variable. Additionally, suppose that μ is ergodic with respect to f. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x)) = \int_X g \, d\mu$$

for μ-almost all $x \in X$.

The reader can find the proof of this theorem on Page 29 of [28].

7.2 Oseledets Theorem

Let us focus on the Ergodic theorem of a discrete-time dynamical system on the metric space we are interested. To apply **Theorem 7.1**, some topological properties of (X, f) are required to ensure an invariant measure.

Theorem 7.2 (Krylov–Bogolyubov Theorem, [28])**.** *Let X be a compact metric space and $f : X \to X$ be a continuous map. Then there exists a probability measure $\mu : \mathcal{B}(X) \to [0, 1]$ that is invariant with respect to f.*

See the proof of this theorem in Page 44 of [28].

Next, we want find an ergodic measure in invariant measures with respect to f. Let us first move back to a more general case that $(X, \mathcal{F})$ is a measurable space and $f : X \to X$ is a measurable map. Define $\mathcal{M}(f)$ as the set of invariant measures for f. We can show $\mathcal{M}(f)$ is a convex set [28]: Suppose that μ_1, μ_2 are two invariant measures with respect to f. Then the convex combination $\mu := \gamma\mu_1 + (1 - \gamma)\mu_2$ for any $\gamma \in [0, 1]$ is also an invariant measure, because of

$$\mu(f^{-1}(A)) = \gamma\mu_1(f^{-1}(A)) + (1-\gamma)\mu_2(f^{-1}(A)) = \gamma\mu_1(A) + (1-\gamma)\mu_2(A) = \mu(A)$$

for all $A \in \mathcal{F}$.

Theorem 7.3 (Ergodic Measures are Extremal Points, [28])**.** *Let $(X, \mathcal{F})$ be a measurable space. Consider a measurable map $f : X \to X$. Then μ is ergodic if and only if it is an extremal point of $\mathcal{M}(f)$, which means if $\mu = \gamma\mu_1 + (1-\gamma)\mu_2$ for some $\mu_1 \neq \mu_2 \in \mathcal{M}(f)$, then necessarily $\gamma = 1$ or $\gamma = 0$.*

See the proof of this theorem in Page 30 of [28].

Now, with the convexity and the compactness on the metric space, there exists an ergodic invariant measure of the dynamical system.

Theorem 7.4 (Existence of an Ergodic Measure, [29])**.** *Let X be a compact metric space and $f : X \to X$ be a continuous map. Then there exists at least one ergodic measure in $\mathcal{M}(f)$.*

See the proof of this theorem in Page 75 of [29].

The proofs of these theorems are closely related to the weak*-topology defined on $\mathcal{M}(f)$ [29], which is beyond the scope of what we learned.

Instead of additive averages, we can also consider multiplicative ones. Informally, set $h := \exp g$, then for μ-almost $x \in X$, we have

$$\lim_{n\to\infty} \Big(\prod_{i=0}^{\infty} h(f^i(x))\Big)^{\frac{1}{n}} = \exp \int_X \log h \, d\mu$$

Theorem 7.5 (Oseledets Theorem, Simple Version, [30] [31])**.** *Let X be a compact subset of $\mathbb{R}^d$, $f : X \to X$ be a continuous map, and μ be an ergodic invariant measure with respect to f. Consider a measurable map $T : X \to \mathbb{R}^{m\times m}$ such that*

$$\int_X \log^+ \|T(x)\| \, d\mu < \infty$$

Here $\log^+(x) := \max(0, \log x)$*. Define* $T^{[n]}(x) = T(f^{n-1}(x))T^{[n-1]}(x)$ *for all* $n \in \mathbb{N}_0$ *recursively, i.e.,* $T^{[n]}(x) = T(f^{n-1}(x))T(f^{n-2}(x))...T(f(x))T(x)$*. Then the following holds:*

1. $\Lambda_{x_0} := \lim_{n\to\infty}\Big((T^{[n]}(x_0))^*T^{[n]}(x_0)\Big)^{\frac{1}{2n}}$ *exists for* μ*-almost* $x_0 \in X$*, where* A^* *denotes the transpose of the matrix* A*.*
2. *there exists* $\lambda_1 > \lambda_2 > ... > \lambda_k \in [-\infty, +\infty)$ *such that for* μ*-almost* $x_0 \in X$*, there exist a sequence of* Λ_{x_0}*-invariant subspaces called the* ***forward Oseledets filtration*** $\mathbb{R}^d = V^1_{x_0} \supset V^2_{x_0} \supset ... \supset V^k_{x_0} \supset V^{k+1}_{x_0} = \{0_v\}$ *such that for any* $u \in V^i_{x_0} \setminus V^{i+1}_{x_0}$*,* $\lim_{n\to\infty} \frac{1}{n} \log\|T^{[n]}(x_0)u\| = \lambda_i$*.*

