Numerical Analysis Project

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I have neither given nor received help (apart from the instructor) to complete this assignment.

Signature:

Title: Variance Reduction Techniques for Efficient Monte Carlo Option Pricing: A Comparative Study of Antithetic and Control Variates for European and Asian Options.

Abstract

This project focuses on developing a Monte Carlo based framework for pricing financial derivatives, with an emphasis on improving numerical efficiency through variance reduction techniques. The analysis begins with standard Monte Carlo simulation under the Black-Scholes model and examines its application to both European and Asian call options.

The framework incorporates antithetic variates and control variates to reduce estimator variance and improve convergence. For European options, closed-form Black-Scholes prices are used as analytical benchmarks to assess estimator accuracy and convergence behavior. For Asian options, which are path-dependent and lack closed-form solutions, the study relies on exact simulation of geometric Brownian motion paths with discrete monitoring.

A key component of the project is the construction and evaluation of alternative control variables. In addition to European payoffs, the geometric Asian call option whose price admits a closed-form expression under Black-Scholes is introduced as a control variate for arithmetic Asian options. Monte Carlo simulations are used to estimate prices, variances, and standard errors across different estimators and sample sizes.

The project aims to analyze how payoff structure, correlation, and control choice affect the effectiveness of variance reduction techniques. By combining stochastic simulation, analytical pricing formulas, and numerical diagnostics, this framework provides a systematic approach to evaluating and improving Monte Carlo methods for option pricing.

1. Introduction

Monte Carlo methods play a central role in modern financial engineering, particularly in the pricing of derivative securities. Under the risk-neutral pricing framework, the value of an option can be expressed as the expected value of its discounted payoff. While closed-form solutions exist for a small class of derivatives, such as European options under the Black–Scholes model, many practically relevant contracts do not admit analytical pricing formulas. In such cases, Monte Carlo simulation provides a flexible and intuitive numerical approach for approximating option prices by simulating asset price paths and averaging the corresponding payoffs.

Despite its generality and ease of implementation, standard Monte Carlo simulation suffers from an important limitation: slow convergence. The accuracy of a Monte Carlo estimator improves at a rate proportional to the inverse square root of the number of simulations, regardless of the dimensionality of the problem. As a result, achieving high accuracy often requires a very large number of simulated paths, leading to significant computational cost. This issue becomes particularly pronounced when pricing derivatives with complex or path-dependent payoffs, where the variance of the estimator can be substantial.

Variance reduction techniques are designed to address this inefficiency by reducing the variability of the Monte Carlo estimator without increasing the number of simulations. By lowering the variance of the estimator, these techniques improve numerical efficiency and lead to tighter confidence intervals for a given computational budget. From a numerical analysis perspective, variance reduction methods are therefore essential tools for improving the performance of Monte Carlo simulations while preserving unbiasedness of the estimator.

In this project, I use European and Asian call options as representative test cases to study the effectiveness of variance reduction techniques. European options serve as a natural benchmark due to their simple payoff structure and the availability of closed-form solutions under the Black-Scholes framework. Asian options, on the other hand, are path-dependent derivatives whose payoff depends on the average of the underlying asset price over time. This path dependence increases the dimensionality of the problem and typically leads to higher estimator variance, making Asian options a particularly suitable setting for evaluating variance reduction methods.

The focus of this study is on two classical variance reduction techniques that I implement and compare: antithetic variates and control variates. Antithetic variates reduce variance by exploiting symmetry in the underlying random variables, while control variates leverage the correlation between the quantity of interest and a related random variable with known expectation.

2. Monte Carlo Framework for Option Pricing

Monte Carlo methods provide a flexible numerical framework for valuing financial derivatives by approximating expectations through random sampling. In option pricing, the key idea is that under the risk-neutral measure, the fair value of a derivative can be expressed as the expected discounted payoff of the option. When analytical solutions are unavailable or impractical, Monte Carlo simulation becomes a natural and widely used tool.

Risk-Neutral Pricing and Expectations

In modern financial theory, the price of a derivative security can be expressed as an expectation under a risk-neutral probability measure. Under this measure, all assets grow on average at the risk-free interest rate, and risk preferences are absorbed into the probability distribution.

Let $S(t)$ denote the price of an underlying asset at time t, and consider a derivative with payoff $\Phi(S(T))$ at maturity T. Under the risk-neutral measure $\mathbb{Q}$, the time-0 price of the derivative is given by

$$V_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT}\,\Phi(S(T))],$$

where r is the continuously compounded risk-free interest rate.

This formulation shows that option pricing reduces to computing an expected value. For example, the payoff of a European call option with strike $K$is

$$\Phi(S(T)) = (S(T) - K)^+ = \max\,(S(T) - K, 0),$$

and its price is the expected discounted payoff under the risk-neutral distribution of $S(T)$.

Monte Carlo methods exploit this representation by approximating the expectation numerically. Instead of evaluating the expectation analytically, which is often impossible for complex or path-dependent options, we simulate many possible realizations of the underlying asset price and average the corresponding discounted payoffs.

Monte Carlo Estimator Definition

The Monte Carlo method is based on a fundamental observation from probability theory: integrals can be interpreted as expectations.

Consider an integral of the form

$$\alpha = \int_0^1 f(x)dx.$$

This integral can be written as an expectation

$$\alpha = \mathbb{E}[f(U)],$$

where $U \sim \text{Uniform}(0,1)$. This identity forms the basis of Monte Carlo integration.