Oseledets theorem is a generalization of Birkhoff's ergodic theorem from an integrable random variable g to the norm of a random matrix T which satisfies the **cocycle property**: if we write $T^{[n]}(x)$ as $T(x, n) : X \times \mathbb{N}_0 \to \mathbb{R}^m \times \mathbb{R}^m$ such that $T(x, 0) = I_m$ and $T(x, n) = T(f^{n-1}(x), 1)T(x, n-1)$ for all $x \in X$ and $n \in \mathbb{N}$.

7.3 Applications of the Ergodic Theorems

In this section, we assume for a discrete-time dynamical system (X, f), X is compact in $\mathbb{R}^d$, f is continuously differentiable, and μ is an ergodic invariant measure with respect to f.

7.3.1 Rigorous Definition of the Lyapunov Exponents

Note that for a discrete system (X, f), define $T : X \times \mathbb{N}_0 \to \mathbb{R}^d \times \mathbb{R}^d$ such that $T(x_0, n) := Df^n(x_0)$. Then T satisfies the cocycle property by the chain rule of the derivatives:

$$T(x_0, n) = Df^n(x_0) = Df(f^{n-1}(x_0))Df^{n-1}(x_0) = T(f^{n-1}(x_0), 1)T(x_0, n-1)$$

Since f is continuously differentiable, $T(x_0, 1) = Df(x_0)$ is continuous and so measurable. X is compact, so $\|T(x_0, 1)\|$ attains its maximum in X. By the inequality $\log^+ x \leq x$ for all $x \in [0, +\infty)$, and monotonicity of integrals,

$$\int_X \log^+ \|T(x_0, 1)\| \, d\mu \leq \int_X \|T(x_0, 1)\| \, d\mu < \infty$$

Now applying Oseledets theorem, we know that there exist $\lambda_1 > \lambda_2 > \cdots > \lambda_k \in [-\infty, +\infty)$ for μ-almost x_0. We define $\lambda_1, \lambda_2, \ldots, \lambda_k$ as the Lyapunov exponents of (X, f) at x_0.

7.3.2 Maximal Lyapunov Exponent

Let us assume that $\dim(X) > 1$. Recall in **Definition 3.3**, assume we have a small perturbation u_0 in any direction at x_0 for the maximal Lyapunov exponent $\lambda(x_0, u_0)$. In the last example, we showed that there exist the maximal Lyapunov exponent and the corresponding subspace such that for any $u_0 \in V^1_{x_0} \backslash V^2_{x_0}$, $\lim_{n\to\infty} \frac{1}{n} \log\|Df^n(x_0)u_0\| = \lambda_1$, where $V^1_{x_0} = \mathbb{R}^d$. If we randomly choose a $u_0 \in \mathbb{R}^d$, it highly likely lies in $V^1_{x_0} \setminus V^2_{x_0}$, instead of $V^2_{x_0}$ which is at most a hypersurface in $\mathbb{R}^d$, i.e. $\dim(V^2_{x_0}) \leq d-1$. Thus it makes the computation of λ_1 easy which is almost independent of the small perturbation u_0.

7.3.3 Review of the Geometrical Definition

Definition 3.1 states that the Lyapunov exponents are related to the growth of the semiaxes of an ellipsoid transformed by the limit of $Df^n(x_0)$. Note that the singular values of any square matrix in $\mathbb{R}^{d\times d}$ can be viewed as lengths of the semiaxes of the d-dimensional ellipsoid transformed from the unit sphere. By the Oseledets theorem and the rigorous definition from **Section 7.3.1**, the eigenvalues of the limit $\Lambda_{x_0} = \lim_{n\to\infty}\Big((Df^n(x_0))^\top Df^n(x_0)\Big)^{\frac{1}{2n}}$ are analogous to the singular values of the limit of $Df^n(x_0)$. Suppose $\mu_1 > \mu_2 > ... > \mu_k$ are distinct eigenvalues of Λ_{x_0}, then $\mu_i = \exp\lambda_i$ where λ_i are the Lyapunov exponents. Suppose U_{μ_i} is the corresponding eigenspace of μ_i, the subspaces in the forward Oseledets filtration $V^i_{x_0} = \bigoplus_{j=i}^k U_{\mu_i}$. Thus, intuitively, the two definitions of the Lyapunov exponents are equivalent.