Given independent and identically distributed samples $U_1, U_2, \ldots, U_n \sim \text{Uniform}(0,1)$, the Monte Carlo estimator of α is defined as the sample mean

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

By construction, this estimator is unbiased, meaning

$$\mathbb{E}[\hat{\alpha}_n] = \alpha.$$

In the context of option pricing, the function $f(\cdot)$ corresponds to the discounted payoff of the option, and the random variables represent simulated realizations of the underlying asset price or price path. The Monte Carlo estimator for an option price is therefore the average of discounted payoffs across simulated scenarios.

Unbiasedness is an important property: although individual simulations may vary significantly, the estimator does not systematically overestimate or underestimate the true option value.

Variance, Standard Error, and Confidence Intervals

While the Monte Carlo estimator is unbiased, it is also random. Its accuracy depends on the variance of the estimator and the number of simulations used.

Let $\sigma_f^2 = \text{Var}[f(U)]$. The variance of the Monte Carlo estimator is

$$\text{Var}(\hat{\alpha}_n) = \frac{\sigma_f^2}{n}.$$

This relationship highlights an important limitation of standard Monte Carlo methods: the error decreases at a rate proportional to $1/\sqrt{n}$. As a result, reducing the estimation error by a factor of two requires approximately four times as many simulations.

By the Law of Large Numbers, the estimator converges almost surely to the true value:

$$\hat{\alpha}_n \longrightarrow \alpha \text{ as } n \to \infty.$$

Furthermore, the Central Limit Theorem implies that for sufficiently large n, the distribution of the estimation error is approximately normal:

$$\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow{d} \mathcal{N}(0, \sigma_f^2).$$

This result allows the construction of confidence intervals. An approximate $(1-\delta)$-confidence interval for α is given by

$$\hat{\alpha}_n \pm z_{\delta/2} \frac{s_f}{\sqrt{n}},$$

where s_f is the sample standard deviation of the simulated payoffs and $z_{\delta/2}$ is the corresponding quantile of the standard normal distribution.

The slow $O(n^{-1/2})$ convergence rate is the principal limitation of standard Monte Carlo methods and provides the primary motivation for variance reduction techniques, which aim to reduce σ_f^2 without increasing the number of simulations.

Simulation of Asset Price Paths

To implement Monte Carlo pricing, it is necessary to specify a stochastic model for the underlying asset price dynamics. A commonly used model is the Geometric Brownian Motion (GBM), which under the risk-neutral measure satisfies the stochastic differential equation

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma\,dW(t),$$

where σ is the volatility and $W(t)$is a standard Brownian motion.

The solution to this equation implies that the terminal stock price is lognormally distributed:

$$S(T) = S(0)\exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right),$$

with $Z \sim \mathcal{N}(0,1)$. For European options, this expression allows direct simulation of $S(T)$without modeling intermediate time steps.

However, for path-dependent options such as Asian options, the payoff depends on the entire trajectory of the asset price. In such cases, the interval $[0, T]$ is divided into m subintervals of length $\Delta t = T/m$, and the asset price is simulated iteratively using an Euler discretization:

$$S(t_{k+1}) = S(t_k)\exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\,Z_k\right),$$

where $Z_k \sim \mathcal{N}(0,1)$ are independent standard normal variables.

Time-stepping introduces additional variance and numerical error, particularly for Asian options whose payoffs involve averages over the simulated path. This characteristic makes Asian options particularly well-suited for the application of variance reduction techniques.

3. Standard Monte Carlo Pricing

I implement standard Monte Carlo methods to price European and Asian call options under the Black-Scholes framework. The goal is to establish baseline Monte Carlo estimators for each option type and to examine their numerical behavior, with particular emphasis on estimator variance. These baseline results provide the motivation for introducing variance reduction techniques.

European Option Pricing via Monte Carlo

Payoff Definition

A European call option gives the holder the right, but not the obligation, to purchase the underlying asset at a fixed strike price Kat maturity T. The payoff of a European call option depends only on the terminal asset price and is given by

$$\Phi(S(T)) = (S(T) - K)^+ = \max(S(T) - K, 0).$$

Under the risk-neutral measure, the terminal stock price follows a lognormal distribution:

$$S(T) = S(0)\exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right),$$

where $Z \sim N(0,1)$, $r$is the risk-free rate, and σ is the volatility.

This closed-form expression allows direct simulation of $S(T)$ without discretizing the price path, making European options computationally straightforward to price using Monte Carlo methods.

Estimator Construction

Given n independent simulations of the terminal stock price $S^{(i)}(T)$, the Monte Carlo estimator for the European call option price is

$$\hat{V}_n^{\text{Euro}} = \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\,(S^{(i)}(T) - K)^+.$$

This estimator is unbiased and converges to the true option price as $n \to \infty$ by the Law of Large Numbers. The standard error of the estimator decreases at a rate proportional to $1/\sqrt{n}$, which implies that a large number of simulations is required to achieve high accuracy.

Because the European option payoff depends only on a single random variable, the variance of the estimator is relatively low compared to path-dependent options. Additionally, the availability of the Black-Scholes closed-form solution allows the Monte Carlo estimates to be benchmarked against an exact price.

Asian Option Pricing via Monte Carlo

Arithmetic Average Payoff

An Asian call option is a path-dependent derivative whose payoff depends on the average value of the underlying asset over a specified set of observation times. In this project, I consider the arithmetic average Asian call option with payoff defined as

$$\Phi(\{S(t_k)\}_{k=1}^{m}) = (\frac{1}{m}\sum_{k=1}^{m} \ (S(t_k) - K)^+,$$

where $\{t_1, t_2, \ldots, t_m\}$ are equally spaced monitoring times over the interval $[0, T]$.

Unlike European options, arithmetic Asian options do not admit a closed-form pricing formula under the Black-Scholes model, making Monte Carlo simulation a natural and widely used pricing approach.

Path Dependence and Simulation

To price an Asian option, the entire asset price path must be simulated. The time interval $[0, T]$is divided into m subintervals of length $\Delta t = T/m$, and the asset price is generated recursively according to

$$S(t_{k+1}) = S(t_k)\exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\, Z_k\right),$$

where $Z_k \sim N(0,1)$ are independent standard normal random variables.

For each simulated path, the arithmetic average of the asset prices is computed and used to evaluate the payoff. The Monte Carlo estimator for the Asian call option price is then given by

$$\hat{V}_n^{\text{Asian}} = \frac{1}{n}\sum_{i=1}^{n} e^{-rT}(\bar{S}^{(i)} - K)^+,$$

where $\bar{S}^{(i)}$denotes the arithmetic average price along the i-th simulated path.

Numerical Issues and Variance Behavior

Why Asian Options Have Higher Variance?

Compared to European options, Asian options typically exhibit higher Monte Carlo estimator variance. This increase in variance arises from several factors. First, the payoff depends on multiple correlated random variables rather than a single terminal value, which increases the dimensionality of the problem. Second, the averaging operation introduces additional variability because fluctuations at each time step contribute to the final payoff. Finally, discretization of the asset price path introduces numerical noise, particularly when the number of monitoring points is large.

As a result, the distribution of Asian option payoffs is often more dispersed, leading to wider confidence intervals for a given number of simulations.

Motivation for Variance Reduction

The higher variance associated with Asian option pricing implies that standard Monte Carlo simulation can be computationally inefficient. This inefficiency motivates the use of variance reduction techniques, which aim to decrease estimator variance without increasing the computational budget.

4. Variance Reduction Techniques

Standard Monte Carlo simulation provides an unbiased estimator for option prices, but its efficiency is often limited by high variance, particularly for path-dependent payoffs. Variance reduction techniques aim to improve numerical efficiency by decreasing estimator variance without increasing the number of simulated paths. In this project, I used two classical variance reduction methods: antithetic variates and control variates.

<u>Antithetic Variates</u>

Definition

The key idea behind antithetic variates is to introduce negative correlation between paired simulations in order to reduce variance. Instead of generating independent samples, the method constructs pairs of dependent samples whose average has lower variability than either sample alone.

In Monte Carlo integration, the estimator is typically based on independent realizations of a random variable. However, if two estimators are negatively correlated, averaging them reduces variance. Antithetic variates exploit symmetry in the underlying random inputs to achieve this effect in a simple and computationally inexpensive manner.

Construction

Let $Z \sim N(0,1)$ be a standard normal random variable and consider a payoff function $f(Z)$. In standard Monte Carlo simulation, the estimator is

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} f(Z_i),$$

where Z_i are independent standard normal draws.

Under the antithetic variates method, for each draw Z_i, a paired draw $-Z_i$ is also used. The antithetic estimator is defined as

$$\hat{\mu}_n^{\text{AV}} = \frac{1}{2n}\sum_{i=1}^{n}[f(Z_i) + f(-Z_i)].$$

Since Z_i and $-Z_i$ have the same distribution, the estimator remains unbiased:

$$\mathbb{E}[\hat{\mu}_n^{\text{AV}}] = \mathbb{E}[f(Z)].$$

Negative Correlation Principle

The effectiveness of antithetic variates depends on the correlation between $f(Z)$ and $f(-Z)$. If the payoff function is monotone in Z, then $f(Z)$ and $f(-Z)$ tend to be negatively correlated. This negative correlation reduces the variance of their average.

Formally, the variance of the antithetic estimator is

$$\text{Var}\left(\frac{f(Z) + f(-Z)}{2}\right) = \frac{1}{4}[\text{Var}(f(Z)) + \text{Var}(f(-Z)) + 2\,\text{Cov}(f(Z), f(-Z))].$$

Since $\text{Var}(f(Z)) = \text{Var}(f(-Z))$, variance reduction occurs whenever

$$\text{Cov}(f(Z), f(-Z)) < 0.$$

Variance Reduction Mechanism

For option pricing problems, particularly under the Black-Scholes model, the payoff is often monotone in the underlying Brownian motion. As a result, antithetic variates are especially effective for European call options and remain useful for Asian options, although the effect may be weaker due to path dependence.

An important advantage of antithetic variates is that the method does not require additional random number generation and introduces negligible computational overhead. This makes it an attractive baseline variance reduction technique.

Control Variates

Definition and Construction

Control variates are a variance reduction technique that exploits the correlation between an estimator of interest and another related random variable whose expectation is known. The method reduces variance without increasing the number of Monte Carlo simulations.