7.3.4 Proof of Theorem 3.2

Recall that in **Section 3.3**, we know $Df^n(x_0) = Q_n(x_0)R^{[n]}(x_0)$ where Q_n is an orthonormal matrix and $R^{[n]}(x_0)$ is a upper triangular matrix using the QR decomposition algorithm. Then we have

$$\begin{aligned}
\Lambda_{x_0} &= \lim_{n\to\infty}\Big((Df^n(x_0))^\top Df^n(x_0)\Big)^{\frac{1}{2n}} \\
&= \lim_{n\to\infty}\Big((R^{[n]}(x_0))^\top (Q_n(x_0))^\top Q_n(x_0)R^{[n]}(x_0)\Big)^{\frac{1}{2n}} \\
&= \lim_{n\to\infty}\Big((R^{[n]}(x_0))^\top R^{[n]}(x_0)\Big)^{\frac{1}{2n}}
\end{aligned}$$

For the upper triangular matrix $R^{[n]} \in \mathbb{R}^{d\times d}$, $\text{diag}(R^{[n]}) = \{r_1^{[n]}, r_2^{[n]}, ..., r_d^{[n]}\}$ equals the set of eigenvalues of $R^{[n]}$. For the large enough n, the eigenvalues are approximately the Lyapunov exponents. Suppose the eigenvalues are distinct, let $\{\hat{v_1}, \hat{v_2}, ..., \hat{v_d}\}$ be the corresponding eigen-basis. Define $V_i = \text{span}(\cup_{j=i}^d\{\hat{v_j}\})$. We can see the sequence $\mathbb{R}^d = V_1 \supset V_2 \supset ... \supset V_d$ which is approximately the forward Oseledets filtration. Thus we can compute the Lyapunov exponents by $\lambda_i = \lim_{n\to\infty}\frac{\log r_i^{[n]}}{n}$. Intuitively, we are applying Cholesky decomposition on Λ_{x_0}, i.e. $\Lambda_{x_0} = (R(x_0))^\top R(x_0)$ where $R(x_0)$ is an upper triangular matrix and the eigenvalues of Λ_{x_0} are equal to the eigenvalues of $R(x_0)$.

7.3.5 Proof of Theorem 3.1

Recall that in **Theorem 3.1**, we use the limit of the average determinant to calculate the sum of the Lyapunov exponents. The existence of this limit is given by the Birkhoff's ergodic theorem. Using the QR decomposition algorithm described above, then we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log|\det(Df^n(x_0))| &= \lim_{n\to\infty}\frac{1}{n}\log|\det(Q_n(x_0)R^{[n]}(x_0)| \\
&= \lim_{n\to\infty}\frac{1}{n}\log|\det(R^{[n]}(x_0)| \\
&= \lim_{n\to\infty}\frac{1}{n}\log|\prod_{i=1}^{d} r_i^{[n]}| \\
&= \sum_{i=1}^{d}\lim_{n\to\infty}\frac{1}{n}\log|r_i^{[n]}| \\
&= \sum_{i=1}^{d}\lambda_i
\end{aligned}$$

where λ_i is the i-th Lyapunov exponent.

8 Conclusion

In this paper, we have provided the reader the essential knowledge and tools to analyze any 2D discrete-time dynamical system. In the first sections, we introduced discrete-time dynamical systems and Lyapunov exponents. We developed the essential analytical and numerical tools to calculate the Lyapunov exponents, which can be applied to any 2D dynamical system. Future work may include the exploration of SVD decomposition - another method of calculating Lyapunov exponents. Furthermore, comparisons with QR decomposition can be made in order to determine when one algorithm is more efficient and accurate than the other.