Suppose the goal is to estimate

$$\mu = \mathbb{E}[X],$$

where X represents the discounted payoff of an option. In standard Monte Carlo simulation, this expectation is estimated by the sample mean

$$m = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Suppose we can find another statistic t, such that

$$\tau = \mathbb{E}[t]$$

is known analytically, and that X and t are strongly correlated. Using t as a control variate, the modified estimator is defined as

$$M = m + c\,(t - \tau),$$

where c is a deterministic control variable.

Since $\mathbb{E}[t - \tau] = 0$, the estimator remains unbiased for any choice of c:

$$\mathbb{E}[M] = \mathbb{E}[m] = \mu.$$

Optimal Control Coefficient

The variance of the control variate estimator is given by

$$\mathrm{Var}(M) = \mathrm{Var}(m) + c^2\mathrm{Var}(t) + 2c\,\mathrm{Cov}(m, t).$$

To minimize the variance, the derivative of $\mathrm{Var}(M)$with respect to $c$is set equal to zero. This yields the optimal control coefficient

$$c^* = -\frac{\mathrm{Cov}(m,t)}{\mathrm{Var}(t)}.$$

Substituting c^* back into the variance expression gives

$$\mathrm{Var}(M) = \mathrm{Var}(m)(1-\rho_{m,t}^2),$$

where $\rho_{m,t}$ denotes the correlation between m and t.

To reduce $\mathrm{Var}(M)$, we want $|\rho_{m,t}|$ not to be 0. Ideally $|\rho_{m,t}|$ should be close to 1.

Choice of Control Variable

The effectiveness of the control variates method depends critically on the choice of the control variable. In this project, a European call option is used as a control variate when pricing an arithmetic Asian call option.

Let:

- m denote the Monte Carlo estimator of the discounted payoff of the Asian call option,
- t denote the Monte Carlo estimator of the discounted payoff of the corresponding European call option,
- $\tau = V_{\mathrm{BS}}^{\mathrm{Euro}}$denote the Black-Scholes price of the European call option.

The control variate estimator for the Asian option price is therefore

$$\hat{V}^{\mathrm{CV}} = m + c^*(V_{\mathrm{BS}}^{\mathrm{Euro}} - t).$$

This choice is motivated by the strong correlation between Asian and European option payoffs, since both depend on the same underlying asset price dynamics. Moreover, the availability of a closed-form Black-Scholes price for the European option allows the expected value of the control variate to be known exactly.

5. Numerical Experiments and Results

All simulations are conducted under the Black-Scholes model, where the underlying asset price S_t follows a geometric Brownian motion:

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t,$$

with constant risk-free rate r and volatility σ.

Parameters

The following parameters are used:

- Initial stock price: $S_0 = 100$
- Strike price: $K = 100$
- Risk-free rate: $r = 5\%$
- Volatility: $\sigma = 20\%$
- Maturity: $T = 1$ year
- Number of monitoring dates (Asian option): $m = 252$ (daily)
- Number of Monte Carlo simulations:

$$N \in \{5{,}000,\ 10{,}000,\ 25{,}000,\ 50{,}000,\ 100{,}000\}$$

All payoffs are discounted using the factor e^{-rT}. Random numbers are generated using a fixed seed to ensure reproducibility.

Performance Metrics

For each estimator, the following quantities are reported:

- Estimated option price: sample mean of discounted payoffs
- Sample variance of the payoff estimator
- Standard error (SE):

$$\mathrm{SE} = \sqrt{\frac{\widehat{\mathrm{Var}}}{N}}$$

While efficiency is often defined as the reduction in variance relative to standard Monte Carlo, the analysis focuses directly on variance and standard error, which fully characterize estimator accuracy and convergence behavior.

Results: European Call Option

The Black-Scholes analytical price of the European call option is:

$$V_{\mathrm{BS}} = 10.4506.$$

This value serves as a benchmark for assessing the accuracy and convergence of the Monte Carlo estimators.

Numerical Results ($N = 100{,}000$)

Method	Price	Variance	Standard Error
Standard MC	10.4986	217.21	0.0466
Antithetic	10.4236	53.92	0.0232
Control Variate (c*≈ -0.676, ρ ≈ 0.925)	10.4241	31.36	0.0177

Analysis of results

The standard Monte Carlo estimator exhibits relatively high variance and slow convergence, as reflected in its wide standard error. While the price estimate is reasonably close to the analytical value, achieving higher precision would require a substantially larger number of simulations.

The antithetic variates method produces a significant variance reduction. By pairing each standard normal draw Z with $-Z$, negative correlation is induced between paired payoffs, leading to a much tighter estimator.

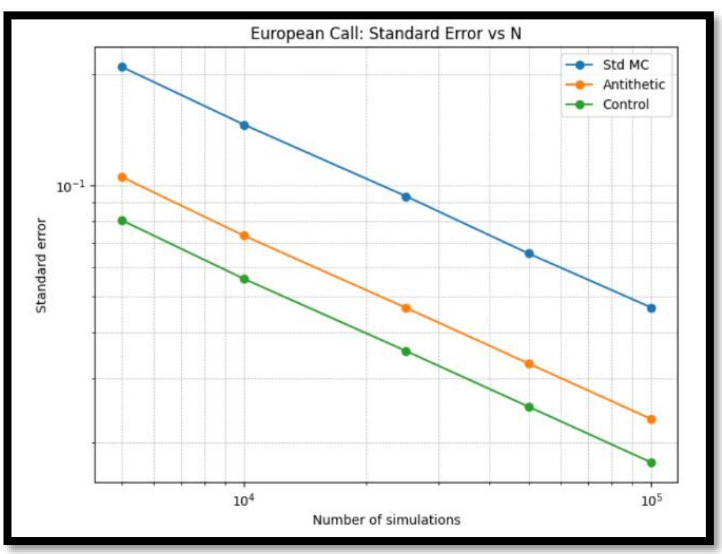

Fig. 1 Log-log plot of standard error versus N (European Call).

This behavior is clearly visible in the log-log plot of standard error versus N, where the antithetic curve lies well below the standard Monte Carlo curve across all sample sizes.

The control variate estimator achieves the strongest variance reduction among the three methods. The control used is the discounted terminal stock price $e^{-rT}S_T$, whose expectation is known analytically:

$$\mathbb{E}[e^{-rT}S_T] = S_0.$$

Because the European call payoff is strongly correlated with the terminal stock price, the control variate effectively removes a large portion of the estimator's variability.

The standard error plot confirms the theoretical $1/\sqrt{N}$ convergence rate for all estimators, with the control variate consistently dominating both standard Monte Carlo and antithetic variates.

Results: Asian Arithmetic Call Option

The discounted payoff of the arithmetic Asian call option is given by:

$$X = e^{-rT}\max\left(\frac{1}{m}\sum_{i=1}^{m} S_{t_i} - K,\ 0\right),$$

where the averaging is performed over $m = 252$ equally spaced monitoring dates.

Unlike the European option, the arithmetic Asian call has no closed-form solution under the Black–Scholes model and must be priced via simulation.

Numerical Results ($N = 100{,}000$)

Method	Price	Variance	Standard Error
Standard MC	5.7888	63.70	0.0252
Antithetic	5.7522	15.19	0.0123
Control Variate ($c^*\approx 0.456$, $\rho\approx 0.839$)	5.7683	18.85	0.0137

Analysis of Results

For the arithmetic Asian option, standard Monte Carlo again exhibits the highest variance, reflecting the path-dependent nature of the payoff.

Antithetic variates perform well for the arithmetic Asian option.

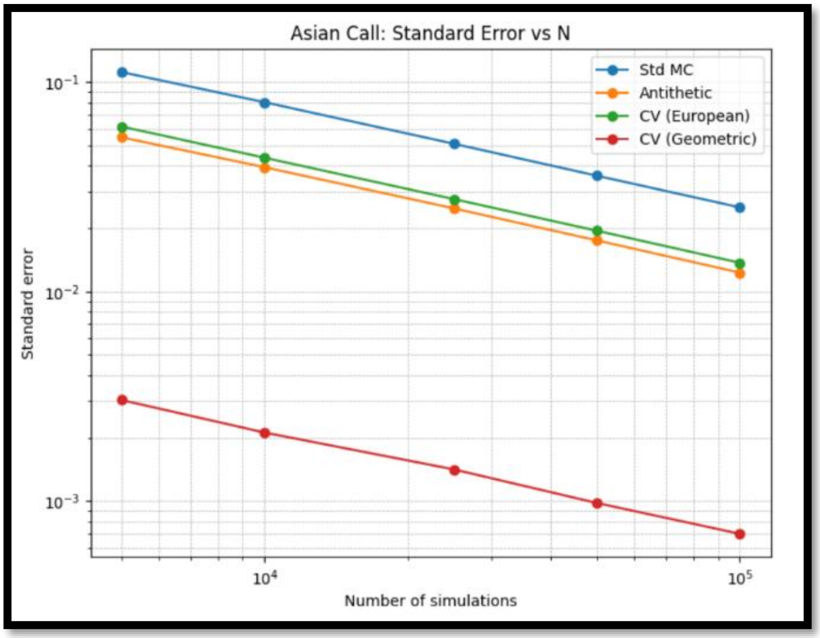

Fig. 2 Log-log plot of standard error versus N (Asian Call).

Because the payoff depends smoothly on the Brownian increments, pairing paths with opposite shocks induces negative correlation and leads to a clear reduction in variance, as reflected in the standard error plot across all values of N.

The control variate uses the discounted European call payoff evaluated at the same terminal stock price, whose expectation is given by the Black–Scholes formula. Although this control is positively correlated with the arithmetic Asian payoff (with correlation around 0.84), the relationship is not sufficiently linear. Consequently, the variance reduction achieved by the control variate is noticeable but does not exceed that of the antithetic estimator.

Importantly, the ordering of variances observed in the results is:

$$\mathrm{Var}_{\mathrm{CV}} < \mathrm{Var}_{\mathrm{MC}}, \text{ but } \mathrm{Var}_{\mathrm{CV}} \not< \mathrm{Var}_{\mathrm{AV}}.$$

Thus, the commonly expected hierarchy

$$\mathrm{Var}_{\mathrm{CV}} \ll \mathrm{Var}_{\mathrm{AV}} < \mathrm{Var}_{\mathrm{MC}}$$

does not hold for the arithmetic Asian option in this project.

Alternative Control Choice: Geometric Asian Call Option

A geometric Asian call option is a path-dependent option whose payoff depends on the geometric average of the underlying asset price over a set of monitoring dates.