We delved into the concepts of chaos and strange attractors, highlighting their roles in discrete-time dynamical systems. We explored their definitions and demonstrated their practical implications through a series of examples. Key differences were noted between chaos and strange attractors, despite their shared ability to generate complex behavior in dynamical systems. Then, we explored dimensions, which help us understand the meaning of a strange attractor better. Almost all initial values in the examples of dynamical systems we gave eventually settle on an attractor. Notice that their Box-Counting and Hausdorff dimension have the same value, and they coincide with their Lyapunov dimension in all cases. This is because the figures in our examples are all self-similar. As a future work, we may examine cases where the figures are not self-similar. Moreover, we can attempt to develop better algorithms for calculating the Box-Counting and Hausdorff dimensions.

We applied all theory to the Bedhead attractor. We analyzed its behaviour in two different cases. This shows how the reader can apply a similar analysis to any 2D discrete-time dynamical system. Finally, we introduced the Ergodic theory, which is essential to prove some of the results established in this paper.

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9 Appendix

9.1 Dynamics and orbits

To analyze discrete-time dynamical systems numerically, we can define a class with attributes evolution function f and its derivative Df. Moreover, we can compute the orbit from any initial state for any finite times of evolution.

```
class Dynamics:
    """Define a discrete-time dynamical system.

    Parameters:
    f: the evolution function of the dynamical system
    Df: the derivative of the evolution function
    """

    def __init__(self, f, Df):
        self.f = f
        self.Df = Df

    def orbit(self, x0: tuple, r: tuple, n=10000) -> list:
        """Return the list of orbit f^i(x0) for 0 <= i <= n-1.

        Parameters:
        x0: the initial value(state) of the orbit in tuple
        r: the parameters for the evolution function f in tuple
        n: the number of evolutions, with default value 10000

        Notes:
        The length of the return list will be n.
        """
        ox = [x0] + [0 for _ in range(n-1)]
        for i in range(1, n):
            ox[i] = self.f(ox[i-1], r)
        return ox
```

9.2 1D dynamics and chaos in the Logistics map

For 1D dynamical systems, we can plot the orbits over evolution from any initial state x_0 for any parameter(s) of the evolution function f. Moreover, to analyse the sensitivity dependence on the initial conditions, we can compare two orbits by plotting the average distance between them over evolution.

```
class Dynamics1D(Dynamics):
    """Define a 1D discrete-time dynamical system.

    Parameters:
```

```
    f: the evolution function of the dynamical system
    Df: the derivative of the evolution function
    """

    def plot_orbit_over_evolution(self, x0, r, n=10000):
        """Plot the orbit over evolution.

        Parameters:
        x0: the initial value
        r: the parameter(s) for f
        n: the number of evolutions
        """
        ox = self.orbit(x0, r, n)
        plt.scatter(
            list(range(10000)), ox, s=.1, c='black', label=rf"$O_{x0}({r})$")
        plt.title(
            rf"The orbit $O_{x0}({r})$",
            loc='left', style='italic', fontsize=18)
        plt.xlabel("n: number of iterations", fontsize=18)
        plt.ylabel(r"$x_n$", fontsize=18)
        plt.show()

    def plot_average_distance_between_orbits(self, x0, x1, r, n=10000):
        """Plot the average distance between two orbits.

        Parameters:
        x0: one initial value
        x1: other initial value
        r: the parameter(s) for f
        n: the number of evolutions
        """
        ox0 = self.orbit(x0, r, n)
        ox1 = self.orbit(x1, r, n)
        sum_dist = abs(x0-x1)
        avg_dist = [abs(x0-x1)] + [0 for _ in range(n-1)]
        for i in range(1, n):
            sum_dist += abs(ox0[i] - ox1[i])
            avg_dist[i] = sum_dist/i
        print(avg_dist)
        plt.figure(figsize=(8, 8))
        plt.title(
            rf"Average distance between $O_{r}({x0})$ and $O_{r}({x1})$",
            loc='left', style='italic', fontsize=18)
        plt.xlabel("n: number of iterations", fontsize=18)
        plt.ylabel(r"$d_n$", fontsize=18)
        plt.legend(loc='upper left', fontsize=5)
```

```
    plt.plot(avg_dist, linewidth=.8, c='black')
    plt.show()
```

We can first define the Logistic model by:

```
logistics = Dynamics1D(lambda x, r: r*x*(1-x), None)
```

Here we define Df =None simply because we do not use it so far. Then to plot Figure 1c:

```
logistics.plot_orbit_over_evolution(0.3, 4, 10000)
```

To plot Figure 2:

```
logistics.plot_average_distance_between_orbits(0.3, 0.3000001, 4, 5000)
```