Results

Its discounted payoff is given by:

$$X_{\text{geo}} = e^{-rT}\max\left(\left(\prod_{i=1}^{m} S(t_i)\right)^{1/m} - K, 0\right).$$

where the product is taken over the same $m = 252$ equally spaced monitoring dates used for the arithmetic Asian option.

Unlike the arithmetic Asian option, the geometric Asian option admits a closed-form pricing formula under the Black-Scholes model. This analytical tractability makes it a natural candidate for use as a control variate.

From a variance reduction perspective, the geometric average is known to be extremely highly correlated with the arithmetic average. This strong structural similarity implies that the geometric Asian payoff can serve as an almost ideal control when pricing arithmetic Asian options. When the control closely mirrors the target payoff, the control variate estimator can eliminate a large fraction of the sampling variability.

For this reason, the geometric Asian option provides a useful benchmark for assessing the theoretical upper bound of variance reduction achievable through control variates.

Numerical Results (N = 100,000)

When the geometric Asian payoff is used as a control variate for the arithmetic Asian option, the following results are obtained:

Method	**Price**	**Variance**	**Standard Error**
Control Variate (Geometric)	5.7823	0.0488	0.000699

The empirical correlation between the arithmetic and geometric Asian payoffs is extremely high, with $\rho \approx 0.9996$.

Analysis of Results

Using the geometric Asian payoff as a control variate leads to an extremely large reduction in variance, far exceeding that achieved by standard Monte Carlo or antithetic variates.

For an optimal control variate,

$$\mathrm{Var}(X_{\mathrm{CV}}) = \mathrm{Var}(X)(1 - \rho^2),$$

where ρ is the correlation between the target payoff and the control. In this case, the empirical correlation is extremely close to one, making $1 - \rho^2$ very small and explaining the dramatic variance reduction observed.

The key advantage of the geometric Asian control lies in structural similarity. Unlike the European payoff, which depends only on the terminal price, both arithmetic and geometric Asian payoffs depend on the full price path and differ mainly in the averaging mechanism.

6. Discussion and Practical Implications of Variance Reduction

Effectiveness of Variance Reduction Methods

The numerical results confirm that variance reduction techniques significantly improve the efficiency of Monte Carlo estimators for both European and Asian options. For European call options, antithetic variates reduce variance by roughly a factor of four, while control variates provide an even larger reduction when the discounted terminal stock price is used as the control. These findings reflect the strong dependence of the European payoff on the underlying Brownian motion.

For arithmetic Asian options, the relative performance of variance reduction methods depends on payoff structure. Antithetic variates outperform the European-based control variate, as the smooth path dependence of the arithmetic average allows paired paths to induce effective negative correlation. In contrast, the geometric Asian payoff provides an almost ideal control variate due to its near-perfect correlation with the arithmetic payoff, resulting in a dramatic reduction in variance.

Numerical Efficiency and Computational Trade-offs

Antithetic variates offer a favorable balance between computational cost and variance reduction. They require no additional model assumptions, introduce minimal overhead, and provide reliable variance reduction across all option types considered. As a result, antithetic variates represent a robust and easily implementable improvement over standard Monte Carlo simulation.

Control variates, while potentially far more powerful, involve additional computation and depends critically on the choice of control. For European options, the cost of implementing a control variate is negligible relative to the efficiency gains. For Asian options, the benefit depends on the availability of a structurally similar control. The geometric Asian control, while computationally more demanding, yields such large variance reductions that the additional cost is easily justified. However, poorly chosen controls may offer only modest improvements.

Limitations and Scope of Applicability

The analysis is conducted entirely within the Black-Scholes framework, which assumes constant volatility, constant interest rates, and lognormal asset dynamics. In more realistic settings involving stochastic volatility or jumps, both the magnitude and reliability of variance reduction techniques may differ. Consequently, the numerical results should be interpreted as model specific.

Another limitation arises from time discretization in the simulation of Asian options. Although exact sampling of geometric Brownian motion is used at discrete monitoring dates, discretization can still affect estimator variance and correlation between payoffs.

Lastly, the success of control variates depends heavily on the availability of suitable controls with known expectations. While the geometric Asian option provides an ideal benchmark, such controls are rarely available for more complex derivatives encountered in practice.

7. Conclusion

This project investigated the performance of standard Monte Carlo simulation and variance reduction techniques in the pricing of European and Asian call options under the Black–Scholes framework. Through a series of numerical experiments, the effectiveness of antithetic variates and control variates was evaluated in terms of variance, standard error, and numerical stability. The results demonstrate that while standard Monte Carlo provides an unbiased and flexible pricing method, its efficiency is limited by slow convergence, particularly for path-dependent options. Variance reduction techniques offer a practical and powerful solution to this limitation.

For European call options, both antithetic variates and control variates significantly improve efficiency relative to standard Monte Carlo. Antithetic variates reduce variance by exploiting symmetry in the underlying normal draws, while control variates achieve even greater variance reduction by leveraging a strongly correlated quantity with known expectation. Using the discounted terminal stock price as a control leads to substantial improvements. These findings confirm that even simple control choices can yield meaningful efficiency gains for options with relatively simple payoff structures.

The benefits of variance reduction become even more pronounced for Asian options, whose payoffs depend on the entire asset price path. Antithetic variates perform particularly well for arithmetic Asian options, reflecting the smooth dependence of the average price on the Brownian increments. Control variates based on the European payoff also reduce variance, but their effectiveness is limited by imperfect correlation and structural differences between the payoffs. This highlights an important insight: correlation alone is insufficient for optimal variance reduction; the control must also closely mirror the structure of the target payoff.

This point is illustrated most clearly by the geometric Asian control variate. Because the geometric and arithmetic Asian payoffs are nearly perfectly correlated and share strong structural similarity, the control variate estimator achieves an extraordinary reduction in variance, effectively eliminating most sampling noise. This result represents an upper bound on the performance of control variates. It underscores the central role of control selection in determining the success of variance reduction techniques.

From a practical perspective, the results of this project are highly relevant to real-world derivative pricing. Monte Carlo simulation is widely used in industry for valuing complex and path-dependent products, where analytical solutions are unavailable. In such settings, computational efficiency is critical. Antithetic variates offer a simple, low-cost improvement that is robust across a wide range of payoffs. Control variates, when appropriate controls are available, can deliver dramatic efficiency gains and substantially reduce computational cost.

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9. Appendix

```
import numpy as np
import matplotlib.pyplot as plt
from math import log, sqrt, exp, erf


def norm_cdf(x: float) -> float:
    return 0.5 * (1.0 + erf(x / sqrt(2.0)))


def bs_european_call_price(S0: float, K: float, r: float, sigma: float, T: float) -> float:
    """Black–Scholes European call price."""
    if T <= 0:
        return max(S0 - K, 0.0)
    d1 = (log(S0 / K) + (r + 0.5 * sigma**2) * T) / (sigma * sqrt(T))
    d2 = d1 - sigma * sqrt(T)
    return S0 * norm_cdf(d1) - K * exp(-r * T) * norm_cdf(d2)


def geometric_asian_call_price_discrete(S0: float, K: float, r: float, sigma: float, T: float, m: int)
-> float:
    if T <= 0:
        return max(S0 - K, 0.0)

    t_bar = T * (m + 1) / (2.0 * m)
    sum_min = T * (m + 1) * (2 * m + 1) / 6.0

    var_G = (sigma**2) * (sum_min / (m**2))
    mu_G = log(S0) + (r - 0.5 * sigma**2) * t_bar
    sig_G = sqrt(var_G)
```

```
    if sig_G == 0:
        G_det = exp(mu_G)
        return exp(-r * T) * max(G_det - K, 0.0)

    d1 = (mu_G - log(K) + var_G) / sig_G
    d2 = d1 - sig_G

    price = exp(-r * T) * (exp(mu_G + 0.5 * var_G) * norm_cdf(d1) - K * norm_cdf(d2))
    return price


def summarize_estimator(payoffs: np.ndarray, name: str = "") -> dict:
    N = len(payoffs)
    price = float(np.mean(payoffs))
    var_payoff = float(np.var(payoffs, ddof=1))
    se = float(np.sqrt(var_payoff / N))
    return {"method": name, "price": price, "var_payoff": var_payoff, "se": se}


def simulate_gbm_paths_exact(S0: float, r: float, sigma: float, T: float, m_steps: int, Z:
np.ndarray) -> np.ndarray:
    dt = T / m_steps
    drift = (r - 0.5 * sigma**2) * dt
    diffusion = sigma * np.sqrt(dt) * Z
    logS = np.cumsum(drift + diffusion, axis=1)
    S = np.empty((Z.shape[0], m_steps + 1))
    S[:, 0] = S0
    S[:, 1:] = S0 * np.exp(logS)
```

```
    return S


def run_experiment(S0, K, r, sigma, T, N, m_steps, seed=12345):
    rng = np.random.default_rng(seed)
    discount = np.exp(-r * T)

    tau_euro = bs_european_call_price(S0, K, r, sigma, T)
    tau_geo  = geometric_asian_call_price_discrete(S0, K, r, sigma, T, m_steps)

    results = {}

    Z_euro = rng.standard_normal(N)
    ST = S0 * np.exp((r - 0.5 * sigma**2) * T + sigma * np.sqrt(T) * Z_euro)
    X_euro = discount * np.maximum(ST - K, 0.0)
    results["Euro Std MC"] = summarize_estimator(X_euro, "Euro Std MC")

    ST_anti = S0 * np.exp((r - 0.5 * sigma**2) * T - sigma * np.sqrt(T) * Z_euro)
    X_euro_av = 0.5 * (X_euro + discount * np.maximum(ST_anti - K, 0.0))
    results["Euro Antithetic"] = summarize_estimator(X_euro_av, "Euro Antithetic")

    t_euro = discount * ST
    cov_e = np.cov(X_euro, t_euro, ddof=1)[0, 1]
    var_X = np.var(X_euro, ddof=1)
    var_t = np.var(t_euro, ddof=1)