9.3 QR decomposition algorithm for 2D dynamical system

```
import numpy as np

'''
This specific code is applied for the classical Henon map.
The reader can apply the same code for any 2D map by changing
n, f, Df, v and r
n_iter - number of iterations
v - initial condition
r - parameters
f, Df - the map and the derivative of the map
'''

n_iter = 1000000
a = 1.4
b = 0.3
v = (0, 0)
r = (a, b)
f = lambda v, r:\
    np.array([1 - r[0]*v[0]**2 + v[1], r[1]*v[0]])
Df = lambda v, r:\
    np.array([[-2*r[0]*v[0], 1], [r[1], 0]])
Q = np.array([[1,0],[0,1]])
LE = np.zeros(n)
for i in range(n_iter):
    v = f(v, r)
    B = Df(v, r)@Q
    Q, R = np.linalg.qr(B)
    LE = LE + np.log(np.absolute(np.diag(R)))
LE = LE/n_iter
print(LE)
```

9.4 Figure 3: Bifurcation of logistic map and its Lyapunov exponents

```
import numpy as np
import matplotlib.pyplot as plt

def logistic_map(x, r):
    return r * x * (1 - x)

def compute_lyapunov(r, x, n):
    lyapunov = 0
    for i in range(n):
        lyapunov += np.log(abs(r - 2*r*x))
        x = logistic_map(x, r)
    return lyapunov / n

n = 1000
r = np.linspace(2.5, 4.0, n)
iterations = 1000
last = 100

x = 1e-5 * np.ones(n)
lyapunov = np.zeros(n)

fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(8, 9), sharex=True)

for i in range(iterations):
    x = logistic_map(x, r)
    lyapunov += np.log(abs(r - 2*r*x))
    if i >= (iterations - last):
        ax1.plot(r, x, ',k', alpha=1.0)

lyapunov /= iterations

ax2.plot(r[lyapunov < 0], lyapunov[lyapunov < 0] / np.log(2), '.k', alpha=1, ms=1)
ax2.plot(r[lyapunov >= 0], lyapunov[lyapunov >= 0] / np.log(2), '.r', alpha=1, ms=1)

ax2.axhline(0, color='blue')  # Add y=0 line

ax1.set_xlim(2.5, 4)
ax1.set_title("Logistic Map")
ax2.set_title("Lyapunov Exponents")
plt.tight_layout()
plt.savefig('logistic_map.png', dpi=100)
plt.show()
```

9.5 Figure 4: The Hénon map

```
import numpy as np
import matplotlib.pyplot as plt

a = 1.4
b = 0.3
x = y = 0
burn_in = 100
N = 100000


points = np.zeros((N-burn_in, 2))
for _ in range(burn_in):
    x, y = y + 1 - a*x**2, b*x

for i in range(N-burn_in):
    x_next = y + 1 - a*x**2
    points[i, :] = [x, x_next]  # Store x and x_next
    x, y = x_next, b*x

plt.figure(figsize=(10, 10))
plt.scatter(points[:, 0], points[:, 1], s=0.1, color='k')
plt.xlabel('$x_n$')
plt.ylabel('$x_{n+1}$')
plt.title('Henon map: $x_{n+1}$ vs $x_n$')
plt.savefig('Henon_map_xn_vs_xn+1.png', dpi=1000)
plt.show()
```

9.6 Figure 5: The difference between two Hénon map with differenct initial conditions

```
import numpy as np
import matplotlib.pyplot as plt

a = 1.4
b = 0.3

x0 = np.array([0, 0.00001])
y0 = np.array([0, 0])

N = 1000
x = np.empty((2, N))
y = np.empty((2, N))

x[:, 0] = x0
```

```
y[:, 0] = y0
for i in range(N-1):
    x[:, i+1] = y[:, i] + 1 - a*x[:, i]**2
    y[:, i+1] = b*x[:, i]
difference = np.abs(x[0, :] - x[1, :])
plt.figure(figsize=(10, 6))
plt.plot(difference, color="black")
plt.yscale('log')
plt.xlabel('Iteration')
plt.ylabel('Difference')
plt.title('Difference between two trajectories of the Henon map')
plt.savefig('Difference between Henon map.png', dpi=1000)
plt.show()
```