    c_star = -cov_e / var_t if var_t > 0 else 0.0
    X_euro_cv = X_euro + c_star * (t_euro - S0)
```

```
cv_res = summarize_estimator(X_euro_cv, "Euro CV")
cv_res["c_star"] = float(c_star)
cv_res["rho"] = float(cov_e / np.sqrt(var_X * var_t)) if (var_X > 0 and var_t > 0) else np.nan
results["Euro CV"] = cv_res

Z_asian = rng.standard_normal((N, m_steps))
S_path = simulate_gbm_paths_exact(S0, r, sigma, T, m_steps, Z_asian)

arith_avg = np.mean(S_path[:, 1:], axis=1)
X_asian = discount * np.maximum(arith_avg - K, 0.0)
results["Asian Std MC"] = summarize_estimator(X_asian, "Asian Std MC")

S_path_anti = simulate_gbm_paths_exact(S0, r, sigma, T, m_steps, -Z_asian)
arith_avg_anti = np.mean(S_path_anti[:, 1:], axis=1)
X_asian_av = 0.5 * (X_asian + discount * np.maximum(arith_avg_anti - K, 0.0))
results["Asian Antithetic"] = summarize_estimator(X_asian_av, "Asian Antithetic")



ST_asian = S_path[:, -1]
t_euro_asian = discount * np.maximum(ST_asian - K, 0.0)

cov1 = np.cov(X_asian, t_euro_asian, ddof=1)[0, 1]
var1 = np.var(t_euro_asian, ddof=1)
c1 = -cov1 / var1 if var1 > 0 else 0.0
X_cv1 = X_asian + c1 * (t_euro_asian - tau_euro)

res1 = summarize_estimator(X_cv1, "Asian CV (European)")
```

```
    res1["c_star"] = float(c1)
    res1["rho"] = float(np.corrcoef(X_asian, t_euro_asian)[0, 1])
    results["Asian CV (European)"] = res1

    geo_avg = np.exp(np.mean(np.log(S_path[:, 1:]), axis=1))
    t_geo = discount * np.maximum(geo_avg - K, 0.0)

    cov2 = np.cov(X_asian, t_geo, ddof=1)[0, 1]
    var2 = np.var(t_geo, ddof=1)
    c2 = -cov2 / var2 if var2 > 0 else 0.0
    X_cv2 = X_asian + c2 * (t_geo - tau_geo)

    res2 = summarize_estimator(X_cv2, "Asian CV (Geometric)")
    res2["c_star"] = float(c2)
    res2["rho"] = float(np.corrcoef(X_asian, t_geo)[0, 1])
    results["Asian CV (Geometric)"] = res2

    return results, tau_euro, tau_geo

def plot_se_vs_N(N_list, se_series, title):
    plt.figure(figsize=(8, 6))
    for method, se_vals in se_series.items():
        plt.plot(N_list, se_vals, marker="o", label=method)
    plt.xscale("log")
    plt.yscale("log")
    plt.xlabel("Number of simulations")
    plt.ylabel("Standard error")
```

```
    plt.title(title)
    plt.legend()
    plt.grid(True, which="both", ls="--", linewidth=0.5)
    plt.show()


if __name__ == "__main__":
    S0, K, r, sigma, T = 100.0, 100.0, 0.05, 0.2, 1.0
    m_steps = 252
    N_list = [5_000, 10_000, 25_000, 50_000, 100_000]

    se_euro = {"Std MC": [], "Antithetic": [], "CV": []}
    se_asian = {"Std MC": [], "Antithetic": [], "CV (European)": [], "CV (Geometric)": []}

    final_results = None
    final_tau_euro = None
    final_tau_geo = None

    for N in N_list:
        results, tau_euro, tau_geo = run_experiment(S0, K, r, sigma, T, N, m_steps, seed=12345 +
N)

        se_euro["Std MC"].append(results["Euro Std MC"]["se"])
        se_euro["Antithetic"].append(results["Euro Antithetic"]["se"])
        se_euro["CV"].append(results["Euro CV"]["se"])

        se_asian["Std MC"].append(results["Asian Std MC"]["se"])
```

```
        se_asian["Antithetic"].append(results["Asian Antithetic"]["se"])
        se_asian["CV (European)"].append(results["Asian CV (European)"]["se"])
        se_asian["CV (Geometric)"].append(results["Asian CV (Geometric)"]["se"])

        if N == N_list[-1]:
            final_results = results
            final_tau_euro = tau_euro
            final_tau_geo = tau_geo

    N_final = N_list[-1]
    print(f"\nBlack–Scholes European Call Price: {final_tau_euro:.6f}")
    print(f"Geometric Asian Call:       {final_tau_geo:.6f}")
    print(f"Results at N = {N_final:,}\n")

    print(f"{'Method':<52} {'Price':>12} {'Var(payoff)':>16} {'SE':>14}")
    print("-" * 94)

    for key in ["Euro Std MC", "Euro Antithetic", "Euro CV"]:
        row = final_results[key]
        name = row["method"]
        if "CV" in key:
            name += f" (c*≈{row['c_star']:.3f}, ρ≈{row['rho']:.6f})"
        print(f"{name:<52} {row['price']:12.6f} {row['var_payoff']:16.4f} {row['se']:14.6f}")

    print("")
    for key in ["Asian Std MC", "Asian Antithetic", "Asian CV (European)", "Asian CV
(Geometric)"]:
```

```
    row = final_results[key]
    name = row["method"]
    if "CV" in key:
        name += f" (c*≈{row['c_star']:.3f}, ρ≈{row['rho']:.6f})"
    print(f"{name:<52} {row['price']:12.6f} {row['var_payoff']:16.4f} {row['se']:14.6f}")

plot_se_vs_N(N_list, se_euro, "European Call: Standard Error vs N")
plot_se_vs_N(N_list, se_asian, "Asian Call: Standard Error vs N")
```