9.7 Box-Counting dimension for the 2D Hénon Map.

The code is from https://github.com/rougier/numpy-100

```
import numpy as np
import matplotlib.pyplot as plt

def henon_map(n, a=1.4, b=0.3, x0=0, y0=0):
    x, y = np.zeros(n), np.zeros(n)
    x[0], y[0] = x0, y0
    for i in range(n-1):
        x[i+1] = 1 - a*x[i]**2 + y[i]
        y[i+1] = b * x[i]
    return x, y

n = 21000
x, y = henon_map(n)
plt.scatter(x, y, s=0.1, color='k')
plt.show()

def box_count(Z, k):
    S = np.add.reduceat(
        np.add.reduceat(Z, np.arange(0, Z.shape[0], k), axis=0),
                           np.arange(0, Z.shape[1], k), axis=1)
    return len(np.where((S > 0) & (S < k*k))[0])

H, xedges, yedges = np.histogram2d(x, y, bins=[np.linspace(min(x),
max(x), 1000), np.linspace(min(y), max(y), 1000)])

sizes = 2**np.arange(0, 11)
counts = []
for size in sizes:
```

```
    counts.append(box_count(H, size))

sizes = [size for size,count in zip(sizes,counts) if count>0]
counts = [count for count in counts if count>0]

c = np.polyfit(np.log(sizes), np.log(counts), 1)
print("The box counting dimension is ", -c[0])
```

9.8 Maximal Lyapunov exponent figure for Bedhead attractor map.

```
import numpy as np
import matplotlib.pyplot as plt

'''
The reader can change the value of a to any desired value
between -1 and 1. This is the code for a = 0.65343.
'''

n_iter = 1000
a = 0.65343
b_vec = np.arange(-1, 1, 0.01)
b_vec_new = np.delete(b_vec, [100])
v = (1, 1)
max_lyap = []
f = lambda x, r:\
        np.array(
            [np.sin(x[0]*x[1]/r[1])*x[1]+np.cos(r[0]*x[0] - x[1]),
            x[0] + np.sin(x[1])/r[1]]
            )
Df = lambda x, r:\
        np.array(
            [[np.cos(x[0]*x[1]/r[1])*x[1]*x[1]/r[1]-

            r[0]*np.sin(r[0]*x[0]-x[1]),
            np.cos(x[0]*x[1]/r[1])*x[0]*x[1]/r[1] + np.sin(r[0]*x[0]-
            x[1]) + np.sin(x[0]*x[1]/r[1])],
            [1,
            np.cos(x[1])/r[1]]])
for b in b_vec_new:
    r = (a, b)
    Q = np.array([[1,0],[0,1]])
    LE = np.zeros(2)
    for i in range(n_iter):
        v = f(v, r)
        B = Df(v, r)@Q
```

```
        Q, R = np.linalg.qr(B)
        LE = LE + np.log(np.absolute(np.diag(R)))
    LE = LE/n_iter
    max_lyap.append(LE[0])

plt.plot(b_vec_new, max_lyap)
plt.ylim([-0.5, 3])
plt.axhline(y=0, color='r', linestyle='-')
plt.title("Maximal_Lyapunov_exponents")
plt.show()
```

9.9 Plot an attractor

```
def plot_attractor(self, x0, r, n=10000, n0=10000):
    """Plot an attractor of a dynamical system.

    Parameters:
    x0: the initial value in tuple
    r: the parameters for the evolution function f in tuple
    n: the number of evolutions
    n0: the number of pre-evolutions before plot
    """
    for _ in range(n0):
        x0 = self.f(x0, r)
    ox = self.orbit(x0, r, n)
    x, y = zip(*ox)
    plt.scatter(x, y, s=.1, c='black')
    plt.show()
```

9.10 Zoom in figure for the maximal Lyapunov exponent

```
def plot_mle(self, x0, r_list, idx, n=1000):
    """Plot the maximal lyapunov exponents over one varied parameter of f.

    Parameters:
    x0: the initial value
    r_list: list of r
    idx: the index of the varied parameter in r
    n: the number of evolution in computing the lyapunov exponents
    """
    mle = [0 for _ in range(len(r_list))]
    i = 0
    for r in tqdm(r_list,
                bar_format="Loading:{l_bar}{bar}[time left:{remaining}]"
                ):
        mle[i] = self.lyapunov_exponents(x0, r, n)[0]
```

```
    i += 1
plt.plot([r[idx] for r in r_list], mle, linewidth=.6, c='black')
plt.axhline(y=0, color='r', linestyle='-')
plt.title("Maximal Lyapunov exponents")
plt.show()
```